セミナー「志村多様体の数論幾何」

題　　目：1. 類体論から志村多様体論へ
2. 局所 Langlands 対応とその幾何学的実現
3. 最近の話題 (1)
4. 最近の話題 (2)

講 演 者：**伊藤哲史，吉田輝義 (京大理・学振)**

日　　時：**・4月11日（月）16：30〜18：00**

・4月12日（火）〜14日（木）15：00〜16：30

場　　所：京都大学理学部3号館(数学教室) 205講義室

アブストラクト：

CM 体や局所体上の GL(n)の Langlands 対応は，ユニタリ型志村多様体の数論幾何を用いて，多くの場合に幾何学的実現が構成されています．この集中セミナーでは，数論幾何学の基本的な概念の復習から始めて，GL(n)の Langlands 対応やユニタリ型志村多様体の数論幾何に関する Harris-Taylor の仕事の概略と，その周辺の最近の話題を解説します．
意欲ある学部生・大学院生の参加も歓迎します．

1. 類体論から志村多様体論へ (吉田)
CM 体上の GL(n)の保型表現に対応した n 次元 Galois 表現の構成(大域 Langlands 対応)は，虚数乗法論(n=1 の場合，類体論の実現)の一般化と考えられる．ユニタリ型志村多様体を用いた Galois 表現の構成(主に Kottwitz，Clozel による) のために必要な数論幾何学を概観する．
2. 局所 Langlands 対応とその幾何学的実現 (伊藤)
Harris-Taylor の仕事は，ユニタリ型志村多様体の悪い還元の様子を詳細に研究し，大域 Langlands 対応の局所的な振る舞いを調べることで局所 Langlands 対応を証明したものである．これは大域類体論から局所類体論を導くことの幾何学化・非可換化にあたるもので，非可換 Lubin-Tate 理論(Carayol のプログラム)と呼ばれる．
3. 最近の話題 (1) (吉田)
分岐が tame の場合の非可換 Lubin-Tate 理論の Deligne-Lusztig 理論を用いた局所的構成，および Harris-Taylor の結果をモノドロミー作用素に関して精密化した，大域・局所 Langlands 対応の整合性の証明について解説する．
4. 最近の話題 (2) (伊藤)
正標数の体上のユニタリ型志村多様体の幾何学，特に高次元井草多様体のコンパクト化とその応用について述べる．また，Rapoport-Zink による p 進一意化理論について解説する．これは，非可換 Lubin-Tate 理論とユニタリ型志村多様体の数論幾何との関係を深く一般化したものである．

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吉田さん

Date 2005 · 4 · 11 No. 1

集中セミナー：志村多様体の数論幾何

Arithmetic Geometry of Shimura varieties

Global Langlands correspondence (Conj.)

$L/\mathbb{Q}$: fin. ext, $n \geq 1$

l : prime, $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\cong} \mathbb{C}$

algebraic automorphic rep'n of $GL_n(\mathbb{A}_L)$ $\longleftrightarrow$ n-dim'l l-adic Galois rep'n of $G_L := Gal(\overline{L}/L)$

$G_L \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$

$\Pi \longmapsto R = R_{l,\iota}(\Pi)$

Theme : Construction of $\Pi \longmapsto R_{l,\iota}(\Pi)$

$n=1$,
- $L=\mathbb{Q}$ Cyclotomic Fields
- $L/\mathbb{Q}$: imaginary quadratic Complex Multiplication of Elliptic Curves
- L : CM field (totally imaginary quad / tot. real) CM of Abelian Var. (Shimura, Taniyama, Weil)

$n=2$,
- $L=\mathbb{Q}$ (holomorphic elliptic modular form) Eichler-Shimura, Deligne (modular curve)
- L : totally real (Hilbert modular form) (Shimura curve)

$n=$ general L : CM field + some conditions

Kottwitz, Clozel (unitary Shimura varieties)

@ l-adic $\overset{\text{étale}}{\text{cohomology}}$ of Shimura varieties $/L$

$\Uparrow$

$(n-1)$-dimensional (quasi-)projective smooth var$/L$

Rem. $\Pi \leftarrow\!\!\dashv R$ (converse)

$n=1$: Global Class Field Theory

$n=2$: $L=\mathbb{Q}$, holom Taniyama - Shimura conj.

(R=T thm)

Cyclotomic theory

K : field,

$n \geq 1$, $\mu_n = \{n\text{-th roots of unity}\} \subset \overline{K}$

$K(\mu_n)$: splitting field of $X^n - 1$.

$(\mathrm{char}\, K, n) = 1 \Rightarrow \mu_n$: cyclic group of order n.

$\mathbb{Z}/n\mathbb{Z} \ni 1 \longmapsto \underbrace{\zeta_n}_{\text{primitive root of unity}} \in \mu_n$

$$\begin{tikzcd}
\mathrm{Gal}(K(\mu_n)/K) \arrow[r, hook] & \mathrm{Aut}(\mu_n) \arrow[d, equal, "\wr", "\text{can.}"'] & \ni (\zeta \mapsto \zeta^a) \quad \forall \zeta \\
& (\mathbb{Z}/n)^\times & \ni a \bmod n \arrow[u, mapsto]
\end{tikzcd}$$

$K=\mathbb{Q}$

$\Rightarrow$ This is isomorphism.

(irreducibility of cyclotomic polynomials)

$({}^{\forall}p \nmid n)$ $\mathrm{Frob}_p \longmapsto p \bmod n$

prime

$(\mathbb{Z}/n)^\times$ ... generated by $p \bmod n$

$\Downarrow$

surjective

※ L/K : Galois , $[K:\mathbb{Q}] < \infty$

$P \in \mathrm{Spec}(\mathcal{O}_K) \setminus \{0\}$

$\uparrow$ integer ring of K

<u>unramified</u> in L $\quad P\mathcal{O}_L = Q_1 \cdots Q_g \quad \underset{(i \neq j)}{(Q_i \neq Q_j)}$

$Q := Q_1$, $k(Q) = \mathcal{O}_L/Q$

$k(P) = \mathcal{O}_K/P$

$\quad \| $

$\quad \mathbb{F}_q$

$$\begin{tikzcd}
\mathrm{Gal}(L/K) & \\
\{\sigma \mid \sigma(Q) = Q\} \arrow[u, phantom, "\cup"] \arrow[r, "\cong"] & \mathrm{Gal}(k(Q)/k(P)) \arrow[d, equal] \\
\sigma \arrow[r, mapsto] & \sigma|_{\mathcal{O}_L} \bmod Q \\
\mathrm{Frob}_Q \arrow[r, leftrightarrow] & \mathrm{Frob}_q \in \langle \mathrm{Frob}_q ; x \mapsto x^q \rangle
\end{tikzcd}$$

$\mathrm{Frob}_P :=$ conjugacy class of Frob_Q

$\hookrightarrow \mathrm{Frob}_p \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$

$p \nmid n$ μ_n has <u>good reduction</u> at p

$\{\bar{1}, \bar{\zeta}_n, \bar{\zeta}_n^2, \cdots, \bar{\zeta}_n^{n-1}\} = \mu_n$

$\cdots$ modulo prime of $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$ above p , $\underset{(i \neq j)}{\bar{\zeta}_n^i \neq \bar{\zeta}_n^j}$

$\downarrow$

$p \nmid n$: unramified in $\mathbb{Q}(\zeta_n)$

$\mathrm{Frob}_p : \zeta \mapsto \zeta^p$

Take limit w.r.t. n $\quad \bigcup_n \mathbb{Q}(\mu_n)$, $m \mid n$

$\mathbb{Q}(\mu_m) \subset \mathbb{Q}(\mu_n)$

$$\begin{tikzcd} & \bigcup_n \mathbb{Q}(\mu_n) \arrow[d, equal] \\ \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \arrow[d, "\cong"] \\ & \varprojlim_n (\mathbb{Z}/n)^\times = \hat{\mathbb{Z}}^\times \end{tikzcd}$$

$$\left(\begin{aligned} \hat{\mathbb{Z}} &:= \varprojlim_n \mathbb{Z}/n = \prod_p \mathbb{Z}_p \\ \mathbb{Z}_p &:= \varprojlim_n \mathbb{Z}/p^n \\ \hat{\mathbb{Z}}^\times &= \prod_p \mathbb{Z}_p^\times \end{aligned} \right.$$

Adelic Formulation

$$\left. \begin{aligned} \mathbb{A}^\infty &:= \hat{\mathbb{Z}} \otimes \mathbb{Q} \\ \mathbb{A} &:= \mathbb{R} \times \mathbb{A}^\infty \end{aligned} \right\} \mathbb{Q}\text{-alg.}$$

$$(x_\infty, \boxed{x^\infty}) \in \mathbb{A}, \quad x^\infty = (x_p)_p$$

$$\begin{tikzcd} \mathbb{A}^\infty \arrow[r, hook] & \prod_p (\mathbb{Z}_p \otimes \mathbb{Q}) \\ & (x_p)_p \quad \mathbb{Z}_p \otimes \mathbb{Q} = \mathbb{Q}_p \end{tikzcd}$$

$$\mathbb{A}^\times \xleftarrow{\cong} \hat{\mathbb{Z}}^\times \cdot \mathbb{Q}^\times \cdot \mathbb{R}^\times_{>0}$$

$\uparrow$ idele group

Algebraic Hecke character := character

$$\mathbb{A}^\times/\mathbb{Q}^\times \xrightarrow[\text{hom}]{\pi} \mathbb{C}^\times \quad \text{continuous}$$

such that

$$\pi\big|_{\mathbb{R}^\times_{>0}}(x_\infty) = x_\infty^k \quad (k \in \mathbb{Z})$$

$$\boxed{\ell : \text{prime}, \quad \iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}}$$

$$\begin{tikzcd}
R_{\ell,\iota}(\pi) : G_{\mathbb{Q}} \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[r, "\text{cyclo}"'] & \hat{\mathbb{Z}}^\times \arrow[dl, bend left, "\cong"'] & \\
& \mathbb{A}^\times/\mathbb{Q}^\times\cdot\mathbb{R}^\times_{>0} \arrow[r] & \mathbb{C}^\times \arrow[r, "\iota^{-1}"] & \overline{\mathbb{Q}}_\ell^\times
\end{tikzcd}$$

$$\uparrow$$

$$\left((x_\infty, (x_p)_p) \longmapsto \pi(x)\cdot x_\infty^{-k}\cdot(\iota x_\ell)^k \right)$$

