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On the equivariant Tamagawa number conjecture for Hecke characters

§1. Introduction

K : number field

$\cup$

$\mathcal{O}_K$: integer ring

(1) $M = h^0(\mathrm{Spec}\, K)$

$$\rightsquigarrow L(M,s) = \underset{\text{Dedekind zeta}}{\zeta_K(s)} = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \neq 0 \text{ ideal}}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

Class number formula

(i) $\underset{s=0}{\mathrm{ord}}\, \zeta_K(s) = r_1 + r_2 - 1 = \mathrm{rank}_{\mathbb{Z}}\, \mathcal{O}_K^\times = \mathrm{rank}_{\mathbb{Z}}\, K_1(\mathcal{O}_K)$

(ii) $L^*(M) := \lim_{s \to 0} \dfrac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\dfrac{R_K \cdot h_K}{w_K}$

where
$r_1 = \#$ of real places
$r_2 = \#$ of complex places
$w_K = \#\, \mathcal{O}_{K,\mathrm{tors}}^\times$
$h_K = \#\, Cl(K)$
$R_K =$ regulator of $K = \left| \det\left(\delta_i \cdot \log |u_i^{\tau_j}|\right) \right|$

$\tau_j : K \hookrightarrow \mathbb{C}$

$1 \leq i, j \leq r_1 + r_2 - 1$

free basis of $\mathcal{O}_K^\times$

$$\delta_i := \begin{cases} 1 & \text{if } \tau_i : \text{real} \\ 2 & \text{if } \tau_i : \text{complex} \end{cases}$$

(2) $M = h^0(\mathrm{Spec}\, K)(i) \quad (i < 0)$

$\rightsquigarrow L(M,s) = \zeta_K(s+i)$

(i) $$\underset{s=0}{\mathrm{ord}}\, L(M,s) = \underset{s=i}{\mathrm{ord}}\, \zeta_K(s) = \begin{cases} r_1 + r_2 & \text{if } i \equiv 0 \bmod 2 \ (i \leq -2) \\ r_2 & \text{if } i \equiv 1 \bmod 2 \end{cases}$$

$$= \mathrm{rank}_{\mathbb{Z}}\, K_{1-2i}(\mathcal{O}_K)$$

(ii) Borel defined the <u>higher regulator map</u> $\mathbb{Q}$-dual

$$\rho_\infty : K_{1-2i}(\mathcal{O}_K) \otimes \mathbb{R} \xrightarrow{\cong} \left(H_B^0\left((\operatorname{Spec} \mathcal{O}_K \otimes \mathbb{C}), \mathbb{Q}(-i) \right)^+ \right)^* \otimes \mathbb{R}$$

$\boxed{\mathbb{Z}\text{-basis of these groups}}$

This is the generalization of

$$\mathrm{Reg} : K_1(\mathcal{O}_K) \otimes \mathbb{R} = \mathcal{O}_K^\times \otimes \mathbb{R} \longrightarrow \mathbb{R}^{r_1+r_2-1} = \left(H_B^0(\operatorname{Spec} \mathcal{O}_K \otimes \mathbb{C}, \mathbb{Q})^+ \right) \otimes \mathbb{R}$$

$$u_i \longmapsto \delta_i \cdot \log |u_i^{\tau_j}|$$

$R_K^{(i)} := \det \rho_\infty$

$$\rightsquigarrow L^*(M) := \lim_{s \to i} \frac{\zeta_K(s)}{(s-i)^{r_i}} \equiv R_K^{(i)} \mod \mathbb{Q}^\times$$

<u>Thm</u> (Huber - Kings)

$K/\mathbb{Q}$: abel. $i \leq -1$, then

$$L^*(M) \doteq R_K^{(i)} \cdot \prod_p \frac{\# H^2(\mathcal{O}_K[\frac{1}{p}], \mathbb{Z}_p(i+1))}{\# H^1(\mathcal{O}_K[\frac{1}{p}], \mathbb{Z}_p(i+1))_{\mathrm{tors}}}$$

up to 2-part

§2 Equivariant TNC

K : imag. quad field with class number one

E/K : elliptic curve with CM by $\mathcal{O}_K$ $(\operatorname{End}_K(E) \cong \mathcal{O}_K)$

ψ : grössencharacter ass. to E

$$\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

$K(\mathfrak{m})$: ray class field of K modulo $\mathfrak{m}$. $\mathfrak{m} \subset \mathcal{O}_K$

