Stochastic Power Law Fluids : Construction of a Weak Solution *

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joint work with

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0. Motivation

• Object: Dynamics of viscous, incompressible fluids with a random perturbation, as a model of **turbulence**.

• The most studied model so far:
The **SNS** (stochastic Navier-Stokes eq.):

▶ $u = (u_i(t,x))_{i=1}^d$: velocity of the fluid,

▶ $\Pi = \Pi(t,x)$: pressure,

▶ $W = (W_i(t,x))_{i=1}^d$: BM in $L^2(dx)^d$ with the trace class cov., div $W = 0$.

1) div $u = 0$ (imcompressible);

1') $$\underbrace{\partial_t u + (u \cdot \nabla)u}_{\text{“acceleration”}} = -\nabla\Pi + \underbrace{\nu\Delta u}_{\text{“friction”}} + \partial_t W.$$

where $\nu > 0$ and $u \cdot \nabla = \sum_{j=1}^d u_j \partial_j$
cf. [Flandoli 2008] and ref.'s therein.

More phys. behind (NS)
– A review of fluid mechanics.

- $u = (u_j)_{j=1}^d$: the velocity field of the fluid.

- The force exerted to the fluid per volume:

$$
\begin{aligned}
& -\nabla \Pi + \underbrace{\mathrm{div}\ \tau(u)}_{\text{friction}} \\
&= \ -\nabla \Pi + \left(\sum_{j=1}^{d} \partial_j \tau_{ij}(u) \right)_{i=1}^{d} \in \mathbb{R}^d,
\end{aligned}
$$

where

$$\tau(u) \in \mathbb{R}^d \otimes \mathbb{R}^d \ \ (\textbf{extra stress tensor})$$

is a function of the (symmetrized) **velocity gradient**:

$$e(u) = \left(\frac{\partial_i u_j + \partial_j u_i}{2} \right) \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

- **Stokes' law:** $\tau(u) = 2\nu e(u)$.

Valid for **Newtonian** fluids (e.g., air, water,...)

$$
\begin{aligned}
& \text{Stokes' law} + \mathrm{div}\ u = 0 \\
& \ \Rightarrow \ \ \mathrm{div}\ \tau(u) = \nu \Delta u \ \ (\text{N.S.eq.}).
\end{aligned}
$$

More general relation of $\tau(u)$ and $e(u)$:

$$\tau(u) = \underbrace{F(|e(u)|)}_{\text{viscosity} > 0} e(u).$$

- Newtonian $\Longleftrightarrow$ $F \equiv$const.

- Many **non-Newtonian** fluids are applied in science and engineering. Typical ones are:

- **Shear thinnig** fluid: (F is $\searrow$ in $|e(u)|$)

"Fluid exhibits shear thinning when dilute polymer melt is added" (**Toms effect** 1948). This is applied to e.g., automobile engine oil, pipelines for crude oil.

- **Shear thickening** fluid: (F is $\nearrow$ in $|e(u)|$)

hydrocarbon fluids, suspension of starch, applications: bullet proof vests, automobile 4WD systems,..).

- **PDE results for non-Newtonian fluids**:

[Málek, Nečas, Rokyta, Růžička, 1996]

$\exists$ sol. and $\exists 1$ sol. in some cases.

Plan for the rest of the talk:

1. The stochastic power law fluid (SPLF)

2. The existence thm for a weak solution

3. The proof of the existence thm

4. Future works

1. The stochastic power law fluid (SPLF)

• Container of the fluid:

▶ $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$

Given the velocity field $u = (u_j)_{j=1}^d$,

▶ $\tau(u) = \underbrace{2(1+|e(u)|^2)^{\frac{p-2}{2}}}_{\text{viscosity}} e(u) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$,

with $p \in (1,\infty)$.

Note

$$p \begin{cases} < 2 & \text{“shear thinning”} \\ & \text{e.g., engine oil (polymer melts), ...} \\ = 2 & \text{“Newtonian”} \\ & \text{e.g., air, water,...} \\ > 2 & \text{“shear thickening”} \\ & \text{e.g., hydrocarbon fluids,} \\ & \text{suspension of starch,...} \end{cases}$$

• <u>SPDE for Stochastic Power Law Fluids</u>:

► $u = (u_i(t,x))_{i=1}^d$: velocity of the fluid,

► $\Pi = \Pi(t,x)$: pressure,

► $W = (W_i(t,x))_{i=1}^d$: BM in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$ with the trace class cov., div $W = 0$.

► $(\text{SPLF})_p$:

2) div $u = 0$ (imcompressible);

2') $$\underbrace{\partial_t u + (u \cdot \nabla) u}_{\text{“acceleration”}} = -\nabla \Pi + \underbrace{\operatorname{div} \tau(u)}_{\text{“friction”}} + \partial_t W.$$

where div $\tau(u) = \left(\sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d$

Rem. $(\text{SPLF})_2$=(SNS), since:

$$p = 2 \Rightarrow \operatorname{div} \tau(u) = \Delta u.$$

2. The existence thm for a weak sol.

- Test Functions:

$$\blacktriangleright\ \mathcal{V} = \text{“div-free smooth vect. fields”}$$
$$= \{v : \mathbb{T}^d \to \mathbb{R}^d\,;\ \text{trigo. polyn.}, \text{div}\, v = 0\}.$$

- Spaces of the solutions:

$$\blacktriangleright\ V_{p,\alpha} = \text{“Sobolev sp. of div-free vect. fields”}$$
$$= \|\cdot\|_{p,\alpha}\text{-completion of } \mathcal{V},$$

where $p \in [1, \infty)$, $\alpha \in \mathbb{R}$ and:

$$\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^p.$$

• <u>The weak solution</u>:

▶ $\mu_0 \in \mathcal{P}(V_{2,0}) =$ prob.'s on $V_{2,0}$.

