Power Law Fluids with Random Forcing[*]

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0. Motivation

- Object: Dynamics of viscous, incompressible fluids with a random perturbation, as a model of **turbulence**.

- The most studied model so far:

The **SNS** (stochastic Navier-Stokes eq.):

▶ $u = (u_i(t,x))_{i=1}^d$: velocity of the fluid,

▶ $\Pi = \Pi(t,x)$: pressure,

▶ $W = (W_i(t,x))_{i=1}^d$: BM in $L^2(dx)^d$ with the trace class cov.

▶ $\nu > 0$: kinematic viscosity

1) $\quad \mathrm{div}\, u = 0,$

1') $\quad \partial_t u + (u \cdot \nabla)u = -\nabla \Pi + \nu \Delta u + \partial_t W.$

where $u \cdot \nabla = \sum_{j=1}^d u_j \partial_j$

cf. [Flandoli 2008] and ref.'s therein.

- (S)NS is valid for the **Newtonian** fluids, i.e.,

$$\text{viscosity} \equiv \nu \quad \text{(e.g., air, water,...)}$$

- On the other hand, many **non-Newtonian** fluids are observed and applied in science and engineering. Typical ones are:

- **Shear thinnig** fluid: (viscosity $\searrow$ when stirred)

"Fluid exhibits shear thinning when dilute polymer melt is added" (**Toms effect** 1948). This is applied to e.g., automobile engine oil, pipelines for crude oil.

- **Shear thickening** fluid:
(viscosity $\nearrow$ when stirred)

hydrocarbon fluids, suspension of starch, applications: bullet proof vests, automobile 4WD systems,..).

- **PDE results for non-Newtonian fluids**:

[Málek, Nečas, Rokyta, Růžička, 1996]
$\exists$ sol. and $\exists 1$ sol. in some cases.
Morally,
Shear thinnig $\longrightarrow$ more difficult than NS.
Shear thickening $\longrightarrow$ less difficult than NS.

Plan for the rest of the talk:

1. The SPDE for the power law fluid
2. The existence theorem for a weak solution
3. Strategy of the proof
4. Future works

1. The SPDE for the power law fluid

- Container of the fluid:

▶ $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$

- Deformation rate tensor:

Given $v : \mathbb{T}^d \to \mathbb{R}^d$ (velocity of the fluid),

▶ $e(v) = \left(\dfrac{\partial_i v_j + \partial_j v_i}{2}\right) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$.

- Extra stress tensor (friction):

▶ $\tau(v) = 2\nu(1 + |e(v)|^2)^{\frac{p-2}{2}} e(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$.

$\nu > 0$: "kinematic viscosity"
$p > 1$: "power"

Note: viscosity $=2\nu(1 + |e(v)|^2)^{\frac{p-2}{2}}$.

$$p \begin{cases} < 2 & \text{"shear thinning"} \\ & \text{e.g., engine oil (polymer melts), ...} \\ = 2 & \text{"Newtonian"} \\ & \text{e.g., air, water,...} \\ > 2 & \text{"shear thickening"} \\ & \text{e.g., hydrocarbon fluids,} \\ & \text{suspension of starch,...} \end{cases}$$

- Dynamics of the fluid (SPDE):

▶ $u = (u_i(t,x))_{i=1}^d$: velocity of the fluid,

▶ $\Pi = \Pi(t,x)$: pressure,

▶ $W = (W_i(t,x))_{i=1}^d$: BM in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$ with the trace class cov.

2) $$\mathrm{div}\, u = 0,$$

2') $$\partial_t u + (u \cdot \nabla)u = -\nabla \Pi + \mathrm{div}\, \tau(u) + \partial_t W.$$

where $\mathrm{div}\, \tau(u) = \left(\sum_{j=1}^d \partial_j \tau_{ij}(u)\right)_{i=1}^d$

Rem.

$$p = 2 \;\Rightarrow\; \mathrm{div}\, \tau(u) = \nu \Delta u$$

(the above eq.=SNS.)

2. The existence thm for a weak sol.

• <u>Test Functions</u>:

► $\mathcal{V} = \{v : \mathbb{T}^d \to \mathbb{R}^d \, ; \text{ trigo. polyn.}, \text{div } v = 0\}$

• <u>Spaces of the solutions</u>:

Let $p \in [1, \infty)$, $\alpha \in \mathbb{R}$,

► $\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1-\Delta)^{\alpha/2} v|^p, \;\; v \in \mathcal{V}.$

► $V_{p,\alpha} = \| \cdot \|_{p,\alpha}$-completion of $\mathcal{V}$.

- The weak solution:

▶ $\mu_0 \in \mathcal{P}(V_{2,0})$ = a prob. meas. on $V_{2,0}$.