<u>ex.</u> $\pi = |\cdot|_{\mathbb{Q}} = \prod_v |\cdot|_{\mathbb{Q}_v}$

$k=1$, $|x_\infty|_{\mathbb{R}}$ ⋯ usual abs. val.

$$\hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times \longrightarrow \mathbb{A}^\times/\mathbb{Q}^\times\cdot\mathbb{R}^\times_{>0} \longrightarrow \mathbb{C}^\times \xrightarrow[\iota^{-1}]{\sim} \overline{\mathbb{Q}}_\ell^\times$$

$$\qquad\qquad\quad \psi \qquad\qquad\qquad\qquad \psi$$

$\boxed{p\neq\ell}$

$$\mathrm{Frob}_p = (p, p, \cdots, p, \underset{\underset{\text{at } p}{\uparrow}}{u}, p, \cdots) \longmapsto (1, \cdots, 1, p^{-1}u, 1, \cdots, 1) \mapsto p$$

$\boxed{p=\ell}$

$$(1, 1, \cdots, 1, \underset{\underset{\text{at } \ell}{\uparrow}}{u}, 1, \cdots) \longmapsto \iota u \mapsto u$$

$$\boxed{\begin{array}{l}(\ell\text{-adic}) \ \underline{\text{cyclotomic character}} \\ \quad G_{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_\ell^\times\end{array}}$$

◦ Cyclotomic Fields as Shimura var.

R: ring

Scheme$/_R$ = covariant functor $(R\text{-alg}) \mapsto (\text{sets})$
satisfying several properties
direct prod. $\rightsquigarrow$ direct sum

e.g. Affine schemes$/_R$ $A \in (R\text{-alg.})$

$$\mathrm{Spec}(A) : R' \mapsto \mathrm{Hom}_R(A, R')$$

Finite scheme$/_R$ if A: finite R-alg. (f.g. R-mod)

ex $R = K$: field
$(\mathrm{char}\, K, n) = 1$
$$\mathrm{Spec}(K(\mu_n)) : R' \longmapsto \underset{\left(\begin{array}{c}\text{primitive } n\text{-th roots of} \\ \text{unity in } R'\end{array}\right)}{\underset{\|}{\mathrm{Hom}_R(K(\mu_n), R')}}$$

Group scheme$/_R$: scheme, such that $(R\text{-alg}) \mapsto (\text{Groups}) \downarrow (\text{Sets})$

Finite group scheme --- finite as a scheme

e.g. $\mathbb{G}_m : R' \longmapsto (R')^\times$
$\uparrow$ multiplicative group.

G: group scheme, $n \geq 1$
$$G[n] : R' \longmapsto G(R')[n]$$

$$\mathbb{G}_m[n] =: \mu_n : R' \mapsto \left\{\begin{array}{c} n\text{-th roots of} \\ \text{unity in } R' \end{array}\right\}$$

$R = K$ field, $(\mathrm{char}\, K, n) = 1$

$$\mu_n = \mathrm{Spec}\Big(\prod_{d|n} K(\mu_d)\Big) \quad \cdots \text{ finite group scheme}$$

$\mathbb{Z}/n\mathbb{Z} : R' \longmapsto \mathbb{Z}/n\mathbb{Z}$ constant group scheme

$\left(\text{if } R' \text{ is not } \cong \underset{\neq 0}{R_1'} \times \underset{\neq 0}{R_2'}\right)$

$\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n)^\times$ $\circlearrowleft$

$$\mathrm{Spec}(K(\mu_n)) : R' \mapsto \overline{\underset{\substack{\uparrow \\ \text{set of isomorphisms}}}{\mathrm{Isom}}\big(\mathbb{Z}/n\mathbb{Z}(R'), \mu_n(R')\big)}$$

primitive n-th roots of unity

$\updownarrow$

level n structure on $\mathbb{G}_m(R')$ $\Longrightarrow$ $\left(\mathbb{Z}/n\mathbb{Z}(R') \xrightarrow{\cong} \mu_n(R') = \mathbb{G}_m(R')[n]\right)$

$$\boxed{\text{Cyclotomic field}/\mathbb{Q} = \text{moduli of level } n\text{-structures on } \mathbb{G}_m/\mathbb{Q}}$$

has $(\mathbb{Z}/n)^\times$- action by definition

$\uparrow$

$$(\mathbb{A}^\infty)^\times \twoheadrightarrow \hat{\mathbb{Z}}^\times$$

$\| \quad GL_1(\mathbb{A}^\infty)$

Shimura varieties (of PEL type)
$\cdots$ has action of $G(\mathbb{A}^\infty)$

◎ Complex Multiplication

(Some problems related to Kronecker's Jugentraum
1976 / R.P. Langlands)

$K/\mathbb{Q}$: imag. quadratic

$\mathcal{O} \subset \mathcal{O}_K$: order $(\mathcal{O} = \mathcal{O}_K)$

moduli of elliptic curves w/ CM by $\mathcal{O}_K$

$$(K\text{-alg.}) \longrightarrow (\text{Sets})$$

$$R' \longmapsto \boxed{\{(E, i)\}/_{\cong}} \quad \left|\begin{array}{l} \circ\ E : \text{elliptic curve } /R' \\ \circ\ i : \mathcal{O}_K \hookrightarrow \operatorname{End}(E) \\ \qquad\qquad\quad \| \\ \qquad\qquad \operatorname{Hom}(E,E)_{\text{gp}_{\text{sch}}} \end{array}\right.$$

(complex multiplication)

$$\left[\begin{array}{c} \underline{\text{ell. curve}} \\ \| \\ \text{gp. scheme, proper, rel. dim} = 1 \\ \text{geom. fiber} = \text{conn.} \end{array}\right]$$

… not representable

$$(\text{coarse moduli}) = \operatorname{Spec}(K(j(E))) \circlearrowleft Cl(\mathcal{O}_K)$$

$$\begin{tikzcd} \operatorname{Spec}(K(j(E))) \arrow[d] & j(E) \\ \operatorname{Spec}(K) & j\text{-invariant} \arrow[u, Rightarrow] \end{tikzcd}$$

◎ $Cl(\mathcal{O}_K)$ acts on $\boxed{}$

ideal class group

$[\mathfrak{a}] \in Cl(\mathcal{O}_K) \quad E \longmapsto E/_{E[\mathfrak{a}]}$

$\left(\begin{array}{c}\mathfrak{a} \subset \mathcal{O}_K \\ \uparrow \text{ ideal}\end{array}\right) \quad E[\mathfrak{a}] : R' \longmapsto \underbrace{E(R')\,[\mathfrak{a}]}_{\mathcal{O}_K\text{-module}}$

proj. $\mathcal{O}_K$-mod. of rank 1.

or

$E \otimes \mathfrak{a} := E/_{E[\mathfrak{a}]} : R' \longmapsto E(R') \otimes_{\mathcal{O}_K} \mathfrak{a}$

analogous to cyclotomic theory

$$\mathrm{Gal}\left(K(j(E))/K\right) \xhookrightarrow{\cong} Cl(\mathcal{O}_K)$$

$$\mathrm{Frob}_{\mathfrak{p}} \longmapsto [\mathfrak{p}^{-1}]$$

+ $\underline{\text{level str.}, \ n \geq 1}$

$$(K\text{-alg}) \longrightarrow (\mathrm{Sets})$$

$$R' \longmapsto \{(E, \iota, \eta)\}/\cong$$

$$\left[\begin{array}{c}\text{level } n\text{-str.}\\ \text{on } E\end{array}\right] \longrightarrow \eta : \mathcal{O}_K/n\,(R') \xrightarrow{\cong} E[n](R')$$

$$\uparrow$$

(isom of $\mathcal{O}_K$-modules)

--- $G(\mathbb{A}^\infty)$-action

$G_{/\mathbb{Q}} = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ alg. group = group scheme of finite type

$$(\mathbb{Q}\text{-alg}) \longrightarrow (\mathrm{Groups})$$

$$R' \longmapsto \mathbb{G}_m(R' \otimes_{\mathbb{Q}} K) = (R' \otimes_{\mathbb{Q}} K)^\times$$

$$G(\mathbb{A}^\infty) = (\mathbb{A}^\infty \otimes_{\mathbb{Q}} K)^\times$$

* * *

G : more general alg gp $/\mathbb{Q}$

$U \subset G(\mathbb{A}^\infty) \rightsquigarrow$ Shimura var. X_U

open compact subgroup

$\updownarrow$

(level str.)

G ⋯ "unitary group" $/\mathbb{Q}$
similitude

$$G \underset{\mathbb{Q}}{\otimes} K = GL_1 \times GL_1$$

F/F^+ $\qquad$ F^+ : totally real.
$\cup$ $\qquad\quad$ F : CM
quadratic ext.

