On motives and ℓ-adic Galois representations associated to automorphic representations of $GSp(4)$

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1. Introduction

This is a write-up of my talk at the Hakuba workshop 2006. In the workshop, I gave two introductory lectures on the Langlands correspondences and ℓ-adic Galois representations. Contrary to the title, I did not talk much about motives.

The aim of this article is to compensate the talk. In this article, I give an informal introduction to the Langlands correspondences and the Principle of Functoriality from the viewpoint of arithmetic geometry. I especially emphasized the role of ℓ-adic Galois representations (instead of complex representations of the hypothetical Langlands dual group) in the theory of automorphic representations. This viewpoint is slightly different from the usual one. I would not be surprised if this article looks strange for 'automorphic' people. However, I still believe this viewpoint is very natural, useful and even necessary in some instances.

Although this is a survey article for non-specialists, I did not even try to give a textbook-style introduction. One reason is that there are so many different aspects in the topics. It is almost impossible (for me, at least) to give a reasonable introduction in a reasonable size. Most 'introductory' books in this area are inevitably too heavy (theoretically and physically). Another reason is that there have been already many excellent articles and books (such as [PSPM33], [PSPM55], [AnnArbor], [Jerusalem], [Toronto], [CKM]). There is no point in writing a similar article any more.

In this article, I do not give precise definitions concerning automorphic representations. I suppose most of them can be found in some articles in this volume (or some references therein). On the other hand, I tried to give basic definitions concerning Galois representations (see §2), which are certainly well-known for arithmetic geometers, but do not seem so for usual 'automorphic' people.

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One thing I tried to stress in this article is that it is very convenient for you to have a viewpoint from arithmetic geometry even if you are not at all interested in it. Namely, "Galois representations are convenient tools to study automorphic representations." Many difficult theorems and problems in this area are easily and precisely explained in terms of Galois representations. Of course, other people may have an opposite viewpoint; "Automorphic representations are convenient tools to study Galois representations."

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2. Galois representations

In this section, we shall recall some basic definitions concerning Galois representations and their L-functions.

2.1. General definitions.

Definition 2.1. Let F be a field. In this paper, the absolute Galois group of F is always denoted by $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ because the usual notation 'G_F' may conflict with reductive groups.

(1) An *Artin representation* of Γ_F is a continuous homomorphism $\rho\colon \Gamma_F \to GL(V)$, where V is a finite dimensional vector space over $\mathbb{C}$.
(2) Let ℓ be a prime number. An *ℓ-adic representation* of Γ_F is a continuous homomorphism $\rho\colon \Gamma_F \to GL(V)$, where V is a finite dimensional vector space over a finite extension of $\mathbb{Q}_\ell$.

The Galois group Γ_F has the Krull topology, and $GL(V)$ has the complex (resp. ℓ-adic) topology for Artin (resp. ℓ-adic) representations. Since Γ_F is a compact group, for an Artin representation ρ, the image $\rho(\Gamma_F)$ is a finite group. By fixing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ (by Axiom of Choice), an Artin representation can be considered as an ℓ-adic representation. The converse is not true. There are many important ℓ-adic representations with infinite image (see below).

Remark 2.2. A continuous homomorphism $\rho\colon \Gamma_F \to GL(n, \overline{\mathbb{Q}}_\ell)$ is also called an *ℓ-adic representation*, where $GL(n, \overline{\mathbb{Q}}_\ell)$ has the ℓ-adic topology. In this case, the image of ρ is automatically contained in $GL(n, K)$ for a finite extension $K/\mathbb{Q}_\ell$. This can be shown as follows. Since Γ_F is compact, the image $H = \mathrm{Im}\,\rho$ is also compact.

Moreover, $H = \bigcup_{[K:\mathbb{Q}_\ell]<\infty} H \cap GL(n, K)$ is a countable union of closed subgroups. By the Baire category theorem, the interior of $H \cap GL(n, K)$ is nonempty for a finite extension $K/\mathbb{Q}_\ell$. Then, $H \cap GL(n, K) \subset H$ is an open subgroup, hence is of finite index. Therefore, $H = H \cap GL(n, K')$ for a finite extension K'/K.

Let us give several examples of Galois representations.

Example 2.3. Let L/F be a quadratic Galois extension. Then, the composite

$$\Gamma_F \longrightarrow \mathrm{Gal}(L/F) \cong \{\pm 1\} \subset \mathbb{C}^\times$$

is a 1-dimensional Artin representation. Similarly, for a finite Galois extension L/F with Galois group G, and a complex representation $G \to GL(n, \mathbb{C})$, the composite

$$\Gamma_F \longrightarrow \mathrm{Gal}(L/F) \cong G \longrightarrow GL(n, \mathbb{C})$$

is an Artin representation. All Artin representations of Γ_F are obtained in this way.

Example 2.4. For an integer $n \geq 1$ invertible in F, let

$$\mu_n := \left\{ x \in \overline{F} \;\middle|\; x^n = 1 \right\}$$

denote the group of n-th roots of unity. For a prime number ℓ invertible in F, Γ_F acts on the inverse limit

$$\mathbb{Z}_\ell(1) := \varprojlim \mu_{\ell^n},$$

which is a free $\mathbb{Z}_\ell$-module of rank 1. This gives us an ℓ-adic character (i.e. 1-dimensional ℓ-adic representation)

$$\chi_\ell \colon \Gamma_F \longrightarrow \mathrm{Aut}_{\mathbb{Z}_\ell}\big(\mathbb{Z}_\ell(1)\big) \cong \mathbb{Z}_\ell^\times \subset GL(1, \mathbb{Q}_\ell),$$

which is called the *ℓ-adic cyclotomic character*. For an ℓ-adic representation V and $n \in \mathbb{Z}$,

$$V(n) := V \otimes \chi_\ell^{\otimes n}$$

is called the *Tate twist* of V.

Example 2.5. For an elliptic curve E over F, and an integer $n \geq 1$ invertible in F, the group of n-torsion points on E

$$E[n] := \{P \in E(\overline{F}) \mid [n](P) = O\}$$

is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2. For a prime number ℓ invertible in F, we define the *ℓ-adic Tate modules* by

$$T_\ell E := \varprojlim E[\ell^n], \qquad V_\ell E := T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Then, $T_\ell E$ is a free $\mathbb{Z}_\ell$-module of rank 2, and $V_\ell E$ is a 2-dimensional vector space over $\mathbb{Q}_\ell$. Since Γ_F acts on $V_\ell E$ continuously, this gives us a 2-dimensional ℓ-adic Galois representation

$$\rho_{E,\ell} \colon \Gamma_F \longrightarrow GL(V_\ell E) \cong GL(2, \mathbb{Q}_\ell).$$

In an arithmetic situation, $\rho_{E,\ell}$ can be highly nontrivial. J.-P. Serre showed that, if F is a number field and E has no complex multiplication, the image $\mathrm{Im}\,\rho_{E,\ell}$ is a finite index subgroup of $GL(T_\ell E) \cong GL(2, \mathbb{Z}_\ell)$ for all ℓ, and $\operatorname{Im} \rho_{E,\ell} = GL(T_\ell E) \cong GL(2, \mathbb{Z}_\ell)$ for all but finitely many ℓ ([Se]).

