Mini-workshop

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Introduction to Local Langlands Corresp. for $GL(n)$ & related topics

LLC $K/\mathbb{Q}_p$: fin, p : prime

$\mathcal{O}_K/\mathfrak{p}_K = k \cong \mathbb{F}_q$

$$K^{ur} := \bigcup_{p \nmid N} K(\mu_N)$$

$$\begin{array}{ccccccc} \varphi : \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow & \mathrm{Gal}(K^{ur}/K) & \xrightarrow[\text{Hensel}]{\sim} & \mathrm{Gal}(\overline{k}/k) & \cong & \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n \\ & \Psi & & \Psi & & \Psi \\ & \mathrm{Frob}_K & \longmapsto & (x \mapsto x^q)^{-1} & \mapsto & 1 \end{array}$$

$W_K := \varphi^{-1}(\mathbb{Z})$ (Weil gp)

$I_K := \mathrm{Ker}\,\varphi$ (inertia gp)

$\Omega := \mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$ ℓ : prime

Def Weil-Deligne rep's $(r, N, V)/\Omega$ of W_K : $\begin{cases} r : W_K \to GL(V) & (\dim_\Omega V < \infty) \\ N \in \mathrm{End}(V) \end{cases}$

s.t. $\begin{cases} \mathrm{Ker}(r|_{I_K}) \triangleleft I_K & : \text{fin. index} \\ r(\sigma) N = \chi(\sigma) N r(\sigma) & (\forall \sigma \in W_K) \end{cases}$ where $\begin{array}{rcl} \chi : W_K & \longrightarrow & \Omega^\times \\ \Psi & & \Psi \\ \sigma & \longmapsto & q^{-\varphi(\sigma)} \end{array}$ (unram. cyclo char)

$\Rightarrow$ $\boxed{N : \text{nilpotent}}$

Thm $\forall \ell \neq p$

$$\left\{ \begin{array}{c} n\text{-dim. WD-rep's}/\overline{\mathbb{Q}_\ell} \\ \text{of } W_K \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{cont. rep.} \\ W_K \to GL_n(\overline{\mathbb{Q}_\ell}) \end{array} \right\}$$

$$(r, N, V) \longmapsto \left(\sigma \mapsto \underset{\substack{\cap \\ GL(V)}}{r(\sigma) \cdot \exp(t(\sigma) N)} \right)$$

$t : I_K \twoheadrightarrow \mathbb{Z}_\ell$

Rem $\{\text{WD-rep}/\overline{\mathbb{Q}_\ell}\} \cong \{\text{WD-rep}/\mathbb{C}\}$

$\forall \ell$ w/. $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$

ex. $Sp_n := (r, N, \langle e_1, \cdots, e_n \rangle)$

$$\begin{cases} r(\sigma) e_i := \chi(\sigma)^{i-1} e_i \\ N e_i := e_{i+1} \quad (e_{n+1} = 0) \end{cases}$$

Prop If r : s.s. (we say (r, N, V) : Frob-s.s.)
semisimple

then $\exists n_i, r_i$ s.t.

$$(r, N) \cong \bigoplus_i Sp_{n_i} \otimes (r_i, 0, V_i)$$

$$(r_i : W_K \longrightarrow GL(V_i) \text{ irred.})$$

$\pi : GL_n(K) \longrightarrow GL(V)$

$V : \Omega$-vect. sp.

Def (π, V) : smooth $\Leftrightarrow$ $\forall v \in V$, $\exists U \subset GL_n(K)$ open compact

s.t. $v \in V^U := \{ v \in V \mid \pi(\sigma) v = v \ \forall \sigma \in U \}$

Th'm (LLC) $n \geqslant 1$

$$\left\{ \begin{array}{l} \text{Frob-s.s.} \\ n\text{-dim WD-rep.}/\Omega \\ \text{of } W_K \end{array} \right\} \xleftrightarrow[\text{bij}]{} \left\{ \begin{array}{l} \text{irred smooth rep.}/\Omega \\ \text{of } GL_n(K) \end{array} \right\}$$

$$r = (r, N, V) \longleftrightarrow \pi$$

satisfying certain properties.

ex. $\chi : W_K \to \mathbb{C}^\times \overset{\text{LCFT}}{\longleftrightarrow} \chi \circ \mathrm{Art}_K$

