Atsushi Ichino, Trilinear forms and the central values of triple product L-functions

<u>Def</u> (Periods of autom. forms)

F: number field

$G \supset H$: ~~reductive~~ semisimple gps $/F$

ϕ : autom. form on $G(\mathbb{A})$

$$\mathcal{P}(\phi) := \int_{H(F)\backslash H(\mathbb{A})} \phi(h)\, dh \quad : \text{period of } \phi \text{ along } H$$

$\in \mathbb{C}$ if it converges

Sometimes, it is related to special values of autom. L-functions

Today, we will consider $(GL_2)^3 \supset GL_2$

$i = 1, 2, 3$

π_i : irred unitary cuspidal autom rep of $GL_2(\mathbb{A})$

ω_{π_i} : central char of π_i

<u>Assume</u> $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = 1$

Write $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$

$$\left(\begin{array}{c} G = PGL_2 \\ \cup \\ H = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} / \mathbb{G}_m \end{array} \right) \rightsquigarrow \int_0^{i\infty} f(\cdot)$$

$L(s, \Pi, r)$: triple product L-function

$r : GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$

<u>Jacquet's conjecture</u> (proved by Harris–Kudla)

$L(\frac{1}{2}, \Pi, r) \neq 0$

$\Leftrightarrow$ $\exists D$: quat. alg $/F$

$\exists \phi_i \in \pi_i^D \xleftrightarrow{JL} \pi_i \quad (i = 1, 2, 3)$

$$\mathcal{P}(\phi_1, \phi_2, \phi_3) := \int_{\mathbb{A}^\times D^\times(F) \backslash D^\times(\mathbb{A})} \phi_1(x) \cdot \phi_2(x) \cdot \phi_3(x)\, d^\times x \neq 0$$

Rem

(i) Such a D is uniquely determined by the following conditions :

$$D_v : \text{split} \iff \varepsilon_v(\tfrac{1}{2}, \Pi_v, r) = 1$$

(ii) Harris-Kudla express $|P(\phi_1, \phi_2, \phi_3)|^2$ using $L(\tfrac{1}{2}, \Pi, r)$ and some local zeta integrals.

(it's not easy to compute)

Want a more precise formula :

- Gross-Kudla $F = \mathbb{Q}$, D_∞ = division, ...
- Böcherer-Schulze-Pillot $F = \mathbb{Q}$, D_∞ : div, ...
- Watson $F = \mathbb{Q}$, D_∞ : split, $\underbrace{\quad\cdots\cdots\quad}_{\text{lots of condition}}$

Want a general version using rep. theory

D : quat. alg. $/F$

$$H := D^\times / F^\times \xhookrightarrow[\text{diag.}]{} G := (D^\times \times D^\times \times D^\times)/F^\times$$

π_i $(i = 1, 2, 3)$ cusp. rep. of $GL_2(\mathbb{A})$ as before

Assume π_i^D : autom. rep. of $D^\times(\mathbb{A})$ $\overset{JL}{\longleftrightarrow} \pi_i$

$$\Pi^D := \pi_1^D \otimes \pi_2^D \otimes \pi_3^D \quad : \text{autom. repr of } G(\mathbb{A})$$

$$\begin{aligned} P : \Pi^D &\longrightarrow \mathbb{C} \quad : H(\mathbb{A})\text{-invariant funct.} \\ \phi = \phi_1 \otimes \phi_2 \otimes \phi_3 &\longmapsto \int_{H(F)\backslash H(\mathbb{A})} \phi_1(x)\, \phi_2(x)\, \phi_3(x)\, d^\times x \end{aligned}$$

where $d^\times x$: Tamagawa measure

$$\begin{aligned} I : \Pi^D \otimes \tilde{\Pi}^D &\longrightarrow \mathbb{C} \\ \phi \otimes \phi' &\longmapsto P(\phi) \cdot P(\phi') \end{aligned}$$

where $\tilde{\Pi}^D$: contragredient of Π^D

$H(\mathbb{A}) \times H(\mathbb{A})$-invariant.

$I \in \mathrm{Hom}_{H(\mathbb{A}) \times H(\mathbb{A})}(\Pi^D \otimes \tilde{\Pi}^D, \mathbb{C})$

Uniqueness of trilinear forms (due to D. Prasad)

$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{H_v}(\Pi_v^D, \mathbb{C}) \leq 1$ for $\forall v$

$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{H_v \times H_v}(\Pi_v^D \boxtimes \tilde{\Pi}_v^D, \mathbb{C}) \leq 1$ for $\forall v$

$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{H(\mathbb{A}) \times H(\mathbb{A})}(\Pi^D \boxtimes \tilde{\Pi}^D, \mathbb{C}) \leq 1$

<u>Th</u> As an element of $\mathrm{Hom}_{H(\mathbb{A}) \times H(\mathbb{A})}(\Pi^D \boxtimes \tilde{\Pi}^D, \mathbb{C})$

$$I = \frac{1}{8} \cdot \zeta_F(2)^2 \cdot \frac{L(\frac{1}{2}, \Pi, r)}{L(1, \Pi, \mathrm{Ad})} \cdot \prod_v I_v \qquad s \leftrightarrow 1-s$$

(completed zeta & L-funct)

where $L(s, \Pi, r)$: triple product L-fn., deg 8

$L(s, \Pi, \mathrm{Ad}) = \prod_{i=1}^{3} L(s, \pi_i, \mathrm{Ad})$ deg 9

$I_v \in \mathrm{Hom}_{H_v \times H_v}(\Pi_v^D \boxtimes \tilde{\Pi}_v^D, \mathbb{C})$ is defined as follows.

