Kazuya Kato

Log abelian varieties (I)

Log geometry

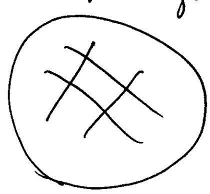
degenerate object $\xrightarrow{\text{(magic of log)}}$ nice object

Beauty and Beast

Beast $\xrightarrow{\text{(Love Of Girl)}}$ nice man

Degenerate abelian variety $\longrightarrow$ log abelian var.

no group structure



with group structure

- polarization
- level str.
  $\mathrm{Ker}(n : A \to A) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$
- End(A)

Joint work with T. Kajiwara

C. Nakayama

Part I preprint ← analytic theory

moduli in analytic theory/$\mathbb{C}$

II ← to apprea in Nagoya Math J.

III } in preparation ← algebraic

IV ← moduli

Tate curves

K : cdvf

E_q : Tate elliptic curve

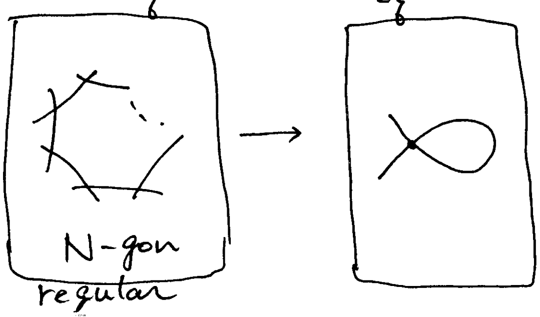
$q \in m_K - \{0\}$

$E_q(K) = K^\times / q^{\mathbb{Z}}$

Integral models

Assume $q = \pi_K^N$

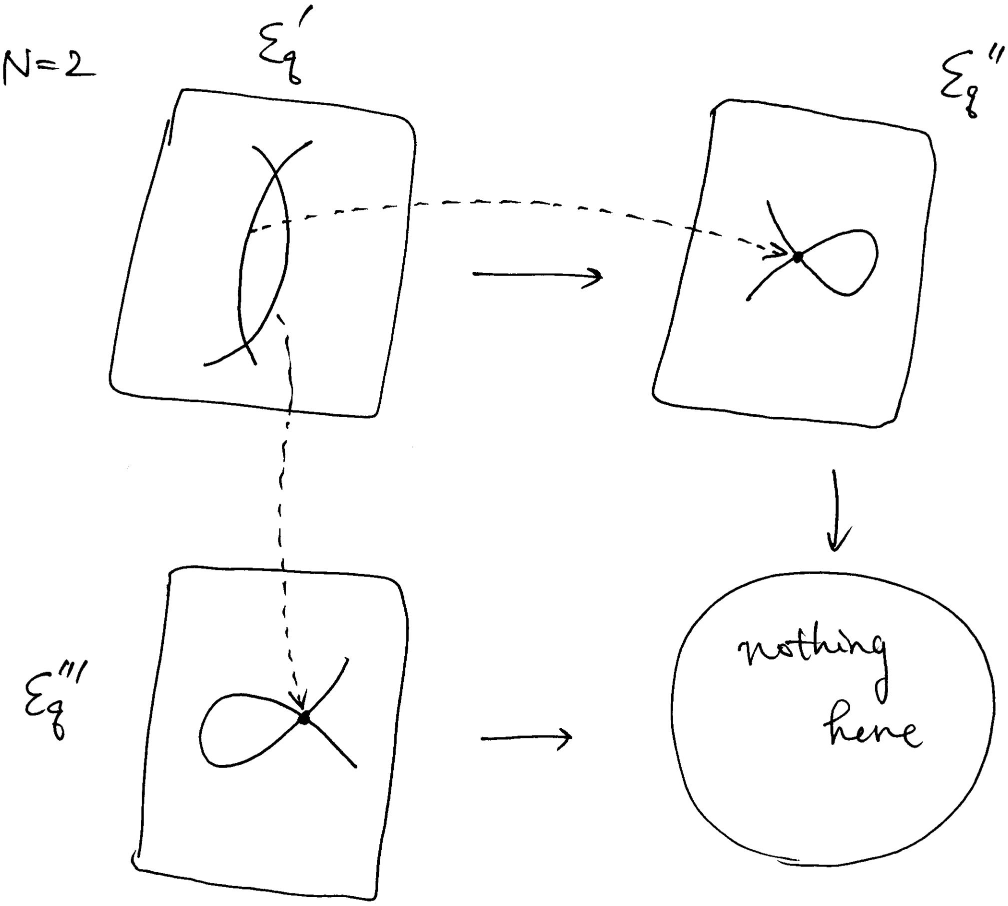
$\mathcal{E}'_q$ $\qquad$ $\mathcal{E}''_q$