GU/F^+ $\qquad \left(GU \underset{F^+}{\otimes} F = GL_n \times GL_1 \right)$
(or its twist)

$$G = \mathrm{Res}_{F^+/\mathbb{Q}}(GU)$$

Moduli of Abel var + $\underset{\substack{\uparrow \\ \text{polarization}}}{P}\underset{\substack{\uparrow \\ \text{endomorphism}}}{E}\underset{\substack{\uparrow \\ \text{level}}}{L}$ str. = Shimura var. of G

$\begin{pmatrix} \text{Ell. curve} \\ \| \\ \text{A.V. of dim 1} \end{pmatrix}$
$\downarrow$
$E/\mathbb{Q}$ $\qquad F^+ = \mathbb{Q}$, $E = F$
quadratic imaginary

represented by $(n-1)$-dim. quasi-proj. smooth var. $/E$

$$(E\text{-alg}) \longrightarrow (\text{sets})$$

$$R' \longmapsto \{(A, \underset{\substack{\uparrow \\ E}}{i}, \underset{\substack{\uparrow \\ P}}{\lambda}, \underset{\substack{\uparrow \\ L}}{\bar{\eta}})\}/\cong$$

A : abelian scheme of dim n $/R'$
i : $\mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ ← (condition on Lie(A))
λ : $A \longrightarrow A^\vee$ polarization such that Rosati involution $\underset{\circlearrowleft}{\overset{\mathrm{End}(A)}{}}\ *_\lambda$
restricts to cpx conj. on $\mathcal{O}_F$

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\phi"'] \arrow[rd, phantom, "\circlearrowleft"] & A^\vee \arrow[d, dashed, "(\phi^{*\lambda})^\vee"] \\ A \arrow[r] & A^\vee \end{tikzcd}$$

$$\phi \longmapsto \phi^{*\lambda}$$
$$\mathrm{End}(A) \xrightarrow[*_\lambda]{} \mathrm{End}(A)$$

($\bar{\eta}$: level U-str. for $U \subset G(\mathbb{A}^\infty)$)

Shimura variety of $\overset{\text{reductive}}{G}/_{\mathbb{Q}}$

$\|$

variety X_U / number field for each $\underset{\text{open cpt}}{U} \subset G(\mathbb{A}^\infty)$, $\mathbb{A}^\infty = \hat{\mathbb{Z}}\otimes\mathbb{Q}$

s.t. $X_U(\mathbb{C})$ = disj. union of $\underset{\text{discrete group.}}{} \backslash$ (herm. symm. space)

"$G(\mathbb{Q}) \cap U$"

$G = GL_2/\mathbb{Q}$ (modular curve), $\mathfrak{h} = SL_2(\mathbb{R})/SO_2(\mathbb{R}) = GL_2^+(\mathbb{R})/\mathbb{R}^\times\cdot SO_2(\mathbb{R})$

$SL_2(\mathbb{Z})\backslash\mathfrak{h}$

$$SL_2(\mathbb{Z})\backslash\mathfrak{h} = GL_2^+(\mathbb{Q})\backslash GL_2(\mathbb{A}^\infty) \times \left(GL_2^+(\mathbb{R})/\mathbb{R}^\times\cdot SO_2(\mathbb{R})\right) \Big/ \underbrace{GL_2(\hat{\mathbb{Z}})}_{\substack{\| \\ \text{max. cpt of} \\ GL_2(\mathbb{A}^\infty)}}$$

$$SL_2(\mathbb{Z}) = GL_2^+(\mathbb{Q}) \cap GL_2(\hat{\mathbb{Z}})$$

There is "no" Shimura var. for GL_n $(n \geq 3)$

$\rightsquigarrow$ use functoriality

$$\boxed{\Pi : \text{cuspidal autom. rep. of } GL_n(\mathbb{A}_L)}$$

$F=L$

① use functoriality

$$\begin{tikzcd} & GL_n/F & \\ GL_n/F^+ \arrow[ur, "\otimes F"] & & GU/F^+ \arrow[ul, "\otimes F"'] \end{tikzcd}$$

② avoid endoscopy (Kottwitz, Clozel)

$$\begin{tikzcd} GL_n \arrow[r, leftrightarrow, "\text{inner form}"'] & B^\times/F \\ & GU/F^+ \arrow[u, "\otimes F"'] \end{tikzcd}$$

B/F : central simple algebra (division) of dim n^2

Jacquet-Langlands correspondence

$$\left\{\begin{matrix}\text{autom.}\\ \text{rep. of}\\ B^\times(\mathbb{A}_F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{autom.}\\ \text{rep. of}\\ GL_n(\mathbb{A}_F)\end{matrix}\right\}$$

$$G = \mathrm{Res}_{F^+/\mathbb{Q}}(GU)$$

$G(\mathbb{A}^\infty) \curvearrowright$ Shimura var.

cohomology of Shimura var $=$

$$\begin{tikzcd} \bigoplus \left(\begin{matrix}\text{irred. adm. rep.}\\ \text{of } G(\mathbb{A}^\infty)\end{matrix}\right) \arrow[d, leftrightarrow, "\text{L.C.}"'] \\ \otimes \left(\begin{matrix}\ell\text{-adic rep}\\ \text{of } G_F\end{matrix}\right) \end{tikzcd}$$

L.C. (Langlands corresp.)

Unitary Shimura var.

$F = E\cdot F^+$, F^+ : tot. real

$E/\mathbb{Q}$: imag. quad. $c \curvearrowright \begin{cases} E \\ F \end{cases}$ cpx. conj.

$B/_F$ div. alg. of dim n^2

$* \curvearrowright B$: positive involution of 2nd kind.
$(*|_F = c)$

B_x (x : place of F)

$$\Vert \begin{cases} M_n(F_x) \text{ (matrix alg.) for } x \neq \overset{\uparrow}{\underset{\text{prime of } F}{z}},\ z^c \\ \text{div. alg. at } x = \underbrace{z,\ z^c}_{\text{Hasse inv } \times(-1)} \end{cases}$$

fix z to do J-L.

$$B \cong B^{op} \otimes_{F, c} F$$

$\longrightarrow$ choose involution $\#$ on B that defines a $*$-herm. form on B $(= V)$

$\Downarrow$

unitary group $/_{F^+}$

$\downarrow$ Res $F^+/\mathbb{Q}$

$$G(R) = \{ (\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \mid g \cdot g^\# = \lambda \}$$

R : $\mathbb{Q}$-algebra

$\Rightarrow$ Shimura var. = moduli of A.V. w/ $\mathcal{O}_B$-action.

dim $= [F^+ : \mathbb{Q}] \times n^2$

BRIEF SUMMARY OF LANGLANDS CORRESPONDENCE

TERUYOSHI YOSHIDA

Global Langlands Correspondence (conjecture)

$[L:\mathbb{Q}]<\infty,\ n\geq 1,\ \ell$: prime, $\iota:\overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$.

$$\begin{array}{rcl}
\text{algebraic automorphic rep. of } GL_n(\mathbb{A}_L) & \longleftrightarrow & n\text{-dim. } \ell\text{-adic rep.}^* \text{ of } G_L=\mathrm{Gal}\,(\overline{L}/L) \\
\Pi & \longleftrightarrow & R_{\ell,\iota}(\Pi) \\
\{\text{ cuspidals }\} & \longleftrightarrow & \{\text{ irreducibles }\}
\end{array}$$

* $G_L \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)\cdots$ (1) unramified at almost all places of L, (2) de Rham at ℓ.

- $R_{\ell,\iota}(\Pi)$ is characterized by : for almost all v of L (with Π_v unramified), $\{$ Satake parameters of $\Pi_v\} = \{$ eigenvalues of $R_{\ell,\iota}(\Pi)(\mathrm{Frob}_v)\in GL_n(\overline{\mathbb{Q}}_\ell)\}$.

Local Langlands Correspondence (theorem)

$[K:\mathbb{Q}_p]<\infty,\ n\geq 1.$

$$\begin{array}{rcl}
\text{irreducible admissible rep. of } GL_n(K) & \longleftrightarrow & n\text{-dim. Weil-Deligne rep. of } W_K \\
\Pi & \longleftrightarrow & \mathrm{rec}(\Pi) \\
\{\text{ cuspidals }\} & \longleftrightarrow & \{\text{ irreducibles }\}
\end{array}$$

$$\begin{array}{ccc}
\{\text{ cuspidals }\} & \Pi \longleftrightarrow r & \{\text{ irreducibles }\} \\
\cap & & \cap \\
\{\text{ square integrables }\} & \mathrm{Sp}_s(\Pi)\longleftrightarrow \mathrm{Sp}_s(r) & \{\text{ indecomposables }\} \\
\cap & & \cap \\
\{\text{ tempered }\} & \longleftrightarrow & \{\text{ pure }\} \\
\cap & & \cap \\
\{\text{ generic }\} & & \{\text{ mixed }\}
\end{array}$$

Date: March 23, 2005.

4/12. T. Ito

Non-abelian Lubin-Tate theory (Carayol's program)

local theory

§1. abelian theory

Recall: class field theory (global/local)

understand G_K^{ab} in terms of autom. rep's.

e.g. (cyclotomic fields)

$K = \mathbb{Q}$. $\mathbb{Q}^{ab} = \bigcup_n \mathbb{Q}(\mu_n)$ (Kronecker-Weber)

$$G_{\mathbb{Q}}^{ab} \xrightarrow[\text{cyclotomic char}]{\cong} \hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$$

$\mathbb{Q} \subset \mathbb{Q}_p$ p-adic completion

($\shortparallel\ \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$). fix $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$\rightsquigarrow G_{\mathbb{Q}_p}^{ab} \subset G_{\mathbb{Q}}^{ab}$ $\Longleftarrow$ $\forall$abel ext. of $\mathbb{Q}_p$ comes from that of $\mathbb{Q}$.

$$\begin{tikzcd} \bigcup_n \mathbb{Q}_p^{ur}(\mu_{p^n}) = \mathbb{Q}_p^{ab} \arrow[d, dash, "\text{difficult.}"] \\ \bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n) =: \mathbb{Q}_p^{ur} \arrow[d, dash] \\ \mathbb{Q}_p \end{tikzcd}$$

local CFT.

$\mathbb{Q}_p^{ur}$ … max. $\boxed{\text{unram}}$ ext. of $\mathbb{Q}_p$.

easy $\longleftrightarrow$ coming from ext. of $\mathbb{F}_p$.

$Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$: gen. by $Frob_p$ ($x \mapsto x^p$)

$K/\mathbb{Q}_p$: fin. ext.

$v_K : K^\times \twoheadrightarrow \mathbb{Z}$. p-adic val. $v_K(ab) = v_K(a) + v_K(b)$

$\mathcal{O}_K$: integer ring of K, $\underset{\mathbb{F}_q}{\underset{\shortparallel}{k(v_K)}}$: residue field

$K^{ur} := \bigcup_{(n,p)=1} K(\mu_n)$ max unram ext. of K.