$\langle , \rangle = \prod_v \langle , \rangle_v$: canonical pairing for $\Pi^D \times \tilde{\Pi}^D$

$d^\times x = \prod_v d^\times x_v$: Tamagawa measure on $H(\mathbb{A})$

$$I_v(\phi_v \otimes \phi_v') := \int_{H_v} \langle \Pi_v^D(x_v)\phi_v, \phi_v' \rangle_v \, d^\times x_v \times \left(\zeta_v(2)^2 \cdot \frac{L_v(\frac{1}{2}, \Pi_v, r)}{L_v(1, \Pi_v, \mathrm{Ad})} \right)^{-1}$$

$\lrcorner$

Rem

(i) Kim-Shahidi estimate $\Rightarrow$ I_v : abs. conv.

(ii) unram. computation $\Rightarrow$ $\prod_v I_v$: well-def.

(iii) compatible with a refined Gross-Prasad conj.
(w/. T. Ikeda) i.e. $\exists$ conjectural formula for periods of autom. forms on

$$G = SO(n+1) \times SO(n) \supset H = SO(n)$$

$\underline{n=3}$

$$SO(3) = PGL(2)$$
$$SO(4) = GL(2) \times GL(2)/\mathbb{G}_m$$

(the case $n=3$ is now a theorem)

(iv) I_v is not really local.

Take arbitrary $\langle , \rangle_v$, $d^\times x_v$

$\exists c > 0$ s.t. $d^\times x = c \cdot \prod_v d^\times x_v$ Tamagawa

Using these, we can define I_v

For $\phi = \bigotimes_v \phi_v \in \Pi^D$, $\phi' = \bigotimes_v \phi_v' \in \tilde{\Pi}^D$ s.t. $\langle \phi, \phi' \rangle \neq 0$

$$\frac{I(\phi \otimes \phi')}{\langle \phi, \phi' \rangle} = \frac{C}{8} \cdot \zeta(2)^2 \cdot \frac{L(\frac{1}{2}, \Pi, r)}{L(1, \Pi, Ad)} \cdot \prod_v \frac{I_v(\phi_v \otimes \phi_v')}{\langle \phi_v, \phi_v' \rangle}$$

Example $F = \mathbb{Q}$, $D = M_2(\mathbb{Q})$ so $\Pi^D = \Pi$

$k_1, k_2, k_3 \in \mathbb{N}$ s.t. $k_1 + k_2 = k_3$

$N_1, N_2, N_3 \in \mathbb{N}$: square free, $N := N_1 \cdot N_2 \cdot N_3$

$f_i \in S_{k_i}(\Gamma_0(N_i))$: primitive form

$\varepsilon_{i,p}$: eigenvalue of f_i for Atkin-Lehner invol at $p \mid N_i$

$$\langle f_1 f_2 , f_3 \rangle := \mathrm{vol}(\Gamma_0(N)\backslash \mathfrak{h})^{-1} \cdot \int_{\Gamma_0(N)\backslash \mathfrak{h}} f_1(z)\cdot f_2(z)\cdot \overline{f_3(z)}\, \mathrm{Im}(z)^{2k_3-2}\, dz$$

<u>Change notation</u>

write $\begin{cases} \xi(s) = \pi^{-s/2}\cdot \Gamma(s/2)\cdot \underset{\substack{\uparrow \\ \text{Riemann zeta fn.}}}{\zeta(s)} \\ \Lambda(s, \cdots) \text{ for completed L-funct.} \end{cases}$

<u>Cor</u>

$$\frac{|\langle f_1 f_2 , f_3\rangle|^2}{\prod_{i=1}^{3} \langle f_i , f_i \rangle} = \frac{1}{2}\cdot \xi(2)\cdot \frac{\Lambda(\frac{1}{2}, f_1\times f_2\times f_3)}{\prod_{i=1}^{3}\Lambda(1, f_i, \mathrm{Ad})}\cdot \prod_{p|N} C_p$$

where
$$C_p = \begin{cases} 0 & \text{if } p|N,\ p^2\nmid N \quad \cdots\cdots (\star) \\ p^{-1} & \text{if } p^2|N,\ p^3\nmid N \\ 2p^{-1}(1+p^{-1}) & \text{if } p^3|N,\ \varepsilon_{1,p}\cdot\varepsilon_{2,p}\cdot\varepsilon_{3,p} = 1 \\ 0 & \text{if } p^3|N,\ \varepsilon_{1,p}\cdot\varepsilon_{2,p}\cdot\varepsilon_{3,p} = -1 . \quad \cdots (\star\star) \end{cases}$$

<u>Rem</u>

(i) $\dim_{\mathbb{C}} \mathrm{Hom}_{H_p}(\Pi_p, \mathbb{C}) = \begin{cases} 1 & (\star) \\ 0 & (\star\star) \end{cases}$

(ii) If $N_1 = N_2 = N_3$,

<u>Cor</u> follows from a result of Watson

$$G = (D^\times \times D^\times \times D^\times)/F^\times$$

$$|\pi_0(Z(\hat{G}^\natural)^\Gamma)| = |Z(\hat{G}^\natural)| = 8$$

$\Gamma = \mathrm{Gal}(\bar{F}/F)$ $\quad SL(2)\times SL(2)\times SL(2)$

$$G^\natural = G/A \leftarrow \text{split comp of center} = G/Z_G(F)$$