Example 2.6. More generally, for an algebraic variety X over F and a prime number ℓ invertible in F, the i-th *ℓ-adic étale cohomology*

$$H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell)$$

is a finite dimensional vector space over $\mathbb{Q}_\ell$ with a continuous action of Γ_F. If X is an elliptic curve,

$$H^1_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell)$$

is isomorphic to the dual of the Tate module $V_\ell X$.

2.2. **Local fields.** Let F be a nonarchimedean local field with residue field $\mathbb{F}_q$. Then, F is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, where p is a prime number and q is a power of p. We have a canonical surjection $\varphi\colon \Gamma_F \twoheadrightarrow \Gamma_{\mathbb{F}_q}$. The kernel $I_F := \operatorname{Ker}(\varphi)$ is called the *inertia group*. Let $\operatorname{Frob}_q \in \Gamma_{\mathbb{F}_q}$ be the *geometric Frobenius element* defined by $\operatorname{Frob}_q(x) = x^{1/q}$ for $x \in \overline{\mathbb{F}}_q$. Then, $\Gamma_{\mathbb{F}_q}$ is topologically generated by Frob_q and isomorphic to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$. The *Weil group* $W_F := \varphi^{-1}(\langle \operatorname{Frob}_q \rangle)$ is defined to be the inverse image by φ of the subgroup generated by Frob_q. We have the following exact sequences :

$$1 \to I_F \to \Gamma_F \to \Gamma_{\mathbb{F}_q} \cong \widehat{\mathbb{Z}} \to 1, \qquad 1 \to I_F \to W_F \to \langle \operatorname{Frob}_q \rangle \cong \mathbb{Z} \to 1$$

We can define a topology on W_F such that $I_F \subset W_F$ is an open subgroup (see [Tat] for details).

Fix a uniformizer $\pi \in F$. Local class field theory gives us a canonical isomorphism called the *reciprocity isomorphism* :

$$\operatorname{Art}\colon F^\times \xrightarrow{\cong} W_F^{ab}.$$

In fact, there are two normalizations of this isomorphism. In this paper, we always use *geometric* normalization. Namely, under this isomorphism, a uniformizer of F corresponds to a lifting of the geometric Frobenius element. Some people working in algebraic number theory use *arithmetic* Frobenius element instead, which is the inverse of the geometric Frobenius element.

Let ℓ be a prime number. As usual, an *ℓ-adic representation* of W_F is a continuous homomorphism $\rho\colon W_F \to GL(V)$, where V is a finite dimensional vector space over a finite extension of $\mathbb{Q}_\ell$. For an ℓ-adic representation ρ of Γ_F, the restriction $\rho|_{W_F}$ is an ℓ-adic representation of W_F. But not all ℓ-adic representations of W_F are constructed in this way because the topology on W_F is stronger than the induced topology from Γ_F (cf. [Tat]). An ℓ-adic representation ρ of W_F (or Γ_F) is said to be *unramified* if $\rho(I_F)$ is trivial.

A typical example of an ℓ-adic representation is

$$|\cdot|\colon W_F \longrightarrow \mathbb{Q}_\ell^\times, \qquad \sigma \mapsto |\sigma| = \left|\mathrm{Art}^{-1}(\sigma)\right|_F$$

where $|\cdot|_F$ denotes the normalized absolute value on F (i.e. $|\pi|_F = q^{-1}$).

Assume that ℓ does not divide q. Then, $|\cdot|$ is nothing but the ℓ-adic cyclotomic character. For an ℓ-adic representation $\rho\colon W_F \to GL(V)$, we define the *L-function* of ρ by

$$L(s,\rho) := \det\bigl(1 - q^{-s}\mathrm{Frob}_q; V^{I_F}\bigr)^{-1}.$$

Remark 2.7. When ℓ divides q, we can define the L-function using p-adic Hodge theory although we do not treat it in this article (cf. Fontaine's D_{pst}-functor, [Fo]. see also [Tay2]).

2.3. **Global fields.** Here we fix a global field F (i.e. F is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$), and a prime number ℓ invertible in F.

Let $\rho\colon \Gamma_F \to GL(V)$ be an ℓ-adic or Artin representation. For a finite place v of F, F_v denotes the completion of F at v and q_v denotes the cardinality of the residue field of F_v. For each v, we fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$ extending $F \hookrightarrow F_v$. This defines an embedding $\Gamma_{F_v} \subset \Gamma_F$. We put $\rho_v := \rho|_{W_{F_v}}$.

The L-function of ρ is defined as

$$L(s,\rho) := \prod_{v\,:\,\text{finite place of }F} L(s,\rho_v).$$

We say ρ is *pure of weight* $w \in \mathbb{Z}$ if there is a finite set S of finite places of F such that, for each finite place $v \notin S$, ρ_v is unramified, and the eigenvalues of $\rho(\mathrm{Frob}_{q_v})$ are algebraic integers whose complex conjugates have complex absolute value $q_v^{w/2}$. Artin representations are pure of weight 0. The ℓ-adic cyclotomic character χ_ℓ is pure of weight -2 (because Frob_{q_v} acts as multiplication by q_v^{-1}). For a projective smooth variety X over F, the i-th ℓ-adic étale cohomology

$$V := H^i_{\text{ét}}\bigl(X_{\overline{F}}, \mathbb{Q}_\ell\bigr)$$

is pure of weight i by basic properties of étale cohomology and the Weil conjectures.

Example 2.8. When $F = \mathbb{Q}$, the L-function of the trivial representation of $\Gamma_\mathbb{Q}$ is the Riemann zeta function :

$$L(s,\mathrm{triv}) = \prod_{p\,:\,\text{prime number}} \frac{1}{1-p^s} = \zeta(s).$$

Similarly, for a primitive Dirichlet character

$$\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

the composite $\chi \circ \varphi_N$ is a 1-dimensional Artin representation of $\Gamma_\mathbb{Q}$, where

$$\varphi_N\colon \Gamma_\mathbb{Q} \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$$

is defined by $\sigma \cdot x = x^{\varphi_N(\sigma)}$ for each $\sigma \in \Gamma_{\mathbb{Q}}$ and $x \in \mu_N$. The L-function $L(s, \chi \circ \varphi_N)$ coincides with the Dirichlet L-function :

$$L(s, \chi \circ \varphi_N) = \prod_{(p,N)=1} \frac{1}{1-\chi(p)p^{-s}} = \sum_{(n,N)=1} \frac{\chi(n)}{n^s} =: L(s, \chi).$$

By the Kronecker-Weber theorem, all 1-dimensional Artin representations of $\Gamma_{\mathbb{Q}}$ are obtained in this way.