$I_K := Gal(\overline{K}/K^{ur})$ inertia gp.

$$\begin{tikzcd} \overline{K} \arrow[d, dash, "I_K"] \\ K^{ur} \arrow[d, dash, "\hat{\mathbb{Z}}"'] \\ K \end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & I_K \arrow[r] \arrow[d, equal] & G_K \arrow[r] & G_{\mathbb{F}_q} \arrow[r] \arrow[d, phantom, "\cong"] & 1 \\
1 \arrow[r] & I_K \arrow[r] & W_K \arrow[r] \arrow[u, phantom, "\cup"] & \hat{\mathbb{Z}} & \\
 & & & \mathbb{Z} \arrow[r] \arrow[u, phantom, "\cup"] & 1.
\end{tikzcd}$$

Weil gp of K ↑ gen. by Frob_q.

<u>local C.F.T.</u> $\exists$ can. isom $\overset{(\text{Artin})}{\mathrm{Art}_K} : K^\times \xrightarrow{\cong} W_K^{ab}$

s.t.

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d, "\cong"'] & K^\times \arrow[r, "v_K"] \arrow[d, "\mathrm{Art}_K"', "\wr\wr"] & \mathbb{Z} \arrow[r] \arrow[d, "\wr\wr"', "\text{mult. by } -1"] & 1 \\
1 \arrow[r] & \text{image of } I_K^{ab} \arrow[r] & W_K^{ab} \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

If $K = \mathbb{Q}_p$. cyclo. char.

<u>Lubin-Tate theory</u> geometric realization of

$$\mathrm{Art}_K|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \xrightarrow{\cong} (\text{image of } I_K)$$

<u>idea</u>: use <u>formal group</u> $/R$ with $\mathcal{O}_K$-action.

functor (Artin R-alg) $\longrightarrow$ (Groups)

$\cap$

(R-alg) $\nearrow$ group scheme.

$(\text{groups}/R) \xrightarrow[\text{restriction}]{} (\text{formal groups}/R)$.

(<u>formal completion</u>)

<u>Rem.</u> Usually

R: noether local

$\mathcal{O}_K$-alg Artin

π : nilpotent in R-alg

(π_K : uniformizer)

<u>Concrete description</u> $\mathfrak{m}_{K^{ur}} \subset \mathcal{O}_{K^{ur}}$: max ideal.

R: local $\mathcal{O}_{K^{ur}}$-alg. s.t. p: nilpotent.

(e.g. $R = \mathcal{O}_{K^{ur}}/\mathfrak{m}_{K^{ur}}^n$. $R = \overline{\mathbb{F}}_q$ if $n=1$)

Def. $F(X.Y) \in R[[X.Y]]$ formal group law

$$\overset{}{\underset{\text{def.}}{\Longleftrightarrow}} \begin{cases} F(X.Y) \equiv X+Y \pmod{\deg 2} \\ F(X.Y) = F(Y.X) \\ F(F(X.Y).Z) = F(X.F(Y.Z)) \end{cases}$$

formal O_K-module $/R$

$$= \begin{cases} F : \text{as above} \\ O_K \hookrightarrow End(F) := \left\{ f(X) \in R[[X]] \;\middle|\; \begin{array}{l} f(0)=0. \\ F(f(X).f(Y)) \\ \quad = f(F(X.Y)) \end{array} \right\} \\ \overset{\rotatebox{90}{$\in$}}{a} \longmapsto \overset{\rotatebox{90}{$\in$}}{[a]} \\ [a](X) \equiv aX \pmod{\deg 2} \quad \ldots \text{ condition on Lie algebra.} \end{cases}$$

$F \rightsquigarrow$ formal group functor $/R$.

A : local R-alg $\longmapsto G$; $G = m_A$ as a set

$\cup$ $\qquad a, b \in G.\ a * b = F(a.b)$

m_A : nile.

Def. R : alg. cl. field. char $p>0$ F : formal O_K-module $/R$.

$[\pi](X) \equiv X^{q^h} \pmod{\deg q^h+1}$ $\quad h \ldots$ height of F

(up to change of parameter) $\quad \rotatebox{90}{$\in$}\ \mathbb{Z}_{\geq 1} \cup \{\infty\}$.

Th $h \geq 1.\ h < \infty$.

(i) $\exists!$ formal O_K-mod. $/\overline{\mathbb{F}_q}$ of ht h $\ldots$ $\Sigma_h/\overline{\mathbb{F}_q}$

(ii) $End_{O_K}(\Sigma_h/\overline{\mathbb{F}_q}) \cong O_D^\times$

$\left(\begin{array}{l} D : \text{central simple alg}/K.\ \dim = h^2.\ \text{inv.} = \frac{1}{h} \\ O_D \subset D : \text{max. order} \end{array} \right)$

(iii) There are $\underline{(h-1) \text{ "directions"}}$ to lift Σ_h to O_K^{ur}

$$\begin{tikzcd} (\text{Artin } \overline{\mathbb{F}_q}\text{-alg}) \arrow[r, "\Sigma_h"] \arrow[d, leftrightarrow] & (\text{Gps}) \\ (\text{Artin } \hat{O_K^{ur}}\text{-alg}) \arrow[ur, dotted, "\text{lift.}"'] & \end{tikzcd}$$

formal moduli

$\cong Spf\ \hat{O_K^{ur}}[[T_1 \ldots T_{h-1}]]$

universal π-action.

$$[\pi](X) \equiv \pi X + \sum_{i=1}^{h-1} T_i X^{q^i} + X^{q^h} \quad (\text{mod deg } q^h+1)$$

$h=1 \rightsquigarrow$ unique lift

$$\widetilde{\Sigma}_1 / \mathcal{O}_{\widehat{K^{ur}}}$$

<u>abelian Lubin-Tate theory</u> $(h=1)$

$$\widetilde{\Sigma}_1[\pi^\infty](\overline{\widehat{K^{ur}}}) \cong K/\mathcal{O}_K$$

$\mathcal{O}_K$-linear action $\circlearrowleft$ $Gal(\overline{K}/K^{ur}) = I_K$

gives LCFT. $I_K \xrightarrow[Art_K^{-1}]{} Aut_{\mathcal{O}_K}(K/\mathcal{O}_K) \cong \mathcal{O}_K^\times$

<u>e.g.</u> $K = \mathbb{Q}_p$. $h=1$.

$F(X,Y) = X+Y+XY$ (formal multiplicative gp)

$F(X,Y)+1 = (X+1)(Y+1)$

$$\widetilde{\Sigma}_1[p^\infty] = \{ a \in \overline{\widehat{\mathbb{Q}_p^{ur}}} \mid \exists n.\ a^{p^n}=1 \} = \bigcup_n \mu_{p^n}$$

$\circlearrowleft I_{\mathbb{Q}_p}$ $\quad (a \in \overline{\mathbb{Q}_p})$ $\quad I_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$ p-adic cyclo. char.

§2. Non-abel. L-T theory

{ local Langlands corresp.
{ local Jacquet-Langlands corresp.

<u>naive idea.</u> { $h=1$. LCFT.
{ h: general ... LLC for $GL(h)$.
need arithmetic geometry.

∘ <u>L.L.C</u> $K/\mathbb{Q}_p$

<u>Def.</u> <u>Weil-Deligne rep</u> of W_K ... triple (V, r, N)

V: $\mathbb{C}$-v.s. fin. dim. $\quad r: W_K \longrightarrow GL(V)$ cont. rep

$N: V \to V$ $\mathbb{C}$-linear. nilpotent s.t.

$\forall \sigma \in W_K . \quad r(\sigma)\cdot N \cdot r(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K \cdot N .$

$$\begin{tikzcd}
W_K^{ab} \arrow[r, "\mathrm{Art}_K^{-1}"'] & K^{\times} \arrow[r, "v_K"] \arrow[rr, bend right, "|\cdot|_K"'] & \mathbb{Z} \arrow[r, "a \mapsto q^{-a}"'] & \mathbb{R}
\end{tikzcd}$$

$(V. r. N)$: Frob. semisimple $\Leftrightarrow$ r : semisimple.

Motivation Grothendieck's monodromy thm.

$$\begin{tikzcd}
(\text{WD rep}) \arrow[r, leftrightarrow, "1:1"'] & (\ell\text{-adic rep}) \\
WD(\rho) & \rho \arrow[l, mapsto] & W_K \xrightarrow{\text{cont.}} GL_n(\overline{\mathbb{Q}}_\ell) \quad (n = \dim V)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(n\text{-dim. F-s.s. WD-rep}\right) \arrow[r, leftrightarrow, "LLC", "1:1"'] & \left(\begin{array}{c}\text{irred. adm. rep'n of} \\ GL_n(K)\end{array}\right) & \\
\cup & \cup & \\
\{\text{indecomposable}\} \arrow[r, leftrightarrow] & \{\text{sq. int}\} \arrow[r, leftrightarrow, "LJLC"'] & \left(\begin{array}{c}\text{irred. adm.} \\ \text{rep. of } D^{\times}\end{array}\right) \\
\cup & \cup & \\
\{\text{irred}\}\ (\Rightarrow N=0) \arrow[r, leftrightarrow] & \{\text{cuspidal}\} & \left(\begin{array}{l}D : \text{central div alg}/K \\ \dim = n^2 . \\ \text{inv} = \frac{1}{n} .\end{array}\right) \arrow[u]
\end{tikzcd}$$

formal moduli

$$\begin{tikzcd}
\widetilde{\Sigma}_h : \text{univ. formal } \mathcal{O}_K\text{-module} \arrow[d] \\
\mathrm{Spf}\ \underbrace{\widehat{\mathcal{O}_K^{ur}}[[T_1 \cdots T_{h-1}]]}_{R_0}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spf}\ R_n \arrow[d] & \cdots \text{classifying space of Drinfeld level str.} & \left(R_n : \text{regular local ring}\right) \\
\mathrm{Spf}\ R_0 & \left(\pi^{-n}\mathcal{O}_K/\mathcal{O}_K\right)^h \arrow[r] & \widetilde{\Sigma}_h[\pi^n] \quad \left(\pi \in \mathcal{O}_K \text{ unif.}\right)
\end{tikzcd}$$

fin. flat Galois covering.