$G_\mathfrak{m} := \operatorname{Gal}(K(\mathfrak{m})/K)$, $\chi : G_\mathfrak{m} \longrightarrow \mathbb{C}^\times$

Fix $a, b \geq 0$, $w = a+b \geq 1$, $\varphi = \psi^a \cdot \overline{\psi}^b$

$$\varphi_\chi := \chi \cdot \varphi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

$$M_\varphi := e_\varphi\left(\bigotimes_{i=1}^{w} h^1(E)\right) \quad : \text{motive ass to } \varphi$$

$\uparrow$ projector ass. to φ

$$M = M_\varphi \otimes_K h^0(\operatorname{Spec} K(m)) \circlearrowleft A := K[G_m] \subset \operatorname{End}(M)$$

$$\mathcal{O} := \mathcal{O}_K[G_m] \subset A$$

$$L({}_A M, s) := \left(L(\varphi_\chi, s)\right)_{\chi \in \widehat{G_m}} \quad : A\text{-equivariant } L\text{-function}$$

$\uparrow$ $\mathbb{C}[G_m]$-valued funct.

Fix $r \in \mathbb{Z}$ s.t. $\begin{cases} -r \leq \min(a,b) & \text{if } a \neq b \\ -r \leq a = b & \text{if } a = b \end{cases}$

<u>non-critical case</u>

$$\rightsquigarrow \quad \operatorname*{ord}_{s=-r} L(\varphi_\chi, s) = 1 \qquad \text{(by funct. eq.)}$$

We want to describe $L^*({}_A M)$ by certain "arithmetic objects".

$$H_f^1(M) = H_{\mathcal{M}}^{w+1}(M, \mathbb{Q}(w+r+1))$$

$$= e_\varphi \cdot K_{w+2r+1}\left(\bigotimes_{i=1}^{w} \mathcal{A}\right)^{(w+r+1)}$$

$\uparrow$ finite part of motivic cohomology

$\uparrow$ $\operatorname{Res}_{K(m)/K} E \times_K \operatorname{Spec} K(m)$

$$H_f^1(M)^{\text{constr}} \quad : \text{constructible subspace (Deninger)}$$

conj "="

$$H_B(M) = H_B^w\left(M \otimes_K \mathbb{C},\ \mathbb{Q}(w+r)\right)$$

$$M_\ell = H_{et}^w\left(M \otimes_K \overline{K},\ \mathbb{Q}_\ell(w+r+1)\right)$$
$$\cup$$
$$T_\ell = H_{et}^w\left(M \otimes_K \overline{K},\ \mathbb{Z}_\ell(w+r+1)\right) \ : \ Gal(\overline{K}/K)\text{-stable lattice}$$

<u>Beilinson regulator map</u> :

$$\rho_\infty : H_f^1(M)^{\mathrm{consth}} \otimes \mathbb{R} \longrightarrow H_{\mathcal{D}}^{w+1}\left(M \otimes_K \mathbb{C},\ \mathbb{R}(w+r+1)\right)$$
$$= M_B \otimes \mathbb{R}$$

<u>Soulé's ℓ-adic regulator map</u> :

$$\rho_\ell : H_f^1(M)^{\mathrm{consth}} \otimes \mathbb{Q}_\ell \longrightarrow H^1(\mathcal{O}_K[\tfrac{1}{m\ell}],\ M_\ell)$$

$\rightsquigarrow$ The fundamental line

A-dual

$$\Xi({}_AM) := \mathrm{Det}_A\left(H_f^1(M)^{\mathrm{consth}}\right)^* \otimes_A \mathrm{Det}_A^{-1}(M_B)^*$$
$$\cap$$
$\mathrm{Inv}(A)$: the category of graded invertible A-modules

ρ_∞ induces
$$ {}_A\theta_\infty : A \otimes \mathbb{R} \xrightarrow{\cong} \Xi({}_AM) \otimes \mathbb{R}$$

<u>Thm</u> (Deninger) $\quad {}_A\theta_\infty\left(\underbrace{L^*({}_AM)^{-1}}_{\substack{\| \\ (L'({}_AM,-r))}}\right) \in \Xi({}_AM) \otimes 1$

$$A_\ell = A \otimes \mathbb{Q}_\ell$$
$$\cup$$
$$\mathfrak{O}_\ell := \mathfrak{O} \otimes \mathbb{Z}_\ell$$

