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On Lefschetz trace formula for adic spaces

§1. Adic spaces

K : valuation field of height 1

$K = \overline{K}$ assume K is complete for simplicity

Building block of adic spaces
(locally of fin. type over K)

$$\mathrm{Spa}(K, \overset{\text{valuation ring of } K}{\underset{\uparrow}{V}})$$

Affinoid algebra

$$A = \underbrace{K\langle T_1, \cdots, T_n \rangle}_{\text{convergent power series}} / (f_1, \cdots, f_m)$$

as topological ring ⌟

$$\mathrm{Spa}(A) = \left\{ v : \text{continuous valuation of } A \quad v(a) \leq 1 \text{ for } a \in \underset{\substack{\uparrow \\ \text{power bounded} \\ \text{elements of } A}}{A^\circ} \right\}$$

$v : A \longrightarrow \Gamma \cup \{0\}$

multiplicative valuation

is continuous

$$\Leftrightarrow \forall \gamma \in \Gamma, \ \exists U \subset A : \text{open nbhd of } 0$$

$$a \in U \Rightarrow v(a) < \gamma$$

Consider the topology on Spa A

generated by $\{ v \in \mathrm{Spa}\, A \mid v(a) \leq v(b) \neq 0 \}$

$(a, b \in A)$

Then Spa A is spectral

* Spa A is NOT tot disconnected ⇒ no need to consider "admissible open" covering

$\exists$ structure sheaf on Spa A

$\rightsquigarrow$ affinoid adic spaces

$\rightsquigarrow$ adic spaces by patching

pseudo-adic space $:=$ a pair (X, S)

$$\begin{cases} X : \text{adic space} \\ S \subset X : \text{subset "not bad"} \end{cases}$$

$$\begin{array}{l} \mathbb{D}^1 = \operatorname{Spa} K\langle T\rangle \\ \cap \\ \mathbb{A}^1 \end{array} \qquad (\mathbb{D}^1(\varepsilon), \overline{\mathbb{D}}^1) \quad \varepsilon = |\pi^{-1}| \quad \pi \in \max$$

$$\partial\mathbb{D}^1 = \overline{\mathbb{D}}^1 \setminus \mathbb{D}^1 = \{v_0\}$$

$$v_0 : K\langle T\rangle \longrightarrow \underset{\uparrow \text{lex}}{(\mathbb{R}\times\mathbb{Z})} \cup \{0\}$$

$$\underset{\substack{\uparrow \\ \text{Gauss norm}}}{\sum a_n T^n} \longmapsto \max\{|a_n|, n\}$$

$\exists$ good theory of étale cohomology for pseudo-adic spaces

§ LTF for open adic spaces

Recall <u>the case of schemes</u>

U : smooth / alg. closed field

$U \underset{j}{\hookrightarrow} X$ compactification

$$\begin{tikzcd} \overline{f} : X \arrow[r] & X \\ \underset{\uparrow \text{proper}}{f} : U \arrow[u, phantom, "\cup"] \arrow[r] & U \arrow[u, phantom, "\cup"] \end{tikzcd} \qquad \operatorname{Fix} f : \text{isolated}$$

$$\begin{tikzcd}
\mathrm{Fix}\, f \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\delta_U"] \\
U \arrow[r, "f \times id"'] & U \underset{k}{\times} U
\end{tikzcd}$$

$\rightsquigarrow$ LVTF $\mathrm{Tr}\left(f^* ; H_c^*(U, \mathbb{Q}_\ell)\right)$

$$= \# \,\mathrm{Fix}\, f + \sum_{\substack{D \in \pi_0(\mathrm{Fix}\, f) \\ D \subset \partial U}} \mathrm{localterm}_D\left(\bar{f}, j_! \mathbb{Q}_\ell\right)$$

with multiplicity

In particular, if $\not\exists$ fixed pt on boundary, then LTF holds. —— ⊛

The case of adic spaces

⊛ is NOT true.