Example 2.9. Let F be a global field, and E an elliptic curve defined over F. Fix a prime number ℓ invertible in F. The L-function of E is defined as

$$L(s, E) := L\big(s, H^1_{\text{ét}}\big(E_{\overline{F}}, \mathbb{Q}_\ell\big)\big).$$

It is known that RHS is independent of the choice of ℓ and $L(s, E)$ is well-defined. For a finite place v of F where E has good reduction, let $\#E(\kappa(v))$ be the number of rational points of the reduction modulo v of E, and put $a_v := q_v + 1 - \#E(\kappa(v))$. Then, the L-function $L(s, E)$ is written as follows :

$$L(s, E) = \left(\prod_{v\,:\,\text{good}} \frac{1}{1 - a_v q_v^{-s} + q_v^{1-2s}} \right) \cdot \big(\text{bad factors}\big).$$

More generally, for a projective smooth variety X over F, the Hasse-Weil zeta function of X (at good places v) can be written as a product of L-functions (and inverses of them) of ℓ-adic representations of Γ_F defined by the étale cohomology $H^i_{\text{ét}}\big(X_{\overline{F}}, \mathbb{Q}_\ell\big)$.

Example 2.10. Let $f = \sum_{n=0}^{\infty} a_n(f) q^n$ be a holomorphic modular form on $\Gamma_1(N)$ of level k and character ε. Assume that f is a normalized cusp form (i.e. $a_0 = 0,\ a_1 = 1$). Assume moreover that f is a common eigenvector of all Hecke operators. Let K be a number field generated by a_n, and λ a finite place of K above ℓ. M. Eichler, G. Shimura, P. Deligne, J.-P. Serre constructed an ℓ-adic representation

$$\rho_{f,\ell} \colon \Gamma_{\mathbb{Q}} \longrightarrow GL(2, K_\lambda)$$

satisfying

$$\mathrm{Tr}(\mathrm{Frob}_p) = a_p(f), \qquad \det(\mathrm{Frob}_p) = \varepsilon(p) p^{k-1}$$

for all p prime to ℓN ([De], [DS]). Usually, $\rho_{f,\ell}$ is called the *ℓ-adic representation associated to f*. If $k = 2$, ε is trivial (i.e. f is a modular form on $\Gamma_0(N)$), and $a_n \in \mathbb{Z}$ for all n, $\rho_{f,\ell}$ is constructed from the Tate module of an elliptic curve over $\mathbb{Q}$ (Eichler-Shimura). For example, let

$$g(q) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2 = \sum_{n=1}^{\infty} a_n(g) q^n$$

be a unique cusp form on $\Gamma_0(11)$. The ℓ-adic representation associated to f is constructed from the Tate module of the elliptic curve

$$E : y^2 + y = x^3 - x^2.$$

Namely, the relation $a_p(g) = p + 1 - \#E(\mathbb{F}_p)$ holds for all $p \neq 11$. Note that the modular curve $X_0(11)$ is an elliptic curve, and isogenous to E.

Remark 2.11. Conversely, when a 2-dimensional ℓ-adic (or Artin) representation ρ is given, it is a very important (but difficult) problem to prove ρ is associated to a modular form (so called *modularity problems*). There are several traditional conjectures of this type. The (strong) *Artin conjecture* predicts that, an irreducible odd 2-dimensional Artin representation $\rho\colon \Gamma_{\mathbb{Q}} \to GL(2,\mathbb{C})$ is associated to a modular form of weight 1 (Here 'odd' means the determinant of complex conjugate is equal to -1). This was known to be true when the image of ρ is solvable (Langlands-Tunnell theorem). Recently, this was (almost) proved by C. Khare and J.-P. Wintenberger (cf. [Kh]). The *Taniyama-Shimura conjecture* predicts that, for an elliptic curve E over $\mathbb{Q}$, the Galois representation obtained from (the dual of) the Tate module of E is associated to some modular form of weight 2 (i.e. All elliptic curves over $\mathbb{Q}$ are *modular*). This was finally proved by C. Breuil-B. Conrad-F. Diamond-R. Taylor ([BCDT]).

3. The Langlands correspondence for $GL(n)$

Let F be a global field (i.e. finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$), and $\mathbb{A}_F$ the adèle ring of F. We fix a prime number ℓ invertible in F. To simplify the notation, we fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ (by Axiom of Choice).

Roughly speaking, the global Langlands correspondence for $GL(n)$ over F is a conjectural correspondence preserving L-functions between

Automorphic side: (algebraic) automorphic representations of $GL(n,\mathbb{A}_F)$, and
Galois side: (geometric[2]) n-dimensional ℓ-adic representations of the absolute Galois group $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$.

An automorphic representation π of $GL(n,\mathbb{A}_F)$ and an n-dimensional ℓ-adic representation ρ of Γ_F are said to be *associated* if there is an integer $w \in \mathbb{Z}$ and a finite set S of finite places of F such that

$$L(s - w/2, \pi_v) = L(s, \rho_v).$$

for each finite place $v \notin S$. (The existence of w is just a matter of normalization.)

Examples of associated pairs (π,ρ) can be obtained via class field theory ($n = 1$), or ℓ-adic representations associated to modular forms ($n = 2$) ([De], [DS]). For each π, there exists at most one ρ associated to π by the Chebotarev density theorem. Also, for each ρ, there exists at most one π associated to ρ by the strong multiplicity one theorem.

[2]in the sense of Fontaine-Mazur (This condition can be made precise using p-adic Hodge theory.)

It is expected that cuspidal automorphic representations correspond to irreducible Galois representations. The Ramanujan conjecture predicts that the ℓ-adic representation associated to a cuspidal automorphic representation is pure.

It has been a serious problem which Galois representations should correspond to automorphic representations. Recently, thanks to the development of p-adic Hodge theory (theory of de Rham representations) and the Fontaine-Mazur conjecture, we can now state a precise conjecture over number fields without assuming other conjectures.

Conjecture 3.1 ([Tay2], Conjecture 3.5)**.** *Let $\rho\colon \Gamma_{\mathbb{Q}} \to GL(V)$ be an irreducible n-dimensional ℓ-adic representation of $\Gamma_{\mathbb{Q}}$ which is unramified at all but finitely many places, and de Rham at $p = \ell$. Then, there is a cuspidal automorphic representation π of $GL(n, \mathbb{A}_{\mathbb{Q}})$ associated to ρ.*

Remark 3.2. In the above statement, the existence of the Langlands dual group '$\mathscr{L}_F$' is not assumed!

Remark 3.3. Conversely, for certain number fields F and certain automorphic representations π of $GL(n, \mathbb{A}_F)$, the ℓ-adic representations associated to π were constructed by L. Clozel, R. Kottwitz, M. Harris-R. Taylor ([Cl], [Ko1], [HT], see also [Tay2]).

Remark 3.4. When F is a global field of characteristic $p > 0$ (i.e. F is a finite extension of $\mathbb{F}_p(t)$), there is also a precise statement of the global Langlands correspondence for $GL(n)$ over F. This was proved by V. Drinfeld (for $GL(2)$) and L. Lafforgue (for $GL(n)$) (cf. [Laf]).