$n=1$ $\qquad \mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$

$$r \otimes \chi \longleftrightarrow \pi \otimes (\chi \circ \mathrm{Art}_K \circ \det)$$

$$r_1 \oplus r_2 \longleftrightarrow \pi_1 \boxplus \pi_2$$

$$\underset{\boxed{\text{irred}}}{r} \mapsto \mathrm{Sp}_n \otimes r \longleftrightarrow \underset{\boxed{\text{cuspidal}}}{\pi} \longmapsto \mathrm{St}_n(\pi)$$

$$\left\{ \begin{array}{l} r : \underline{\text{unram.}} \\ (r|_{I_K} = \mathrm{id},\ N=0) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \pi : \underline{\text{spherical}} \\ \underline{\text{i.e.}} \text{ gen. by } \underset{\text{1-dim}}{\pi^U} \\ U = GL_n(\mathcal{O}_K) \end{array} \right\}$$

$$\mathbb{C}[U \backslash GL_n(K) / U]$$

$$\underset{\text{Satake isom}}{\cong} \mathbb{C}[X_1^\pm, \dots, X_n^\pm]^{\mathfrak{S}_n} \quad \text{symm gp.}$$

eigenvalue of Frob_K $\downarrow$

$$(\mathbb{C}^\times)^n / \mathfrak{S}_n \leftarrow$$

eigenvalue of $X_1, \dots, X_n \curvearrowright V^U$
(Satake parameters)

Rem LLC should be characterized by

- $n=1$ $\quad \chi \longleftrightarrow \chi \circ \mathrm{Art}_K$
- $\oplus \longleftrightarrow \boxplus$
- $\mathrm{Sp}_n(r) \longleftrightarrow \mathrm{St}_n(\pi)$
- Induction $\longleftrightarrow$ Automorphic Induction
  (no local def'n yet)

$\boxed{\text{GLC}}$ $[L:\mathbb{Q}] < \infty$

$G_L := \mathrm{Gal}(\bar{L}/L)$ $v \mid p \Rightarrow [L_v : \mathbb{Q}_p] < \infty$

Ⓐ Galois rep'n

$$R : G_L \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell) \quad \text{irred. continuous}$$
$$\begin{matrix} G_L \\ \cup \\ G_{L_v} \end{matrix}$$

s.t. $R|_{G_{L_v}}$: $\begin{cases} \text{i) } \underline{\text{unram}} \text{ for almost all } v \\ \quad (\text{id. on } I_{L_v}) \\ \text{ii) } \underline{\text{de Rham}} \text{ at } v \mid p \end{cases}$

Ⓑ Cuspidal Autom. Rep'n

$\Pi = \bigotimes_v \Pi_v$ of $GL_n(\mathbb{A}_L)$ in $L^2\left(GL_n(L)\backslash GL_n(\mathbb{A}_L)\right)^{cusp}$

$\left[\Pi_v : \text{irred smooth rep. of } GL_n(L_v)\right]$

s.t. Π_v : algebraic for $v \mid \infty$

Conj (GLC)

$\exists$ bij $\{R\} \longleftrightarrow \{\Pi\}$

$\forall\, \iota : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$

s.t. $\iota(R|_{W_{L_v}}) \overset{LLC}{\longleftrightarrow} \Pi_v$ $(\forall v \nmid \ell)$

Rem

- $\begin{cases} R \text{ determined by } R|_{W_{L_v}} \text{ for a.a. } v \quad (\text{Chebotarev density}) \\ \Pi \text{ ———— } \Pi_v \text{ ———— } \quad (\text{Multiplicity One}) \end{cases}$

- GLC $\Rightarrow$ $\left\{\begin{matrix} \ell\text{-adic, irred.} \\ \text{Galois rep'n} \end{matrix}\right\}$ is indep. of ℓ

  but we don't have any ℓ-indep. notion for global Galois reps.

  (like WD-rep.)

$\boxed{\text{Shimura Varieties}}$

Known constructions of $\begin{cases} \text{GLC} \\ \text{LLC} \end{cases}$

$$\text{Ⓐ} \longleftrightarrow \text{Ⓑ}$$
Gal   Autom.

all make use of Sh. V.

proof of LLC by Harris-Taylor uses GLC in special cases

$\overset{c}{\curvearrowleft}$ cpx conj

F : CM-field $F = E.\underset{\text{tot real}}{F^+}$, $F^+ = F^c$

B : div. alg $/F$ $[F:F^+] = [E:\mathbb{Q}] = 2$

$\dim_F B = n^2$ $* \curvearrowright B$ involution $*|_F = c$

$\langle , \rangle : B \times B \longrightarrow \mathbb{Q}$ alt. pairing.