N-gon

regular

N=2



In the log world

(in the cat. of fs-log schemes)

$$
\begin{tikzcd}
\mathrm{Mor}(\ ,\mathcal{E}_q') \arrow[r, "\subset"] \arrow[d, "\cap"'] & \mathrm{Mor}(\ ,\mathcal{E}_q') \arrow[d, phantom, "\bigcap"] \\
\mathrm{Mor}(\ ,\mathcal{E}_q''') \arrow[r, phantom, "\subset"] & \mathcal{E}_q
\end{tikzcd}
$$

log elliptic curve

<u>log str.</u>

a log str. on a scheme S
is a sheaf M of commutative monoids on $S_{\text{ét}}$
endowed with a hom

$$\alpha : M \longrightarrow \mathcal{O}_S$$

regard here as a multiplicative monoid

such that

$$\alpha^{-1}(\mathcal{O}_S^\times) \xrightarrow[\alpha]{\cong} \mathcal{O}_S^\times$$

$$\alpha^{-1} : \mathcal{O}_S^\times \hookrightarrow M$$

"fs log str" $\quad f = \underline{f}\text{ine}$, $\underline{f}\text{initely generated}$ $\quad s = \underline{s}\text{aturated}$

M : fs log str.

$$\Rightarrow \quad M \hookrightarrow M^{gp} = \{ fg^{-1} \mid f, g \in M \}$$

fs log scheme = scheme with an fs log str.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
M_X \arrow[d] & f^{-1}(M_Y) \arrow[l, dashed] \arrow[d] \\
\mathcal{O}_X & f^{-1}(\mathcal{O}_Y) \arrow[l]
\end{tikzcd}$$

$$S = \operatorname{Spec} \mathcal{O}_K$$

$$S_n = \operatorname{Spec}(\mathcal{O}_K / m_K^n)$$

$$M_S = \mathbb{G}_m \{ \pi_K^n \mid n \geq 0 \}$$

$$M_S \big|_{\operatorname{Spec}(K)} = \mathbb{G}_m$$

$$M_{S,\bar{s}} = \mathcal{O}_{S,\bar{s}}^\times \times \underset{\substack{\uparrow \\ \text{generated by } \pi_K}}{\mathbb{N}}$$

$s \in \operatorname{Spec} \mathcal{O}_K$
s, closed point

$$\boxed{f \mid g \underset{\text{def}}{\Leftrightarrow} \frac{g}{f} \in M \text{ in } M^{gp}}$$

$$\begin{tikzcd}
M_{S_n} = \mathbb{G}_m \times \underset{\substack{\uparrow \\ \text{generated by } \pi_K}}{\mathbb{N}} \subset M_{S_n}^{gp} \ni \pi_K^{-1} \arrow[d, "\text{injective}"] \\
\mathcal{O}_{S_n}
\end{tikzcd}$$