▶ $(u, W) = ((u_t, W_t))_{t \geq 0}$: a process s.t.

$$\begin{array}{rcl} u & \in & L_{p,\text{loc}}(\mathbb{R}_+ \to V_{p,1}) \bigcap L_{\infty,\text{loc}}(\mathbb{R}_+ \to V_{2,0}) \\ & & \bigcap C(\mathbb{R}_+ \to V_{2,-\beta}), \ \text{for } \exists \beta > 0, \\ W & : & \text{BM in } L^2(\mathbb{T}^d \to \mathbb{R}^d), \text{ trace class cov. } \Gamma. \end{array}$$

▶ (u, W) is a **weak sol.** to SPDE 2)–2') with initial law μ_0, if

3) $P(u_0 \in \cdot) = \mu_0$,

3')
$$\begin{array}{rcl} \langle \varphi, u_t - u_0 \rangle & = & \int_0^t \langle (u_s \cdot \nabla)\varphi, u_s \rangle ds \\ & - & \int_0^t \langle e(\varphi), \tau(u_s) \rangle ds + \langle \varphi, W_t \rangle. \end{array}$$

for $\forall \varphi \in \mathcal{V}$.

Rem.

- 2),2') $\overset{\text{formal IBP}}{\Longrightarrow}$ 3'), when Π disappears, since

$$\langle \varphi, \nabla \Pi \rangle = -\langle \text{div } \varphi, \Pi \rangle = 0.$$

Theorem Suppose:

- $$\frac{3d}{d+2} \vee \frac{3d-4}{d} < p < \frac{2d}{d-2}$$

(e.g., $p > \frac{3}{2}$ for $d = 2$, $\frac{9}{5} < p < 6$ for $d = 3$).

- $\mu_0 \in \mathcal{P}(V_{2,1})$, $\int \|v\|_{2,1}^2 \mu_0(dv) < \infty$.

- $\Delta\Gamma = \Gamma\Delta$, $\{\Gamma, \Gamma\Delta\} \subset$ trace class.

Then, $\exists$ weak sol. (u, W) to SPDE 2)–2') with initial law μ_0. Moreover, the following a priori bound holds:

$$E\left[\sup_{t\leq T} \|u_t\|_2^2 + \int_0^T \|u_t\|_{p,1}^p dt\right] \leq (1+T)C < \infty.$$

Rem.'s

- $d = 2, 3$, $p = 2 \Rightarrow$ result for SNS cf. [Flandoli 2008] and ref.'s therein.

- $W \equiv 0 \Rightarrow$ PDE result [Málek et al. '96].

- Pathwise uniqueness: seems OK for $p \geq \frac{d+2}{2}$. (looks VERY hard for $p < \frac{d+2}{2}$, e.g. 3D NS.)

3. Strategy of the proof

Step 1 (Garelkin approximation)

- Take subspaces $\mathcal{V}^{(n)} \nearrow \mathcal{V}$, $\dim \mathcal{V}^{(n)} < \infty$.

- Solve an approx. eq. of 3)-3') with the unique sol. $u^{(n)}$ with values in $\mathcal{V}^{(n)}$.

Step 2 (A priori bounds)

- Establish some a priori bds for $u^{(n)}$ unif in n. We need several ones, of which the following is the most basic:

$$E\left[\sup_{t\le T}\|u_t^{(n)}\|_2^2 + \int_0^T \|u_t^{(n)}\|_{p,1}^p dt\right] \le (1+T)C < \infty.$$

Technique: Itô calculus, Sobolev imbedding,...

Step 3 (<u>Tightness</u>)

- Prove the tightness of $u^{(n)}$, $n \geq 1$ (in an Sobolev-type space, say X) so that

$$u^{(n)} \overset{n\to\infty}{\longrightarrow} \exists u \text{ in law along a subseq.}$$

To this end, we choose $X_1 \subset X$ s.t.

a) $X_1 \hookrightarrow X$ <u>compactly</u> .
(cpt imbedding thm's for Sobolev sp's).

b) $\exists \delta > 0$, $\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}] \leq C_T$
(A priori bds used here).

Then, the desired tightness in X can be seen as follows:

$$\{u \in X_1 \,;\, \|u\|_{X_1} \leq R\} \overset{\text{a)}}{\subset\subset} X$$

and

$$\begin{aligned}\sup_n P(\|u^{(n)}\|_{X_1} > R) &\leq \frac{\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}]}{R^\delta} \\ &\overset{\text{b)}}{\leq} \frac{C_T}{R^\delta} \overset{R\to\infty}{\longrightarrow} 0.\end{aligned}$$

Step 4 (Verification of SPDE)

- Let

$*$) $\qquad u^{(n)} \xrightarrow{n\to\infty} \exists u$ in law along a subseq.

and define a linear functional W_t by

$$\begin{aligned}
&\langle \varphi, W_t \rangle \\
&\quad = \langle \varphi, u_t - u_0 \rangle - \int_0^t \langle (u_s \cdot \nabla)\varphi, u_s \rangle ds \\
&\qquad + \int_0^t \langle e(\varphi), \tau(u_s) \rangle ds
\end{aligned}$$

Then, verify that W_t is a BM in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$.

To do so, realize $*$) as a.s. conv. (Skrokhod), then use apriori bds to exchange the order of $n \to \infty$ and $\int_0^t ...ds$

4. Future works

- Invariant measure
(an example of “non-equilibrium steady state”)

- Ergodicity
(a starting point to discuss the turbulence)

- (In the distant future ??) Approach to Kolmogorov’s K41 theory, Onsager conjecture