s.t. $\langle bx, y\rangle = \langle x, b^* y\rangle$

$G := \mathrm{Aut}_B(B, \langle , \rangle)$ unitary similitude group $/\mathbb{Q}$

$G_0 := \mathrm{Ker}(G \longrightarrow \mathbb{Q}^\times)$

$G_0(\mathbb{R}) \cong U(1, n-1) \times U(0,n) \times \cdots \times U(0,n)$

$\mathbb{A}^\infty := \hat{\mathbb{Z}} \otimes \mathbb{Q}$

$G(\mathbb{A}^\infty) \supset U$ small, open compact subgp (level)

$\Rightarrow X_U/F$: <u>Shimura var</u> $\cdots$ projective smooth of dim $\underline{n-1}$

moduli of Abel var. of dim $[F^+:\mathbb{Q}] \cdot n^2$ $\hat{\mathbb{Z}} \curvearrowright$ $B \otimes \mathbb{A}^\infty \curvearrowleft$

w/ $\begin{cases} \text{Pol} & \lambda : A \to A^\vee \\ \text{End} & i : B \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q} \\ \text{Level} & \eta : U\text{-orbit of } \mathrm{Isom}_{\substack{B\otimes\mathbb{A}^\infty \\ \langle , \rangle}}(B \otimes \mathbb{A}^\infty, VA) \end{cases}$

$$\left(\varprojlim_m A[m]\right) \otimes_{\mathbb{Z}} \mathbb{Q} \;\; \Vert$$

$$H(X) := \varinjlim_{U:\text{smaller}} H^{n-1}_{ét}(X_U \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell)$$

$$\circlearrowright \qquad \circlearrowright$$

$$G(\mathbb{A}^\infty) \times G_F$$

Thm

$$H(X) \supset \bigoplus_{\pi} \pi \otimes R_\ell(\pi)$$

$\pi = \bigotimes_p \pi_p$: cusp. autom. rep. of $G(\mathbb{A}^\infty)$

$R_\ell(\pi)$: n-dim irred cont rep.

$$G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

s.t. Base change of π to $GL_n/F \underset{GLC}{\longleftrightarrow} R_\ell(\pi)$

$v | p$ $\quad B_v \cong M_n(F_v)$

p splits in $E \Rightarrow G(\mathbb{Q}_p) \cong \underbrace{GL_n(F)}_{\cup} (\times \cdots)$

$$U = U_p \times U^p$$

$$H^{n-1}_{ét}(X_U \otimes_F \overline{F}_v, \overline{\mathbb{Q}}_\ell)$$

$$\circlearrowright$$

$$G_{F_v} \times \overline{\mathbb{Q}}_\ell [U_p \backslash GL_n(F_v) / U_p]$$

Hecke alg. at v

$$\pi_p^{U_p} \otimes R_\ell(\pi)|_{G_{F_v}}$$

LLC

$$H^{n-1}(X_U \otimes \overline{F}_v, \overline{\mathbb{Q}}_\ell)$$

$$\Uparrow$$

$$\mathbb{H}^{n-1}(\mathfrak{X}_U \otimes_{\mathcal{O}_{F_v}} \overline{k}_v, \underline{R\Psi \mathbb{Q}_\ell})$$

($\uparrow$ regular integral model of X_U over $\mathcal{O}_{F_v}$)

regular integral model

complete local ring = deformation space of 1-dim. formal $\mathcal{O}_K$-module
Spec R (w/ Drinfeld level str.)

stalk of $R\Psi\overline{\mathbb{Q}}_\ell$ = $\underbrace{\text{cohomology of } \operatorname{Spec}(R \otimes_{\mathcal{O}_{F_v}} \overline{\mathbb{F}}_r)}_{\text{where LLC is realized.}}$

Ψ

Further Topic

- Find a local proof of the fact that LLC is realized in Ψ (Non-abel. Lubin-Tate theory)

- Other sh. V.
  - remove cond "B : div. alg" $\Rightarrow$ Get GLC in greater generality (Shin)

    introduce endoscopy

  - consider other unitary sh. V.

    than $G_0(\mathbb{R}) \cong U(1, n-1) \times \underbrace{U(0,n) \times \cdots \times U(0,n)}_{\text{compact gp}}$

    proper sh. V.

    this can't happen when $F^+ = \mathbb{Q}$

    geometry of non-proper sh. V. (Lan)

  - more difficult integral models
    local models of deform. sp. for higher dim BT gps (Nicole)