Remark 3.5. Usually, the 'statement' of the Langlands correspondence (or conjecture) uses complex representations of the (hypothetical) Langlands dual group '$\mathscr{L}_F$' (cf. [Co1]). It is expected that $\mathscr{L}_F$ is a topological group and is an extension of the absolute Galois group Γ_F by a compact group :

$$1 \longrightarrow \big(\text{compact group}\big) \longrightarrow \mathscr{L}_F \longrightarrow \Gamma_F \longrightarrow 1.$$

A tricky point is that the group $\mathscr{L}_F$ is yet to be defined for a global field F! Perhaps, when $\operatorname{char} F > 0$, we could define (a candidate of) $\mathscr{L}_F$ *using* the global Langlands correspondence proved by V. Drinfeld and L. Lafforgue. There is a definition of $\mathscr{L}_F$ for a local field F. If F is nonarchimedean, we define $\mathscr{L}_F := W_F \times SU(2)$. On the other hand, If F is archimedean (i.e. $\mathbb{R}$ or $\mathbb{C}$), $\mathscr{L}_F$ is the same as the Weil group W_F, which is not treated in this paper (cf. [Tat], [Kn]).

Remark 3.6. You might be irritated with the 'statement' of the 'Langlands conjectures' in a usual literature. A prescription for this is, whenever you read (or write) articles containing $\mathscr{L}_F$, you replace 'complex representations of $\mathscr{L}_F$' by 'ℓ-adic representations of Γ_F' and use p-adic Hodge theory (secretly in your mind). It seems that the notion of ℓ-adic representations of Γ_F is essential to state precise conjectures (without assuming other conjectures such as the existence of $\mathscr{L}_F$).

3.1. **The local Langlands correspondence for $GL(n)$.** Let F be a nonarchimedean local field (i.e. finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$). We fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ as before.

An ℓ-adic representation ρ of W_F is said to be *Frobenius semisimple* if, for a lifting $\sigma \in W_F$ of the geometric Frobenius element, $\rho(\sigma)$ is semisimple (i.e. diagonalizable). There is a notion of an *irreducible admissible representation* (over $\mathbb{C}$) of $GL(n, F)$ (for precise statements, see [LRS], [HT], [He]. see also [Ku]).

Theorem 3.7 (Local Langlands Correspondence for $GL(n)$). *There exists a 'natural' bijection between the equivalence classes (over $\overline{\mathbb{Q}}_\ell$) of Frobenius semisimple n-dimensional ℓ-adic representations of the Weil group W_F, and the equivalence classes (over $\mathbb{C}$) of irreducible admissible representations of $GL(n, F)$.*

The local Langlands correspondence for $GL(1)$ is nothing but the local class field theory. Of course, one has to explain what should be a 'natural' bijection. There is a precise statement involving L-functions and ε-factors for pairs, and it was proved by G. Laumon-M. Rapoport-U. Stuhler when $\operatorname{char} F > 0$, and by M. Harris-R. Taylor, G. Henniart when $\operatorname{char} F = 0$

Remark 3.8. Usually, the local Langlands correspondence for $GL(n)$ is formulated in terms of complex representations of the (local) Langlands dual group $\mathscr{L}_F := W_F \times SU(2)$ (or the Weil-Deligne group, equivalently). The above statement is equivalent to the usual one (see [Tat], Theorem 4.2.1).

Remark 3.9. In contrast to the global case, over a local field, L-functions are not strong enough to characterize representations. For example, for $n \geq 2$ and a supercuspidal representation (which corresponds to an irreducible Galois representation) π of $GL(n, F)$, the local L-function $L(s, \pi)$ is always 1.

Remark 3.10. It is worth noting that although Theorem 3.7 is a local statement, no local proof was known in general. Even for $GL(2)$, a purely local proof was obtained quite recently (cf. [BH]). All the proofs given in [LRS], [HT], [He] use (global) trace formula arguments and the étale cohomology of Shimura varieties (or their function field analogues).

Remark 3.11. In this article, we do not treat archimedean places (cf. [Kn]). There is a statement of the local Langlands correspondence (for general reductive groups) over archimedean local fields (i.e. $\mathbb{R}$ or $\mathbb{C}$), and it was proved by R. Langlands himself (Langlands classification, [Lan]).

4. The case of general reductive groups

For general reductive groups, the statement of the Langlands correspondence is much more ambiguous. In this section, let F be a global or local field, and G a connected reductive group over F.

4.1. *L*-groups and dual groups. The *L-group* ${}^LG$ of G is of the form

$$ {}^LG = \widehat{G} \rtimes \Gamma_F, $$

where $\widehat{G}$ is a connected reductive group over $\mathbb{C}$ called the *dual group* of G, and $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ is the absolute Galois group of F as usual. Instead of giving the definition, we give several examples (for a precise definition, see [Bo], [Co2]).

G	$\widehat{G}$
$GL(n)$	$GL(n, \mathbb{C})$
$SL(n)$	$PGL(n, \mathbb{C})$
$PGL(n)$	$SL(n, \mathbb{C})$
$SO(2n)$	$SO(2n, \mathbb{C})$
$SO(2n+1)$	$Sp(2n, \mathbb{C})$
$Sp(2n)$	$SO(2n+1, \mathbb{C})$
$GSp(2n)$	$GSpin(2n+1, \mathbb{C})$

Roughly speaking, $\widehat{G}$ is obtained from G by interchanging the roots and coroots. If G is simply connected (resp. adjoint), the dual group $\widehat{G}$ is adjoint (resp. simply connected).

Since an inner automorphism acts trivially on roots and coroots, if G' is an inner form of G, we have canonical identifications ${}^LG = {}^LG'$, $\widehat{G} = \widehat{G'}$.

The action of Γ_F on $\widehat{G}$ is obtained from the action of Γ_F on the set of roots and coroots of G. Therefore, if G is an inner form of a split group, the action of Γ_F on $\widehat{G}$ is trivial, and ${}^LG$ is just a direct product of $\widehat{G}$ and Γ_F :

$$ {}^LG = \widehat{G} \times \Gamma_F. $$

4.2. The Langlands correspondence for G. Let F be a global field (although the local version should exist as well). Roughly speaking, the Langlands correspondence for G over F is a conjectural correspondence between

Automorphic side: (certain) automorphic representations of $G(\mathbb{A}_F)$, and
Galois side: (certain) homomorphisms $\rho\colon \mathscr{L}_F \to {}^LG$, called *L-parameters*, such that the composite $\mathrm{pr}_2 \circ \rho\colon \mathscr{L}_F \to \Gamma_F$ is the canonical one.

It is not bijective for general G. It is also conjectured that the set of equivalence classes of automorphic representations of $G(\mathbb{A}_F)$ should be partitioned into a set of

'*L-packets*', and an L-packet should correspond to an L-parameter. This is a vague statement because neither the notion of L-packets nor L-parameters are defined. (The Langlands dual group $\mathscr{L}_F$ is yet to be defined!)

To obtain a reasonable (and practical) statement, it seems necessary to consider ℓ-adic representations of Γ_F (instead of complex representations of '$\mathscr{L}_F$').

For simplicity, we assume G is an inner form of a split group. Then, the L-group ${}^LG$ is the product of $\widehat{G}$ and Γ_F. Since the complex reductive group $\widehat{G}$ is defined over $\mathbb{Z}$, we consider the set of $\overline{\mathbb{Q}}_\ell$-rational points of it. We denote it by $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ here. Then, a continuous homomorphism

$$\rho\colon \Gamma_F \longrightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

is an 'ℓ-adic avatar' of $\rho\colon \mathscr{L}_F \to {}^LG$.