▶ $(u, W) = ((u_t, W_t))_{t \geq 0}$: a process s.t.

$$\begin{array}{rcl} u & \in & L_{p,\mathrm{loc}}(\mathbb{R}_+ \to V_{p,1}) \bigcap L_{\infty,\mathrm{loc}}(\mathbb{R}_+ \to V_{2,0}) \\ & & \bigcap C(\mathbb{R}_+ \to V_{p' \wedge 2, -\beta}), \ \text{for } \exists \beta > 0, \\ W & : & \text{BM in } V_{2,0}, \text{ trace class cov. } \Gamma. \end{array}$$

▶ (u, W) is a **w-sol.** to $(\mathrm{SPLF})_p$ with init. law μ_0, if:

3) $P(u_0 \in \cdot) = \mu_0$,

3')
$$\begin{array}{rcl} \langle \varphi, u_t - u_0 \rangle & = & \int_0^t \langle u_s, (u_s \cdot \nabla)\varphi \rangle ds \\ & - & \int_0^t \langle e(\varphi), \tau(u_s) \rangle ds + \langle \varphi, W_t \rangle. \end{array}$$
for $\forall \varphi \in \mathcal{V}$.

Rem.

• 2),2') $\overset{\text{IBP}}{\Longrightarrow}$ 3'), when Π disappears, since

$$\langle \varphi, \nabla \Pi \rangle = -\langle \operatorname{div} \varphi, \Pi \rangle = 0.$$

Theorem 1 Suppose:

- $p \in \exists I_d$
(e.g., $I_2 = (3/2, \infty)$, $I_3 = (9/5, \infty)$, $I_4 = (2, \infty)$,...)

- $\mu_0 \in \mathcal{P}(V_{2,1})$, $\int \|v\|_{2,1}^2 \mu_0(dv) < \infty$.

- $\Delta\Gamma = \Gamma\Delta$, $\{\Gamma, \Gamma\Delta\} \subset$ trace class.

Then, $\exists$ w-sol. (u, W) to (SPLF)$_p$ with init. law μ_0. Moreover,

$$E\left[\sup_{t \le T} \|u_t\|_2^2 + \int_0^T \|u_t\|_{p,1}^p dt\right] \le (1+T)C < \infty.$$

Rem.'s

- $d = 2, 3$, $p = 2 \Rightarrow$ result for SNS cf. [Flandoli 2008] and ref.'s therein.

- $W \equiv 0 \Rightarrow$ PDE result [Málek et al. '96].

- Pathwise uniqueness: OK for $p \ge \frac{d+2}{2}$.
(looks VERY hard for $p < \frac{d+2}{2}$, e.g. 3D NS.)

Technical difference: (S)PLF $\leftrightarrow$ (S)NS

• (S)PLF is L_p (Banach sp.)-theory as opposed to L_2 (Hilbert sp.)-theory for (S)NS.

• Extra non-linearity in the friction term: the proofs of some a priori bounds are much more difficult to get.

3. The proof of the existence thm.

Step 1 (Galerkin approximation)

Set up:

- Subsp.'s $\mathcal{V}^{(n)} \nearrow \mathcal{V}$, $\dim \mathcal{V}^{(n)} < \infty$.

- An approx. eq. $\Rightarrow$ unique sol. $u^{(n)} \in \mathcal{V}^{(n)}$.

Step 2 (A priori bds)

- Establish some a priori bds for $u^{(n)}$ unif in n, e.g.,

$$E\left[\sup_{t\le T}\|u_t^{(n)}\|_2^2 + \int_0^T \|u_t^{(n)}\|_{p,1}^p dt\right] \le (1+T)C < \infty.$$

Technique:
Itô calculus, Martingale ineq.'s (e.g.,B-D-G), Sobolev imbedding.

Step 3 (Tightness)

• Prove the tightness of $u^{(n)}$, $n \geq 1$ in Sobolev sp.'s of the form:

$$X = L_{p_1,\alpha_1}([0,T] \to V_{p_2,\alpha_2})$$

so that

$$u^{(n)} \stackrel{n\to\infty}{\longrightarrow} \exists u \text{ in law along a subseq.}$$

To this end, we choose $X_1 \subset X$ s.t.

a) $X_1 \hookrightarrow X$ compactly .
(cpt imbedding thm's for Sobolev sp's).

b) $\exists \delta > 0$, $\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}] \leq C_T$
(A priori bds used here).

Then, the desired tightness in X can be seen as follows:

$$\{u \in X_1 \,;\, \|u\|_{X_1} \leq R\} \stackrel{\text{a)}}{\subset\subset} X$$

and

$$\begin{aligned}\sup_n P(\|u^{(n)}\|_{X_1} > R) &\leq \frac{\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}]}{R^{\delta}} \\ &\stackrel{\text{b)}}{\leq} \frac{C_T}{R^{\delta}} \stackrel{R\to\infty}{\longrightarrow} 0.\end{aligned}$$

Step 4 (Verification of SPDE)

- By Step 3,

$$u^{(n)} \overset{n\to\infty}{\longrightarrow} \exists u \text{ in law along a subseq.}$$

Then, $\exists$ BM W s.t. (u, W) is a w-sol. to $(\mathrm{SPLF})_p$.

4. Future works

- Invariant measure

(an example of “non-equilibrium steady state”)

- Ergodicity

(a starting point to discuss the turbulence)

- (In the distant future ??)

Approach to Kolmogorov’s K41 theory,
Onsager conjecture