Obvious counterexample

$$U = \mathbb{D}^1 , \quad f : z \longmapsto z+1$$

$$\mathrm{Tr}\left(f^* ; H_c^*(\mathbb{D}^1, \mathbb{Q}_\ell)\right) = 1 \neq \# \,\mathrm{Fix}\, f = 0$$

$$\begin{tikzcd}
\mathbb{D}^1 \arrow[r, hook] \arrow[d, "f"'] & \mathbb{P}^1 \arrow[d, "\bar{f}"] \\
\mathbb{D}^1 \arrow[r, hook] & \mathbb{P}^1
\end{tikzcd}$$

$$\mathrm{Fix}\, \bar{f} = \underset{\substack{\not\cap \\ \partial U}}{\{\infty\}}$$

$\ast$ $\{v_0\} \in \partial U$

$f(v_0) = v_0$

Observation

$$\begin{tikzcd}
j : U \arrow[r, hook] \arrow[d, phantom, sloped, "="] & X \arrow[d, phantom, sloped, "="] \\
\mathbb{D}^1 & \overline{\mathbb{D}}^1
\end{tikzcd}$$

(proper)

$$(1\times j)_! \, (j\times 1)_* \, \mathbb{Q}_\ell \xrightarrow{\tau} (j\times 1)_* \, (1\times j)_! \, \mathbb{Q}_\ell$$

$$\begin{tikzcd}
X \times_S U \arrow[r, hook, "1\times j"] & X \times_S X \\
U \times_S U \arrow[u, hook, "j\times 1"] \arrow[r, hook, "1\times j"] & U \times_S X \arrow[u, hook, "j\times 1"']
\end{tikzcd}$$

is NOT isom!

* τ is isom for schemes

$$(1\times j)_! \, R(j\times 1)_* \, \mathbb{Q} \xrightarrow[\tau]{\cong} R(j\times 1)_* \, (1\times j)_! \, \mathbb{Q}_\ell$$

$\because$ LHS $= Rj_* \mathbb{Q}_\ell \overset{L}{\boxtimes} j_! \mathbb{Q}_\ell =$ RHS

by Künneth formula $\lrcorner$

* $\delta_X : X \to X \times_S X$

$$\begin{tikzcd}
X \arrow[d, phantom, sloped, "\supset"] \\
\partial U \arrow[d, phantom, sloped, "="] \\
\{v_0\}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_0 \arrow[r, mapsto] & v_1 \\
v_0' \arrow[u, squiggly] \arrow[r, mapsto] & v_1' \arrow[u, squiggly]
\end{tikzcd}$$

the unique max generalization of v_0

$$\left((1\times j)_! \, (j\times 1)_* \, \mathbb{Q}_\ell \right)_{v_1} = 0$$

$$\left((j\times 1)_* \, (1\times j)_! \, \mathbb{Q}_\ell \right)_{v_1} = \left((1\times j)_! \, \mathbb{Q}_\ell \right)_{v_1'} = \mathbb{Q}_\ell$$

* After taking $R\Gamma(X \times_S X, -)$, τ is isom.

(by quasi-compact / generalizing base change)

We may calculate

$\mathrm{Tr}(f^*; H_c^*(U, \mathbb{Q}_\ell))$ by the diagram

$d = \dim X$

$$\begin{tikzcd}
 & H^{2d}(X \times_S X, (1\times j)_!(j\times 1)_* \mathbb{Q}_\ell) \arrow[dl] \arrow[rr, "\tau", "\cong"'] \arrow[dd, "\delta_X^*"] & & H^{2d}(X \times_S X, (j\times 1)_*(1\times j)_! \mathbb{Q}_\ell) \arrow[dl] \\
\Gamma_{\bar f} & & \Gamma_{\bar f} & \mathrm{cl}(\Gamma_f) \arrow[u, phantom, "\in"] \arrow[ddl, bend left] \\
\Delta_U \cap \Gamma_f = \Delta_X \cap \Gamma_{\bar f} & H^{2d}(X, j_! \mathbb{Q}_\ell) \arrow[l] \arrow[d, equal] & & \\
 & H_c^{2d}(U, \mathbb{Q}_\ell) \arrow[r, "\mathrm{Tr}"'] & \mathbb{Q}_\ell \arrow[r, phantom, "\ni"] & \mathrm{Tr}(f^*)
\end{tikzcd}$$

If τ was isom, we had LTF by localizing everything.