For an automorphic representation π of $G(\mathbb{A}_F)$ and a representation of $\widehat{G}$ as a reductive group

$$r\colon \widehat{G} \longrightarrow GL(n),$$

the theory of Satake parameters enable us to define the L-function $L(s, r, \pi_v)$ for all but finitely many v (cf. [Bo]).

In the Galois side, L-functions are defined as follows. For each finite place v of F, we fix an embedding $\overline{F} \subset \overline{F}_v$. This gives us an embedding $\Gamma_{F_v} \subset \Gamma_F$ as usual. For each $\rho\colon \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ and $r\colon \widehat{G} \to GL(n)$, the composite

$$(r \circ \rho)|_{\Gamma_{F_v}} \colon \Gamma_{F_v} \xrightarrow{\ \rho\ } \widehat{G}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\ r\ } GL(n, \overline{\mathbb{Q}}_\ell)$$

is an n-dimensional ℓ-adic representation of Γ_{F_v}. Then, the L-function $L(s, r, \rho_v)$ is defined to be the L-function of $(r \circ \rho)|_{\Gamma_{F_v}}$:

$$L(s, r, \rho_v) := L\big(s,\ (r \circ \rho)|_{\Gamma_{F_v}}\big).$$

Then, an ℓ-adic version of the Langlands correspondence for G over F is a conjectural correspondence between

Automorphic side: (L-packets of) automorphic representations π of $G(\mathbb{A}_F)$, and

Galois side: continuous homomorphisms $\rho\colon \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$.

such that, for all $r\colon \widehat{G} \to GL(n)$, there exists an integer $w \in \mathbb{Z}$ satisfying the equality

$$L(s - w/2, r, \pi_v) = L(s, r, \rho_v)$$

for all but finitely many v (If it is satisfied, we say π and ρ are *associated*). This is still an ambiguous statement because the notion of 'L-packets' is yet to be defined (even locally). However, it makes sense to conjecture the following direction :

Conjecture 4.1. *For an automorphic representation π of $G(\mathbb{A}_F)$, there exist a continuous homomorphism $\rho\colon \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ associated to π.*

Of course, once this is established, it is natural to consider the correspondence $\pi_v \leftrightarrow \rho_v$ for all v (including bad places).

Remark 4.2. There should exist a similar statement over a local field F. When F is archimedean, it was formulated and proved by Langlands (Langlands classification, [Lan]). Also, when F is nonarchimedean, it was formulated and proved for unramified representations (Satake parameters, [Bo]).

5. Principle of Functoriality

Let F be a global field (although the local version should exist as well), and G, G' connected reductive groups over F. For simplicity, we assume G, G' are inner forms of split groups as before.

Let us believe (an ℓ-adic version of) the Langlands correspondence explained in §4.2 for the moment. For an automorphic representation π of $G(\mathbb{A}_F)$, there should exist a $\widehat{G}$-valued ℓ-adic representation

$$\rho\colon \Gamma_F \longrightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

associated to π. For a homomorphism $f\colon \widehat{G} \to \widehat{G'}$ of reductive groups, the composite

$$\Gamma_F \xrightarrow{\ \rho\ } \widehat{G}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\ f\ } \widehat{G'}(\overline{\mathbb{Q}}_\ell)$$

is a $\widehat{G'}$-valued ℓ-adic representation. Then, by the Langlands correspondence for G', $f \circ \rho$ should be associated to (an L-packet containing) an automorphic representation $f_*\pi$ of $G'(\mathbb{A}_F)$.

By this way, it is expected that each homomorphism $f\colon \widehat{G} \to \widehat{G'}$ should induce an operation $\pi \mapsto f_*\pi$ from the set of (L-packets of) automorphic representations of $G(\mathbb{A}_F)$ to that of $G'(\mathbb{A}_F)$ satisfying

$$L(\pi, r \circ f, s) = L(f_*\pi, r, s)$$

for all $r\colon \widehat{G'} \to GL(n)$. This is called the *Principle of Functoriality*, and such $f_*\pi$ is called the *lifting* (or *transfer*) of π. Note that, although the above explanation uses (an ℓ-adic version of) the Langlands correspondence, the (conjectural) operation '$\pi \mapsto f_*\pi$' can be described without ℓ-adic representations (at least for all but finitely many v) (cf. [Co2]).

The full version of the Principle of Functoriality is actually described in terms of L-groups rather than dual groups. The Langlands correspondence itself can be considered as a special case of the Principle of Functoriality (between the trivial group and G). To establish some cases of the Principle of Functoriality is one of the most important problems in this area.

We give several (conjectural) examples (see [Co2]).

Example 5.1. For a finite extension F'/F, the restriction $\rho \mapsto \rho|_{\Gamma_{F'}}$ should induce an operation from automorphic representations of $G(\mathbb{A}_F)$ to automorphic representations of $G(\mathbb{A}_{F'})$. This operation is called *base change*.

Example 5.2. For simplicity, consider the case of $GL(n)$. For a finite extension F'/F of degree d, the induction $\rho \mapsto \mathrm{Ind}_{\Gamma_{F'}}^{\Gamma_F} \rho$ should induce an operation from automorphic representations of $GL(n, \mathbb{A}_{F'})$ to automorphic representations of $GL(nd, \mathbb{A}_F)$. This operation is called *automorphic induction*.

Example 5.3. If G is an inner form of G', the L-groups (and dual groups) of G, G' are the same. Assume moreover that G' is quasi-split. The identity $\widehat{G} = \widehat{G'}$ should induce an operation from automorphic representations of $G(\mathbb{A}_F)$ to automorphic representations of $G'(\mathbb{A}_F)$. This operation is called *transfer to quasi-split inner forms*. When $G' = GL(2)$ and $G = B^\times$ for a quaternion algebra B over F, this operation is also called the *Jacquet-Langlands(-Shimizu) correspondence*. Note that this operation is not a bijection. For example, an automorphic representation π of $GL(2, \mathbb{A}_F)$ comes from $B^\times(\mathbb{A}_F)$ if and only if π_v is discrete series for all v such that $B \otimes_F F_v$ is a division algebra.

Example 5.4. Let $G = GL(2)$ and $G' = GL(n+1)$. There is a symmetric power homomorphism

$$\mathrm{Sym}^n \colon GL(2, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C}).$$

This should induce an operation from automorphic representations of $GL(2, \mathbb{A}_F)$ to automorphic representations of $GL(n+1, \mathbb{A}_F)$. This operation is called the *symmetric power lifting*.

Why is the Principle of Functoriality important? There are several reasons. One reason is that it explains many difficult theorems in a uniform way. It also predicts many difficult conjectures. The Principle of Functoriality gives us a collection of (very difficult) problems with an answer sheet. (However, naturally, it does not tell us how to solve them.) Another reason is that, since the Principle of Functoriality is widely believed, it gives us a strong evidence to other number theoretic problems. For example, it has been known that the (weak) Artin conjecture (i.e. entireness of $L(s, \rho)$ for irreducible nontrivial Artin representations ρ) is a consequence of the Langlands correspondence for $GL(n)$ for Artin representations, Fermat's Last Theorem is a consequence of the Taniyama-Shimura conjecture, and the Sato-Tate conjecture is a consequence of the Taniyama-Shimura conjecture and the existence of symmetric power liftings (as in Example 5.4, see also [Tay3]).

