The renormalization for parabolic fixed points and their perturbation

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Dedicated to the memory of Adrien Douady

Abstract

For holomorphic maps of one variable with a parabolic fixed point, the parabolic renormalization $\mathcal{R}_0$ is defined in terms of Fatou coordinates and horn maps. A class $\mathcal{F}_1$ of such maps is proposed so that it is invariant under $\mathcal{R}_0$, which acts as a uniform contraction with respect to a certain metric. The near-parabolic renormalization $\mathcal{R}$ is also defined for the perturbation of these maps, and it amounts to taking a first return map on a certain fundamental region. It is also shown that $\mathcal{R}$ is hyperbolic on the space of maps whose multiplier is sufficiently close to 1 . These results will help us to analyze the behavior of orbits of near the fixed points, especially irrationally indifferent ones. Buff and Chéritat [BC] used our result as one of main tools in their construction of a quadratic polynomial with Julia set of positive Lebesgue measure.

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Introduction

Let $f(z)$ be a holomorphic function defined near $z_0 \in \mathbb{C}$ and suppose z_0 is a fixed point. Its *multiplier* is $\lambda = f'(z_0)$ and the fixed point z_0 is called *parabolic* if λ is a root of unity. We will mainly consider the case $\lambda = 1$. In this case, for simplicity we say z_0 is *1-parabolic* and we call it *non-degenerate* if $f''(z_0) \neq 0$.

Near a non-degenerate 1-parabolic point z_0, the orbits are attracted towards z_0 on one side and repelled away on the other side. The *parabolic basin*

$$Basin(z_0) = \{z : \{f^n\}_{n=0}^{\infty} \text{ converges uniformly to } z_0 \text{ in a neighborhood of } z\}$$

is an open set containing z_0 on the boundary and occupies most of area near z_0. So the local dynamics is relatively simple. However, once perturbed, it becomes the source of rich and delicate bifurcation phenomena. The points in the basin of unperturbed map can now escape through the "gate" between the bifurcated fixed points, thus new recurrent orbits may be created. These "new" orbits depend extremely sensitively on the perturbation, and this causes a drastic change of dynamics or the discontinuity of Julia sets. Also the perturbation into certain direction, such as z_0 turning into irrationally indifferent fixed point (i.e. $|\lambda| = 1$ but λ is not a root of unity), can create highly recurrent behavior, which leads into delicate questions, e.g. the linearizability problem or Cremer Julia sets which are not locally connected.

The main tool to analyze such bifurcation is Fatou coordinates and horn maps, which were developed by Douady–Hubbard [DH1, DH2] and Lavaurs [La]. In order to trace escaping or recurrent orbits, a croissant-shaped "fundamental region" is defined near the fixed points and the first return map to this region is described by the horn map. By gluing the boundary curves by the dynamics, we obtain a cylinder which is isomorphic to $\mathbb{C}/\mathbb{Z}$, and the return map induces a holomorphic map defined near the ends of the cylinder. A brief review on this theory will be given in §§ 1 and 2. It was first used in the study of the landing of external rays at the Mandelbrot set, the discontinuity of the Julia sets and the straightening of polynomial-like maps, and the non-local connectivity of the connectedness locus of cubic polynomials. There are subsequent applications of these techniques, for example, [Do], [Sh1], [So], [Hi], [Ou], [KN].

When we study irrationally indifferent fixed points whose rotation number has continued fraction with large coefficients, it becomes important to carry out successive construction of return maps. This leads to the definition of parabolic and near-parabolic renormalizations $\mathcal{R}_0$ and $\mathcal{R}$ which will be described in §3. In fact, in [Sh1], such a notion was already introduced and its second iterate played a crucial role in the proof of the fact that a parabolic point can be perturbed so that the Hausdorff dimension of the Julia set is arbitrarily close to 2. A class $\mathcal{F}_0$ of 1-parabolic maps was introduced there and proved to be invariant under the parabolic renormalization $\mathcal{R}_0$. However, in order to study their perturbation, for example, irrationally indifferent fixed points, we need a class where near-parabolic renormalization $\mathcal{R}$ can be iterated (with control) infinitely many times. It turns out that $\mathcal{F}_0$ cannot serve for this purpose, and the main goal of this paper is to propose the class $\mathcal{F}_1$ (defined in §4) which fulfills the requirements.

Maps in this class are written as $f = P \circ \varphi^{-1}$, where $P(z) = z(1+z)^2$, φ is a normalized univalent function defined in a domain V.

Main results in this paper (stated in §4) are as follows: Main Theorem 1 states that $\mathcal{F}_1$ is invariant under $\mathcal{R}_0$, and the renormalized map has a slightly better extension property. Main Theorem 2 relates $\mathcal{F}_1$ to the Teichmüller space of a punctured disk and asserts that the induced map is a uniform contraction with respect to the Teichmüller metric. In Main Theorem 3, we obtain the invariance of $\mathcal{F}_1$ for the "fiber" renormalization $\mathcal{R}_\alpha$ for small α, which implies the hyperbolicity of near-parabolic renormalization $\mathcal{R}$. As a corollary, we conclude that if $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ has all continued fraction coefficients sufficiently large, then $e^{2\pi i\alpha}z + z^2$ cannot have a critical orbit which is dense in its Julia set.

There is a remarkable application of our results:

Theorem (Buff–Chéritat [BC])**.** *There exists an irrational number α such that $f(z) = e^{2\pi i\alpha}z + z^2$ has Julia set of positive Lebesgue measure.*

There are two renormalization theories which are closely related to ours – Yoccoz's and McMullen's. Yoccoz's renormalization was used in his proof [Yo] of Siegel-Bruno Theorem on the linearization of irrationally indifferent fixed points. His renormalization and our renormalization produce sequences which are locally conjugate. Yoccoz's renormalization is defined for any univalent function with any rotation number and corresponds to taking the first return map to a sector with a vertex at the fixed point. The renormalized map becomes again a univalent function after cutting off the domain of definition, and in this sense, an upper bound on its non-linearity is given. On the other hand, our renormalization is restricted to small rotation number and the class $\mathcal{F}_1$, but it includes the critical point in the domain of definition and gives a lower bound as well as upper bound on the non-linearity. When the rotation number is small, our domain of definition is substantially larger than Yoccoz's.

McMullen's renormalization [Mc1] deals with Siegel disks of quadratic polynomials for which the rotation number is of bounded type. He shows the convergence of scaled return maps near the critical point. This result can be recovered from our results when the rotation number has large coefficients for the continued fraction expansion.

There is also a similar renormalization theory for critical circle maps by Epstein-Yampolsky [Ya], [EY]. Their cylinder renormalization also uses the Ecalle-Voronin cylinder (see §1) to induce the renormalization for parabolic or near-parabolic fixed points of critical circle maps. In their setting, they do not encounter the difficulties discussed at the end of §3, therefore a class similar to $\mathcal{F}_0$ was sufficient. For Feigenbaum-Coullet-Tresser type renormalizations, see Sullivan [Su] (especially for the first attempt to use the Teichmüller theory for renormalizations), Lyubich [Ly] and McMullen [Mc2]. There is also a computer assisted work for period-tripling bifurcation by Golberg-Sinai-Khanin [GSK].

Some words about the proof of Main Theorem 1: It is difficult to calculate $\mathcal{R}_0 f$ explicitly, since the construction involves transcendental steps, such as constructing Fatou coordinates or uniformizing the quotient cylinders. In order to define an invariant class, we need a way to conclude that $\mathcal{R}_0 f$ belongs to this class. We will characterize a map in $\mathcal{F}_1$ (or $\mathcal{F}_2^P$ defined in §5.A) by its covering property, i.e. regard its domain as an abstract Riemann surface and see how it covers the range which is the complex plane. It is helpful to partition the range into several domains, take the connected components of their inverse images and see how these components are glued together along their boundary curves. This will be carried out for the horn map E_f in §5.M.

We needed to check a number of inequalities, and some of them (26 inequalities) have been checked numerically with computer. These inequalities are about elementary functions evaluated at explicit values. Initial estimates were done using Maple, and rigorous checking was done using MATLAB together with INTLAB. See [IS] for actual calculations.

Another important ingredient is the theory of univalent functions. In particular, Theorem 5.12, which is a consequence of Golusin inequalities, allowed us to derive sharp bounds on the Fatou coordinates (Proposition 5.6).

Main Theorem 2 relates $\mathcal{F}_1$ to the Teichmüller space of $\mathbb{C} \smallsetminus \overline{V}$ which is a punctured disk. In fact, the quasiconformal extension of φ determines a point in the Teichmüller space, and the induced renormalization there is holomorphic, therefore does not expand the Teichmüller distance, by Royden-Gardiner Theorem. The extra extension property in Main Theorem 1 gives a contracting factor. We show that an inclusion map between punctured disks induces a contraction between corresponding Teichmüller spaces (Theorem 6.3). This is shown via the estimates in the "pre-dual" space, which is the space of integrable holomorphic quadratic differentials, and it is a consequence of the modulus-area inequality (Theorem 6.6) which in turn follows from the isoperimetric inequality for quadratic differentials on a punctured disk (Theorem 6.4).

Main Theorem 3 is derived from the continuity of the construction.

Organization of paper. This paper is organized as follows: In §§1 and 2, we review the theory of Fatou coordinates and horn maps for a parabolic fixed point and its perturbation. In §3, we will define the parabolic and near-parabolic renormalizations $\mathcal{R}_0$ and $\mathcal{R}$, then discuss how these renormalizations can be used in order to understand the dynamics of maps with irrationally indifferent periodic points. We will also mention a previously known invariant class $\mathcal{F}_0$ for $\mathcal{R}_0$. In §4, we state the main theorems and corollaries. The section §5 is devoted to the proof of Main Theorem 1, whose outline is given in §5.A. In §6, we state the properties of the Teichmüller space of punctured disk and prove Main Theorem 2. In §7, we prove Main Theorem 3 and corollaries. Several facts on the Univalent functions are summarized in Appendix.

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Notation. The sets of all natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. Denote the Riemann sphere by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the unit disk by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, a disk in general by $\mathbb{D}(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ and its closure by $\overline{\mathbb{D}}(a,r)$. Let $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$. The set of positive (resp. negative) real numbers is denoted by $\mathbb{R}_+$ (resp. $\mathbb{R}_-$). For a complex number $z \neq 0$, $\arg z$ denotes its argument. In this paper, an inequality involving log or arg means that it holds for a suitably chosen branch of log or arg. For a hyperbolic Riemann surface X, $d_X(\cdot,\cdot)$ denotes the Poincaré distance on X, which is induced from the Poincaré metric $\frac{2|dz|}{1-|z|^2}$ on $\mathbb{D}$. We denote $\mathbb{D}_X(a,r) = \{z \in X : d_X(z,a) < r\}$. The spherical distance on $\widehat{\mathbb{C}}$ is denoted by $d_{\widehat{\mathbb{C}}}(\cdot,\cdot)$. For a function $f(z)$, we denote $f^\star(z) = \overline{f(\bar{z})}$.

1 Parabolic fixed points, Fatou coordinates and horn maps

In this section and next section, we review the theory of Fatou coordinates and horn maps, which was developed by Douady-Hubbard-Lavaurs [DH1, DH2, La]. For the proof of the statements and more details, refer to [Sh1, Sh2].

Let $f(z)$ be a holomorphic function with a non-degenerate 1-parabolic fixed point at $z = 0$, i.e.

$$f(z) = z + a_2 z^2 + O(z^3),$$

with $a_2 \neq 0$. Introduce a coordinate change $w = -\frac{1}{a_2 z}$, which sends the fixed point to ∞. The dynamics in this coordinate is

$$F(w) = -\frac{1}{a_2 f(-\frac{1}{a_2 w})} = w + 1 + \frac{b_1}{w} + O(\frac{1}{w^2})$$

near ∞, with some constant $b_1 \in \mathbb{C}$. See Figure 1.

