

T. Yoshida, Compatibility of local and global Langlands correspondence (with Richard Taylor)

Langlands corresp.

l : prime, $\iota: \bar{\mathbb{Q}}_l \xrightarrow{\cong} \mathbb{C}$ (as fields)

GLC $[L:\mathbb{Q}] < \infty, n \geq 1$

algebraic autom. rep. of $GL_n(\mathbb{A}_L)$

$\left\{ \begin{array}{l} \text{unram. at almost all } p \\ \text{de Rham at } l \end{array} \right.$
 n -dim'l l -adic rep'n of $G_L := \text{Gal}(\bar{L}/L)$

$\Pi \leftrightarrow R_{l,2}(\Pi)$

$G_L \rightarrow GL_n(\bar{\mathbb{Q}}_l)$

$n=1$ --- global class field theory (compatible family)

ex. $L = \mathbb{Q}$ $n=1$ $\Pi|_{\mathbb{R}_{>0}^\times}(\alpha) = x^k$ ($k \in \mathbb{Z}$)

alg. Hecke char. $\Pi: \mathbb{A}^\times/\mathbb{Q}^\times \rightarrow \mathbb{C}^\times$

$G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow G_{\mathbb{Q}}^{\text{ab}} \xrightarrow{(\cong)} \hat{\mathbb{Z}}^\times$
 cyclotomic theory $\hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$
 \mathbb{Q} -alg. $\left\{ \begin{array}{l} \mathbb{A}^\infty := \hat{\mathbb{Z}} \otimes \mathbb{Q} \\ \mathbb{A} := \mathbb{A}^\infty \times \mathbb{R} \end{array} \right.$

$N \geq 1$

$\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\cong} (\mathbb{Z}/N)^\times$
 $p \nmid N$ Frobp $\mapsto p \pmod N$

$\hat{\mathbb{Z}}^\times \xrightarrow{\cong} \mathbb{A}^\times/\mathbb{Q}^\times \cdot \mathbb{R}_{>0}^\times \xrightarrow{\Pi'} \mathbb{C}^\times \xrightarrow{\cong} \bar{\mathbb{Q}}_l^\times$

$x \mapsto \Pi(x) \cdot (x_\infty)^{-k} \cdot (2x_l)^k$

$\hat{\mathbb{Z}}^\times = \prod_l \mathbb{Z}_l^\times$

$\Rightarrow R_{l,2}(\Pi): G_{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_l^\times$

ex. L : imag. quad / CM field, $n=1$
 --- CM theory

ex. $n=2, L=\mathbb{Q}$ (holom.)

cuspidal autom rep'n = elliptic cusp. modular form (Hecke eigen)

$$\rightsquigarrow G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}_p})$$

Eichler-Shimura, Deligne

↑
l-adic cohomology of modular curves

L.L.C. $[K:\mathbb{Q}_p] < \infty, n \geq 1$
(Thm)

irred. admissible rep'n of $GL_n(K)$

↔ n-dim'l WD-rep'n of W_K (F-ss)

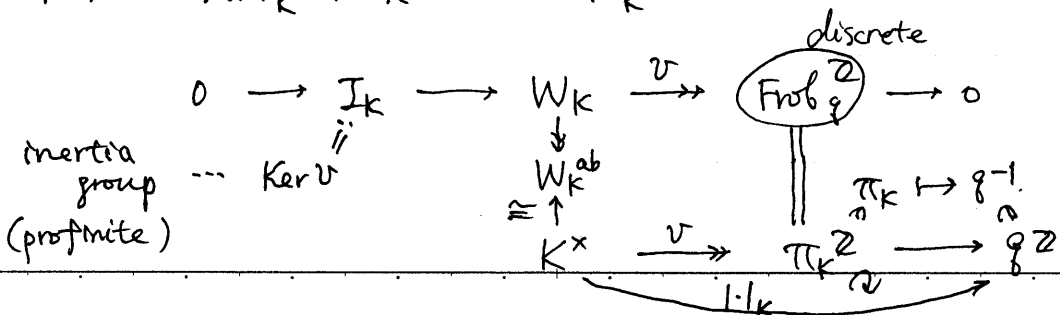
$$\pi \mapsto \text{rec}(\pi)$$

$$\left(\begin{array}{l} K \supset \mathcal{O}_K, v: K^\times \rightarrow \mathbb{Z} \\ \downarrow \\ \mathcal{O}_K/\pi_K = k(v) \cong \mathbb{F}_q \end{array} \right)$$