Contrary to its name, there is no 'principle' to establish the Principle of Functoriality. One reason seems that we do not know many methods to construct automorphic representations from group theoretic data. Nevertheless, today, there are several techniques to construct automorphic representations, and most of them can be used to establish the Principle of Functoriality in some cases. Let us list a few of them.

- Some automorphic forms are explicitly constructed (e.g. Eisenstein series, Ramanujan's Δ-function). For $GL(n)$, Eisenstein series corresponds to a direct sum of Galois representations. Although explicit construction is important in its own right, there seems no hope to construct all automorphic representations by this way.
- Theta series (or theta correspondence) is a traditional and powerful method to construct automorphic representations. There are many important and impressive results concerning this. It is applicable only for special (pairs of) groups.
- For $GL(n)$, once we know *nice* analytic properties of L-functions, we can construct automorphic representations of $GL(n, \mathbb{A}_F)$ by the Converse Theorem. Recently, this method is studied in detail to establish the Principle of Functoriality for 'generic' representations of classical groups ([CKPSS], [AS1], [AS2]). This is also used in Lafforgue's work ([Laf]).
- Trace formula (stabilized or twisted) is a general powerful method. Up to now, it works well only for $GL(n)$ and related groups. One of the major difficulty is so called the Fundamental Lemma. Contrary to its name, it is not a lemma, but a collection of conjectural identities between orbital integrals. Recently, great progress has been made with (some versions of) this Lemma for $GSp(4)$ ([Wh1], [Wh2]) and unitary groups ([LN]). It would be expected that, thanks to these results, many cases of the Principle of Functoriality will be established in the near future.
- We can sometimes construct automorphic representations *using* Galois representations. For example, we can construct automorphic representations of $GL(n)$ using (known cases of) the Artin conjecture (e.g. Langlands-Tunnell theorem). A non-Galois (global) automorphic induction, which was constructed by M. Harris *using* Galois representations, is crucially used in the proof of the local Langlands correspondence for $GL(n)$ over p-adic fields ([HT], [He]). After Taylor-Wiles ([TW], [Wi]), the '$R = T$ method' becomes a very powerful tool in arithmetic geometry. By this method, A. Wiles proved some cases of the Taniyama-Shimura conjecture (which is a special case of the Langlands correspondence for $GL(2)$) and proved Fermat's Last Theorem. L. Clozel, M. Harris, N. Shepherd-Barron, R. Taylor established the existence of (a 'potential' version of) the (even dimensional) symmetric power liftings and proved many cases of the Sato-Tate conjecture ([HSBT], [CHT], [Tay3]). C. Khare and J.-P. Wintenberger proved (almost) all cases of the Artin conjecture for $GL(2)$ over $\mathbb{Q}$ (cf. [Kh]). The '$R = T$ method' for $GSp(4)$ has been studied by A. Genestier and J. Tilouine ([GT], [Ti]).

6. Motivating examples — the case of $GSp(4)$

In this section, we give several results and expectations concerning automorphic representations of $GSp(4)$ from the viewpoint of ℓ-adic Galois representations and the Principle of Functoriality. We hope that the content of this section would motivate the participants of the workshop (and the readers of this volume) to study the case of $GSp(4)$ in detail.

6.1. Definition of $GSp(4)$.

To fix the notation, let $J \in M_4(\mathbb{Q})$ be the 4×4 matrix defined by

$$J := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

We define the *symplectic similitude group* $GSp(4)$ over $\mathbb{Q}$ by

$$GSp(4, R) := \{(X, \lambda) \in M_4(R) \times R^\times \mid {}^tXJX = \lambda J\},$$

where R is a $\mathbb{Q}$-algebra.

Remark 6.1. There are several different conventions in the literature according to different backgrounds. The group $GSp(4)$ is also written as $GSp(2)$ because its semisimple rank is equal to 2. Perhaps it may be a natural scene in a workshop that one asks "What is the size of the matrices for your GSp?", "Is your $GSp(2)$ the same as my $GSp(4)$?",... etc. Instead of J, one can also use

$$J' := \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$$

to obtain another group, which is isomorphic to $GSp(4)$. This seems the *official* definition of $GSp(4)$ in this workshop according to the webpage [3]. In fact, to discuss the relation between automorphic representations and classical Siegel modular forms, J' seems more convenient than J.

6.2. The Langlands correspondence for $GSp(4)$.

Let F be a global or a local field. Let us consider $GSp(4)$ as an algebraic group over F. The dual group $\widehat{GSp(4)}$ of $GSp(4)$ is isomorphic to $GSpin(5, \mathbb{C})$. Since $GSpin(5, \mathbb{C})$ is isomorphic to $GSp(4, \mathbb{C})$ via the spinor representation provided by Clifford algebra theory, we have :

$$\widehat{GSp(4)} = GSpin(5, \mathbb{C}) \cong GSp(4, \mathbb{C}), \qquad {}^LGSp(4) \cong GSp(4, \mathbb{C}) \times \Gamma_F.$$

The spinor representation

$$\mathrm{spin}\colon \widehat{GSp(4)} = GSpin(5, \mathbb{C}) \cong GSp(4, \mathbb{C}) \subset GL(4, \mathbb{C})$$

[3] http://math01.sci.osaka-cu.ac.jp/~furusawa/Hakuba2006/Hakuba%202006.html

is denoted by spin. For an automorphic representation π of $GSp(4,\mathbb{A}_F)$, the L-function $L(s,\mathrm{spin},\pi)$ is called the *spinor L-function*. On the other hand, a natural map $GSpin(5,\mathbb{C})\to GL(5,\mathbb{C})$ is denoted by st (standard). The L-function $L(s,\mathrm{st},\pi)$ is called the *standard L-function*.

Let F be a global field. We fix a prime number ℓ invertible in F, and an isomorphism $\mathbb{Q}_\ell\cong\mathbb{C}$. Then, the global Langlands correspondence for $GSp(4)$ is a conjectural correspondence between

Automorphic side: (L-packets of) automorphic representations π of $GSp(4,\mathbb{A}_F)$, and

Galois side: continuous homomorphisms $\rho\colon \Gamma_F\to GL(4,\overline{\mathbb{Q}}_\ell)$.

If ρ corresponds to π, one hopes the following equality holds

$$L(s,\rho_v)=L(s,\mathrm{spin},\pi_v)$$

for all but finitely many v.

It seems to the author that, up to now, the local Langlands correspondence for $GSp(4)$ is not completely known although great progress has been made in recent years when π is *generic* ([AS1], [AS2], [JS]). The local L-factor $L(s,\mathrm{spin},\pi_v)$ is yet to be defined for all v (without assuming other conjectures). A notion of L-packets for $GSp(4)$ is yet to be defined (even locally). Once everything is defined and the local Langlands correspondence is established for $GSp(4)$, it is natural to hope that the L-factors are equal for all v, and, moreover, the correspondence $\rho_v\leftrightarrow\pi_v$ is *the* local Langlands correspondence for all v (i.e. compatibility of the local and global Langlands correspondences).