Gal. gp $= \mathrm{Aut}_{\mathcal{O}_K}\left(\left(\pi^{-n}\mathcal{O}_K/\mathcal{O}_K\right)^h\right) \cong GL_h(\mathcal{O}_K/\pi^n)$

take $\varprojlim_n$. $\quad$ "$\varprojlim_n \mathrm{Spf}\, R_n$" $=: LT_\infty$

$$\begin{tikzcd}
LT_\infty \arrow[d, "{)\, GL_h(\mathcal{O}_K)}"] \\
\mathcal{O}_D^{\times} \cong \mathrm{Aut}(\Sigma_h) \curvearrowright \mathrm{Spf}\ R_0
\end{tikzcd}$$

non-abel. L-T theory : fix $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$

$$\Psi_h = \left(\ell\text{-adic formal vanishing cycle of } LT_\infty \right)$$

(Berkovich)

$$\text{``} H^*_{et}(LT_{\infty\,\bar{\eta}}, \overline{\mathbb{Q}}_\ell)\text{''}$$

$$\begin{tikzcd} \cong \bigoplus \begin{pmatrix}\text{rep of}\\ I_K\end{pmatrix} \arrow[rr, bend right, leftrightarrow, "\text{LLC}"'] & \otimes & \begin{pmatrix}\text{rep of}\\ GL_h(\mathcal{O}_K)\end{pmatrix} \arrow[rr, bend right, leftrightarrow, "\text{LJLC}"'] & \otimes & \begin{pmatrix}\text{rep of}\\ \mathcal{O}_D^\times\end{pmatrix} \end{tikzcd}$$

3日目

∘ NALT (local theory)

(II) $[K:\mathbb{Q}_p]<\infty$

Harris-Taylor chap. $\mathcal{O}_K,\ \pi,\ \mathcal{O}_K/\pi = k(v) \cong \mathbb{F}_q$

$s := \operatorname{Spec}\overline{\mathbb{F}}_q$

$\Sigma_{n/s}$: "the" formal $\mathcal{O}_K$-module of height n (≥ 1)

$R_0^{(n)} = \widehat{\mathcal{O}_{K^{ur}}}[[T_1,\cdots,T_{n-1}]]$ represents

$$\mathcal{C}_K := \begin{pmatrix}\text{Artin local } \widehat{\mathcal{O}_{K^{ur}}}\text{-alg.}\\ \text{w/ res. field} \cong \overline{\mathbb{F}}_q\end{pmatrix} \longrightarrow (\text{sets})$$

$$(A,m) \longmapsto \mathrm{Def}_A(\Sigma_n)$$

$$\mathrm{Def}_A(\Sigma_n) = \left\{\left(\Sigma/A,\ i : \Sigma\otimes_A A/m \cong \Sigma_n\right)\right\}/\cong$$

$$\cong \mathrm{Hom}_{\mathcal{C}_K}(R_0^{(n)}, A) \ni \varphi \mapsto \Sigma = \widetilde{\Sigma}_n \otimes_{R_0^{(n)},\varphi} A$$

i.e. $\exists\, \widetilde{\Sigma}_n / R_0^{(n)}$

$R_m^{(n)}/R_0^{(n)}$ pro-represents

fin. flat $GL_n(\mathcal{O}_K/\pi^m)$-covering

$$(A,m) \longmapsto \mathrm{Def}_A(\Sigma_n, \eta_s) = \{(\Sigma/A,\ i,\ \eta)\}/\cong$$

$$\eta : (\mathcal{O}_K/\pi^m)^n \longrightarrow \Sigma[\pi^m]$$

$$\text{s.t.} \prod_{x\in(\mathcal{O}_K/\pi^m)^n} (X-\underset{\Sigma(A)=m}{\underset{\cap}{\eta(x)}}) = \underset{A[[X]]}{\underset{\cap}{[\pi^m](X)}}\cdot(\text{unit})$$

… Drinfeld level π^m-str. on Σ

<u>Rigidity</u>: If $\eta = 0$, then A : $\overline{\mathbb{F}}_q$-alg, $\Sigma = \Sigma_n \otimes_{\overline{\mathbb{F}}_q} A$

$\Downarrow$

$$X_i := \eta^{\mathrm{univ}}(\underset{\substack{\uparrow \\ \text{canonical basis } 1 \leq i \leq n \\ \text{of } (\mathcal{O}_K/\pi^m)^n}}{e_i}) \in R_m^{(n)} \quad \text{are regular parameters of } R_m^{(n)}$$

<u>m=1</u>

$$R_1^{(n)} = \overbrace{\mathcal{O}^{\wedge}_{K^{ur}}[\![X_1, \cdots, X_n]\!]}^{(n+1)\text{-dim}} \Big/ \Big(\pi - (\text{unit}) \cdot \prod_{\underline{a} \in (\mathbb{F}_q)^n \setminus \{\underline{0}\}} \big(\overbrace{[a_1](X_1) \underset{\widetilde{\Sigma}}{+} \cdots \underset{\widetilde{\Sigma}}{+} [a_n](X_n)}^{\widetilde{\Sigma}_n / R_0^{(n)} \text{ operation}}\big)\Big)$$

We can compute the vanishing cycle cohomology by blow-up

Deligne-Lusztig theory

$$\mathrm{DL} : \prod_{\underline{a} \in (\mathbb{F}_q)^n \setminus \{\underline{0}\}} (a_1 X_1 + \cdots + a_n X_n) = 1$$

$\circlearrowleft$

$GL_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times}$ ... smooth affine var $/\overline{\mathbb{F}_q}$

$\zeta \in \mathbb{F}_{q^n}^{\times}$ $\quad X_i \mapsto \zeta X_i$ $\qquad (n-1)$-dim

<u>Thm</u> (Y)

$$X_{\overline{\eta}} = \mathrm{Spec}(R_1^{(n)} \otimes \widehat{K^{ur}})$$

$$H^*(S) := \sum_i (-1)^i H^i_{et}(S, \overline{\mathbb{Q}_\ell})$$

$$H^*(X_{\overline{\eta}}) = H^*_c(\mathrm{DL}) \quad \text{in } \underset{\substack{\uparrow \\ \text{Grothendieck group of} \\ \text{fin. dim rep's } / \overline{\mathbb{Q}_\ell}}}{\underline{\mathrm{Groth}}}(GL_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times})$$

$$\begin{tikzcd}
GL_n(\mathcal{O}_K) \arrow[d] & \times & I_K \arrow[d] \\
\pi_\theta \quad GL_n(\mathbb{F}_q) & \times & \mathbb{F}_{q^n}^{\times} \quad \theta
\end{tikzcd}$$

$\circlearrowleft$ $H^*(X_{\overline{\eta}})$

- Depth 0 L.L.C. = Deligne-Lusztig corresp.

<u>cf.</u> work of [illegible] Pan

cf. DeBacker's lectures (next week)

depth 0 rep. of G (general)

$\xleftrightarrow[\text{open problem}]{}$ DL-theory.

$$\Psi_n := \varinjlim_{m} \underset{\circlearrowleft}{H^*}(\mathrm{Spec}(R_m^{(n)} \otimes \widehat{K^{ur}}), \overline{\mathbb{Q}}_\ell)$$

$$\bigoplus_{\mathbb{Z}} \qquad \mathrm{Ker}(\underset{\substack{\uparrow \\ \text{div.alg.}/K \\ \text{Hasse inv. } \frac{1}{n}}}{D_n^\times} \times GL_n(K) \times \underset{\substack{\uparrow \\ \text{Weil gp.}}}{W_K} \longrightarrow \mathbb{Z})$$

LLC (NALT)

π : cuspidal rep'n of $GL_n(K)$

JL : $\mathrm{Rep}(D_n^\times) \longrightarrow \mathrm{Rep}(GL_n(K))$

$$\mathrm{rec}(\pi) := \Psi_n[\underbrace{JL^{-1}(\pi) \otimes \pi}_{\text{isotypic part}}] \cdots \text{irred } n\text{-dim'l rep'n of } W_K$$

- Unitary Shimura var. (global theory)

Π : regular alg. cuspidal autom rep. of $GL_n(\mathbb{A}_F)$

$F/\underset{d}{F^+}/\mathbb{Q}$: CM field, $F = F^+ \cdot E$, $E/\mathbb{Q}$: imag. quad

assume $\boxed{\text{discrete series at one place } \exists z}$

$\boxed{\Pi \text{ is assumed to be conjugate self-dual } \Pi^\vee = \Pi^c}$

JL : $\underset{\substack{\uparrow \\ \text{autom rep}}}{\mathcal{A}(B^\times)} \longrightarrow \mathcal{A}(GL_n(\mathbb{A}_F))$

B/F : central div. alg. of dim n^2

$* \circlearrowright B$, $*|_F = c$, x : place of F

$B^{op} \simeq B \otimes_{F,c} F$, $B_x = M_n(F_x)$ $x \neq z, z^c$

$$\# \leftrightarrow \beta$$

<u>Descent</u> $\# \overset{\text{inv}}{\circlearrowleft} B$, $g^{\#} = \beta \cdot g^{*} \cdot \beta^{-1}$

(VI) $(\beta \in B^{*=-1})$

$$B \times B \longrightarrow \mathbb{Q}$$

$(x_1, x_2) \longmapsto tr(x_1 \beta x_2^{*})$ $*$-Hermitian alt. form.

$$G(R) = \{ (\lambda, g) \in R^{\times} \times (B^{op} \otimes R)^{\times} \mid g \cdot g^{\#} = \lambda \}$$

R: $\mathbb{Q}$-alg $\quad G \subset GL_1 \times \mathrm{Res}_{F/\mathbb{Q}} (B^{op})^{\times}$

$G_1 := \mathrm{Ker}(G \longrightarrow GL_1) \; / \; \mathrm{Spec}\, F^{+}$

$(\lambda, g) \mapsto \lambda$

$$G_1 \otimes_{F^+} \mathbb{R} = U(1, n-1) \times \underbrace{U(0,n) \times \cdots \times U(0,n)}_{d-1}$$

for suitable β

<u>cf</u> CM-type, $n=1$ $(+, -, \cdots, -)$

$$BC : \mathcal{A}(G) \longrightarrow \mathcal{A}(B^{\times})$$

<u>Thm</u> (Clozel, Labesse)

$\Pi = JL(BC(\pi))$ for some autom rep π of $G(\mathbb{A})$

∘ <u>Shimura var.</u>

(III) $U \subset G(\mathbb{A}^{\infty})$: open compact

X_U : represents

$$\{ \text{scheme}/F \} \longrightarrow \{ \text{sets} \}$$

$$S \longmapsto \{ \underset{(P),(E),(L)}{(A, \lambda, i, \bar{\eta})} \} / \sim_{\text{isog.}}$$

- A/S : abelian scheme, $\dim = dn^2$
- $i : B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$

  $\mathrm{Lie}(A) = \underbrace{\mathrm{Lie}^+(A)}_{rk\ n} \oplus \underbrace{\mathrm{Lie}^-(A)}_{rk\ dn^2 - n}$

  $\mathrm{Lie}^+(A) = \mathrm{Lie}(A) \otimes_{E \otimes \mathcal{O}_S} \mathcal{O}_S \ {}^{\nearrow 1\otimes 1}$

  $F^+ \subset B$ acts via $\mathcal{O}_S$
- $\lambda : A \to A^\vee$, $*_\lambda|_B = *$
- $\bar\eta$: level U-str.

