Number theory seminar @ Osaka Univ.

T. Yoshida, Compatibility of local and global Langlands correspondence (with Richard Taylor)

Langlands corresp.

ℓ : prime, $\quad \iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$ (as fields)

GLC $\quad [L:\mathbb{Q}] < \infty, \; n \geq 1$

$\left\{\begin{array}{l}\text{unram. at almost all } p\\ \text{de Rham at } \ell\end{array}\right. \downarrow$

algebraic autom. rep. of $GL_n(\mathbb{A}_L)$ $\longleftrightarrow$ n-dim'l ℓ-adic rep'n of $G_L := \mathrm{Gal}(\overline{L}/L)$

$\Pi \longleftrightarrow R_{\ell,\iota}(\Pi)$

$G_L \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

$n=1$ ⋯ global class field theory $\qquad$ (compatible family)

ex. $L = \mathbb{Q} \quad n = 1$

alg. Hecke char. $\Pi : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$

$\Pi|_{\mathbb{R}^\times_{>0}}(x) = x^k \; (k \in \mathbb{Z})$

$\hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$

$$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow G_{\mathbb{Q}}^{ab} \xrightarrow{(\cong)} \hat{\mathbb{Z}}^\times$$

cyclotomic theory

$\mathbb{Q}$-alg. $\left\{\begin{array}{l}\mathbb{A}^\infty := \hat{\mathbb{Z}} \otimes \mathbb{Q}\\ \mathbb{A} := \mathbb{A}^\infty \times \mathbb{R}\end{array}\right.$

$N \geq 1$

$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N)^\times$

$p \nmid N \qquad \mathrm{Frob}_p \mapsto p \bmod N$

$$\hat{\mathbb{Z}}^\times \xrightarrow{\cong} \mathbb{A}^\times/\mathbb{Q}^\times \cdot \mathbb{R}^\times_{>0} \xrightarrow{\Pi'} \mathbb{C}^\times \xrightarrow[\cong]{\iota^{-1}} \overline{\mathbb{Q}}_\ell^\times$$

$$x \longmapsto \Pi(x) \cdot (x_\infty)^{-k} \cdot (\iota x_\ell)^k$$

$\hat{\mathbb{Z}}^\times = \prod_\ell \mathbb{Z}_\ell^\times$

$\Rightarrow \quad R_{\ell,\iota}(\Pi) : G_{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

ex. L : imag. quad / CM field, $\; n = 1$

⋯ CM theory

ex. $n = 2$, $L = \mathbb{Q}$ (holom.)

cuspidal autom rep'n = elliptic cusp. modular form (Hecke eigen)

$\rightsquigarrow G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}_\ell})$

Eichler-Shimura, Deligne

$\uparrow$ ℓ-adic cohomology of modular curves

L.L.C. (Thm) $[K : \mathbb{Q}_p] < \infty$, $n \geq 1$

irred. admissible rep'n of $GL_n(K)$ $\longleftrightarrow$ n-dim'l WD-rep'n of W_K (F-ss)

$\pi \mapsto \mathrm{rec}(\pi)$

$$\left(\begin{array}{l} K \supset \mathcal{O}_K \ni \pi_K, \quad v : K^\times \twoheadrightarrow \mathbb{Z} \\ \mathcal{O}_K/\pi_K = k(v) \cong \mathbb{F}_q \end{array} \right)$$

$$v : G_K = \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \mathrm{Gal}(\overline{k(v)}/k(v)) = \hat{\mathbb{Z}} = \langle \overset{\text{geom}}{\mathrm{Frob}_q} \rangle$$

$(x \mapsto x^q)^{-1}$

$W_K := v^{-1}(\mathbb{Z})$ ⋯ Weil gp.