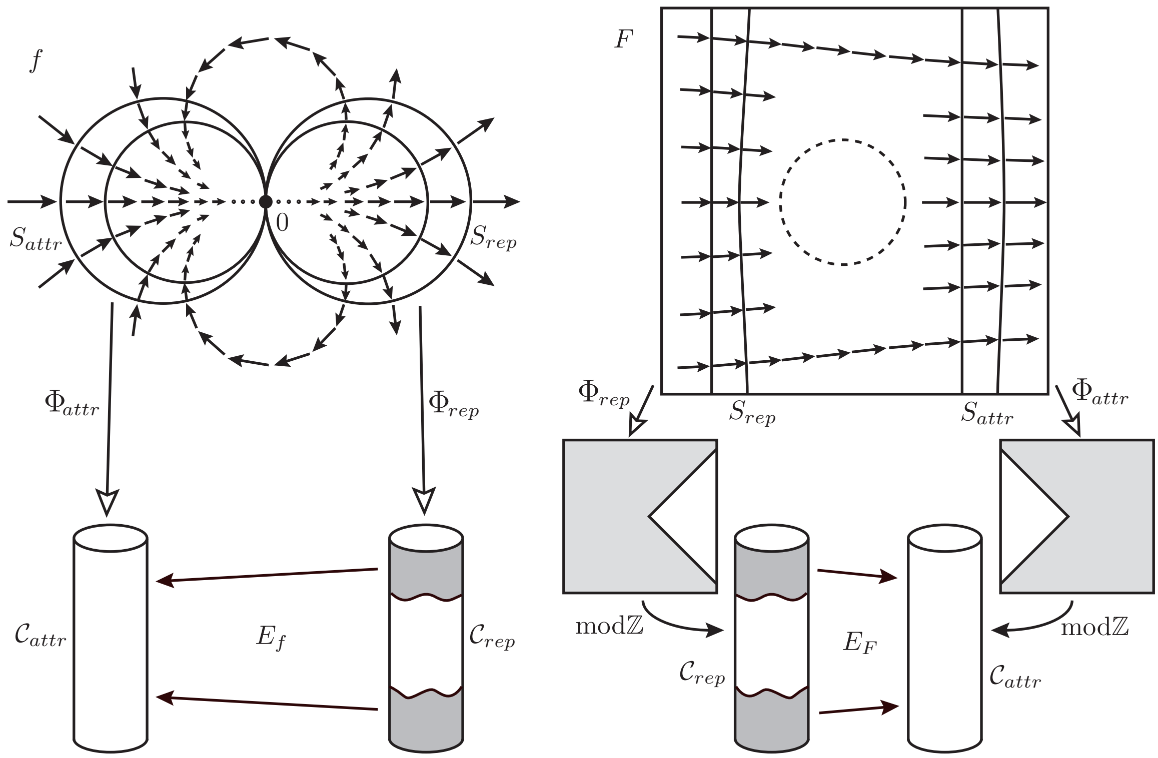

Figure 1: Parabolic fixed point with nearby orbits, fundamental regions, Fatou coordinates, Ecalle-Voronin cylinders and horn maps for f (left) and for F (right).

Theorem 1.1. (a) *For a sufficiently large L, there exist injective holomorphic functions $\Phi_{attr} = \Phi_{attr,F} : \{w : \text{Re } w > L\} \to \mathbb{C}$ and $\Phi_{rep} = \Phi_{rep,F} : \{w : \text{Re } w < -L\} \to \mathbb{C}$ such that they satisfy the functional equation*

$$\Phi_s(F(w)) = \Phi_s(w) + 1 \quad (s = attr, rep) \tag{1.1}$$

in the region where both sides are defined.

(b) *Φ_{attr} and Φ_{rep} are unique up to addition of constant.*

(c) *Using (1.1), Φ_{attr} and Φ_{rep} can be extended to $\{w : \text{Re}\, w - L' > -|\text{Im}\, w|\}$ and $\{w : \text{Re}\, w + L' < |\text{Im}\, w|\}$ respectively with large L'.*

(d) *In the above regions, Φ_{attr} and Φ_{rep} have asymptotic expansion* $w - b_1 \log w + const + o(1)$ *as* $w \to \infty$.

Definition. The functions Φ_{attr} and Φ_{rep} are called *attracting* and *repelling Fatou coordinates* respectively. They are considered to be coordinates for half-neignborhoods (*"petals"*) of the fixed point such that the dynamics is conjugated to the translation $T : z \mapsto z+1$. In the regions $V_\pm = \{w : \pm \operatorname{Im} w > |w| + L'\}$, both Fatou coordinates are defined. Now define the *horn map* E_F on $\Phi_{rep,F}(V_\pm)$ to be

$$E_F = \Phi_{attr} \circ \Phi_{rep}^{-1} \tag{1.2}$$

(which will be extended by Theorem 1.2 below).

Theorem 1.2. (a) *There exists $L'' > 0$ such that $\{z : -1 \le \operatorname{Re} z \le 1,\ |\operatorname{Im} z| \ge L''\}$ is contained in $\Phi_{rep}(V_\pm)$ therefore E_F is defined there.*
(b) *For $-1 \le \operatorname{Re} z \le 0,\ |\operatorname{Im} z| \ge L''$, E_F satisfies*

$$E_F(z+1) = E_F(z) + 1, \tag{1.3}$$

which implies that $E_F(z) - z$ is periodic with period 1. Therefore E_F extends holomorphically to $\{z :\ |\operatorname{Im} z| \ge L''\}$ and satisfies (1.3) there.
(c) *There exist constants c_{upper} and c_{lower} such that*

$$E_F(z) - z \to c_{upper} \text{ as } \operatorname{Im} z \to +\infty \quad \text{and} \quad E_F(z) - z \to c_{lower} \text{ as } \operatorname{Im} z \to -\infty,$$

and $c_{lower} - c_{upper} = 2\pi i b_1$.

Interpretation via fundamental regions and quotient cylinders: Let $\ell = \{w : \operatorname{Re}\, w = \xi\}$ be a vertical line with sufficiently large $|\xi|$. Then ℓ and $F(\ell)$ (which is on the right hand side of ℓ) bound an open region S and F is injective in a neighborhood of $\overline{S}$. The closed strip $\overline{S}$ is often called a *fundamental region* for F, because, when $|\xi| > L+2$ with L large, any maximal orbit of F within $\{w : \operatorname{Re}\, w > L\}$ ($\xi > 0$) or $\{w : \operatorname{Re}\, w < -L\}$ ($\xi < 0$), extended forward and backward until they it leaves the half plane, passes $\overline{S}$ exactly once, except those which pass ℓ and $F(\ell)$. The quotient $\overline{S}/\sim$, where $\ell \ni w \sim F(w) \in F(\ell)$, is a topological cylinder and is called *attracting* (resp. *repelling*) *Ecalle-Voronin cylinder* $\mathcal{C}_{attr}$ (resp. $\mathcal{C}_{rep}$) when $\xi \gg 0$ (resp. when $\xi \ll 0$). Since the identification F is analytic near ℓ, the cylinder has a natural structure as a Riemann surface. In fact, the Fatou coordinates induce isomorphisms from attracting/repelling cylinders onto $\mathbb{C}/\mathbb{Z}$, via the natural projection $\bmod \mathbb{Z} : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$.

As for the horn map E_F, it induces via $\bmod \mathbb{Z}$ a map on $\mathbb{C}/\mathbb{Z}$ defined only in the neighborhoods of both ends $\pm i\infty$. By abuse of notation, we also denote the induced map by E_F. This map allows the following interpretation (or an alternative definition). Let S_{attr} and S_{rep} be fundamental regions on attracting and repelling sides. If $w \in \overline{S}_{rep}$ with $|\operatorname{Im} z|$ sufficiently large, then its orbit will eventually land on $\overline{S}_{attr}$. This induces a map from a neighborhood of an upper or lower end of $\mathcal{C}_{rep}$ to $\mathcal{C}_{attr}$. It may appear that the map can be discontinuous when $w \in \partial S_{rep}$ or its orbit arrives in ∂S_{attr}, however it is well-defined and continuous because of the identification on the boundary. This map is exactly the one induced by E_F via the Fatou coordinates.

Normalization: The Fatou coordinates are only determined up to additive constant. It is convenient to make a normalization for the Fatou coordinates. If there is a special point z_* of interest on the attracting side, we normalize Φ_{attr} so that $\Phi_{attr}(z_*) = 0$. In this paper, we always have a special point which is a specific critical point cp, so the normalization is $\Phi_{attr}(cp) = 0$. For Φ_{rep}, instead of choosing another special point, we will normalize it so that $c_{upper} = 0$, i.e.

$$E_F(z) = z + o(1) \text{ as } \operatorname{Im} z \to +\infty. \tag{1.4}$$

Before the normalization, the horn map was determined up to pre- and post-composition of translations (i.e. adding constants before and after E_F). In fact, the horn map modulo this ambiguity classifies completely the local analytic conjugacy class of F or f, and called *Ecalle-Voronin invariant* (see [Vo]).

Global extension: The functional equation (1.1) allows us to extend the Fatou coordinates by the dynamics. Suppose, for example, F is a rational map. Then Φ_{attr} extends to $\Phi_{attr} : Basin(\infty) \to \mathbb{C}$ by setting $\Phi_{attr}(w) = \Phi_{attr}(F^m(w)) - m$ when $F^m(w) \in \{\operatorname{Re} w > L\}$ (such an $m \in \mathbb{N}$ must exist for $w \in Basin(\infty)$). After the extension, Φ_{attr} is not injective any more, but is a branched covering map such that w is a critical point of Φ_{attr} if and only if the forward orbit of w passes through a critical point of F. Similarly Φ_{rep}^{-1} can be extended to a map from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. The horn map E_F will be extended to $\Phi_{rep}^{-1}(Basin(\infty))$ so that it is also a branched covering onto $\mathbb{C}$, such that it is only branched over Φ_{attr}-image of critical orbits of F.

For the original map f, which has the parabolic fixed point at $z = 0$, we can define Fatou coordinates $\Phi_{attr,f}$, $\Phi_{rep,f}$ and horn map E_f through the coordinate change $w = -\frac{1}{a_2 z}$. In the original z-coordinate, the fundamental regions are "croissant-shaped" regions whose both "horns" point at the fixed point 0. The horn map E_f is induced by the orbits going from the horns of $S_{rep,f}$ to $S_{attr,f}$. See Figure 1.

To discuss the continuity, we need:

Definition. For a function f, its domain of definition is denoted by $Dom(f)$. A *neighborhood* of f is

$$\mathcal{N} = \mathcal{N}(f; K, \varepsilon) = \left\{ g : Dom(g) \to \widehat{\mathbb{C}} \;\middle|\; K \subset Dom(g) \text{ and } \sup_{z \in K} d_{\widehat{\mathbb{C}}}(g(z), f(z)) < \varepsilon \right\},$$

where K is a compact set contained in $Dom(f)$ and $\varepsilon > 0$. We say a sequence $\{f_n\}$ (for which f_n are not necessarily defined on the same domain) *converges* to f *uniformly on compact sets* if for any neighborhood $\mathcal{N}$ of f, there exists an n_0 such that $f_n \in \mathcal{N}$ for $n \geq n_0$.