$$\mathbb{G}_{m,\log}(T) = \Gamma(T, M_T^{gp})$$

$$q \in \Gamma(S, M_S) = \mathcal{O}_K - \{0\}$$
$$\pi_K \in$$
$$q = \pi_K^N$$

(fs/S_n) = category of fs log schemes over S_n

$$\mathrm{Mor}(\ \ , \mathcal{E}_q' \times_S S_n) = \{ t \in \mathbb{G}_m^{\log} \mid \pi_K^k \mid t \mid \pi_K^{k+1} \ \exists k \in \mathbb{Z} \} / q^{\mathbb{Z}}$$

$$\cap$$

$$\mathrm{Mor}(\ \ , \mathcal{E}_q'' \times_S S_n) = \{ t \in \mathbb{G}_m^{\log} \mid q^k \mid t \mid q^{k+1} \ \exists k \in \mathbb{Z} \} / q^{\mathbb{Z}}$$

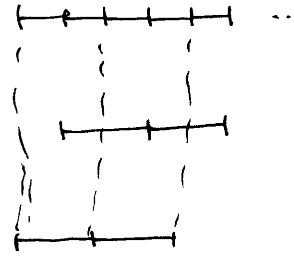
$N=2$

$$\mathrm{Mor}(\ \ ,\ \mathcal{E}_q''' \times_S S_n) = \left\{ t \in \mathbb{G}_m^{\log} \ \middle|\ \pi_K^{2k-1} \,|\, t \,|\, \pi_K^{2k+1} \ \exists k \in \mathbb{Z} \right\} \Big/ q^{\mathbb{Z}}$$

$\mathcal{E}_q'$

$\mathcal{E}_q''$

$\mathcal{E}_q'''$



on (fs/S_n)

$\mathcal{E}_q$ a sheaf of abelian groups on (fs/S)

$$\mathcal{E}_q \Big|_{(fs/S_n)} = \left\{ t \in \mathbb{G}_{m,\log} \ \middle|\ q^i \,|\, t \,|\, q^j \ \exists i,j \in \mathbb{Z} \right\} \Big/ q^{\mathbb{Z}}$$

this is a group sheaf

$$q^i \,|\, t \,|\, q^j,\ q^{i'} \,|\, t' \,|\, q^{j'} \Rightarrow q^{i+i'} \,|\, tt' \,|\, q^{j+j'}$$

$$\begin{array}{ccc} \mathcal{E}_q' & \subset & \mathcal{E}_q'' \\ \cap & & \cap \\ \mathcal{E}_q''' & \subset & \mathcal{E}_q \end{array} \quad \text{ghost}$$

2. Log abelian variety

fs log scheme

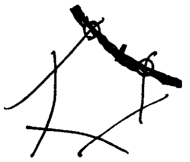
A <u>log abelian variety</u> over S is a sheaf of abelian groups such that

① $\exists\, G \subset A$ (subgroup sheaf)

G: semi-abelian scheme over the scheme S (in the usual sense)

$$\mathrm{Mor}_{(fs/S)}(S', G) = \mathrm{Mor}_{(Sch/S)}(S', G)$$

with inverse image of log of S — log is forgotten

$(G \subset E_q)$ 

and locally on S,

$\exists\, X, Y$: free $\mathbb{Z}$-modules of finite rank

and a pairing

$$\langle\,,\rangle : X \times Y \longrightarrow \mathbb{G}_{m,\log}/\mathbb{G}_m$$

such that

$$0 \longrightarrow G \longrightarrow A \longrightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y \longrightarrow 0$$

is exact

and $\forall s \in S$, $\exists \phi : Y_{\bar{s}} \longrightarrow X_{\bar{s}}$

satisfying $\quad \langle \phi(y), z\rangle = \langle \phi(z), y\rangle \quad \forall y, z \in Y$

② ——

③ ——

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} = \{ \varphi \in \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \mid$$

$\forall x \in X$, locally on S,
$\exists\, y, y' \in Y$ such that
$\langle x, y\rangle \mid \varphi(x) \mid \langle x, y'\rangle \}$

