Y. Mieda A comparison result in rigid geometry

Fujiwara space

$\big\downarrow$ pass everything is locally noetherian

adic space

§. Coherent Fujiwara space

CFS := cat of coherent formal schemes (quasi-cpt)

CFuj := cat of coherent Fujiwara space

$:= (\text{adm blow-up})^{-1}\ \mathrm{CFS}$

$$\begin{array}{ccc} & & \mathfrak{X}' \\ & & \downarrow{\scriptstyle \text{adm. blow-up}} \\ Y \underset{\text{closed}}{\subset} \mathfrak{X}^{red} & \hookrightarrow & \mathfrak{X} \end{array}$$

object : coherent formal schemes

morphism : $\mathrm{Hom}_{\mathrm{CFuj}}(\mathfrak{X}, \mathfrak{Y}) := \varinjlim_{\substack{\mathfrak{X}' \to \mathfrak{X} \\ \text{adm}}} \mathrm{Hom}_{\mathrm{CFS}}(\mathfrak{X}', \mathfrak{Y})$

$$\mathrm{CFS} \xrightarrow{\ rig\ } \mathrm{CFuj} \qquad \mathfrak{X}^{rig} \longrightarrow \mathfrak{Y}^{rig}$$
$$\mathfrak{X} \longmapsto \mathfrak{X}^{rig}$$

$$\begin{tikzcd} \mathfrak{X}' \arrow[d, "\text{adm}"'] \arrow[dr] & & \mathfrak{Y}' \arrow[dl, "\text{adm}"'] \arrow[dr] & \\ \mathfrak{X} & \mathfrak{Y} & & \mathfrak{Z} \end{tikzcd}$$

"<u>underlying top. space</u>"

$X \in \mathrm{CFuj} \Rightarrow \boxed{\langle X \rangle} := \varprojlim_{\mathfrak{X}' \to \mathfrak{X}} \mathfrak{X}'$ q-cpt Zariski top

$X = \mathfrak{X}^{rig}$

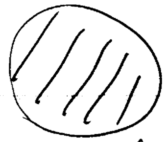
(<u>Zariski-Riemann sp.</u>)

$\mathfrak{X} = \mathrm{Spf}\, A$

V

<u>e.g.</u> V : CDVR , $K = \mathrm{Frac}\, V$

$\mathbb{D}^1_K := (\mathrm{Spf}\ \underline{V\langle T \rangle})^{rig}$



usual top (π-adic top)

$X \to Y$ in CFuj is <u>open immersion</u>

$\underset{\text{def}}{\Leftrightarrow}$ $\exists\ \mathcal{X} \hookrightarrow \mathcal{Y}$ open imm in CFS

s.t. $(\mathcal{X} \hookrightarrow \mathcal{Y})^{\mathrm{rig}} = (X \to Y)$

$X \hookrightarrow Y$ open imm $\underset{\text{Thm}}{\Rightarrow}$ $\langle X \rangle \hookrightarrow \langle Y \rangle$ open imm. of top spaces

underlying top sp

$\rightsquigarrow$ CFuj becomes a site.

Covering $(U_i \xhookrightarrow[\text{open imm}]{} X)$ s.t. $\langle X \rangle = \bigcup_i \langle U_i \rangle$

§. General Fujiwara spaces

---- defined by patching in $\mathrm{CFuj}^{\sim}$ (sheaves on CFuj)

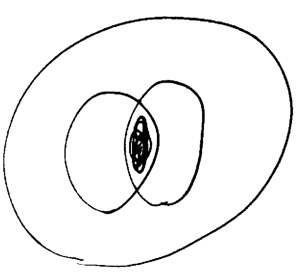
<u>Fact</u> $\mathrm{CFuj} \hookrightarrow \mathrm{CFuj}^{\sim}$

<u>Def</u> $X \in \mathrm{CFuj}^{\sim}$ is <u>Fujiwara space</u>

$\Leftrightarrow$ $\exists$ surj. $\coprod_i Y_i \longrightarrow X$ where $Y_i \in \mathrm{CFuj}$

s.t. $\forall i, j$, $Y_i \times_X Y_j \longrightarrow Y_i$ is

represented by an increasing sys. of open imm. in CFuj.