The construction $f \rightsquigarrow E_f$ is continuous and holomorphic in the following sense (see [Sh2] for the proof):

Theorem 1.3 (Continuity and holomorphic dependence)**.** (a) *Let f be a holomorphic map with a non-degenerate 1-parabolic fixed point at $z = 0$. Given a neighborhgood $\mathcal{N}$ of its horn map E_f, there exists a neighborhgood $\mathcal{N}'$ of f such that if $g \in \mathcal{N}'$ and g has a 1-parabolic fixed point at 0, then its horn map E_g can be defined so that $E_g \in \mathcal{N}$.*

(b) *Suppose $f_\lambda(z)$ is holomorphic in $(\lambda, z) \in \Lambda \times \mathcal{U}$, where Λ is a complex manifold and $\mathcal{U} = Dom(f_\lambda) \subset \widehat{\mathbb{C}}$. Assume that f_λ always have a non-degenerate 1-parabolic fixed point at $z = 0$. Then for $\lambda_* \in \Lambda$ and an open set $\mathcal{V} \subset \mathbb{C}$ whose closure is compact and contained in $Dom(E_{f_{\lambda_*}})$, there exists a neighborhood Λ_1 of λ_* in Λ such that $(\lambda, z) \mapsto E_{f_\lambda}(z)$ is defined and holomorphic in $\Lambda_1 \times \mathcal{V}$.*

Here the normalization of the horn maps should be understood as follows: fix one point in either attracting or repelling half neighborhood where one of Fatou coordinates is defined. Normalize this Fatou coordinate so that the marked point is sent to 0 (or maybe 1). Adjust the other Fatou coordinate so that the horn map satisfies (1.4). The marked point can be chosen so that it depends continuously or holomorphically on f or λ.

2 Bifurcation of parabolic fixed points

Let f_0 be a holomorphic function with a non-degenerate 1-parabolic fixed point at $z = 0$, and consider its perturbation f which is close to f_0 in a neighborhood of 0. Since $z = 0$ has multiplicity 2 as a solution of $f_0(z) - z = 0$, f has two fixed points (or 1-parabolic fixed point) near 0. After a small shift of coordinate, we may suppose that $z = 0$ is still a fixed point of f. Its multiplier is close to 1, so it can be written as $e^{2\pi i\alpha}$ with small $\alpha \in \mathbb{C}$. It is well known that complicated and interesting bifurcation phenomena occur when α is in the tangential direction to $\mathbb{R}$. So we restrict our perturbation to the direction $|\arg\alpha| < \frac{\pi}{4}$ or $|\arg(-\alpha)| < \frac{\pi}{4}$. The latter case reduces to the former by a complex conjugation, i.e. $\overline{f(\bar{z})}$ has corresponding angle within $(-\frac{\pi}{4}, \frac{\pi}{4})$ and is a perturbation of $\overline{f_0(\bar{z})}$.

Thus we will consider a perturbation f of the form:

$$f(z) = e^{2\pi i\alpha} z + O(z^2) \quad \text{where } \alpha = \alpha(f) \text{ is small and } |\arg\alpha| < \tfrac{\pi}{4}. \tag{2.1}$$

Let $\sigma = \sigma(f)$ be the other fixed point of f near 0 (set $\sigma(f) = 0$ if $\alpha(f) = 0$). Then it can be shown that $\sigma(f)$ has asymptotic expansion $\sigma(f) = -2\pi i\alpha/a_2 + o(\alpha)$ when f converges to f_0 in a fixed neighborhood of 0 (and hence $\alpha(f) \to 0$), where $a_2 = f_0''(0)/2$.

Theorem 2.1. *Suppose f_0 has a non-degenerate 1-parabolic fixed point at $z = 0$. Then there exists a neighborhood $\mathcal{N} = \mathcal{N}(f_0; K, \varepsilon)$ (0 should be contained in int$K$) such that if $f \in \mathcal{N}$ and f satisfies* (2.1)*, then the fundamental regions $S_{attr,f}$, $S_{rep,f}$ are defined near those of f_0, except that the horns of $S_{attr,f}$ and $S_{rep,f}$ now point to distinct fixed points* 0 *and $\sigma(f)$ (if $\alpha(f) \neq 0$). Moreover the Fatou coordinates $\Phi_{attr,f}$ and $\Phi_{rep,f}$ are also defined in a neighborhood of $\overline{S}_{attr,f} \smallsetminus \{0, \sigma(f)\}$ and $\overline{S}_{rep,f} \smallsetminus \{0, \sigma(f)\}$ so that they induce isomorphisms from the quotient cylinders $\mathcal{C}_{attr,f}$, $\mathcal{C}_{rep,f}$ onto $\mathbb{C}/\mathbb{Z}$. The horn map E_f is similarly defined.*

After a suitable normalization as in §1, $\Phi_{attr,f}$, $\Phi_{rep,f}$ and E_f depend continuously and holomorphically on f.

For more description of domains etc, see [Sh1]. See Figure 2 for the content of this theorem and the next. For the perturbation with $f'(0) \neq 0$, there are new type of global orbits.

Theorem 2.2. *Let f be as in the previous theorem and assume $f'(0) \neq 1$. Then for any orbit starting from $\overline{S}_{attr,f} \smallsetminus \{0, \sigma(f)\}$ eventually lands on $\overline{S}_{rep,f} \smallsetminus \{0, \sigma(f)\}$. Such a correspondence induces an isomorphisim χ_f from $\mathcal{C}_{attr,f}$ onto $\mathcal{C}_{rep,f}$. By identifying these cylinders with $\mathbb{C}/\mathbb{Z}$ by the Fatou coordinates, χ_f can be expressed as*

$$\chi_f(z) = z - \frac{1}{\alpha(f)} \quad \text{on } \mathbb{C}/\mathbb{Z}, \tag{2.2}$$

provided that the horn map E_f is normalized so that $E_f(z) = z + o(1)$ as $\operatorname{Im} z \to +\infty$.

The composition $h = \chi_f \circ E_f$ corresponds to the first return map of f to the region $\overline{S}_{rep,f} \smallsetminus \{0, \sigma(f)\}$ near the horns, i.e., if $z \in \overline{S}_{rep,f} \smallsetminus (\{0, \sigma(f)\} \cup$ "inner boundary") and $w = \Phi_{rep,f}(z) \in \mathbb{C}/\mathbb{Z}$ has sufficiently large $|\operatorname{Im} w|$, then there is a smallest $n \geq 1$ such that $f^n(z) \in \overline{S}_{rep,f} \smallsetminus \{0, \sigma(f)\}$ such that $\Phi_{rep,f}(f^n(z)) = h(w) = \chi_f \circ E_f(w)$ in $\mathbb{C}/\mathbb{Z}$.

We call $h = \chi_f \circ E_f$ the *return map* of f. However, when we extend h to a larger region by analytic continuation, h may not necessarily correspond to the "first" return map, but still represents an orbit relation induced from f. The advantage of considering the return map is that extremely high iterates of f near the fixed point can be replaced by a single iterate of h. The above theorem enables us to decompose h into non-linear but stable part E_f and simple (linear) but sensitive part χ_f. If α is an irrational real number, this suggests a successive construction of return maps, which leads into the renormalization defined in the next section.

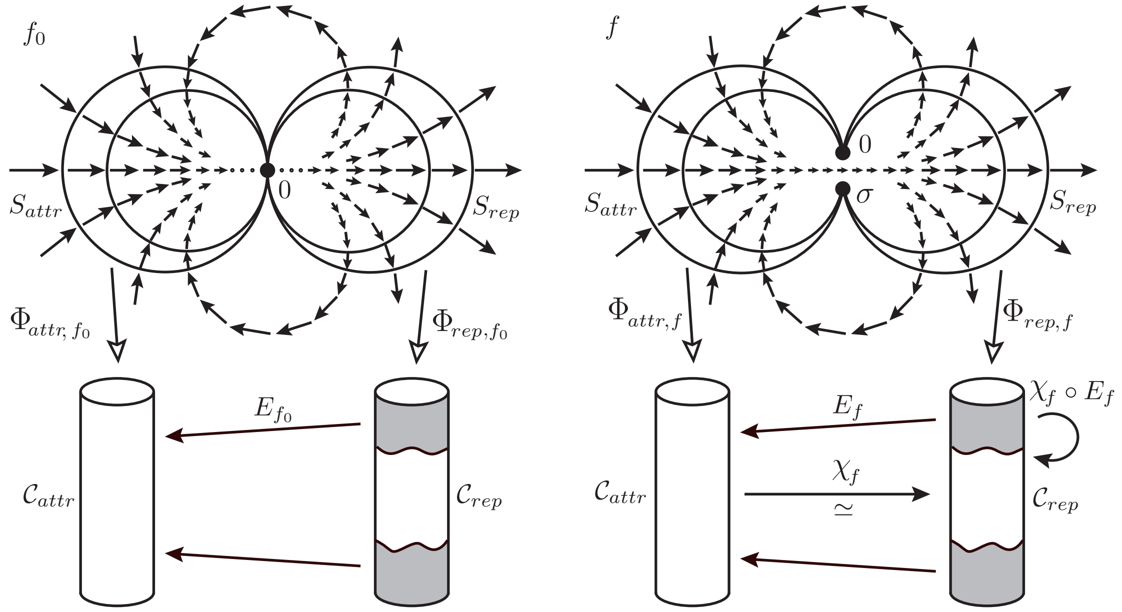

Figure 2: Perturbation of parabolic fixed point: before (left) and after (right)

3 Parabolic and near-parabolic renormalizations

Now we define our main objects, the parabolic and near-parabolic renormalizations.

Definition. Denote $\mathrm{Exp}^\sharp(z) = e^{2\pi i z}$ and $\mathrm{Exp}^\flat(z) = e^{-2\pi i z}$. Both functions induce isomorphisms from $\mathbb{C}/\mathbb{Z}$ onto $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$; $\mathrm{Exp}^\sharp$ sends upper end $+i\infty$ to 0 and lower end $-i\infty$ to ∞, and for $\mathrm{Exp}^\flat$, the role of the ends is interchanged.

Suppose f has a non-degenerate parabolic fixed point at 0. Its *parabolic renormalization* is defined to be

$$\mathcal{R}_0 f = \mathcal{R}_0^\sharp f = \mathrm{Exp}^\sharp \circ E_f \circ \left(\mathrm{Exp}^\sharp\right)^{-1}, \tag{3.1}$$

where E_f is the horn map of f, defined in §1 and normalized as $E_f(z) = z + o(1)$ as $\mathrm{Im}\, z \to +\infty$. Then $\mathcal{R}_0 f$ extends holomorphically to 0 and $\mathcal{R}_0 f(0) = 0$, $(\mathcal{R}_0 f)'(0) = 1$. So 0 has again a 1-parabolic fixed point at 0. See Figure 3.

Similarly the parabolic renormalization for lower end is defined as

$$\mathcal{R}_0^\flat f = c\, \mathrm{Exp}^\flat \circ E_f \circ \left(\mathrm{Exp}^\flat\right)^{-1}, \tag{3.2}$$

where $c \in \mathbb{C}^*$ is chosen so that $(\mathcal{R}_0^\flat f)'(0) = 1$.