$\Rightarrow X_U$: proj. smooth of dim $(n-1)$

$(U$: suff. small$)$

$\Rightarrow$ representable

- Hida, p-adic Autom. forms on Shimura var. (2004)

$H^*(X) := \varinjlim_{U} H^*(X_U, \overline{\mathbb{Q}}_\ell)$

$\circlearrowleft \quad \circlearrowleft \qquad\qquad \wr$

$\underbrace{G(\mathbb{A}^\infty)}$ $G_F = \mathrm{Gal}(\bar F/F)$ $\left(H^*(S, \mathcal{F}) := H^*_{et}(S \times_F \bar F, \mathcal{F}) \right)$

Hecke corresp.

$g^{-1} U' g \subset U$

$\Rightarrow g : X_{U'} \longrightarrow X_U$

$$\begin{tikzcd}
 & X_{(gUg^{-1}) \cap U} \arrow[dl, "can"'] \arrow[dr, "g"] & \\
X_U \arrow[rr, dashed, "corresp."'] & & X_U
\end{tikzcd}$$

GLC (Kottwitz, Clozel) $\Pi \in \mathcal{A}(GL_n(\mathbb{A}_F))$ $\left\{\begin{array}{l} \circ \text{ regular} \\ \circ\ \Pi^\vee = \Pi^c \\ \circ \text{ disc. series at } \exists z \end{array}\right.$

$$\Pi = JL(BC(\pi))$$

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}^\infty)$$

$$\pi = \pi_\infty \times \boxed{\pi^\infty} \text{ irred adm. rep.}$$

$$a \cdot R_{\ell,\iota}(\Pi) = H^*(X)[\pi^\infty] \quad (a \geq 0)$$

$$R_{\ell,\iota}(\Pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

Compatibility $\forall v$: prime of F, $v \nmid \ell$

(up to s.s. Harris-Taylor

N : Taylor-Yoshida

$$WD\left(R_{\ell,\iota}(\Pi)\big|_{W_{F_v}}\right)^{F\text{-s.s.}} = rec(\Pi_v)$$

Π_v : unram. principal series

$$\Rightarrow \left\{\begin{array}{c}\text{Satake param.} \\ \text{of } \Pi_v\end{array}\right\} = \left\{\begin{array}{c}\text{Frobenius eigenvalues} \\ \text{of } R_{\ell,\iota}(\Pi) \text{ at } v\end{array}\right\}$$

Cor $z \nmid \ell \Rightarrow R_{\ell,\iota}(\Pi)$: irreducible

Integral model at v $(v \neq z, z^c)$

Ⅲ $v\big|_E = u$, $p = u \cdot u^c$

$*$-stable

Fix max. $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of B.

$$B_v^{op} \cong M_n(F_v)$$

$$\mathcal{O}_B^{op} \otimes \mathcal{O}_{F,v} \cong M_n(\mathcal{O}_{F,v})$$

$$G(\mathbb{Q}_p) = GL_n(F_v)\underbrace{\left(\times \prod_{w\neq v} (B_w^{op})^\times \times \mathbb{Q}_p^\times\right)}_{\text{ignore}}$$

Rem $\Pi = JL(BC(\pi))$

$\Pi_v = \pi_p$

$$\begin{array}{ccccc} U_0 & := & U^p & \times & GL_n(\mathcal{O}_{F,v}) \\ \cap & & \cap & & \cap \\ G(\mathbb{A}^\infty) & = & G(\mathbb{A}^{\infty,p}) & \times & G(\mathbb{Q}_p) \end{array}$$

X_{U_0} represents ($\mathcal{O}_{F,v}$-schemes) $\longrightarrow$ (Sets)

(Kottwitz, 1992, JAMS)

$S \longmapsto \{(A,\lambda,i,\bar{\eta})\}/\sim$ prime-to-p isog.

- A/S : dim dn^2
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$
  $\mathcal{O}_{F^+}$ acts via $\mathcal{O}_S$ on Lie(A)
  Lie$^+(A)$: rank n
- λ : prime-to-p, $*_\lambda|_{\mathcal{O}_B} = *$
- $\bar{\eta}$: level U^p-str.

X_{U_0} : proper, $\mathcal{A} \downarrow X_{U_0}$ universal A.V.

p-divisible group (cf. Tate, p-divisible groups)
(Barsotti-Tate gp)

$\lambda : \mathcal{A}[u^\infty] \to (\mathcal{A}[(u^c)^\infty])'$
Cartier dual.

$$\underset{\substack{\mathbb{Z}_p\text{-ht } 2dn^2 \\ \circlearrowleft \\ \mathcal{O}_{B^{op}}}}{\boxed{\mathcal{A}[p^\infty]}} = \underbrace{\mathcal{A}[u^\infty]}_{\substack{\| \\ \mathcal{A}[v^\infty] \oplus \bigoplus_{\substack{w\neq v \\ w|u}} \mathcal{A}[w^\infty]}} \oplus \mathcal{A}[(u^c)^\infty]$$

$$\mathcal{O}_B^{op} \otimes \mathcal{O}_{F,v} = M_n(\mathcal{O}_{F,v}) \curvearrowright \boxed{\mathcal{A}[v^\infty]}$$

$\mathbb{Z}_p$-ht , $[F_v : \mathbb{Q}_p] \cdot n^2$

$\underbrace{\mathrm{Lie}^+ A}_{\text{rank } n} = \mathrm{Lie}\, \mathcal{A}[u^\infty] \circlearrowleft \mathcal{O}_F$ acts via $\mathcal{O}_s / \mathcal{O}_{F,v}$

$$\Rightarrow \begin{cases} \mathcal{A}[w^\infty] : \text{étale for } w \neq v,\ w | u \\ \mathcal{A}[v^\infty] : \begin{cases} \mathcal{O}_{F,v}\text{-ht } n^2 \\ \dim\ n \end{cases} \end{cases}$$

$\mathcal{G}_{/X_{U_0}} = e \cdot \mathcal{A}[v^\infty]$ ... $\mathcal{O}_{F,v}$ BT-gp $\begin{cases} \text{ht } n \\ \dim 1 \end{cases}$

$$e = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}$$

$s : \mathrm{Spec}\, \overline{\mathbb{F}}_q \underset{\text{geom. pt}}{\hookrightarrow} X_{U_0}$

$h(s)$: étale $\mathcal{O}_{F,v}$-ht of $\mathcal{G}_s$

$$0 \to \underbrace{\Sigma_{n-ht(s)}}_{\substack{\text{formal} \\ \text{ht } n-ht(s)}} \to \underbrace{\mathcal{G}_s}_{\text{ht } n} \to \underbrace{\mathcal{G}_s^{\text{ét}}}_{\text{ht } h(s)} \to 0$$

$\hat{\mathcal{O}}_{X_{U_0}, s}$ = completion of étale local ring at s

represents $\mathcal{C}_{F_v} \longrightarrow (\text{Sets})$

$$\begin{tikzcd} s \arrow[r] \arrow[d] & U \arrow[d, "\text{étale}"] \\ \mathrm{Spec}\, R \arrow[r] \arrow[ur, dashed] & X_{U_0} \end{tikzcd}$$

$(R, m) \longmapsto \mathrm{Def}_R(A, \lambda, \iota, \bar{\eta})_s$

$\overset{\boxed{\text{Serre-Tate thm}}}{=} \mathrm{Def}_R(A[p^\infty], \lambda, \iota, \bar{\eta})_s$

$\overset{\lambda,\ \mathcal{A}[w^\infty] : \text{étale},\ \bar{\eta} : \text{étale}}{=} \mathrm{Def}_R(A[v^\infty], \iota)_s$

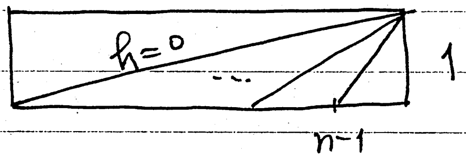
$\overset{\text{Morita equiv}}{=} \mathrm{Def}_R(\mathcal{G}_s)$

$$\therefore\ \hat{\mathcal{O}}_{X_{U_0},s} = R_0^{(n-h(s))}\underbrace{[[T_1,\cdots,T_{h(s)}]]}_{\text{deformation of } \mathrm{Ext}^1}$$

$\therefore\ X_{U_0}/\mathcal{O}_{F,v}$: smooth rel. dim $n-1$.