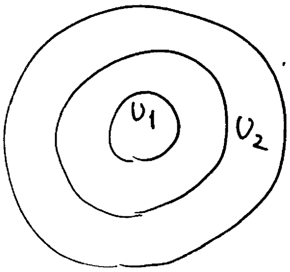


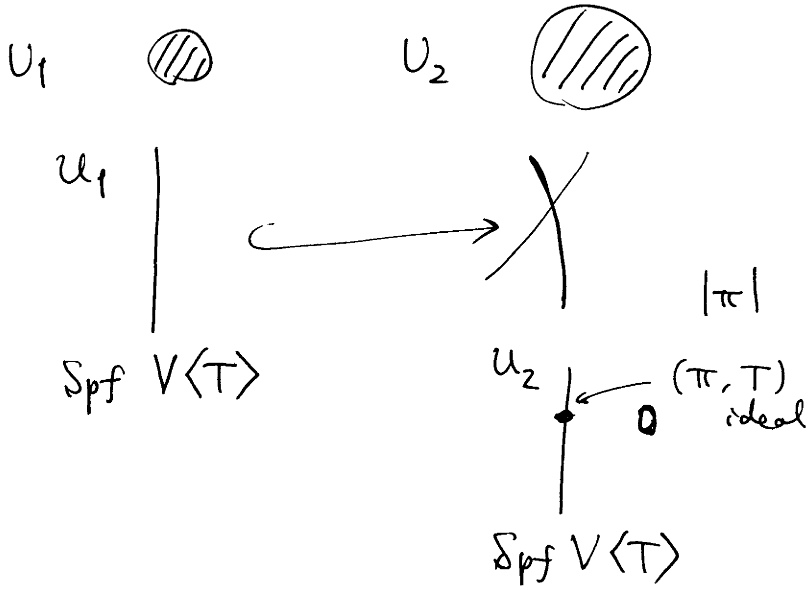
$\varinjlim U_k$ $\quad U_k \xhookrightarrow[\text{open}]{} Y_i$

$\boxed{\mathrm{Fuj}}$: cat of Fujiwara space

<u>Rem</u> $X \in \mathrm{Fuj} \Rightarrow \langle X \rangle$ by patching

e.g. $\mathbb{A}^1_K = \varinjlim \mathbb{D}^1_K$





§. Comparison functor

$\mathfrak{S} := (S, S_0, S^\circ)$

(in Fujiwara's work: S : affine)

$$\begin{cases} S : \text{q-cpt scheme} \\ S_0 \subset S \quad : \text{closed} \\ S^\circ := S \setminus S_0 \end{cases}$$

$$\mathcal{S} := S^{\wedge}\big|_{S_0}$$

$\rightsquigarrow$ (separated S°-sch. of finite type) $\longrightarrow$ (Fuj sp/ $\mathcal{S}^{rig}$)

$X \longmapsto X^{\mathfrak{S}\text{-an}}$

<u>Construction of $X^{\mathfrak{S}\text{-an}}$</u>

$\mathcal{C}_{X/S}$: cat of diagrams

$$\begin{tikzcd} X \arrow[r, hook, "\text{open dense}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \overline{X} \arrow[d, "\text{proper}"] \\ S^\circ \arrow[r, hook] & S \end{tikzcd}$$

For $\overline{X} \in \mathcal{C}_{X/S}$

$$U_{\overline{X}} := \overline{X} \setminus \overline{\left((\overline{X} \times_S S^\circ) \setminus X \right)}$$ closure

$U^\wedge_{\bar X}$ completion of $U_{\bar X}$ along $U_{\bar X} \times_S S_0$

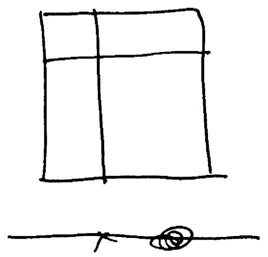
$$X^{\mathfrak{S}\text{-an}} := \varinjlim_{\bar X \in \mathcal{C}_{X/S}} \left(U^\wedge_{\bar X}\right)^{\mathrm{rig}} \qquad \left(\begin{array}{c}\text{transition}\\ = \text{open}\end{array}\right)$$