Conj (Weak Leopoldt conj)

$$H^2\left(\mathcal{O}_K\left[\tfrac{1}{m\ell}\right], M_\ell\right) = 0$$

Assuming WLC for M_ℓ, ρ_ℓ induces

$${}_A\vartheta_\ell : \Xi({}_AM) \otimes \mathbb{Q}_\ell \xrightarrow{\ \cong\ } \mathrm{Det}_{A_\ell} R\Gamma_c\left(\mathcal{O}_K\left[\tfrac{1}{m\ell}\right], M_\ell\right)$$

invertible A_ℓ-module

$$\begin{tikzcd}
A \otimes \mathbb{R} \arrow[r, "\cong"', "{}_A\vartheta_\infty"] & \Xi({}_AM)\otimes\mathbb{R} & \\
L^*({}_AM)^{-1} \arrow[u, phantom, "\in" sloped] \arrow[dr, mapsto] & \Xi({}_AM)\otimes 1 \arrow[u, phantom, "\subset" sloped] \arrow[r, phantom, "\subset"] & \Xi({}_AM)\otimes\mathbb{Q}_\ell \arrow[dl, bend left, "{}_A\vartheta_\infty"] \\
& {}_A\vartheta_\infty\left(L^*({}_AM)^{-1}\right) \arrow[u, phantom, "\in" sloped] \arrow[dd, mapsto, bend right] & \\
& \mathrm{Det}_{A_\ell} R\Gamma_c\left(\mathcal{O}_K\left[\tfrac{1}{m\ell}\right], M_\ell\right) \arrow[r, phantom, "\supset"] & \mathrm{Det}_{\mathcal{O}_\ell} R\Gamma_c\left(\mathcal{O}_K\left[\tfrac{1}{m\ell}\right], T_\ell\right) \\
& {}_A\vartheta_\ell\left({}_A\vartheta_\infty\left(L^*({}_AM)^{-1}\right)\right) \arrow[u, phantom, "\in" sloped] &
\end{tikzcd}$$

zeta value

as $\mathcal{O}_\ell$-lattice

$L'(\varphi_x, -r)$
$-r \leq 0$

Conj (ETNC)
For $\forall \ell$: prime, we have

$$\mathcal{O}_\ell \cdot \underbrace{{}_A\vartheta_\ell}_{\text{Kings}}\left(\underbrace{{}_A\vartheta_\infty}_{\text{Deninger}}\left(L^*({}_AM)^{-1}\right)\right) = \underbrace{\mathrm{Det}_{\mathcal{O}_\ell} R\Gamma_c\left(\mathcal{O}_K\left[\tfrac{1}{m\ell}\right], T_\ell\right)}_{\text{Iwasawa Main Conj for imag. quad fields}}$$

Rubin

Thm Assume $\ell \neq 2$, WLC for M_ℓ, then Conj holds.

Rem (1) The case of $\underline{m=1}$, this is proved by Kings and Bars.

(2) For almost all r, the WLC hold using Rubin's result.

(3) For critical case, Tsuji proved TNC for Hecke char.
(Flach dealt with ETNC for $w=1$)

§3 Another results for TNC

$$f = \sum_{n=1}^{\infty} a_n(f) \cdot q^n \in S_k(\Gamma_1(N)) \quad \begin{pmatrix} k \geq 2 \\ N \geq 5 \end{pmatrix}$$

newform

$T_{f,\ell}$: ℓ-adic Gal. rep ass to f

$\pi = \pi_f$: autom rep. ass to f

Thm (Gealy)

$r > 0$ $\quad \ell$: odd prime

Assume (1) local rep at 2 is not supercuspidal

(2) $H^2(\mathbb{Z}[1/N\ell], T_{f,\ell}(-r)) \otimes \mathbb{Q}_\ell = 0$

(3) If $k = 2$ or 4, $L(f, k/2) \neq 0$

(4) Iwasawa Main Conj for f^*

$\Rightarrow$ ℓ-part of TNC holds for $M_f(-r)$

Thm (Diamond, Flach, Guo)

$\ell \nmid Nk!$, $F/\mathbb{Q}$: quad ext. s.t. $F \subset \mathbb{Q}(\zeta_\ell)$

Assume $\bar{\rho}_{f,\ell}|_{G_F} : G_F \longrightarrow GL_2(\bar{\mathbb{F}}_\ell)$: absolutely irreducible

Then, ℓ-part of TNC for $M = Ad^\circ M_f$, $Ad^\circ M_f(1)$

(use Taylor-Wiles methods)