$$v: G_K = \text{Gal}(\overline{K}/K) \longrightarrow \text{Gal}\left(\frac{\overline{k(v)}}{k(v)}\right) = \hat{\mathbb{Z}} = \langle \overset{\text{geom}}{\text{Frob}}_q \rangle \quad (\alpha \mapsto \alpha^q)^{-1}$$

$$W_K := v^{-1}(\mathbb{Z}) \dots \text{Weil gp.}$$

LCFT $\text{Art}_K: K^\times \xrightarrow{\cong} W_K^{\text{ab}}$



WD-rep'n of W_K / \mathbb{C} or \mathbb{Q}_ℓ = (V, r, N)

- $\dim_{\mathbb{C}} V < \infty$
- $r: W_K \rightarrow GL(V)$
- $N \in \text{End}(V)$

$$r(\sigma) \cdot N \cdot r(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)|_K \cdot N$$

... ℓ -independent notion for ℓ -adic rep. of G_K

- $r|_{I_K}$ factors thru finite quotients (s.s.)
- $(V, r, N)^{F\text{-ss}} = (V, r^{F\text{-ss}}, N)$
- can make Frobenius act semisimply

ℓ -adic rep'n of G_K $\xleftrightarrow{1:1}$ WD-rep'n / $\overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$
 (Frob \mapsto e.v. are ℓ -adic units) $\rho \mapsto \text{WD}(\rho)$

$$\rho(\sigma) = r(\sigma) \cdot \exp(t_\ell(\sigma) \cdot N) \leftarrow (V, r, N)$$

($t_\ell: I_K \rightarrow \mathbb{Z}_\ell$ tame character)

JAMS '92 IHS '92

Thm (Kottwitz, Clozel)

L : CM field, Π : autom. rep'n of $GL_n(\mathcal{O}_L)$
 c : cpx conj. cuspidal

$\exists R_{\mathcal{O}_L, L}(\Pi):$
 $G_L \rightarrow GL_n(\overline{\mathbb{Q}_\ell})$
 (image of Shimura var.)

- regular algebraic
- $\Pi^v = \Pi^c$... (descends to unitary group / L^+)
- discrete series at a fin place \mathcal{Z} of L

(image of Jacquet-Langlands corres. from div. alg.)

• For a.a. (almost all) v

$$\Pi_v : \text{unram. principal series} \Rightarrow \left\{ \begin{array}{l} \text{Satake param} \\ \text{of } \Pi_v \end{array} \right\} = \left\{ \begin{array}{l} \text{eigenvalues} \\ \text{of } R_{\ell,2}(\Pi)(\text{Frob}_v) \end{array} \right\}$$

Thm $\forall v \neq l$, $\text{WD}(R_{\ell,2}(\Pi)|_{G_{L_v}})^{F\text{-ss}} = \text{rec}(\Pi_v)$
 compatibility up to semisimplification: Harris-Taylor

for N (monodromy): Taylor-Yoshida
 see arXiv

Construction of $R_{\ell,2}(\Pi) \subset \underline{H^*(X)}$ ℓ -adic $\check{\text{etale}}$ cohom.
 X : Shimura var.

$L = E/\mathbb{Q}$: mag. quad. field

Unitary Shimura var (of PEL-type)

G : certain unitary similitude group $\leftarrow U(1, n-1)/\mathbb{R}$
 $G \otimes_{\mathbb{Q}} E = GL_n \times GL_1$

$U \subset G(\mathbb{A}^{\infty})$, X_U : represents
small open cpt. (scheme/ E) \rightarrow (Sets)
 E -alg $S \mapsto \left\{ (A, \lambda, i, \bar{\eta}) \right\} / \sim$
PEL diag.

$$\left\{ \begin{array}{l} A/S : \text{abel. scheme of dim } n. \quad \text{Lie } A : \text{loc free } \mathcal{O}_S\text{-mod} \\ i : E \hookrightarrow \text{End}(A) \otimes \mathbb{Q} \quad \text{of rank } n \\ E \text{ acts via } E \rightarrow \mathcal{O}_S \text{ on } \text{Lie}^+(A) : \text{rk } 1 \\ \quad \quad \quad E \hookrightarrow \mathcal{O}_S \text{ on } \text{Lie}^-(A) : \text{rk } n-1 \\ \quad \quad \quad \text{Lie}(A) = \text{Lie}^+(A) \oplus \text{Lie}^-(A) \\ \lambda : A \rightarrow A^\vee \quad \text{Rosati } *_{\lambda}|_E = c \\ \eta : \text{level } U\text{-str.} \end{array} \right.$$