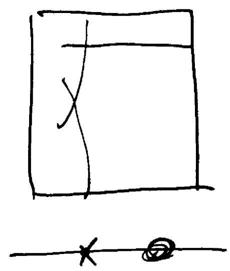
e.g. $X \longrightarrow S^0$ proper $\Rightarrow$ $X^{\mathfrak{S}\text{-an}} = (X^\wedge)^{\mathrm{rig}}$

e.g. $\mathfrak{S} = (\operatorname{Spec} V, \operatorname{Spec} k, \operatorname{Spec} K)$, $X = \mathbb{A}^1_K$

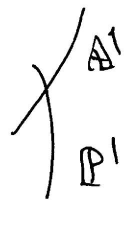
$\Rightarrow$ $X^{\mathfrak{S}\text{-an}} = \mathbb{A}^1_K$

- $\bar X = \mathbb{P}^1_V$ $\qquad$ $\bar X$ = blowup of $\mathbb{P}^1_V$ at ∞

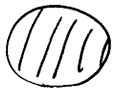




$U^\wedge_{\bar X} = \operatorname{Spf} V\langle x\rangle$ $\qquad$ $U^\wedge_{\bar X}$



$\rightsquigarrow$



$\rightsquigarrow$ disk with larger radius

<u>Another description</u>

Fix $X \hookrightarrow \bar X$, compactif. $/S$

$$\underset{\text{Raynaud-Gruson}}{\rightsquigarrow} \quad X^{\mathfrak{S}\text{-an}} = \varinjlim_{\substack{\bar X' \to \bar X \\ \text{blowup outside } X}} U^\wedge_{\bar X'} \qquad \rightsquigarrow \mathcal{C}_{X/S}$$

depends only on $\bar X$ (not on S)

<u>Prop</u> $\mathcal{S}'$: proper $/ \mathcal{S}$, $\quad \mathcal{S}_0' := \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0$, $\quad \mathcal{S}'^{\circ}$

$\mathfrak{S}' := (\mathcal{S}', \mathcal{S}_0', \mathcal{S}'^{\circ})$

X : sep of fin. type $/_{\mathcal{S}'}$ $\Rightarrow$ $X^{\mathfrak{S}'\text{-an}} = X^{\mathfrak{S}\text{-an}}$

Other properties we need:

<u>Prop</u> $\mathcal{S}' \hookrightarrow \mathcal{S}$: open, $\mathfrak{S}'$, $X/_{\mathcal{S}}$. $X' := X \times_{\mathcal{S}} \mathcal{S}'$

$\Rightarrow$ $(X')^{\mathfrak{S}'\text{-an}}$ is open in $X^{\mathfrak{S}\text{-an}}$ and

$$
\begin{tikzcd}
\langle X'^{\mathfrak{S}'\text{-an}} \rangle \arrow[r, hook] \arrow[ddd] \arrow[dddr, phantom, "\square"] & \langle X^{\mathfrak{S}\text{-an}} \rangle \arrow[d] \\
 & \langle \mathcal{S}^{\circ\, \mathfrak{S}\text{-an}} \rangle \arrow[d, equal] \\
 & \langle \mathcal{S}^{rig} \rangle \arrow[d, "sp_{\mathcal{S}}"] \\
\mathcal{S}_0' \arrow[r, hook, "\text{open}"'] & \mathcal{S}_0
\end{tikzcd}
$$

<u>Prop</u> $\mathcal{S} = \operatorname{Spec} A$, $\quad \mathcal{S}_0 = V(I)$

$\mathcal{S}' := \operatorname{Spec} \hat{A}$, $\mathfrak{S}'$, $X/_{\mathcal{S}}$, X'

$\Rightarrow$ $(X')^{\mathfrak{S}'\text{-an}} \cong X^{\mathfrak{S}\text{-an}}$

<u>Rem</u> By patching, we can extend $\mathfrak{S}$-an to
($\mathcal{S}^{\circ}$-scheme loc. of fin. type) $\lrcorner$

§ . Etale cohomology

local iso
$(\mathrm{Spf}\, B)^{\mathrm{rig}} \longrightarrow (\mathrm{Spf}\, A)^{\mathrm{rig}}$
(B/A : ~~etale~~ etale outside ...)