Remark. (a) Both attracting and repelling Fatou coordinates are determined up to additive constants. After the normalization of E_f, there still remains a degree of freedom, which amounts to the conjugacy by a translation for E_f, or the conjugacy by a linear map $z \mapsto az$ for $\mathcal{R}_0 f$. Therefore we should consider that $\mathcal{R}_0 f$ is determined up to linear conjugacy $\underset{\text{linear}}{\sim}$. From next section, we will deal with the case where there is a unique (or preferred) critical value. In that case, we can choose a representative of each linear conjugacy class by fixing the position of the critical value.
(b) There is ambiguity on how far the domain of E_f should be extended. If we shrink the domain of definition of f, the domain of E_f will also be shrunk. So $\mathcal{R}_0$ can be considered as

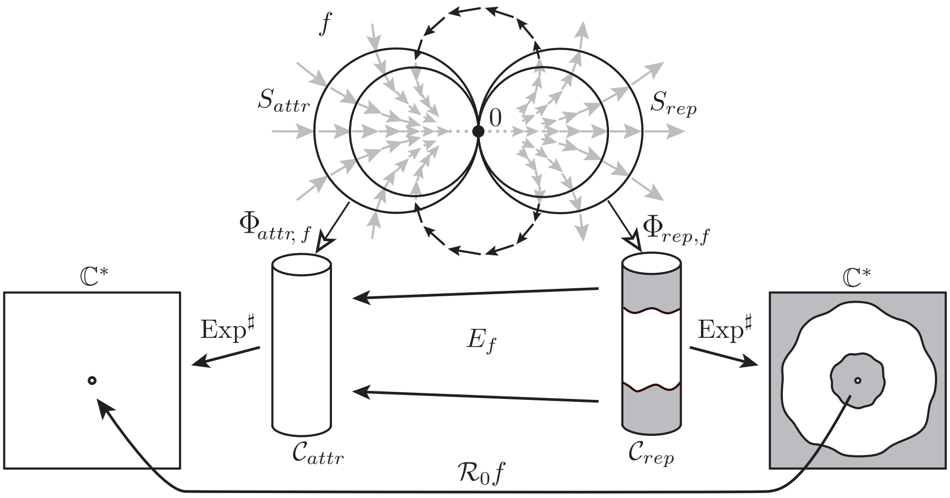

Figure 3: Parabolic renormalization

acting on germs of holomorphic function with 1-parabolic fixed points. On the other hand, in Main Theorem 1 in next section, for $f \in \mathcal{F}_1$, we will associate a specific domain of definition to $\mathcal{R}_0 f$.
(c) Note also that the parabolic renormalization of two locally holomorphically conjugate germs will give the same germ (up to linear conjugacy). This is because the conjugacy induces the conformal isomorphisms between $\mathcal{C}_{attr}$'s and between $\mathcal{C}_{rep}$'s respectively, and these isomorphisms relate the two renormalizations via pre- and post-composition of linear maps, and by the normalization $(\mathcal{R}_0 f)'(0) = 1$, they must give a linear conjugacy. On the other hand, the parabolic renormalization of two topologically (or quasiconformally) conjugate germs are not necessarily topologically conjugate. They are related by pre- and post-composition of two (usually distinct) homeomorphisms (or quasiconformal maps).

Definition. Suppose that $f(z) = e^{2\pi i\alpha} z + O(z^2)$ with $\alpha \neq 0$ and has fundamental domains and return map $h = \chi_f \circ E_f$ as in Theorems 2.1 and 2.2 (hence α is supposed to be small and $|\arg \alpha| < \frac{\pi}{4}$). Its *near-parabolic renormalization* (or also called cylinder renormalization) is defined by

$$\mathcal{R}f = \mathcal{R}^\sharp f = \mathrm{Exp}^\sharp \circ \chi_f \circ E_f \circ \left(\mathrm{Exp}^\sharp\right)^{-1}. \tag{3.3}$$

Then $\mathcal{R}f$ extends to 0 and $\mathcal{R}f(0) = 0$, $(\mathcal{R}f)'(0) = e^{-2\pi i \frac{1}{\alpha}}$. See Figure 4.

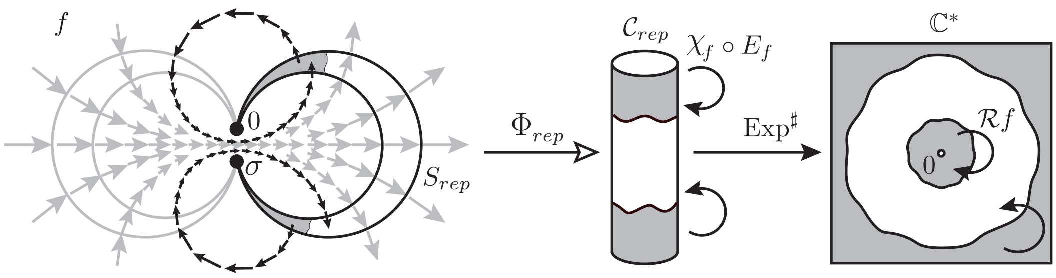

Figure 4: Near-parabolic renormalization and first return map

For small α with $|\arg(-\alpha)| < \frac{\pi}{4}$, the above construction can be applied to $f^\star(z) = \overline{f(\bar{z})}$ and define $\mathcal{R}f = \mathcal{R}^\sharp f = \mathrm{Exp}^\sharp \circ \chi_{f^*} \circ E_{f^*} \circ \left(\mathrm{Exp}^\sharp\right)^{-1}$ etc.

For lower end, define $\mathcal{R}^\flat f$ replacing $\mathrm{Exp}^\sharp$ by $\mathrm{Exp}^\flat$ in the definition of $\mathcal{R}^\sharp$. This map, restricted to a neighborhood of 0, corresponds to the return map near the fixed point $\sigma(f)$.

Remark. The above remarks (a) and (b) apply to this case.
(c') If two holomorphic maps as above are locally holomorphically conjugate around 0, then their first return maps are also locally holomorphically conjugate around 0. Moreover if the original conjugacy is holomorphic and univalent in a neighborhood of the closure of the repelling fundamental domain S_{rep}, the resulting conjugacy with be linear.
(d) Theorems 2.1 and 2.2 state that if f_0 with a non-degenerate 1-parabolic point is given, then the construction can be carried out for f sufficiently close to f_0. However when f is given first (i.e. not given as a perturbation of some f_0), it is not clear whether $\mathcal{R}f$ can be defined or not. Main Theorem 3 will try to answer this question at least uniformly for class $\mathcal{F}_1$ and small α.

Continued fraction: Any irrational number $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ can be written as an accelerated continued fraction of the form:

$$\alpha = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots}}}, \quad \text{where} \quad \begin{array}{l} a_n \in \mathbb{Z}, \ \ \varepsilon_n = \pm 1 \ (n = 0, 1, 2, \dots), \\ a_n \geq 2 \ (n \geq 1). \end{array} \tag{3.4}$$

Denote by $\boldsymbol{a}(x)$ the closest integer to $x \in \mathbb{R}$ (a convention: $\boldsymbol{a}(m + \frac{1}{2}) = m$ for $m \in \mathbb{Z}$) and $T(x) = \boldsymbol{a}(\frac{1}{|x|}) - \frac{1}{|x|}$. Then $\alpha_n \in (-\frac{1}{2}, \frac{1}{2})$ and a_n and ε_n are determined by

$$a_0 = \boldsymbol{a}(\alpha),\ \alpha_0 = \alpha - a_0,\ \varepsilon_0 = sign\,\alpha_0;\ \ a_{n+1} = \boldsymbol{a}(\frac{1}{|\alpha_n|}),\ \alpha_{n+1} = T(\alpha_n),\ \varepsilon_n = -sign\,\alpha_n. \tag{3.5}$$

Successive renormalizations: Let $f(z) = e^{2\pi i\alpha} z + O(z^2)$ with $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ as above. We are interested in the construction of successive renormalizations:

$$f_0(z) = f(z) \qquad f_{n+1}(z) = \begin{cases} \mathcal{R}f_n(z) & (\alpha_n > 0) \\ \mathcal{R}f_n^\star(z) & (\alpha_n < 0) \end{cases} \quad (n \geq 0), \tag{3.6}$$

where $f_n(z) = e^{2\pi i\alpha_n} z + O(z^2)$ with α_n defined by (3.5). This means that each f_n has a fundamental domain S_{rep,f_n} joining 0 and $\sigma(f_n)$ and the return map of f_n or $f_n^\star$ to S_{rep,f_n} defines (via Φ_{rep,f_n} and $\mathrm{Exp}^\sharp$) the next map f_{n+1}. If such a construction is possible, we hope that the dynamics of f, whose irrationally indifferent fixed point causes recurrent behavior for nearby orbits, can be studied through the sequence $\{f_n\}$. In fact, problems involving high iterates of f_n often reduce to simpler problems on fewer iterates of f_{n+1}. The geometric structure near recurrent orbits may be "magnified" by the renormalization process. Hence it is natural to ask:

Question. When is it possible to define the sequence (3.6)?

Main Theorem 3 gives an answer (a sufficient condition) to this question. It will be important to find a space of maps where the renormalization can be iterated infinitely many times.

We will write f as $f(z) = e^{2\pi i\alpha} h(z)$, where $h(0) = 0$ and $h'(0) = 1$, thus identifying f with the pair (α, h). Under this identification, the near-parabolic renormalization can be expressed as a skew product:

$$\mathcal{R} : (\alpha,\, h) \longmapsto (T(\alpha),\ \mathcal{R}_\alpha h)\,, \tag{3.7}$$

where $\mathcal{R}_\alpha h$ is the *renormalization in fiber direction* defined by

$$\mathcal{R}_\alpha h = \begin{cases} \mathrm{Exp}^\sharp \circ E_{(e^{2\pi i \alpha} h)} \circ \left(\mathrm{Exp}^\sharp\right)^{-1} & \text{if } \alpha \in (0, \frac{1}{2}] \\ \mathrm{Exp}^\sharp \circ E_{(e^{-2\pi i \alpha} h)^\star} \circ \left(\mathrm{Exp}^\sharp\right)^{-1} & \text{if } \alpha \in (-\frac{1}{2}, 0). \end{cases} \tag{3.8}$$

In many renormalization theory, we often expect to see hyperbolic behavior, which usually has consequences such as universality in bifurcation structures and phase spaces. (See [Su].) In our case, α-direction is obviously expanding.

Conjecture. The renormalization $\mathcal{R}$ is hyperbolic on a certain space of maps. More specifically, the fiber renormalization $\mathcal{R}_\alpha$ is contracting.

Main Theorem 3 will also give an answer to this question. See Figure 5.

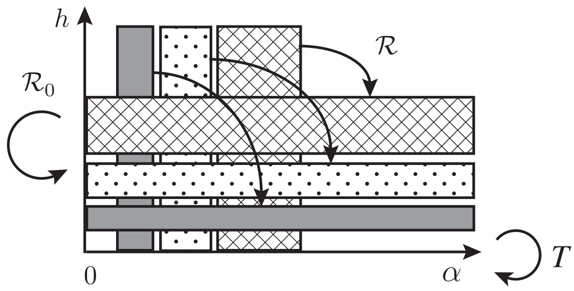

Figure 5: Hyperbolicity of renormalization and limit at $\alpha = 0$ (only the part $\alpha \geq 0$ is drawn)

By the continuity of horn map, we have $\mathcal{R}_\alpha h \to \mathcal{R}_0 h$ when $|\arg \alpha| < \frac{\pi}{4}$ and $\alpha \to 0$. (On the other hand, $\mathcal{R}_\alpha h \to \mathcal{R}_0 h^\star$ when $|\arg(-\alpha)| < \frac{\pi}{4}$ and $\alpha \to 0$.) So we are led to the study of the limiting case: the parabolic renormalization $\mathcal{R}_0$. For $\mathcal{R}_0$, an invariant class was already known in [Sh1], to which we refer for the proofs of Lemma 3.1 and Theorem 3.2 below.

Definition (Class $\mathcal{F}_0$)**.** Let

$$\mathcal{F}_0 = \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open } \subset \mathbb{C}, \quad f \text{ is holomorphic in } Dom(f), \\ f(0) = 0,\ f'(0) = 1,\ f : Dom(f) \smallsetminus \{0\} \to \mathbb{C}^* \text{ is a branched} \\ \text{covering map with a unique critical value } cv_f,\ \text{all critical} \\ \text{points are of local degree 2} \end{array} \right\}.$$

Examples: The quadratic polynomial $z + z^2$ and the Koebe function $f_{Koebe}(z) = z/(1-z)^2$ belong to $\mathcal{F}_0$.