<u>Example</u> $\quad \mathcal{E}_q$ over $S = \mathrm{Spec}(\mathcal{O}_K/m_K^n)$

$G = \mathbb{G}_m$

$$X = Y = \mathbb{Z} \qquad X \times Y \longrightarrow \mathbb{G}_{m\log}/\mathbb{G}_m$$

$$(m, n) \longmapsto q^{mn}$$

$$\mathcal{H}om(X, \mathbb{G}_{m\log}/\mathbb{G}_m)^{(Y)} = \{ t \in \mathbb{G}_{m,\log}/\mathbb{G}_m \mid q^i \mid t \mid q^j \ \exists\, i, j \in Y \}$$

$$0 \to \mathbb{G}_m \to \mathcal{E}_q \to \left(\mathbb{G}_{m,\log}/\mathbb{G}_m\right)^{(Y)}\Big/ Y \to 0$$

$$\mathcal{E}_q = \left\{ t \in \mathbb{G}_{m,\log} \;\middle|\; q^i \mid t \mid q^j \;\; \exists i,j \right\} \Big/ q^{\mathbb{Z}}$$

② $A \xrightarrow{\text{diagonal}} A \times A$

is represented by finite morphisms

③ $\forall s \in S$, the pullback of A to $(fs/\bar{s})$ has the following shape.

$$A\Big|_{(fs/\bar{s})} = \mathrm{Coker}\left(Y \to G_{\log}^{(Y)}\right)$$

for some $Y \xrightarrow{h} G_{\log}$ $\quad$ T: torus, B: abelian variety

satisfying some condition $(*)$ $\quad$ $\boxed{\text{polarization condition}}$

$$\begin{array}{ccccccccc} 1 & \to & T & \to & G & \to & B & \to & 1 \\ & & \cap & & \cap & & \| & & \\ 1 & \to & T_{\log} & \to & G_{\log} & \to & B & \to & 1 \end{array}$$

$$T_{\log} = \mathcal{H}om(X, \mathbb{G}_{m,\log}) \cong \mathbb{G}_{m,\log}^r$$

$$\begin{array}{ccc} X & = & \mathcal{H}om(T, \mathbb{G}_m) \\ \wr\| & & \wr\| \\ \mathbb{Z}^r & & \mathbb{G}_m^r \end{array}$$

$$\left(\begin{array}{l} \text{Case of } \mathcal{E}_q\Big|_{\mathrm{Spec}(O_K/m_K)} \\ T = \mathbb{G}_m = G \\ T_{\log} = \mathbb{G}_{m\,\log} \end{array} \right)$$

$Y \xrightarrow{h} G_{\log}$ <u>log 1-motive</u>

$X \xrightarrow{h^*} G^*_{\log}$ dual log 1-motive

$$1 \to T^* \to C \to B^* \to 1$$

$$T = \mathcal{H}om(Y, \mathbb{G}_m)$$

(*) $\exists\ p_1 : Y \to X$

$p_0 : G_{\log} \to G^*_{\log}$

such that $p_0 : B \to B^*$ is a polarization of B^*

$p_0 : T \to T^*$ induces p_{-1}

$$G^{(Y)}_{\log} = \text{the inverse image of } \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$$

$$\begin{tikzcd}
G_{\log}/G = T_{\log}/T \arrow[r, phantom, "\cong"] & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \\
Y \arrow[u] & Y \arrow[u]
\end{tikzcd}$$

$q \mid t \mid q^2$ ... cone decomposition $\longrightarrow$ representable objects

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] & A \arrow[r] \arrow[dr, phantom, "\square"] & \text{strange thing} \arrow[r] & 1 \\
 & & P \arrow[u, phantom, "\cup"] \arrow[r] & I \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

これは

cone decomp.