Etale morph, Etale site.

comparison functors

$$\varepsilon_{\mathfrak{S}} : (X^{\mathfrak{S}\text{-an}})_{\text{et}} \longrightarrow X_{\text{et}}$$

morphism of sites

$$\begin{pmatrix} Y^{\mathfrak{S}\text{-an}} \\ \downarrow \\ X^{\mathfrak{S}\text{-an}} \end{pmatrix} \longleftarrow\!\shortmid \begin{pmatrix} Y \\ \downarrow \\ X \end{pmatrix}$$

Main thm k : field, l : prime, $(l, \mathrm{char}(k)) = 1$

S is fin. type $/k$, S_0, S°, $\mathfrak{S}$

X, Y : S°-sch. of fin. type

$f : X \longrightarrow Y$ S°-mor

$\mathcal{F} \in D_c^b(X, \mathbb{Z}/l^n)$

$$\Rightarrow \boxed{\varepsilon_{\mathfrak{S}}^* Rf_* \mathcal{F} \xrightarrow{\cong} Rf_*^{\mathfrak{S}\text{-an}} \varepsilon_{\mathfrak{S}}^* \mathcal{F}}$$

$$\begin{tikzcd} X_{\text{et}}^{\mathfrak{S}\text{-an}} \arrow[r, "f^{\mathfrak{S}\text{-an}}"] \arrow[d, "\varepsilon_{\mathfrak{S}}"] \arrow[dr, phantom, "\circlearrowright"] & Y_{\text{et}}^{\mathfrak{S}\text{-an}} \arrow[d, "\varepsilon_{\mathfrak{S}}"] \\ X_{\text{et}} \arrow[r, "f"'] & Y_{\text{et}} \end{tikzcd}$$

[requires only proper base change]

Rem

- f : proper $\Rightarrow$ true for any $l, \mathfrak{S}$
- S : Spec of DVR, S_0 : closed pt of it
  (l is invertible on S°) open (by Fujiwara)
  $\Rightarrow$ true

Key of proof ... 3 properties of $\mathfrak{S}$-an I already gave

<u>lem 1</u> $\quad S = \bigcup_i S_i$ open cov.

$$\mathfrak{S}_i := (S_i,\ S_i \times_S S_0,\ S_i \times_S S^\circ)$$

Comparison true for $\mathfrak{S}_i \Rightarrow$ so is S.

($\because$ 2nd property)

<u>lem 2.</u> $\quad S' \longrightarrow S$ fin. type, $\mathfrak{S}'$

Comparison true for $\mathfrak{S} \Rightarrow$ so is $\mathfrak{S}'$

$$\left(\begin{array}{l} \because\ S' \longrightarrow S \text{ proper} \Rightarrow \text{by 1st property} \\ \quad S' \hookrightarrow S \text{ open} \Rightarrow \text{by 2nd} \quad " \end{array} \right)$$

<u>proof of main thm</u>

$\mathfrak{S} = (S, S_0, S^\circ)$

blow up along $\mathfrak{S}$ $\underset{\text{1st prop.}}{\Longrightarrow}$ WMA $\quad S_0 \subset S$ <u>loc.</u> defined by 1 eq.

<u>Lem 1</u>

$\Longrightarrow$ WMA $\quad S_0 \subset S$ defined by 1 eq.

<u>i.e.</u>

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{A}^1 \\ \cup & & \cup \\ S_0 & \longrightarrow & \{0\} \end{array}$$

<u>Lemma 2</u>

$\Longrightarrow$ WMA $\quad S = \mathbb{A}^1, \quad S_0 = \{0\}$

Fujiwara

$\underset{\text{3rd prop}}{\Longrightarrow}$ WMA $\quad S = \operatorname{Spec} k[[T]], \quad S_0 = V(T)$