Lemma 3.1. *For $f \in \mathcal{F}_0$, $f''(0) \neq 0$ and f has only one petal. The critical value belongs to the immediate basin of the parabolic fixed point. The dynamics in the basin is conjugate to that of $z + z^2$ in its basin.*

Theorem 3.2. *The class $\mathcal{F}_0$ is invariant under $\mathcal{R}_0$. Moreover any map in the image $\mathcal{R}_0(\mathcal{F}_0)$ can be expressed as $g_{Koebe} \circ \varphi^{-1}$, where $g_{Koebe} = \mathcal{R}_0(f_{Koebe})$, which is defined on $\mathbb{D}$, and $\varphi : \mathbb{D} \to \mathbb{C}$ is a univalent function with $\varphi(0) = 0, \varphi'(0) = 1$.*

Remark. Since $\mathcal{R}_0(\mathcal{F}_0)$ has one to one correspondence to $\mathcal{S}$ (see Appendix), which is compact with respect to the topology of uniform convergence on compact sets (by Koebe distortion theorem).

Unfortunately this class $\mathcal{F}_0$ cannot be invariant for the fiber renormalization $\mathcal{R}_\alpha$ for $\alpha \neq 0$. As soon as $f \in \mathcal{F}_0$ is perturbed into near-parabolic $e^{2\pi i\alpha} f$, the simple covering structure of horn map is destroyed, hence there may be infinitely many critical values, or it may not be a branched covering at all. It was for this reason that we had to look for another class which is invariant under $\mathcal{R}_\alpha$. The main goal of this paper is to propose a new invariant class $\mathcal{F}_1$, which will be defined in the next section.

Periodic points and mixed renormalization. The fixed points 0 and $\sigma(f)$ correspond to the upper and lower ends $\pm i\infty$ of $\mathcal{C}_{rep}$, which in turn correspond to 0 and ∞ of $\mathbb{C}^*$ via $\mathrm{Exp}^\sharp$ (or to ∞ and 0 via $\mathrm{Exp}^\flat$). If $f_1 = \mathcal{R} f = \mathcal{R}^\sharp f$ has $f_1''(0) \neq 0$ and $f_1'(0)$ is close to 1, then f_1 has another fixed point $\sigma(f_1)$ near 0 and this corresponds to a periodic orbit for f, whose multiplier is equal to $f_1'(\sigma(f_1))$. Hence if the sequence $\{f_n\}$ is given by (3.6), then the fixed points $\sigma(f_n)$ correspond to periodic points near 0 for the original f.

We can consider the near-parabolic renormalization for f_1 near $\sigma(f_1)$, and this can be carried out by considering $\mathcal{R}^\flat f_1 = \mathcal{R}^\flat \mathcal{R}^\sharp f$ (which is the return map of f_1 near $\sigma(f_1)$) and further $\{(\mathcal{R}^\sharp)^n \mathcal{R}^\flat \mathcal{R}^\sharp f\}_{n=0}^\infty$. For example, if $\mathcal{R}^\flat f_1$ has an indifferent fixed point at 0, then $\sigma(f_1)$ is indifferent and f has an indifferent periodic orbit bifurcated from fixed point. In the case where $f(z)$ is a quadratic polynomial $z^2 + c$ (after the coordinate change so that 0 is fixed), the parameter c will be on the boundary of a satellite hyperbolic component attached to the main cardioid.

Similarly, one can also consider mixed iteration of $\mathcal{R}^\sharp$ and $\mathcal{R}^\flat$. This corresponds to an infinite "satellite renormalization" when $\mathcal{R}^\flat$ appears infinitely often.

4 A new class $\mathcal{F}_1$ and main results

In this section, we define our class $\mathcal{F}_1$ and state main results.

Definition (P and Class $\mathcal{F}_1$)**.** Let $P(z) = z(1+z)^2$. The polynomial P has a parabolic fixed point at 0 and critical points $-\frac{1}{3}$ and -1 with $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$. Let V be a domain of $\mathbb{C}$ containing 0 and define

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent, } \varphi(0) = 0,\ \varphi'(0) = 1 \\ \text{and } \varphi \text{ has a quasiconformal extension to } \mathbb{C} \end{array} \right\},$$

where univalent means holomorphic and injective. Note that if $f \in \mathcal{F}_1$, 0 is a 1-parabolic fixed point of f. If $-\frac{1}{3} \in V$, then $cp_f := \varphi(-\frac{1}{3})$ is a critical point and $-\frac{4}{27}$ is a critical value of f.

Main Theorem 1 (Invariance of $\mathcal{F}_1$)**.** *There exist a Jordan domain V containing 0 and $-\frac{1}{3}$ with a smooth boundary and an open set V' containing $\overline{V}$ such that the above $\mathcal{F}_1$ satisfies the following:*

(a) $f''(0) \neq 0$ *(in fact, $|f''(0) - 4.91| \leq 1.14$). $cp_f \in Basin(0)$.*

(b) $\left(\mathcal{F}_0 \smallsetminus \{\textit{quadratic polynomial}\}\right) / \underset{linear}{\sim}$ *can be naturally included into $\mathcal{F}_1$.*

(c) *$\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$. That is, for $f \in \mathcal{F}_1$, the parabolic renormalization $\mathcal{R}_0 f$ is well-defined so that $\mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_1$. Moreover ψ extends to a univalent function from V' to $\mathbb{C}$.*

(d) *$\mathcal{R}_0$ is holomorphic in the following sense: Suppose a family $f_\lambda = P \circ \varphi_\lambda^{-1}$ is given by a holomorphic function $\varphi_\lambda(z)$ in two variables $(\lambda, z) \in \Lambda \times V$, where Λ is a complex manifold. Then the renormalization can be written as $\mathcal{R}_0 f_\lambda = P \circ \psi_\lambda^{-1}$ with $\psi_\lambda(z)$ holomorphic in $(\lambda, z) \in \Lambda \times V'$.*

Remark. When f is defined in a larger domain and its restriction $f|_U$ to a domain U belongs to $\mathcal{F}_1$, the theorem asserts that its renormalization $\mathcal{R}_0(f) = P \circ \psi^{-1} : \psi(V') \to \mathbb{C}$ can be defined only using the iterates of f within U.

As we will see later, P and V are symmetric with respect to the complex conjugation. Therefore $f \in \mathcal{F}_1$ if and only if $f^\star \in \mathcal{F}_1$.

This theorem is central in this paper and will be proved in §5. The outline of the proof as well as the explicit definition of V and V' will be presented in §5.A. Here it is important that $\overline{V} \subset V'$, i.e., the new domain for ψ is strictly larger than that of original φ (*analyticity improving*), which was not achieved with class $\mathcal{F}_0$. This fact leads to Main Theorems 2 and 3.

Main Theorem 2 (Contraction)**.** *There exists a one to one correspondence between $\mathcal{F}_1$ and the Teichmüller space of $\mathbb{C} \smallsetminus \overline{V}$. Let $d(\cdot,\cdot)$ be the distance on $\mathcal{F}_1$ induced from the Teichmüller distance, which is complete. Then $\mathcal{R}_0$ is a uniform contraction;*

$$d(\mathcal{R}_0(f), \mathcal{R}_0(g)) \le \lambda\, d(f,g) \quad \textit{for } f,g \in \mathcal{F}_1$$

where $\lambda = e^{-2\pi \operatorname{mod}(V' \smallsetminus \overline{V})} < 1$. The convergence with respect to d implies the uniform convergence on compact sets (but not vice versa).

The proof will be given in §6 and basic facts about the Teichmüller space is also summarized there. An immediate consequence, together with Theorem 3.2, is the following:

Corollary 4.1. *The parabolic renormalization $\mathcal{R}_0$ on $\mathcal{F}_1$ has a unique fixed point, which belongs to $\mathcal{F}_0$. For any $f \in \mathcal{F}_1$, $\{\mathcal{R}_0^n f\}_{n=0}^\infty$ converges to the fixed point exponentially fast with respect to the metric defined in Main Theorem 2. Moreover, if $f \in \mathcal{F}_0$, then the renormalizations $\mathcal{R}_0^n f$ considered as elements of $\mathcal{F}_0$ converge to the fixed point uniformly on compact sets in the sense of §1.*

We can derive similar results for the near-parabolic renormalization $\mathcal{R}$ and the fiber renormalization $\mathcal{R}_\alpha$ defined in the previous section, provided that α is small.

Definition. For $\alpha_* > 0$, denote $\mathcal{I}(\alpha_*) = (-\alpha_*, \alpha_*) \smallsetminus \{0\}$ and

$$e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1 = \{e^{2\pi i\alpha} h(z) \mid \alpha \in \mathcal{I}(\alpha_*),\ h \in \mathcal{F}_1\}.$$

The distance on this space is defined by $d(f,g) = d(\frac{1}{f'(0)} f, \frac{1}{g'(0)} g) + |f'(0) - g'(0)|$, where d on the right hand side is the one for $\mathcal{F}_1$ defined in Main Theorem 2.

For an integer N, let $\mathit{Irrat}_{\ge N}$ be the set of irrational numbers α such that the continued fraction expansion (3.4) has coefficients $a_n \ge N$.

Main Theorem 3 (Invariance of $\mathcal{F}_1$ under $\mathcal{R}_\alpha$ and hyperbolicity)**.** *There exists $\alpha_* > 0$ such that if $\alpha \in \mathbb{C}$, $|\arg \alpha| < \pi/4$ (or $|\arg(-\alpha)| < \pi/4$) and $0 < |\alpha| \le \alpha_*$, then $\mathcal{R}_\alpha$ can be defined in $\mathcal{F}_1$ so that (c) and (d) of Main Theorem 1 hold for $\mathcal{R}_\alpha$. Moreover $\mathcal{R}_\alpha$ is a contraction as in Main Theorem 2 with the same λ. Hence the renormalization $\mathcal{R}$ is hyperbolic in $e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1$.*

In particular, there exists an integer $N \ge 2$ for which the following holds:
If $f(z) = e^{2\pi i\alpha} h(z)$ with $h \in \mathcal{F}_1$ and $\alpha \in \mathit{Irrat}_{\ge N}$, then the sequence of renormalizations (3.6) *can be defined and f_n's belong to $e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1$. If $g(z)$ is another map of the same type with the same α, then $d(\mathcal{R}^n f, \mathcal{R}^n g) \to 0$ as $n \to \infty$ exponentially fast.*

The proof of this theorem and the corollaries below will be given in §7. We obtain these α_* and N by a continuity argument, so we do not have explicit bounds. It will be important to know how big α_* can be.

Corollary 4.2. *There exists an N (may be larger than the one in Main Theorem 3) such that if $f(z) = e^{2\pi i\alpha}h(z)$ with $h \in \mathcal{F}_1$ and $\alpha \in Irrat_{\geq N}$, then the critical orbit of f stays in the domain of definition of f and can be iterated infinitely many times. Moreover there exists an infinite sequence of periodic orbits to which the critical orbit does not accumulate.*

The same conclusion holds for $f(z) = e^{2\pi i\alpha}z + z^2$ provided that $\alpha \in Irrat_{\geq N}$ and α itself is sufficiently small. Hence the critical orbit is not dense in J_f.