6.3. **Results of R. Taylor, G. Laumon, R. Weissauer.** When $F=\mathbb{Q}$, the following results are known ([Tay1], [Lau1], [Lau2], [Wei]).

Theorem 6.2 (R. Taylor, R. Weissauer, G. Laumon (see [Wei], Theorem I))**.** *Let π be a cuspidal automorphic representation of $GSp(4,\mathbb{A}_\mathbb{Q})$ such that π_∞ is a holomorphic discrete series of weight (k_1,k_2), $k_1\geq k_2\geq 3$ (This means, in a classical language, the automorphy factor of the corresponding (vector valued if $k_1>k_2$) Siegel modular form is $\mathrm{Sym}^{k_1-k_2}\otimes\det^{\otimes k_2}$ as a representation of $GL(2,\mathbb{C})$). For each prime number ℓ, there exists a finite extension $E_{\pi,\ell}$ of $\mathbb{Q}_\ell$ and a 4-dimensional ℓ-adic representation*

$$\rho=\rho_{\pi,\ell}\colon\Gamma_\mathbb{Q}\longrightarrow GL(4,E_{\pi,\ell})$$

such that the equality

$$L(s,\rho_p)=L\big(s-(k_1+k_2-3)/2,\mathrm{spin},\pi_p\big)$$

holds for all but finitely many prime number p.

We say ρ *is an ℓ-adic representation associated to* π (this explains a part of the title of this article).

R. Taylor constructed $\rho_{\pi,\ell}$ under certain hypothesis using the congruence relation. On the other hand, G. Laumon and R. Weissauer (independently) calculated the Hasse-Weil zeta function of Siegel 3-folds by comparing the Selberg and Lefschetz trace formulae (Ihara-Langlands methods). Then, they constructed $\rho_{\pi,\ell}$ from the étale cohomology of Siegel 3-folds. Note that the Fundamental Lemma in this case was proved by T. Hales and R. Weissauer.

Remark 6.3. It is natural to expect that the image of $\rho_{\pi,\ell}$ should be contained in $GSp(4, E_{\pi,\ell})$. This does not seem to be known, up to now. However, this would be a consequence of the Poincaré duality for the étale cohomology of Siegel 3-folds, and the multiplicity one property of automorphic representations of $GSp(4)$ (see [Wei], Theorem IV). One also hopes that, in the near future, the development of the (twisted stabilized) trace formula for $GSp(4)$ would enable us to establish this property ([Ar2]).

Remark 6.4. Conversely, when a 4-dimensional ℓ-adic representation of Γ_F is given, it is natural to ask whether it is associated to an automorphic representation of $GSp(4, \mathbb{A}_F)$. In [RS], D. Ramakrishnan and F. Shahidi studied this problem for Galois representations obtained as a symmetric cube of the Tate module of an elliptic curve. In [Ti], J. Tilouine studied 4-dimensional ℓ-adic representations of $\Gamma_{\mathbb{Q}}$ defined by the Tate modules of abelian surfaces.

6.4. **Transfers concerning $GSp(4)$.** We give several important examples of transfers, which are predicted by the Principle of Functoriality, concerning $GSp(4)$ over a global field F. (Of course, there should be a local analogue. But we do not consider it here.)

Example 6.5 (Transfer from $GSp(4)$ to $GL(4)$)**.** Consider the spinor representation

$$\mathrm{spin}\colon \widehat{GSp(4)} = GSpin(5, \mathbb{C}) \cong GSp(4, \mathbb{C}) \subset GL(4, \mathbb{C}).$$

The Principle of Functoriality predicts that, for an automorphic representation π of $GSp(4, \mathbb{A}_F)$, there should exist an automorphic representation τ of $GL(4, \mathbb{A}_F)$ satisfying

$$L(s, \mathrm{spin}, \pi) = L(s, \tau).$$

If it exists, it is called the *transfer of π to $GL(4)$*. In recent years, great progress has been made in the construction of this transfer either by the Converse Theorem and the Langlands-Shahidi method ([AS1], [AS2]), or the (twisted stabilized) trace formula ([Wh1], [Wh2]).

Example 6.6 (Transfer from an inner form of $GSp(4)$ to $GSp(4)$)**.** Let G be an inner form of $GSp(4)$. Such G is called a *quaternion unitary group*, and constructed from a quaternion algebra as follows. Let B be a quaternion algebra over F, and

$B \ni b \mapsto \overline{b} \in B$ the main involution. Then, the reduced trace of $b \in B$ is $\overline{b} + b$ and the reduced norm of b is $\overline{b} \cdot b$. When B is split (i.e. isomorphic to the matrix algebra $M_2(F)$), the main involution is

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}.$$

The *quaternion unitary group* G_B is defined as

$$G_B(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B \otimes_F R) \;\middle|\; \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

for an F-algebra R. When $B = M_2(F)$, G_B is isomorphic to $GSp(4)$. The L-groups of G_B and $GSp(4)$ are the same. Since $GSp(4)$ is quasi-split (in fact, it is split), for an automorphic representation π of $G_B(\mathbb{A}_F)$, there should exist an automorphic representation τ of $GSp(4, \mathbb{A}_F)$ whose L-functions are the same. This is an analogue of the Jacquet-Langlands-Shimizu correspondence.

Example 6.7 (Endoscopic transfer)**.** Let

$$H := \big(GL(2) \times GL(2)\big)/GL(1)$$

be a unique nontrivial elliptic endoscopic group of $GSp(4)$, where the embedding of $GL(1)$ into $GL(2) \times GL(2)$ is given by

$$GL(1) \ni x \mapsto \left(\begin{pmatrix} x & \\ & x \end{pmatrix}, \begin{pmatrix} x^{-1} & \\ & x^{-1} \end{pmatrix} \right) \in GL(2) \times GL(2).$$

The dual group $\widehat{H}$ of H is given by

$$\widehat{H} := \big\{ (A, B) \in GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \;\big|\; \det A = \det B \big\},$$

and it is embedded into $\widehat{GSp(4)} = GSp(4, \mathbb{C})$ by

$$\widehat{H} \ni \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & & & b_1 \\ & a_2 & b_2 & \\ & c_2 & d_2 & \\ c_1 & & & d_1 \end{pmatrix} \in \widehat{GSp(4)}.$$

Let π be an automorphic representation of $H(\mathbb{A}_F)$, which is identified with a pair (π_1, π_2) of automorphic representations of $GL(2, \mathbb{A}_F)$ with same central character. Then, the Principle of Functoriality predicts there should exist an automorphic representation τ of $GSp(4, \mathbb{A}_F)$ satisfying

$$L(s, \mathrm{spin}, \tau) = L(s, \pi_1) \cdot L(s, \pi_2).$$

Examples of such representations are obtained via the *Yoshida lifting* (or *Saito-Kurokawa lifting* when one of π_1, π_2 is an Eisenstein series). In terms of the associated Galois representations, this operation corresponds to a direct sum :

$$\rho_{\tau,\ell} = \rho_{\pi_1,\ell} \oplus \rho_{\pi_2,\ell}.$$

6.5. **Arthur's classification for** $GSp(4)$**.** Finally, we show, for a global field F, how Arthur's (conjectural) classification (cf. [Ar2]) of automorphic representations of $GSp(4, \mathbb{A}_F)$ in the discrete spectrum of $L^2\big(GSp(4,F)\backslash GSp(4,\mathbb{A}_F)\big)$ can be explained in terms of Galois representations. This is a 'non-canonical' explanation. Usually, as in [Ar2], it is explained in terms of 'A-packets' (for A-packets, see Hiraga's article [Hi]).