- $\overline{X}_{U_0} := X_{U_0}\times k(v)$ has <u>Newton stratification</u>

$$\overline{X}_{U_0} = \coprod_{h=0}^{n-1} \overline{X}^{(h)}_{U_0} \quad \left\{ \begin{array}{l} \overline{X}^{(h)}_{U_0} \text{ : smooth of dim } h \\ (\text{locus where } h(s)=h) \end{array} \right.$$



- level str. at v

X_{U_m} := moduli of Drinfeld level v^m-str. on $\mathcal{G}/X_{U_0}$

$$\left\{ \begin{array}{l} \hat{\mathcal{O}}_{X_{U_m},s} = R_m^{(n-ht(s))}[[T_1,\cdots,T_{h(s)}]] \\ \overline{X}^{(h)}_{U_m} := X_{U_m}\times_{X_{U_0}} \overline{X}^{(h)}_{U_0} \ \cdots \text{ smooth of dim } h \end{array} \right.$$

- <u>Igusa var.</u>

$I_m^{(h)}$: moduli of (étale) level v^m-str. on $\mathcal{G}^{et}/\overline{X}^{(h)}_{U_0}$

$$\begin{tikzcd} I_m^{(h)} \arrow[d, "\text{finite étale } GL_h(\mathcal{O}_{F,v}/v^m)\text{-covering}"] \\ \overline{X}^{(h)}_{U_0} \end{tikzcd}$$

$$\overline{X}^{(h)}_{U_m} \xleftarrow{\cong} \coprod_{P_h(\mathcal{O}_{F,v}/v^m)\backslash GL_n(\mathcal{O}_{F,v}/v^m)} I_m^{(h)}$$

$$P_h = L_h \cdot U_h \,, \quad L_h = GL_{n-h} \times GL_h$$

$$P_h \;\|\; \begin{pmatrix} \boxed{n-h} & \boxed{*} \\ 0 & \boxed{h} \end{pmatrix}$$

- compute cohomology (in Groth. gp.)

(IV)

$$\underset{\substack{\circlearrowleft \\ G(\mathbb{A}^{\infty}) \times Gal(\overline{F}/F) \\ \| \\ G(\mathbb{A}^{\infty,p}) \times GL_n(F_v)}}{H^*(X)}\Big|_{W_{F_v}} \overset{\text{proper smooth base change}}{=} \varinjlim_{U} H^*(X_U \times_{F_v} \overline{F}_v \,, \overline{\mathbb{Q}}_\ell)$$

$$\underset{\substack{\text{vanishing cycle}\\ \text{spectral seq.} \\ + \text{ stratify}}}{=} \varinjlim_{U} \sum_{h=0}^{n-1} H^*_c(\overline{X}_U^{(h)}, \underbrace{R\Psi\overline{\mathbb{Q}}_\ell}_{\substack{\| \\ \Psi_{n-h} \otimes (\text{local system})}})$$

$$= \sum_{h=0}^{n-1} \mathrm{Ind}_{P_h(F_v)}^{GL_n(F_v)} \; H^*_c(I^{(h)}, R\Psi\overline{\mathbb{Q}}_\ell)$$

$$\| \quad \Big(\varinjlim_{U^p,\, m} H^*_c(I_m^{(h)}, R\Psi\overline{\mathbb{Q}}_\ell) \Big)$$

Berkovich theory

$$= \sum_{h=0}^{n-1} \mathrm{Ind}_{P_h(F_v)}^{GL_n(F_v)} \Big[\underbrace{H^*(I^{(h)}, (\text{local system}))}_{\substack{\circlearrowleft \\ G(\mathbb{A}^{\infty,p}) \times GL_h(F_v) \times D^\times_{n-h}}} \otimes_{D^\times_{n-h}} \underbrace{\Psi_{n-h}}_{\substack{\circlearrowleft \\ D^\times_{n-h} \times GL_{n-h}(F_v) \times W_{F_v}}} \Big]$$

LLC

(IV 2.9)

- Counting points (Trace Formula)

(V)

(V.5.4) $$H^*(I^{(h)}, (\text{local system})) = \mathrm{Red}^{(h)}(H^*(X))$$

where $\mathrm{Red}^{(h)} = JL^{-1}_{n-h} \circ J_{U_h}$

$$\begin{tikzcd}
GL_h \times D^{\times}_{n-h} & GL_h \times GL_{n-h} \arrow[l, "JL^{-1}"'] & GL_n \arrow[l, "J_{U_h}"']
\end{tikzcd}$$

(Jacquet module adjoint of $\mathrm{Ind}_{P_h}^{GL_n}$)

∘ $R_{l,\iota}(\Pi)\big|_{W_{F_v}} = H^*(X)[\Pi^\infty]\big|_{W_{F_v}}$

(VII) $= \left(H^*(X)[\Pi^p]\big|_{W_{F_v}}\right)[\Pi_p]$

But,

$$H^*(X)[\Pi^p]\big|_{W_{F_v}} \overset{\theta}{=} \sum_h \mathrm{Ind}_{P_h}^{GL_n}\left(\mathrm{Red}^{(h)}(\pi_p)\otimes\Psi_{n-h}\right)$$

(multiplicity one
$[\Pi^p]$-isotypic part is $\underset{\overset{\|}{\Pi_v}}{\pi_p}$-isotypic ~~part~~)

$$= \sum_h \mathrm{Ind}_{P_h}^{GL_n}\Big(\big(\underset{\substack{\| \\ \pi_p^{(n-h)}\otimes\pi_p^{(h)}}}{J_{U_h}(\pi_p)}\big)\otimes\Psi_{n-h}\left[JL^{-1}(\pi_p^{(n-h)})\otimes\pi_p^{(n-h)}\right]\Big)$$

$$= \pi_p \otimes \sum_h \text{"rec"}(\pi_p^{(n-h)})$$

$$= \pi_p \otimes \mathrm{rec}(\pi_p)$$

$\therefore\ R_{l,\iota}(\Pi)\big|_{W_{F_v}} = \mathrm{rec}(\pi_p)$ (in Groth. up to s.s.)

@ $\underset{\substack{\uparrow \\ \text{Geometry}}}{\underline{\text{Mixed}}}$ + π_p: generic $\underset{\substack{\uparrow \\ \text{(Weil implies \underline{Ramanujan})}}}{\Longrightarrow}$ tempered (even if π_p: ramified)

$\Rightarrow$ $H^*(X)[\pi] = 0$ unless $* = n-1$.

Prove

$WD(H^{n-1}(X)|_{W_{F_v}})$ is <u>pure</u> (Friday)

— suffices to show when π_p has Iwahori fixed vector

$\downarrow$ (base change)

X_U : semistable reduction

$\downarrow$

Each term of weight spectral seq. can be computed via

$H^*(I^{(h)}) = \mathrm{Red}^{(h)}(H^*(X))$

$\downarrow$

solve weight-monodromy conj.

T. Ito 4/14

Recent topics.

geometry of Shimura var. after Harris-Taylor

abelian Lubin-Tate.

$GL(1)$. $K/\mathbb{Q}_p$: fin. 1-dim. formal $\mathcal{O}_K$-mod. ht $= 1$.

○ NALT. 1-dim formal $\mathcal{O}_K$-mod. ht $= n$.

$GL(n)$

"dual"

$$\begin{tikzcd} LT_\infty \arrow[d, "\text{level str}"'] & GL_n(\mathcal{O}_K) \\ \mathrm{Aut}(\Sigma_n) \curvearrowright \mathrm{Spf}\, \hat{\mathcal{O}}_K^{ur}[[T_1 \cdots T_{n-1}]]. & \end{tikzcd}$$

"$\mathcal{O}_D^\times$". Shimura var. at v s.t. $G(F_v) = GL_n(F_v)$

○ NALT(2). n-dim formal $\mathcal{O}_D$-mod. "ht $= 1$".

$$\begin{tikzcd} \Omega^n_\infty \arrow[d] & \mathcal{O}_D^\times \\ & \end{tikzcd}$$

D: central div. alg $/F_v$. dim n^2. inv $= 1/n$

$GL_n(K) \curvearrowright \Omega^n$: Drinfeld upper half space. dim $n-1$.

↙ Shimura var. at v s.t. $G(F_v) = D^\times$.

p-adic unif.

$X \cong \coprod_i \Gamma_i \backslash \Omega^n$. $(\Gamma_i \subset GL_n(F_v))$

⇒ application (I).

can prove compatibility of N at Steinberg places ← (geometric proof.)

◎ generalization "Rapoport-Zink sp." { Ann of Math Studies 141 (1996) / Asterique 291 (2004)

(general group / sign ← $U(1, n-1) \times U(0, n) \times \cdots \times U(0, n)$)

◎ Faltings' period isom.

"$LT_\infty \cong \Omega^n_\infty$" as "limit rigid spaces"

(exchange level str. & Hodge filtration).

◎ Study geometry of Shimura varieties mod p.

<u>Motivation 1</u> (classical)

<u>Deuring</u> p : prime.

B : quat. alg. $/\mathbb{Q}$ ramified at p, ∞

$$\#\left\{ E : \underset{\substack{\uparrow \\ E[p](\overline{\mathbb{F}}_p)=\{0\}}}{\underline{\text{supersing.}}} \text{ ell. curve}/\overline{\mathbb{F}}_p \right\}/\cong \; = (\text{class number of } B)$$

<u>Igusa</u> ('58)

$$LHS = \frac{p-1}{2} + \frac{1}{3}\left\{1-\left(\frac{-3}{p}\right)\right\} + \frac{1}{4}\left\{1-\left(\frac{-4}{p}\right)\right\}$$

use <u>Hasse invariant</u> ... mod p modular form, wt $p-1$

($\rightsquigarrow$ algebraic proof of Eichler's formula for RHS.)

<u>Motivation 2</u>

$k=\overline{\mathbb{F}}_p$. $(N,p)=1$. $N\geq 3$. $g\geq 1$.

$\mathcal{A}_{g,N}/k$: moduli sp. of p.p. AV dim. g w/ level N str.

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift left, "\text{univ. A.V.}"] \\ \mathcal{A}_{g,N} \arrow[u, shift left, bend left, "e"] \end{tikzcd} \qquad \omega = \det\left(e^*\Omega^1_{\mathcal{A}/\mathcal{A}_{g,N}}\right) \text{ ... ample line bundle on } \mathcal{A}_{g,N}$$

$H^0(\mathcal{A}_{g,N}, \omega^{\otimes w})$ ... space of Siegel modular form mod p wt $w \in \mathbb{Z}_{\geq 0}$

<u>Raynaud's trick</u> : A : ordinary AV$/k$ dim g
(i.e. $A[p](k) \cong (\mathbb{Z}/p)^g$)

$$G = A[p]. \quad \begin{cases} F : G \longrightarrow G^{(p)} := G \times_{k, \, p\text{-th power}} k & \text{Frobenius} \\ V : G^{(p)} \longrightarrow G & \text{Verschiebung (dual of } F) \end{cases}$$

$$0 \longrightarrow \underbrace{(\mu_p)^g}_{\text{conn.}} = \mathrm{Ker}\,F \longrightarrow G \longrightarrow \underbrace{(\mathbb{Z}/p)^g}_{\text{étale}} \longrightarrow 0$$

C: proper curve / k $\mathcal{A}/C$: family of ordinary AV dim. g, w/ level N str.