Log abelian varieties (II)

Analytic moduli

1. $\Gamma \backslash D$

V : $\mathbb{Q}$-vector space, $\dim = 2g$

$\psi : V \times V \longrightarrow \mathbb{Q}$ nondegenerate anti-symmetric pairing

$R \subset \mathrm{End}_{\mathbb{Q}}(V)$ semi simple $\mathbb{Q}$-alg. such that for each $a \in R$ $\exists a^* \in R$ for which $\psi(ax, y) = \psi(x, a^* y)$ $\forall x, y \in V$

$$D = \left\{ F \;\middle|\; \begin{array}{l} R_{\mathbb{C}}\text{-submodule of } V_{\mathbb{C}} \text{ such that} \\ \psi(F, F) = 0 \\ V_{\mathbb{C}} = F \oplus \overline{F} \\ F \times F \longrightarrow \mathbb{C} : (x, y) \mapsto \psi(x, \overline{y}) \\ \quad \text{is positive definite} \end{array} \right\}$$

If $R = \mathbb{Q}$, then $D \cong \mathfrak{h}_g$

Fix $V_{\mathbb{Z}}$: $\mathbb{Z}$-submodule of V such that

$\mathbb{Q} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} = V$

$\psi(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subset \mathbb{Z}$

Fix $\Gamma \subset G_{\mathbb{Q}} = \mathrm{Aut}_R(V, \psi)$ such that
subgroup

- $\gamma V_{\mathbb{Z}} = V_{\mathbb{Z}}$ $\forall \gamma \in \Gamma$
- Γ is <u>neat</u> $\left\{ \begin{array}{l} \forall \gamma \in \Gamma \text{ the subgroup of } \mathbb{C}^\times \\ \text{gen by. eigenvalues of } \gamma \text{ is} \\ \text{torsion free} \end{array} \right.$

$$\Phi_\Gamma : (\mathrm{an}) \longrightarrow (\mathrm{Sets})$$

$(\mathrm{an}) = \{\text{analytic space}/\mathbb{C}\}$

$A \to S$

$\Phi_\Gamma(S) = \{(A, \iota, p, k) \mid$ A : abelian variety over S

$\iota : R \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ ring hom

$p : A \longrightarrow A^*$ polarization

$k \in \Gamma(S, \Gamma \backslash \mathrm{Isom}_{R,\psi}(\mathcal{H}_1(A, \mathbb{Z}), V_\mathbb{Z}))\}$

k : level str.

$\mathcal{H}_1(A, \mathbb{Z})$: sheaf on (an/S)

$\mathrm{Isom}_{R,\psi}$: after $\otimes \mathbb{Q}$, compatible with the action of R, sends $p : \mathcal{H}_1(A, \mathbb{Z}) \times \mathcal{H}_1(A, \mathbb{Z}) \to \mathbb{Z}$ to ψ

Well known:

$$\Phi_\Gamma \cong \mathrm{Mor}(\ \ , \Gamma\backslash D) \quad \text{on } (\mathrm{an})$$

2. Analytic log abelian varieties

S : fs log analytic space

$(\mathrm{fsan}/S) = \{\text{fs log analytic space over } S\}$

<u>Log abelian variety</u> (of dimension g) over S

is a sheaf A of abelian groups on (fsan/S)

which is locally on S,

$\exists X, Y$ free $\mathbb{Z}$-modules of rank g

$\exists X \times Y \longrightarrow \mathbb{G}_{m,\log}$ satisfying certain conditions

such that $A = \mathrm{Coker}(Y \to \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)})$

$\mathcal{H}_1^{\log}(A, \mathbb{Z})$

$$S^{\log} = \left\{ (s, h) \;\middle|\; s \in S,\ h : M^{gp}_{S,s} \xrightarrow{\text{hom}} S^1 = \{ z \in \mathbb{C}^\times \mid |z| = 1 \} \right\}$$

such that

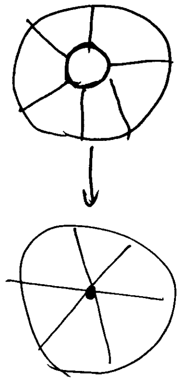
$$h(u) = \frac{u(s)}{|u(s)|} \quad \forall u \in \mathcal{O}^\times_{S,s}$$