Remark. Main Theorems 1 and 2 also hold for the lower-end renormalization $\mathcal{R}^\flat$. This can be easily seen by taking a complex conjugation $f \mapsto f^\star$. As for the near-parabolic renormalization, we can formulate as follows: there exists $\alpha'_* > 0$ such that if $f = e^{2\pi i\alpha}h$ (with $\alpha \in \mathbb{C}$, $|\alpha| < \alpha'_*$ and $h \in \mathcal{F}_1$) has a fixed point $\sigma(f)$ whose multiplier is $e^{2\pi i\beta}$ satisfying $0 < |\beta| < \alpha'_*$ and $|\arg\beta| < \frac{\pi}{4}$ (or $|\arg(-\beta)| < \frac{\pi}{4}$), then $\mathcal{R}^\flat f$ can be defined so that

$$\mathcal{R}^\flat f = e^{2\pi i\gamma}h_1 \quad \text{with } \gamma = -\frac{1}{\beta} \text{ (or } \frac{1}{\bar{\beta}}\text{)} \quad \text{and } h_1 \in \mathcal{F}_1.$$

In fact, when we prove the first part of Main Theorem 3, 0 and $\sigma(f)$ play symmetric role in the analysis of perturbation. (See §7.) Therefore we have a hyperbolicity of mixed iteration of $\mathcal{R}^\sharp$ and $\mathcal{R}^\flat$ (see the previous section) similar to Main Theorem 3, except that the domain of definition does not have a simple characterization by the multiplier of a single fixed point.

5 Proof of Main Theorem 1 – Invariance of $\mathcal{F}_1$

5.A Outline of the proof

Strategy: Our main goal is to prove (c) of Main Theorem 1, i.e., to find ψ such that $\mathcal{R}_0 f = \Psi_0 \circ E_f \circ \Psi_0^{-1} = P \circ \psi^{-1}$, where $\Psi_0(z) = c\,\mathrm{Exp}^\sharp(z)$ with some constant $c \in \mathbb{C}^*$. Then ψ should be *formally* written as

$$\psi = \Psi_0 \circ \Phi_{rep} \circ {\Phi_{attr}}^{-1} \circ \Psi_0^{-1} \circ P = \Psi_0 \circ \Phi_{rep} \circ f^{-n} \circ {\Phi_{attr}}^{-1} \circ \Psi_0^{-1} \circ P. \tag{5.1}$$

Here the equality on the right is a tautology, because $\Phi_{rep}(f(z)) = \Phi_{rep}(z) + 1$ and $\Psi_0(z+1) = \Psi_0(z)$. But the right hand side has following interpretation: Φ_{attr} and Φ_{rep} are first defined in attracting and repelling half-neighborhoods of 0 (corresponding to $\{\mathrm{Re}\, z > L\}$ and $\{\mathrm{Re}\, z < -L\}$ for F as in Theorem 1.1), then the inverse branch f^{-n} "maps" part of attracting half-neighborhood to repelling one. It is important that the multi-valuedness and branching of f^{-n} should be balanced by three-to-one map P at the beginning of composition.

In order to carry out various estimates, we move the fixed point to ∞ and reduce the problem to a map F which has a parabolic fixed point at ∞ ($\mathcal{F}_1^Q$ defined below, cf. Propositions 5.2 and 5.3). On the repelling side of the fixed point, we construct a Riemann surface X with a projection $\pi_X : X \to \mathbb{C}$ and a map $g : X \to X$ so that g corresponds to an inverse branch of f and the repelling Fatou coordinate is defined on X (Propositions 5.4 and 5.5). As for the attracting Fatou coordinate, Proposition 5.6 gives an estimate on Φ_{attr} in the region $\mathrm{Re}\,\Phi_{attr}(z) \geq 1$ (under normalization $\Phi_{attr}(cv) = 1$), especially it gives bounds on the location of $D_1 = \Phi_{attr}^{-1}(\{z : 1 < \mathrm{Re}\, z < 2 \text{ and } |\mathrm{Im}\, z| < \eta\})$ and $D_1^\sharp$ (corresponding to $\mathrm{Im}\, z > \eta$). We trace specific inverse images of D_1 and $D_1^\sharp$ and obtain domains D_0, D'_0, D_{-1}, D''_{-1} and $D_0^\sharp$, which can be lifted to X (Proposition 5.7). We partition the domain of P according to D_1 and $D_1^\sharp$ and define ψ in each component so that (5.1) is defined through one of the above domains (Proposition 5.8). The resulting ψ is consistent on the boundary of the components and yields $\mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_2^P$.

Now we move on to more details of the proof. To start with, the following proposition explains why $P(z) = z(1+z)^2$ is important in our results.

Proposition 5.1 (Subcover like P)**.** *Let $f \in \mathcal{F}_0$ and suppose that f is not a quadratic polynomial. After a linear conjugacy, one may suppose that its unique critical value is $-\frac{4}{27}$. Then there exists a confomal mapping φ from $\mathbb{C} \smallsetminus (-\infty, -1]$ onto an open subset $U \subset Dom(f)$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$ and*

$$f = P \circ \varphi^{-1} \quad \text{on } U.$$

The proof of this proposition, given in §5.C, uses the idea that the maps are regarded as a (partial) branched covering over the range, and this covering structure is common up to certain "sheets". This view motivates the definition of $\mathcal{F}_1$ (or $\mathcal{F}_2^P$ defined below), characterizing the maps by their covering property over the range. (See Figure 10 there.)

Definition (Mapping Q)**.** Define

$$Q(z) = z\,\frac{\left(1+\frac{1}{z}\right)^6}{\left(1-\frac{1}{z}\right)^4}, \quad \psi_1(z) = -\frac{4z}{(1+z)^2} = 4f_{Koebe}\left(-\frac{1}{z}\right), \quad \psi_0(z) = -\frac{4}{z}.$$

In §5.D, we will see that Q is related to P by $Q = \psi_0^{-1} \circ P \circ \psi_1$ and ψ_1^{-1} "opens up" the slit $(-\infty, -1]$ to the unit disk so that $\psi_1(\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}) = \widehat{\mathbb{C}} \smallsetminus (-\infty, -1]$ with $\psi_1(\infty) = 0$.

Definition ($V' = U_\eta^P$ and U_η^Q)**.** Let $\eta > 0$ and $cv_P = -\frac{4}{27}$ (which is a critical value of P) and define

$$\begin{aligned} V' = U_\eta^P &= P^{-1}\left(\mathbb{D}(0, |cv_P|\, e^{2\pi\eta})\right) \\ &\quad \smallsetminus \left((-\infty,-1] \cup \left(\text{the component of } P^{-1}\left(\mathbb{D}(0,|cv_P|\, e^{-2\pi\eta})\right) \text{ containing } -1\right)\right). \end{aligned}$$

See Figure 6. Let $U_\eta^Q = \psi_1^{-1}(U_\eta^P) \smallsetminus \overline{\mathbb{D}}$.

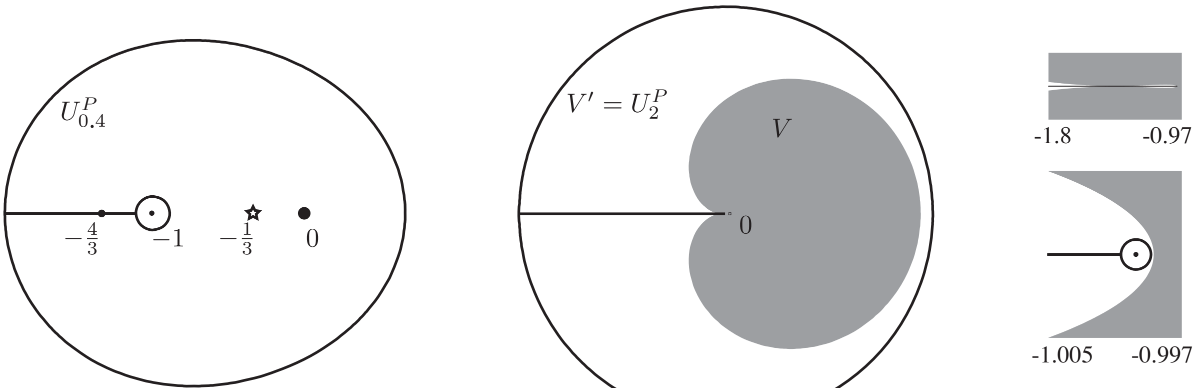

Figure 6: Left: U_η^P for $\eta = 0.4$ (this η was chosen so that the deleted component around -1 is visible); Middle: U_η^P for $\eta = 2$ and V. The outer boundary of U_η^P looks like a circle with radius about 35; Right: successive blow-ups of U_2^P and V near -1.

Definition (Ellipse E and V)**.** Let $x_{_E} = -0.18$, $a_{_E} = 1.24$, $b_{_E} = 1.04$ and define

$$E = \left\{ x + iy \in \mathbb{C} : \left(\frac{x - x_{_E}}{a_{_E}} \right)^2 + \left(\frac{y}{b_{_E}} \right)^2 \leq 1 \right\}$$

and $V = \psi_1(\widehat{\mathbb{C}} \smallsetminus E)$.

Proposition 5.2 (Relation between $\widehat{\mathbb{C}} \smallsetminus intE$ and U^Q_η)**.** *Let $\eta = 2$. Then we have*

$$\widehat{\mathbb{C}} \smallsetminus intE \subset U^Q_\eta \subset \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}.$$

Hence

$$\overline{V} \subset V' = U^P_\eta \subset \mathbb{C} \smallsetminus (-\infty, -1].$$

The proof is given in §5.E. The constant $\eta = 2$ and the ellipse E will be used throughout this paper.

Definition (Classes $\mathcal{F}^P_2$, $\mathcal{F}^Q_1$)**.** From now on, we denote the class $\mathcal{F}_1$ by $\mathcal{F}^P_1$. We now define two more classes of maps:

$$\mathcal{F}^P_2 = \left\{ f = P \circ \varphi^{-1} \;\middle|\; \varphi : V' \to \mathbb{C} \text{ is univalent, } \varphi(0) = 0,\ \varphi'(0) = 1 \right\}$$

$$\mathcal{F}^Q_1 = \left\{ F = Q \circ \varphi^{-1} \;\middle|\; \begin{array}{c} \varphi : \widehat{\mathbb{C}} \smallsetminus E \to \widehat{\mathbb{C}} \smallsetminus \{0\} \text{ is a normalized univalent mapping} \\ \text{and has a quasiconformal extension to } \widehat{\mathbb{C}} \end{array} \right\}$$

Here a *univalent mapping* is a holomorphic and injective mapping (in general it is allowed to take value ∞); it is called *normalized* if $\varphi(0) = 0$ and $\varphi'(0) = 1$ when 0 is in the domain, or if $\varphi(\infty) = \infty$ and $\lim_{z \to \infty} \frac{\varphi(z)}{z} = 1$ when ∞ is in the domain instead.

Proposition 5.3 (Relation between $\mathcal{F}^P_1$, $\mathcal{F}^P_2$, $\mathcal{F}^Q_1$ and $\mathcal{F}_0$)**.** *We have the relation*

$$\left((\mathcal{F}_0 \smallsetminus \{\textit{quadratic polynomials}\}) / \underset{linear}{\sim} \right) \subset \mathcal{F}^P_2 \subset \mathcal{F}^P_1 \cong \mathcal{F}^Q_1.$$

More precisely it is formulated as follows:

(a) *There is a natural injection* $\left((\mathcal{F}_0 \smallsetminus \{\textit{quadratic polynomials}\}) / \underset{linear}{\sim} \right) \hookrightarrow \mathcal{F}^P_2$.