LCFT $\mathrm{Art}_K : K^\times \xrightarrow{\cong} W_K^{ab}$

$$\begin{tikzcd}
0 \arrow[r] & I_K \arrow[r] \arrow[d, equal] & W_K \arrow[r, two heads, "v"] \arrow[d] & \boxed{\mathrm{Frob}_q^{\mathbb{Z}}}^{\text{discrete}} \arrow[r] \arrow[dd, equal] & 0 \\
\text{inertia group (profinite)} \cdots & \mathrm{Ker}\, v & W_K^{ab} & & \pi_K \mapsto q^{-1} \\
& & K^\times \arrow[u, "\cong"] \arrow[r, two heads, "v"] \arrow[rr, bend right, "|\cdot|_K"'] & \pi_K^{\mathbb{Z}} \wr \arrow[r] & q^{\mathbb{Z}}
\end{tikzcd}$$

<u>WD-rep'n of W_K</u> / $\mathbb{C}$ or $\mathbb{Q}_\ell$ $= (V, r, N)$

- $\dim_{\mathbb{C}} V < \infty$
- $r : W_K \longrightarrow GL(V)$
- $N \in \mathrm{End}(V)$

$$r(\sigma)\cdot N\cdot r(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K \cdot N$$

… ℓ-independent notion for ℓ-adic rep. of G_K

- $r|_{I_K}$ factors thru finite quotients (s.s.)

$$(V, r, N)^{F\text{-}ss} = (V, \boxed{r^{F\text{-}ss}}, N)$$

- can make Frobenius act semisimply

ℓ-adic rep'n of G_K (Frob $\mapsto$ e.v. are ℓ-adic units) $\xleftrightarrow{1:1}$ WD-rep'n / $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$

$\ell \neq p$

$$\rho \mapsto WD(\rho)$$

$$\rho(\sigma) = r(\sigma)\cdot\exp(t_\ell(\sigma)\cdot N) \mapsfrom (V, r, N)$$

$$(t_\ell : I_K \twoheadrightarrow \mathbb{Z}_\ell \text{ tame character})$$

JAMS '92 IHES '92

<u>Thm</u> (Kottwitz, Clozel)

L : CM field, Π : autom. rep'n of $GL_n(\mathbb{A}_L)$ cuspidal

$c \curvearrowright$

c : cpx conj.

- regular algebraic
- $\Pi^\vee = \Pi^c$ … (descends to unitary group / L^+)
- <u>discrete series</u> at a fin place z of L (image of Jacquet-Langlands corres. from div. alg.)

$\exists R_{\ell,\iota}(\Pi) : G_L \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

(cohom. of Shimura var.)

- For a.a. (almost all) v

$$\Pi_v : \underset{\text{series}}{\text{unram. principal}} \Rightarrow \left\{ \begin{array}{c} \text{Satake param} \\ \text{of } \Pi_v \end{array} \right\} = \left\{ \begin{array}{l} \text{eigenvalues} \\ \text{of } R_{\ell,\iota}(\Pi)(\mathrm{Frob}_v) \end{array} \right\}$$

$\underline{\text{Thm}}$ $\forall v \nmid \ell$, $\iota WD\left(R_{\ell,\iota}(\Pi)\big|_{G_{L_v}}\right)^{F\text{-ss}} = \mathrm{rec}(\Pi_v)$

compatibility

$\underline{\text{up to semisimplification}}$: Harris-Taylor

$\underline{\text{for } N \text{ (monodromy)}}$: Taylor-Yoshida

see arXiv

Construction of $R_{\ell,\iota}(\Pi) \subset \underbrace{H^*(X)}$ ℓ-adic $\overset{\text{étale}}{\text{cohom.}}$

X : Shimura var.

$L = E/\mathbb{Q}$: imag. quad. field

Unitary Shimura var (of PEL-type)

$\to$ $G/\mathbb{Q}$: certain unitary similitude group $\leftarrow U(1, n-1)/\mathbb{R}$

$$G \underset{\mathbb{Q}}{\otimes} E = GL_n \times GL_1$$

$\underset{\text{small open cpt.}}{U} \subset G(\mathbb{A}^\infty)$, X_U : represents

$$\begin{array}{ccc} (\text{scheme}/E) & \longrightarrow & (\text{Sets}) \\ E\text{-alg } S & \longmapsto & \{(A, \underset{P}{\underset{\uparrow}{\lambda}}, \underset{E}{\underset{\uparrow}{i}}, \underset{L}{\underset{\uparrow}{\bar{\eta}}})\} \big/ \underset{\text{isog.}}{\sim} \end{array}$$