Let π be a (discrete) automorphic representation of $GSp(4, \mathbb{A}_F)$. As usual, we expect there should exist a 4-dimensional ℓ-adic Galois representation $\rho = \rho_{\pi,\ell}$ associated to π. Then, it seems natural to classify π according to the decomposition of ρ into irreducible representations as follows.

(1) (General type) ρ is 4-dimensional, irreducible, pure of weight w.
(2) (Yoshida type) $\rho = \tau \oplus \tau'$ is a direct sum of τ and τ'. Both are 2-dimensional, irreducible, and pure of weight w.
(3) (Soudry type) $\rho = \tau \oplus \tau(1)$, where $\tau(1)$ is the Tate twist of τ, and τ is 2-dimensional, irreducible, pure of weight $w+1$. Hence, $\tau(1)$ is pure of weight $w-1$.
(4) (Saito-Kurokawa type) $\rho = \tau \oplus \chi \oplus \chi(1)$, where τ is 2-dimensional, irreducible, pure of weight w, and χ is 1-dimensional, pure of weight $w+1$.
(5) (Howe, Piatetski-Shapiro type) $\rho = \chi \oplus \chi(1) \oplus \chi' \oplus \chi(1)$, where χ, χ' are 1-dimensional and pure of weight $w+1$.
(6) (1-dimensional type) $\rho = \chi \oplus \chi(1) \oplus \chi(2) \oplus \chi(3)$, where χ is 1-dimensional and pure of weight $w+3$.

These names are taken from [Ar2]. The readers should not confuse 'general type' with 'generic', which is a different notion concerning the existence of Whittaker models (see Ichino's article [Ic]). One naturally expects that, geometrically, the above classification should correspond to the Lefschetz decomposition of the étale cohomology of Siegel 3-folds (cf. [Ar1], §9, see also [Ko2]).

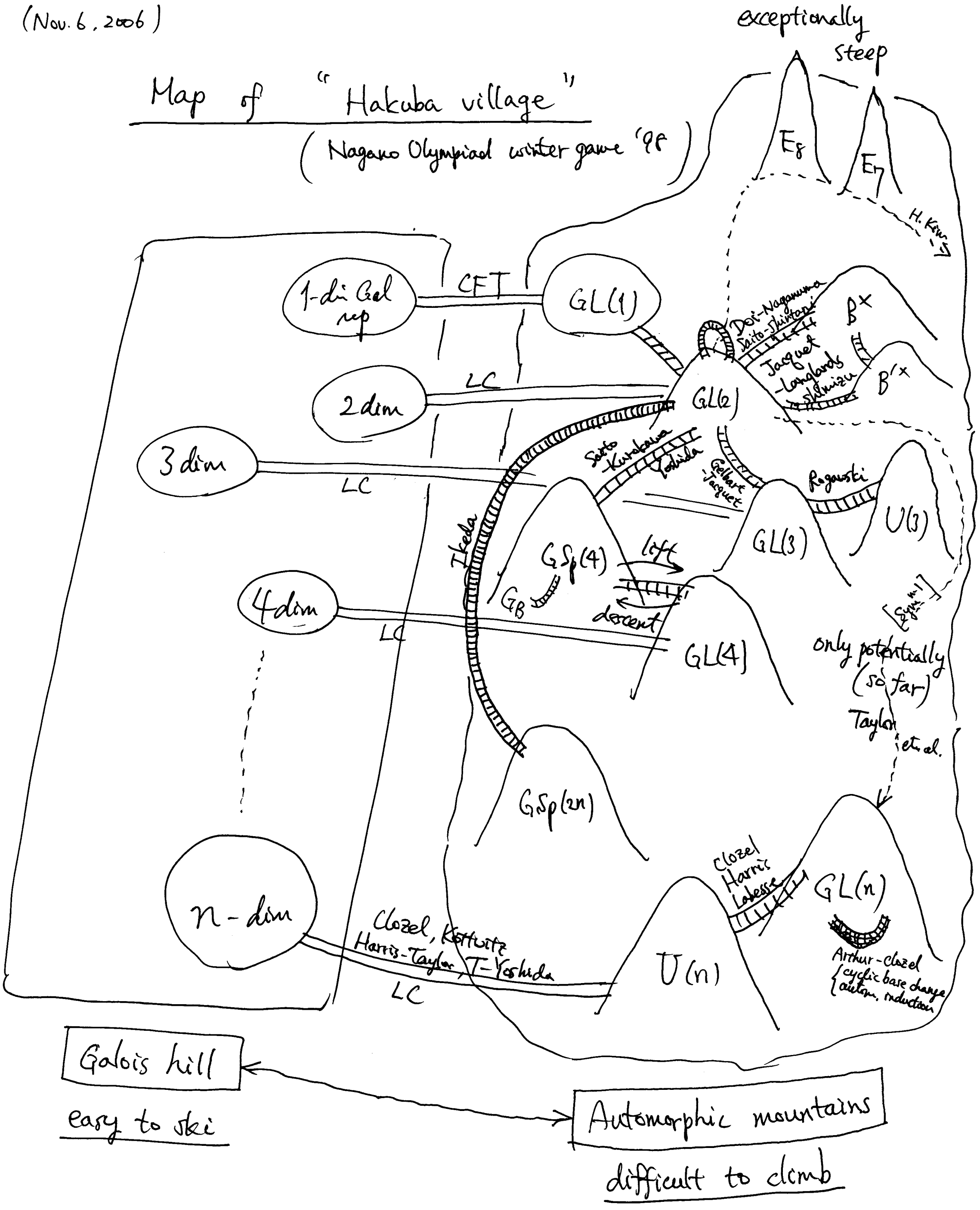
In contrast to the case of $GL(n)$, many *cuspidal* automorphic representations of $GSp(4)$ correspond to reducible Galois representations. Some of them do not even satisfy the Ramanujan conjecture (i.e. the associated 4-dimensional Galois representations are not pure). Examples of such representations are obtained via the *Saito-Kurokawa lifting* (cf. [Sch]).

An automorphic representation π of type (1) or (2) satisfies the Ramanujan conjecture. Sometimes, π of type (2) is called *endoscopic* (examples are obtained via the *Yoshida lifting*), and π of type (3),(4), or (5) is called *CAP* (Cuspidal Associated with Parabolic). Precisely speaking, π of type (3) is associated with the Klingen parabolic subgroup, π of type (4) is associated with the Siegel parabolic subgroup, and π of type (5) is associated with the Borel subgroup.

(Nov. 6, 2006)

Map of "Hakuba village"

(Nagano Olympiad winter game '98)



Conclusion :

- Langlands corresp = highway (difficult & expensive to construct but, convenient to use)
- We need lifts!

[arith. geom. is used in crucial ways (Shimura var, ell. curves, HBAV, Calabi-Yau)]

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