(b) *There is a natural injection $\mathcal{F}^P_2 \hookrightarrow \mathcal{F}^P_1$, defined by the restriction of φ to V for $f = P \circ \varphi^{-1} \in \mathcal{F}^P_2$.*

(c) *There exists a one to one correspondence between $\mathcal{F}^P_1$ and $\mathcal{F}^Q_1$, defined by*

$$\mathcal{F}^P_1 \ni f = P \circ \varphi^{-1} \longmapsto F = \psi_0 \circ f \circ \psi_0^{-1} = \psi_0^{-1} \circ P \circ \psi_1 \circ \psi_1^{-1} \circ \varphi^{-1} \circ \psi_0 = Q \circ \hat{\varphi}^{-1} \in \mathcal{F}^Q_1,$$

with associated correspondence $\varphi \longmapsto \hat{\varphi} = \psi_0^{-1} \circ \varphi \circ \psi_1$. In this case, if $\hat{\varphi}(z) = z + c_{_0} + O(\frac{1}{z})$ near ∞, then $f''(0) = 5 - \frac{c_0}{2}$.

The proof will be given in §5.D.

The above (a) is implied by Proposition 5.1 and implies (b) of Main Theorem 1. The first half of Main Theorem 1 (a) follows from the above (c) and $|c_{_0} - 0.18| \leq 2.28$, which is proved in Lemma 5.22 (a) in §5.G. In order to show (c) of Main Theorem 1, it suffices to prove that if $F = Q \circ \varphi^{-1} \in \mathcal{F}^Q_1$ (instead of $\mathcal{F}^P_1$), then the parabolic renormalization $\mathcal{R}_0 F$ (which is defined similarly as in §3) belongs to $\mathcal{F}^P_2$.

Assumption: Let $F = Q \circ \varphi^{-1} \in \mathcal{F}^Q_1$. Therefore $\varphi : \widehat{\mathbb{C}} \smallsetminus E \to \widehat{\mathbb{C}} \smallsetminus \{0\}$ is a normalized univalent mapping. We do not need to assume the existence of quasiconformal-extension, which is needed only for Theorems 2 and 3. Basic estimates on Q, φ and F will be given in §§5.E, 5.F, 5.G and 5.I.

Definition (Riemann surface X)**.** Let $cv = cv_Q = 27$ (which is a critical value of Q), $R = 266$ and $\rho = 0.05$. Define four "sheets" by

$$X_{1\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im} z \geq 0,\ |z| > \rho \text{ and } \frac{\pi}{6} < \pm \arg(z - cv) \leq \pi\},$$
$$X_{2\pm} = \{z \in \mathbb{C} : z \notin \mathbb{R}_-,\ \pm \operatorname{Im} z \geq 0,\ \rho < |z| < R \text{ and } \frac{\pi}{6} < \pm \arg(z - cv) \leq \pi\}.$$

Here these "sheets" are considered to be lying in disjoint copies of $\mathbb{C}$ and let $\pi_{i\pm} : X_{i\pm} \to \mathbb{C}$ ($i = 1, 2$) be the natural projection. Now we glue them together to construct a Riemann surface X as follows: X_{1+} and X_{1-} are glued along negative real axis (i.e., for $x < -\rho$, $\pi_{1+}^{-1}(x) \in X_{1+}$ and $\pi_{1-}^{-1}(x) \in X_{1-}$ are identified), X_{1+} and X_{2-} are glued along positive real axis and X_{1-} and X_{2+} are also glued along positive real axis. The projection $\pi_X : X \to \mathbb{C}$ is defined as $\pi_X = \pi_{i\pm}$ on $X_{i\pm}$. The complex structure is given through the projection. See Figure 7.

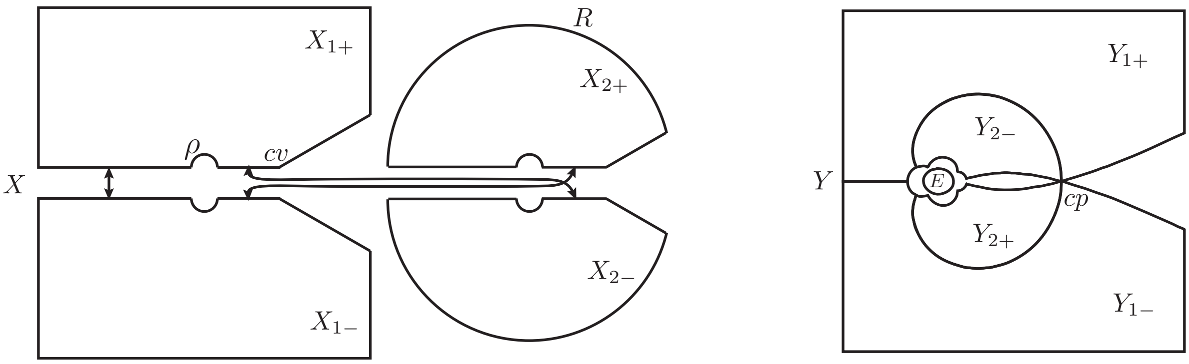

Figure 7: Riemann Surface X (left) and Domain Y (right)

Proposition 5.4 (Lifts of Q and φ to X)**.** *There exists an open subset $Y \subset \mathbb{C} \smallsetminus (E \cup \mathbb{R}_+)$ with the following properties:*

(a) *There exists an isomorphism $\widetilde{Q} : Y \to X$ such that $\pi_X \circ \widetilde{Q} = Q$ on Y and $\widetilde{Q}^{-1}(z) = \pi_X(z) - 10 + o(1)$ as $z \in X$ and $\pi_X(z) \to \infty$;*

(b) *The map φ restricted to Y can be lifted to a univalent holomorphic map $\tilde{\varphi} : Y \to X$ so that $\pi_X \circ \tilde{\varphi} = \varphi$ on Y.*

This will be proved in §5.H. The Riemann surface X allows us to lift $F^{-1} = \varphi \circ Q^{-1}$ to a single-valued branch, so that it is easy to iterate without falling out of the domain of definition of φ. Therefore if some inverse images of a set arrives in X then it can be safely iterated by this specific branch of F^{-1}.

Definition. Let $g = \tilde{\varphi} \circ \widetilde{Q}^{-1} : X \to X$.

Proposition 5.5 (Repelling Fatou coordinate on X)**.** *The map g satisfies $F \circ \pi_X \circ g = \pi_X$. There exists an injective holomorphic mapping $\widetilde{\Phi}_{rep} : X \to \mathbb{C}$ such that $\widetilde{\Phi}_{rep}(g(z)) = \widetilde{\Phi}_{rep}(z) - 1$. Moreover in $\{z : \operatorname{Re} z < -R\}$, $\widetilde{\Phi}_{rep} \circ \pi_X^{-1}$ is a repelling Fatou coordinate for $F = Q \circ \varphi^{-1}$.*

This will be proved in §5.J.

Definition. For $z_0 \in \mathbb{C}$ and $\theta > 0$, denote $\mathbb{V}(z_0, \theta) = \{z : z \neq z_0, |\arg(z - z_0)| < \theta\}$, $\overline{\mathbb{V}}(z_0, \theta) =$ the closure of $\mathbb{V}(z_0, \theta)$. Define

$$\mathrm{W}_1 = \mathbb{V}(cv, \tfrac{2\pi}{3}) \smallsetminus \overline{\mathbb{V}}(F(cv), \tfrac{\pi}{3}) = \{z : |\arg(z - cv)| < \tfrac{2\pi}{3} \text{ and } |\arg(z - F(cv)) - \pi| < \tfrac{2\pi}{3}\}.$$

We will see in Lemma 5.28 that $\operatorname{Re} F(cv) > 30$ hence W_1 is connected. Finally, let $u_0 = \frac{25}{\sqrt{3}}$ ($\doteqdot$ 14.43...) and $R_1 = 239$.

Proposition 5.6 (Attracting Fatou coordinate and shape of D_1)**.** (a) *The F maps $\mathbb{V}(u_0, \frac{2\pi}{3})$ into itself and $\mathbb{V}(u_0, \frac{2\pi}{3})$ is contained in $Basin(\infty)$. There exists an attracting Fatou coordinate $\Phi_{attr} : \mathbb{V}(u_0, \frac{2\pi}{3}) \to \mathbb{C}$ such that $\Phi_{attr}(F(z)) = \Phi_{attr}(z) + 1$ and $\Phi_{attr}(cv) = 1$. Moreover Φ_{attr} is injective in $\mathbb{V}(u_0, \frac{2\pi}{3})$ and $\Phi_{attr}(\mathbb{V}(u_0, \frac{2\pi}{3}))$ contains $\{z : \operatorname{Re} z > 1\}$.*

(b) *There are domains $D_1, D_1^\sharp, D_1^\flat \subset \mathrm{W}_1 (\subset \mathbb{V}(u_0, \frac{2\pi}{3}))$ such that*

$$\Phi_{attr}(D_1) = \{z : 1 < \operatorname{Re} z < 2,\ -\eta < \operatorname{Im} z < \eta\} \text{ and } D_1 \subset \mathbb{D}(cv, R_1);$$

$$\Phi_{attr}(D_1^\sharp) = \{z : 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z > \eta\} \text{ and } D_1^\sharp \subset \{z : \tfrac{\pi}{6} < \arg(z - cv) < \tfrac{2\pi}{3}\};$$

$$\Phi_{attr}(D_1^\flat) = \{z : 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z < -\eta\} \text{ and } D_1^\flat \subset \{z : -\tfrac{2\pi}{3} < \arg(z - cv) < -\tfrac{\pi}{6}\}.$$

This is the most delicate estimate and will be proved in 5.K. The key estimate in the proof is Theorem 5.12. In fact, in this proposition, η can be replaced by 13.0 while still using the same R_1. The above (a) implies that cv and also $cp_F = \varphi(cp_Q)$ are in $Basin(\infty)$, which is the second half of Main Theorem 1(a). Normalize $\widetilde{\Phi}_{rep}$ by adding a constant so that $\widetilde{\Phi}_{rep}(z) - \Phi_{attr}(\pi_X(z)) \to 0$ when $z \in X$, $\pi_X(z) \in D_1^\sharp$ and $\operatorname{Im} \pi_X(z) \to +\infty$.

Proposition 5.7 (Domains around critical point)**.** *There exist disjoint Jordan domains $D_0, D_0', D_{-1}, D_{-1}''$ and a domain $D_0^\sharp$ such that*

(a) *the closures $\overline{D}_0, \overline{D}_0', \overline{D}_{-1}, \overline{D}_{-1}''$ and $\overline{D}_0^\sharp$ are contained in $Image(\varphi) = Dom(F)$;*

(b) *$F(D_0) = F(D_0') = D_1$, $F(D_{-1}) = F(D_{-1}'') = D_0$, $F(D_0^\sharp) = D_1^\sharp$;*

(c) *F is injective on each of these domains;*

(d) *$cp_F = \varphi(cp_Q) \in \overline{D}_0 \cap \overline{D}_0' \cap \overline{D}_{-1} \cap \overline{D}_{-1}''$, $\overline{D}_0 \cap \overline{D}_1 \neq \emptyset$, $\overline{D}_0^\sharp \cap \overline{D}_1^\sharp \neq \emptyset$, $\overline{D}_{-1} \cap \overline{D}_0^\sharp \neq \emptyset$;*

(e) *$\overline{D}_0 \cup \overline{D}_0' \cup \overline{D}_{-1} \cup \overline{D}_{-1}'' \setminus \{cv\} \subset \pi_X(X_{2+}) \cup \pi_X(X_{2-}) = \mathbb{D}(0, R) \setminus (\overline{\mathbb{D}}(0, \rho) \cup \mathbb{R}_- \cup \overline{\mathbb{V}}(cv, \frac{\pi}{6}))$ and $\overline{D}_0^\sharp \subset \pi_X(X_{1+})$.*

This will be proved in §5.L, by bounding the regions which contain inverse images of D_1. Much of efforts are put into proving $(\overline{D}_0 \cup \overline{D}_0' \cup \overline{D}_{-1} \cup \overline{D}_{-1}'') \cap \mathbb{R}_- = \emptyset$. See Figure 8, for the shape of these domains in the case of $\varphi = id$.