$G := \mathcal{A}[p]$ $(p.N) = 1$.

$H := \mathrm{Ker}(F : G \to G^{(p)})$

$\mathrm{Lie}(\mathcal{A}) = \mathrm{Lie}(\mathcal{A}[p]) = \mathrm{Lie}(H)$

$V : H^{(p)} \xrightarrow{\cong} H$ ($\Leftarrow$ ordinary).

$\mathrm{Lie}(H)^{(p)} \xrightarrow{\cong} \mathrm{Lie}(H)$

$\Rightarrow (\det \mathrm{Lie}(\mathcal{A})^\vee)^{(p)} = \det \mathrm{Lie}(\mathcal{A})$

$\parallel$

$(\det \mathrm{Lie}(\mathcal{A})^\vee)^{\otimes p} \xrightarrow[V]{\cong}$ section = <u>Hasse invariant</u>

$\Rightarrow (\det \mathrm{Lie}(\mathcal{A})^\vee)^{\otimes p-1}$: trivial (w/ can. trivialization)

$\mathcal{A}/C \rightsquigarrow \varphi_{\mathcal{A}} : C \longrightarrow \mathcal{A}_{g,N}$ moduli map.

$\therefore (\varphi_{\mathcal{A}}^* \underbrace{\omega}_{\text{ample}}{}^{\otimes p-1})$: trivial l.b. on C.

$\therefore \varphi_{\mathcal{A}}(C)$: 0-dim. $\therefore \mathcal{A}/C$: trivial family

<u>Key</u>: $\exists$ <u>can. section</u> of $\omega^{\otimes p-1}$ on ordinary locus] not only / curves
↖ "Hasse inv."

[cf. K3. Calabi-Yau. van der Geer / Katsura.]

- Hasse inv. for Shimura var. of H-T type (work in progress)

$$\begin{tikzcd}[arrows=dash] & & F := F^+E \arrow[d] \\ u.u^c \arrow[d] & E \arrow[d] \arrow[ur] & \\ p & \mathbb{Q} \arrow[r] & F^+ \end{tikzcd}$$

imag. quad.

tot. real.

$v | u$: place of F.

B/F : central div alg dim n^2

$\{ \cdot\ B_v \cong M_n(F_v)$

$\ \ \cdot\ * \curvearrowright B$: 2nd kind inv.

etc.

$\rightsquigarrow$ G : unitary similitude group

$\rightsquigarrow$ X_{U_0} : Shimura var. / $\mathcal{O}_{F,v}$.　　U_0 : maximal at v

fix $\mathcal{O}_B$.　(moduli sp. of A.V., PEL).　　↳ X_{U_0} : good red

$$\begin{tikzcd} \mathcal{A} \arrow[d] & \text{univ. A.V} \\ X_{U_0} & \end{tikzcd}$$

$M_n(\mathcal{O}_{F,v})$　　$w|u.\ w\neq v$

$$\mathcal{O}_{B\otimes\mathbb{Q}_p}^{\times} \curvearrowright \mathcal{A}[p^\infty] \cong \boxed{\mathcal{A}[v^\infty] \oplus (\text{étale})} \oplus (\text{dual of } \boxed{})$$

$\uparrow$ v^c w^c

$$e = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$$

$$\begin{tikzcd} \mathcal{G} = e\cdot\mathcal{A}[v^\infty] \arrow[d, "\text{1-dim. } \mathcal{O}_{F,v}\text{-BT-gp. ht } n."] \\ X_{U_0} \end{tikzcd}$$

$k = \overline{k(v)}$.　　Newton stratification :　　　smooth. dim $= h$.

$$\overline{X}_{U_0} := X_{U_0} \times_{\mathcal{O}_{F,v}} k = \coprod_{h=0}^{n-1} \overline{X}_{U_0}^{(h)}$$

$\Downarrow$

$\uparrow$ étale ht $= h$ locus

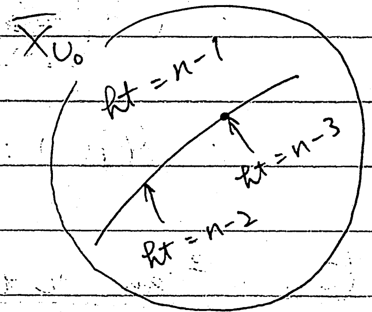
(NALT for $GL(n-h)$)

cf. (classical)

$$\begin{pmatrix}\text{mod. curve}\\ \text{mod } p\end{pmatrix} = \begin{pmatrix}\text{ordinary}\\ \text{locus}\end{pmatrix} \amalg \begin{pmatrix}\text{s.s.}\\ \text{pts}\end{pmatrix}$$

1-dim.　　$\uparrow$ 0-dim.

$\#(\text{s.s}) = (\text{class } \#)$



Known :

$\#(\overline{X}_{U_0}^{(0)}) = (\text{some class number})$

(H-T. V.4.5) $\to \mathbb{X}_0$

Problem : $\# \pi_0(\overline{X}_{U_0}^{(h)}) = \text{?}$　　　$h = n-1$ … use theory / $\mathbb{C}$

@ apply Ekedahl - Oort stratification theory

$G := \mathcal{G}[v]$ fin. flat gp scheme, rank q^n　($q = \#k(v)$)

$$\begin{cases} F : G \longrightarrow G^{(q)} & q\text{-th power Frob.} \\ V : G^{(q)} \longrightarrow G & \text{dual.} \end{cases}$$

Over $\overline{X}^{(h)}_{U_0}$ $\leftarrow$ $\overset{\mathcal{G}:}{\text{étale}}$ ht = h

$$0 \longrightarrow \overset{\text{conn.}}{H} \longrightarrow G \longrightarrow \overset{\text{étale}}{G^{\text{ét}}} \longrightarrow 0$$

$\overset{\|}{\mathrm{Ker}}(F^{n-h}) \cong \Sigma_{n-h}[v]$ at geom. pt.

$F : H \longrightarrow H^{(q)}$ nilpotent

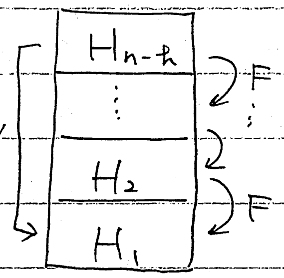
Kernel filtration : $0 \subset \mathrm{Ker}\, F \subset \mathrm{Ker}\, F^2 \subset \cdots \subset \mathrm{Ker}\, F^{n-h} = H$

↳ (can. filtr. in Ekedahl-Oort theory)

$H_i := \mathrm{Ker}\, F^i / \mathrm{Ker}\, F^{i-1}$ … rk q

$$H_{n-h} \xrightarrow[\cong]{F} H^{(q)}_{n-h-1} \xrightarrow[\cong]{F} \cdots \xrightarrow[\cong]{F} H_1^{(q^{n-h-1})}$$

$$V : H^{(q)}_{n-h} \xrightarrow{\cong} H_1.$$



Over $\overline{X}^{[h]}_{U_0} := \begin{pmatrix} \text{étale ht} \\ \text{of } \mathcal{G} \leq h \end{pmatrix} = \begin{pmatrix} \text{closure} \\ \text{of } \overline{X}^{(h)}_{U_0} \end{pmatrix}$

↖ proj. smooth, dim = h (H-T ; Mantovan (w/ level))

(deformation theory at boundary $\overline{X}^{[h]}_{U_0} \setminus \overline{X}^{(h)}_{U_0}$)

$$0 \longrightarrow \mathrm{Ker}\, F^{n-h} \longrightarrow G \longrightarrow \underbrace{G'}_{\text{étale on } \overline{X}^{(h)}_{U_0}} \longrightarrow 0$$

define $H_i := \mathrm{Ker}\, F^i / \mathrm{Ker}\, F^{i-1}$

$$H_1^{(q^{n-h})} \xleftarrow[\cong]{F} H_2^{(q^{n-h-1})} \xleftarrow[F]{\cong} \cdots \xleftarrow[F]{\cong} H^{(q)}_{n-h} \xrightarrow{V} H_1$$

⇑
- isom on $\overline{X}^{(h)}_{U_0}$
- zero map on the boundary

⇝ can. map $H_1^{(q^{n-h})} \overset{\cong \text{ on } \overline{X}^{(h)}_{U_0}}{\longrightarrow} H_1$

⇝ can. section of $((\mathrm{Lie}\, H_1))^{\otimes(1-q^{n-h})}$ over $\overline{X}^{[h]}_{U_0}$

$\mathrm{Lie}\, H_1 = \mathrm{Lie}\, \mathcal{G}$.

⇝ can. section of $((\mathrm{Lie}\, \mathcal{G})^\vee)^{\otimes(q^{n-h}-1)}$

zero = boundary.

Conj. : 1) $(\mathrm{Lie}\, \mathcal{G})^\vee$: ample

2) order of zero = 1.

(true on $\overline{X}^{(n-1)}_{U_0}$)

$\left(\det\left((\mathrm{Lie}\, \mathcal{G})^\vee \oplus (\mathrm{Lie}\, \mathcal{G}^\vee)^\vee\right) \text{ : ample} \right.$

(Raynaud's trick))

Th (Assume Conj.) $\overline{X}_{U_0}^{(h)}$: affine var. $\forall h$.

boundary $\overline{X}_{U_0}^{[h]} \setminus \overline{X}_{U_0}^{(h)}$: smooth ample div.

$\downarrow$ (weak Lefschetz)

$$\begin{cases} \# \pi_0(\overline{X}_{U_0}^{(h)}) : \text{independent of } h \text{ for } 1 \leq h \leq n-1 \\ \overline{X}_{U_0}^{(0)} \neq \emptyset \qquad (h = n-1 \rightsquigarrow \text{known}/\mathbb{C}) \end{cases}$$

Problems. ∘ cohomological application

- ∘ autom. interpretation — GL_2 case
- ∘ mod p modular form "$q\frac{d}{dq}$" — Kodaira-Spencer + Hasse inv.
- ∘ p-adic modular form (Katz)