§3 Results

① The simplest case

Thm X : pseudo-adic space proper $/ S = \mathrm{Spa}(K, V)$

$$\begin{array}{rcl} \bar f : X & \longrightarrow & X \\ \cup & & \cup \\ \underset{\text{proper}}{\textcircled{f}} : U & \longrightarrow & U \end{array}$$

$$\mathbb{D}^1 = (\mathbb{D}^1, \mathbb{D}^1)$$
$$j \downarrow$$
$$X = (\mathbb{A}^1, \underset{\mathbb{P}^1}{\bar{\mathbb{D}}^1})$$

$\boxed{\text{Assume Fix } f \text{ is proper} / S}$

If $\forall x \in \partial U$, $\exists W_x$: constructible closed subset of X

s.t. $x \in W_x$

$f(W_x) \cap W_x = \phi$

$\Rightarrow$ We have $\mathrm{Tr}\left(f^* ; H_c^*(U, \mathbb{Q}_\ell)\right) = \# \mathrm{Fix}\, f$

$\uparrow$ by l-adic cohom

$$\begin{tikzcd}
cl(\Gamma_f) \arrow[ddd, mapsto] & \in & H^{2d}_{\Gamma}(U \times U) \arrow[d, "\delta_U^*"] \\
 & & H^{2d}_{\Delta_U \cap \Gamma}(U) \arrow[dd, "\mathrm{Tr}"] \\
 & & \\
\# \mathrm{Fix}\, f & \in & \mathbb{Q}_\ell
\end{tikzcd}$$

[• comparison result
  • étale local

<u>Idea of proof</u>

closed

enlarge $\Gamma_{\bar f}$ to $\Gamma' \subset X \times_S X$

so that
$$\begin{cases} \bullet\ H^{2d}_{\Gamma'}(\tau) \text{ is isom} \\ \bullet\ \Delta_X \cap \Gamma' = \Delta_U \cap \Gamma_f \end{cases}$$

<u>Key</u> For $W_1, W_2 \subset X$ constructible closed subsets

$R\Gamma_{W_1 \times W_2}(\tau)$ is isom. ┘

Application

- LTF for contracting map

$$\underset{\text{proper}}{f : X \longrightarrow X} \qquad f : \text{contracting}$$

$$\Rightarrow \mathrm{Tr}(f^*) \text{ is naive}$$

② Separation of the contribution from boundary (in progress)

Assume $\dim X = 1$ for simplicity

$$\partial U = \underset{\substack{\text{finite}\\ \text{discrete}}}{\{x_i\}_{i \in I}}$$

$$I_{fix} := \{ i \in I \mid f(x_i) = x_i \}$$

Assume Fix f is proper. $\Gamma' := \Gamma_{\bar{f}} \cup \bigcup_{i \in I} \{x_i\} \times_S \{x_i\}$

$$\rightsquigarrow \quad \Delta_X \cap \Gamma' = (\Delta_U \cap \Gamma_f) \amalg \coprod_{i \in I_{fix}} \Delta_X \cap (\{x_i\} \times_S \{x_i\})$$

$\rightsquigarrow$ We may define "the contribution of x_i" for $i \in I_{fix}$ by:

$$\begin{tikzcd}
H^2_{\Gamma'}(X \times_S X, (1 \times j)_! (j \times 1)_* \mathbb{Q}_\ell) \arrow[d, "\delta_X^*"] \arrow[r, "\cong"] & H^2_{\Gamma'}(X \times_S X, (j \times 1)_* (1 \times j)_! \mathbb{Q}_\ell) \\
H^2_{\Delta_U \cap \Gamma_f}(U, \mathbb{Q}_\ell) \oplus H^2_{\Delta_X \cap \{x_i\} \times_S \{x_i\}}(X, j_! \mathbb{Q}_\ell) \arrow[d] & cl(\Gamma_f) \arrow[u, phantom, sloped, "\in"] \arrow[d, mapsto] \\
H^2_{\Delta_X \cap \{x_i\} \times_S \{x_i\}}(X, j_! \mathbb{Q}_\ell) \arrow[r, "\mathrm{Tr}"] & \mathbb{Q}_\ell \ni loc(i)
\end{tikzcd}$$

<u>**Prop**</u>

$$\mathrm{Tr}\left(f^* \,;\, H_c^*(U, \mathbb{Q}_\ell)\right) = \# \mathrm{Fix}\, f + \sum_{i \in I_{\mathrm{fix}}} \mathrm{loc}(i)$$

Huber

$\downarrow$

swan conductor