$A/_S$: abel. scheme of dim n. Lie A : loc free $\mathcal{O}_S$-mod of rank n

$i : E \hookrightarrow \mathrm{End}(A)\otimes\mathbb{Q}$

E acts via $E \to \mathcal{O}_S$ on $\mathrm{Lie}^+(A)$: rk 1

$E \xrightarrow{c} \mathcal{O}_S$ on $\mathrm{Lie}^-(A)$: rk $n-1$

$\mathrm{Lie}(A) = \mathrm{Lie}^+(A) \oplus \mathrm{Lie}^-(A)$

$\lambda : A \longrightarrow A^\vee$ Rosati $*_\lambda|_E = c$

$\bar\eta$: level U-str.

X_U : (quasi-proj.) smooth var $/_E$ of dim $n-1$

(cf. Hida, p-adic autom forms on Shimura var. (2004))

$$H^*(X_U) := \varinjlim_U H^*_{et}(X_U \times_E \bar E, \bar{\mathbb{Q}}_\ell)$$

$\circlearrowleft$

$G(\mathbb{A}^\infty) \times \mathrm{Gal}(\bar E/E)$

Hecke <u>corresp.</u>

$$\begin{tikzcd} & X_{(gUg^{-1})\cap U} \arrow[dl, "can"'] \arrow[dr, "g"] & \\ X_U \arrow[rr, dashed, "corresp."'] & & X_U \end{tikzcd}$$

$\pi = \pi_\infty \times \overset{G(\mathbb{A}^\infty)}{\textcircled{π^∞}}$: autom. rep'n of $G(\mathbb{A})$

$\nwarrow$

$$\boxed{\underbrace{H^*(x)[\pi^\infty]}_{\text{isotypic part}} \doteqdot R_{\ell,\iota}(\Pi)}$$

$\Pi : GL_n(\mathbb{A}_E)$

$\to R_{\ell,\iota}(\Pi)|_{G_{E_v}}$ --- $H^*_{et}(X_U \times_E \bar E, \bar{\mathbb{Q}}_\ell) \circlearrowleft G_E$ } restriction

$\times \bar E_v$ $\circlearrowleft G_{E_v}$

v : place of E

$$\begin{cases} \text{smooth} \to \text{unram.} \\ \text{bad} \to \underline{\text{ramify}} \end{cases}$$

$X_U / \mathcal{O}_{E,v}$

integral model

↑

moduli problem $/\mathcal{O}_{E,v}$

$$H^*_{et}(X_U \times_E \overline{E}_v, \overline{\mathbb{Q}}_\ell)$$

$\Vert$ in Grothendieck gp.

$$H^*_{et}(\underbrace{X_U \times_{\mathcal{O}_{E,v}} \overline{k(v)}}_{\text{special fiber}}, \underbrace{R\Psi\overline{\mathbb{Q}}_\ell}_{\text{vanishing cycle sheaf}})$$

- <u>Local properties</u> of integral model

--- deformation theory of A.V.

↕ ← Serre-Tate

deform. of p-div. gp (Barsotti-Tate gp.)

$p = u \cdot u^c$ in E

$$A[p^\infty] = \underbrace{A[u^\infty]}_{\substack{ht=n,\ dim=1 \\ \text{p-div. gp.}}} \oplus A[(u^c)^\infty]$$

- <u>count points</u> on $X_U \times_{\mathcal{O}_{E,v}} \overline{k(v)}$ ⇒ Lefschetz trace formula

Honda-Tate theory "mod p"

↕ compare

Arthur-Selberg trace formula

$$(V, r, \underset{\substack{\uparrow \\ \text{monodromy}}}{N})$$

- weight spectral sequence (Rapoport-Zink)

semistable reduction

⇒ weight monodromy conj. (purity)

H-T.

Π_v: tempered

Galois | Autom.

-- | -- Generalized Ramanujan Conj.

Deligne :