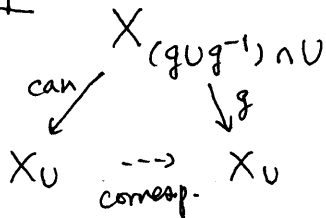
X_U : (quasi-proj.) smooth var/ E of dim $n-1$

(cf. Hida, p -adic autom forms on Shimura var. (2004))

$$H^*(X_U) := \varinjlim_U H_{\text{ét}}^*(X_U \times_{\mathbb{F}} \overline{\mathbb{F}}, \overline{\mathbb{Q}_\ell})$$

$$G(A^\infty) \times \text{Gal}(\overline{\mathbb{F}}/E)$$

Hecke corresp.



$$\pi = \pi_\infty \times \pi^\infty : \text{autom. rep'n of } G(A) \uparrow$$

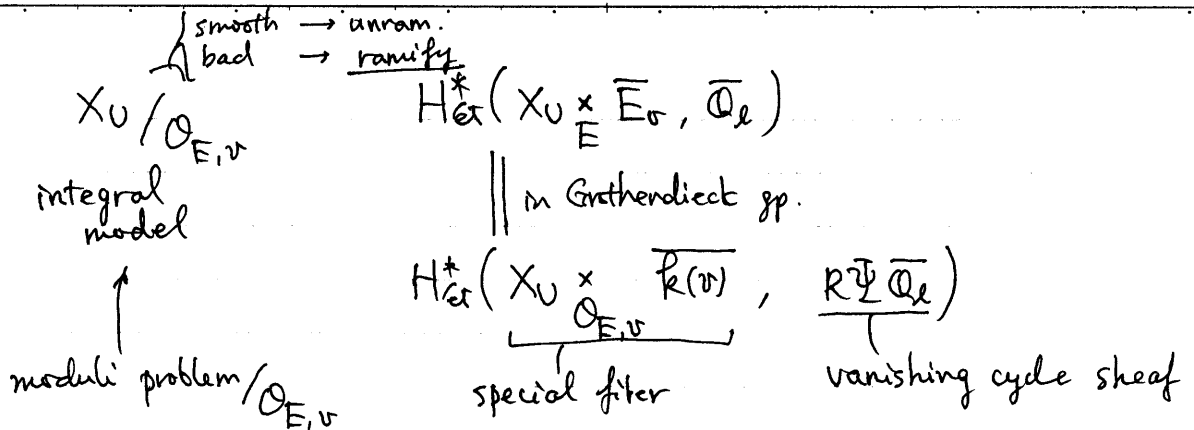
$$H^*(X) [\pi^\infty] \doteq R_{\ell,1}(\Pi)$$

isotypic part

$$\Pi : GL_n(\mathcal{O}_E)$$

$$\rightarrow R_{\ell,2}(\Pi) \Big|_{G_{E_v}} \quad \dots \quad H_{\text{ét}}^*(X_U \times_{\mathbb{F}} \overline{\mathbb{F}}, \overline{\mathbb{Q}_\ell}) \Big|_{G_{E_v}} \hookrightarrow G_{E_v} \text{ restriction}$$

v : place of E



• Local properties of integral model

--- deformation theory of A.V.
 \updownarrow Serre-Tate
 deform. of p -div. gp (Barsotti-Tate gp.)

$$p = u \cdot u^c \text{ in } E$$

$$A[p^\infty] = \underbrace{A[u^\infty]}_{\substack{\text{ht} = n, \text{dim} = 1 \\ p\text{-div. gp.}}} \oplus A[(u^c)^\infty]$$

• count points on $X_U \times_{\mathcal{O}_{E, \nu}} \overline{R}(\nu)$ \Rightarrow Lefschetz trace formula
 Honda-Tate theory "mod p " \updownarrow compare
 Arthur-Selberg trace formula

(V, r, N)
 \uparrow
 monodromy

• weight spectral sequence (Rapoport-Zink) H-T.
 semistable reduction II ν : tempered
 \Rightarrow weight monodromy conj. Galois | Autom.
---|--- Generalized Ramanujan Conj.