If $S = \Delta = \{ q \in \mathbb{C} \mid |q| < 1 \}$ with log str.

$$M = \{ f \in \mathcal{O}_\Delta \mid f \text{ is invertible outside } 0 \}$$

then

$$\begin{array}{c} S^{\log} \\ \downarrow \\ S \end{array}$$



$$M_{\Delta,0} = \mathcal{O}^\times_{\Delta,0} \times \mathbb{N}$$

$$(\text{fsan}/S)^{\log} = \left\{ (S', U) \;\middle|\; U \underset{\text{open}}{\subset} (S')^{\log} \right\}$$

$\tau \downarrow$ topology $\{ (S'_i, U'_i) \}_i$ is covering of (S, U)

if $S'_i \underset{\text{open}}{\subset} S$, $\bigcup_i U'_i = U$

(fsan/S)

A <u>log abelian variety</u> $/S$

- $\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}) \underset{\text{locally}}{\cong} \mathbb{Z}^{2g}$

$$\mathcal{H}_1^{\log}(A, \mathbb{Z}) := \mathcal{H}om\left(\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}), \mathbb{Z} \right)$$

3. Log moduli functors

$$\Phi_\Gamma : (\mathrm{fsan}) \xrightarrow[\text{log}]{\text{forget}} (\mathrm{an}) \xrightarrow{\Phi_\Gamma} (\mathrm{Sets})$$

$(\mathrm{fsan}) = \{\text{fs log analytic space}\}$

$$\Phi_\Gamma \subset \overline{\Phi}_\Gamma : (\mathrm{fsan}) \longrightarrow (\mathrm{Sets})$$

$$\overline{\Phi}_\Gamma(S) = \left\{ (A, \iota, p, k) \;\middle|\; \begin{array}{l} A : \text{log abelian var} / S \\ \iota : R \xrightarrow{\text{hom}} \mathrm{End}(A) \otimes \mathbb{Q} \\ p : A \longrightarrow A^* \text{ polarization} \\ k \in \Gamma(S^{\log}, \Gamma \backslash \mathrm{Isom}_{R,\psi}(\mathcal{H}_1^{\log}(A, \mathbb{Z}), V_{\mathbb{Z}})) \end{array} \right\} / \cong$$

level str.

Σ : <u>cone decomposition</u> of

$$\{ N : V \to V \mid N \text{ is } R\text{-linear}, \ \psi(x, Ny) + \psi(Nx, y) = 0 \ \forall x, y \in V \}$$

$$\Phi_\Gamma \subset \overline{\Phi}_{\Gamma,\Sigma} \subset \overline{\Phi}_\Gamma$$

$$\overline{\Phi}_{\Gamma,\Sigma}(S) = \{ (A, \iota, p, k) \in \overline{\Phi}_\Gamma(S) \mid$$

$\forall s \in S$, $\exists \underset{\text{cone}}{\sigma} \in \Sigma$ such that

if $t \in \tau^{-1}(s)$, $\gamma \in \pi_1(\tau^{-1}(s), t)$

$\boxed{\tau : S^{\log} \longrightarrow S}$

$\tilde{k}_t : \mathcal{H}_1^{\log}(A, \mathbb{Z})_t \xrightarrow{\cong} V_{\mathbb{Z}}$ (representative of k_t)

$\log(\gamma) : \mathcal{H}_1^{\log}(A, \mathbb{Z})_t \longrightarrow \mathcal{H}_1^{\log}(A, \mathbb{Z})_t$ is in σ, via $\tilde{k}_t$ $\}$

$$\begin{array}{ccc} V_{\mathbb{Z}} & \overset{\tilde{k}_t}{\cong} & \mathcal{H}_1^{\log}(A, \mathbb{Z})_t \\ \downarrow & & \downarrow \log(\gamma) \\ V_{\mathbb{Z}} & \cong & \mathcal{H}_1^{\log}(A, \mathbb{Z}) \end{array}$$

$\overline{\Phi}_{\Gamma,\Sigma}$ = part of $\overline{\Phi}_{\Gamma}$ consisting of A whose local monodromy is in the direction of Σ

4. <u>Thm</u>

$$\overline{\Phi}_{\Gamma,\Sigma} \cong \mathrm{Mor}\left(\quad , (\Gamma\backslash D)_\Sigma\right) \quad \text{on (fsan)}$$

$$\begin{array}{c} (\Gamma\backslash D)_\Sigma \\ \cup \ \text{open} \\ \Gamma\backslash D \end{array}$$

partial toroidal compactification of $\Gamma\backslash D$ with log $\{f \in \mathcal{O} \mid f \text{ is invertible on } \Gamma\backslash D\}$

5. Rough Proof

Fix X, Y $(\cong \mathbb{Z}^g)$ and $\phi : Y \to X$

Locally,

$$\{X \times Y \xrightarrow{\text{pairing}} \mathbb{G}_m\} \subset \{X \times Y \xrightarrow{\text{pairing}} \mathbb{G}_{m,\log}\}$$

$\cup$ moduli of abelian varieties $\subset \bullet \subset$ $\boxed{\text{toric var}}$ $\supset$ moduli of log abelian variety $\cup$

defined by Σ

[moduli of log abel var with local monodromy in the direction of Σ]

toric variety $\mathrm{Spec}(\mathbb{C}[S])_{an}$

S = integral cone (= integral pts in a rational cone)

$$\begin{array}{ccccc} \mathrm{Spec}(\mathbb{C}[S^{gp}])_{an} & \underset{\text{open}}{\subset} & \mathrm{Spec}(\mathbb{C}[S])_{an} & & \mathbb{G}_{m,\log}^r \\ \| & & \| & & \wr\| \\ \mathcal{H}om(S^{gp}, \mathbb{G}_m) & \subset & \mathcal{H}om(S, M) & \subset & \mathcal{H}om(S^{gp}, \mathbb{G}_{m,\log}) \\ \| & & & & \| \\ \mathbb{G}_m^r & & M(S) = \Gamma(S, M) & & M^{gp} \end{array}$$

$S^{gp} \cong \mathbb{Z}^r$

6. Satake-Baily-Borel compactification $(\Gamma\backslash D)_{SBB}$

$\Gamma\backslash D \subset_{\text{open}}$

log str is

$= \{ f \in \mathcal{O} \mid f \text{ is invertible on } \Gamma\backslash D \}$

(usually $= \mathcal{O}^\times$)

$$\begin{tikzcd}
(\Gamma\backslash D)_\Sigma \arrow[r] \arrow[d, phantom, "\parallel\wr"] & (\Gamma\backslash D)_{SBB} \\
\overline{\Phi}_{\Gamma,\Sigma} \arrow[r, phantom, "\subset"] & \overline{\Phi}_\Gamma \arrow[u]
\end{tikzcd}$$

<u>Vague thm</u>

$(\Gamma\backslash D)_{SBB}$ is the coarse moduli space of $\overline{\Phi}_\Gamma$

<u>Thm</u> If $R = \mathbb{Q}$,

$(\Gamma\backslash D)_{SBB}$ is <u>universal</u> among Hausdorff fs log analytic spaces P endowed with $\overline{\Phi}_\Gamma \to \mathrm{Mor}(\ ,P)$

$$\{\text{log abel var}\} \cong \left\{ \begin{matrix} \text{log Hodge str} \\ \text{(with condition)} \end{matrix} \right\}$$

cat. equiv.

nilpotent orbit thm (W. Schmid)

fs log p-div. gps