Proposition 5.8 (Relating E_F to P)**.** *The parabolic renormalization $\mathcal{R}_0 F$ belongs to the class $\mathcal{F}_2^P$ (possibly after a linear conjugacy). In fact, we prove the following.*

Regard D_0, D_0', D_{-1}'', $D_0^\sharp$ as subsets of $X_{1+} \cup X_{2-} \subset X$ and let

$$U = \text{the interior of } \bigcup_{n=0}^{\infty} g^n \left(\overline{D}_0 \cup \overline{D}_0' \cup \overline{D}_{-1}'' \cup \overline{D}_0^\sharp \right).$$

Then there exists a surjective holomorphic mapping $\Psi_1 : U \to U_\eta^P \setminus \{0\} = V' \setminus \{0\}$ such that

(a) *$P \circ \Psi_1 = \Psi_0 \circ \widetilde{\Phi}_{attr}$ on U, where $\Psi_0 : \mathbb{C} \to \mathbb{C}^*$, $\Psi_0(z) = cv_P e^{2\pi i z} = cv_P \operatorname{Exp}^\sharp(z)$, and $\widetilde{\Phi}_{attr} : U \to \mathbb{C}$ is the natural extension of the attracting Fatou coordinate to U;*

(b) *$\Psi_1(z) = \Psi_1(z')$ if and only if $z' = g^n(z)$ or $z = g^n(z')$ for some integer $n \geq 0$;*

(c) *$\psi = \Psi_0 \circ \widetilde{\Phi}_{rep} \circ \Psi_1^{-1} : V' \setminus \{0\} \to \mathbb{C}^*$ is well-defined and extends to a normalized univalent function on V';*

(d) *on $\psi(V' \setminus \{0\})$, the following holds*

$$P \circ \psi^{-1} = P \circ \Psi_1 \circ \widetilde{\Phi}_{rep}^{-1} \circ \Psi_0^{-1} = \Psi_0 \circ \widetilde{\Phi}_{attr} \circ \widetilde{\Phi}_{rep}^{-1} \circ \Psi_0^{-1} = \Psi_0 \circ E_F \circ \Psi_0^{-1};$$

(e) *we have the holomorphic dependence as in Main Theorem 1 (d).*

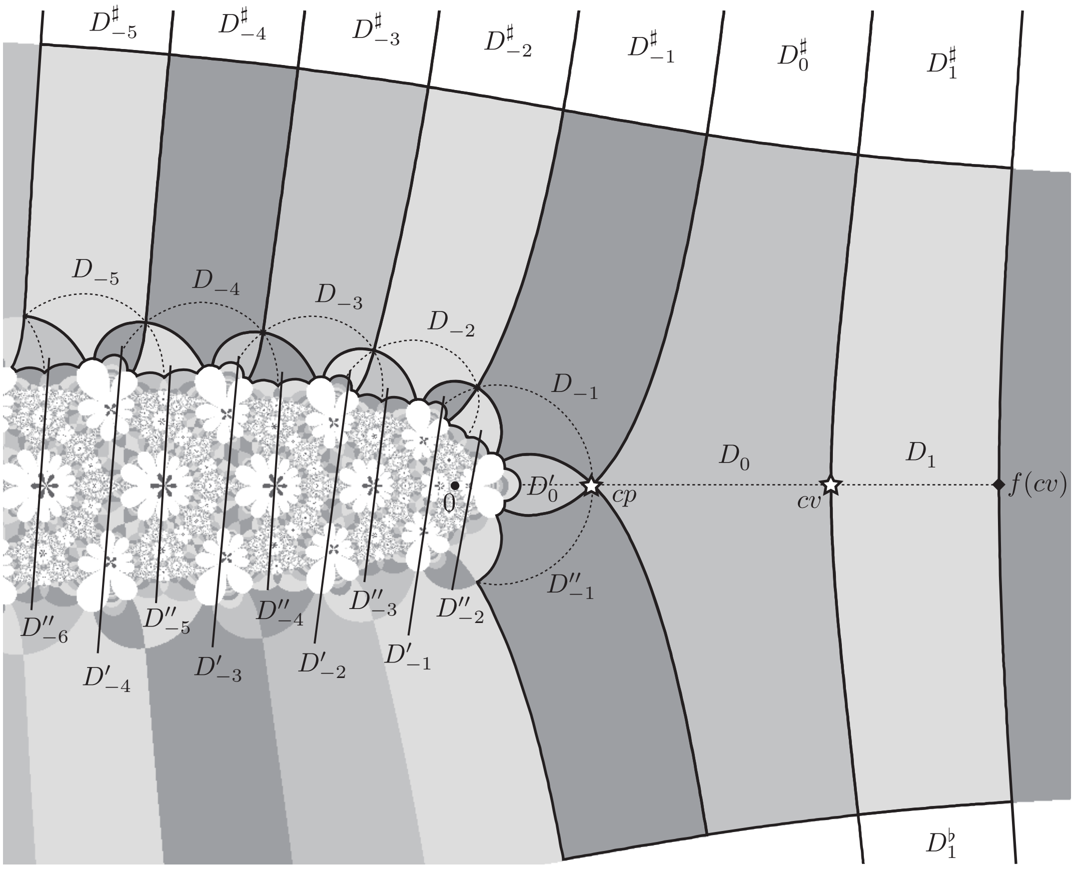

Figure 8: D_1, D_0 etc. for $F = Q$ ($\varphi = id$). Further inverse images are denoted by $D_{-n} = g^n(D_0)$, $D'_{-n} = g^n(D'_0)$, $D''_{-n} = g^{n-1}(D''_{-1})$, $D^\sharp_{-n} = g^n(D^\sharp_0)$, and their projection by π_X are drawn.

This will be proved in §5.M. The Ψ_1 is defined by choosing an appropriate branch of $P^{-1} \circ \Psi_0 \circ \widetilde{\Phi}_{attr}$ on each domain $D_{-n} = g^n(D_0)$ etc. Its consistency can be observed by comparing Figure 8 and Figure 9. Thus, by setting

$$\mathcal{R}_0 F = P \circ \psi^{-1} \in \mathcal{F}^P_2 \quad \text{for } F = Q \circ \varphi^{-1} \in \mathcal{F}^Q_1 \ (\simeq \mathcal{F}^P_1),$$

we have obtained (c) and (d) of Main Theorem 1, via $\mathcal{F}^P_2 \hookrightarrow \mathcal{F}^P_1$ in Proposition 5.3. This concludes the proof of Main Theorem 1.

5.B Preparation

We prepare some lemmas and notation for the proof.

Lemma 5.9. (a) *If* $a, b \in \mathbb{C}$ *and* $|a| > |b|$, *then* $|\arg(a+b) - \arg a| \le \arcsin\left(\frac{|b|}{|a|}\right)$.
(b) *If* $0 \le x \le \frac{1}{2}$, *then* $\arcsin x \le \frac{\pi}{3}\, x$.

Proof. (a) The tangent from 0 to $\partial\mathbb{D}(a, |b|)$ has angle $\arcsin\left(\frac{|b|}{|a|}\right)$ with respect to the vector $\overrightarrow{0a}$.
(b) This follows from the concavity of $\sin\theta$ in $0 \le \theta \le \frac{\pi}{6}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$. $\square$

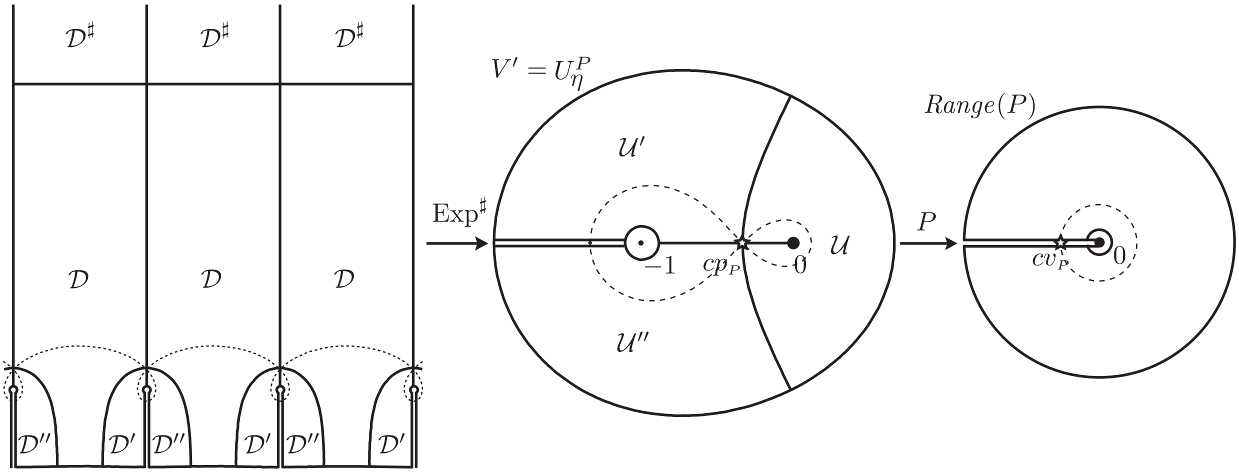

Figure 9: U^P_η and its log lift (inverse image by $\mathrm{Exp}^\sharp$). To emphasize the details, $\eta = 0.4$ for U^P_η and $\eta = 0.2$ for $Range(P)$ were used.

Lemma 5.10. *Let $e_1 = 1.14$, $e_0 = -0.18 = x_E$, $e_{-1} = 0.1$ and define*

$$\zeta(w) = e_1 w + e_0 + \frac{e_{-1}}{w}.$$

Then ζ is a conformal map from $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ onto $\mathbb{C} \smallsetminus E$, and sends $\{w : |w| = r\}$ onto ∂E_r, where $E_r = \left\{x + iy : \left(\frac{x-e_0}{a_E(r)}\right)^2 + \left(\frac{y}{b_E(r)}\right)^2 \leq 1\right\}$ with $a_E(r) = e_1 r + \frac{e_{-1}}{r}$ and $b_E(r) = e_1 r - \frac{e_{-1}}{r}$. For $r = 1$, we have $a_E(1) = a_E$, $b_E(1) = b_E$ and $E_1 = E$, which are defined in §5.A.

Proof. If $w = re^{i\theta}$, then $\zeta(w) = e_0 + a_E(r)\cos\theta + ib_E(r)\sin\theta$. □

Lemma 5.11. (a) *If $\mathrm{Re}(ze^{-i\theta}) > t > 0$ with $\theta \in \mathbb{R}$, then*

$$\frac{1}{z} \in \mathbb{D}\left(\frac{e^{-i\theta}}{2t}, \frac{1}{2t}\right);$$

(b) *If $H = \{z : \mathrm{Re}(z\,e^{-i\theta}) > t\}$ and $z_0 \in H$ with $u = \mathrm{Re}(z_0 e^{-i\theta}) - t$, then*

$$\mathbb{D}_H(z_0, s(r)) = \mathbb{D}\left(z_0 + \frac{2ur^2 e^{i\theta}}{1-r^2}, \frac{2ur}{1-r^2}\right),$$

where the right hand side is an Euclidean disk and $s(r) = d_{\mathbb{D}}(0, r) = \log\frac{1+r}{1-r}$.

Proof. (a) Immediate from the property of Möbius transformation $\frac{1}{z}$ or a simple calculation.
(b) When $\theta = 0$, $t = 0$ and $z_0 = 1$ (hence $u = 1$), $\mathbb{D}\left(z_0 + \frac{2ur^2 e^{i\theta}}{1-r^2}, \frac{2ur}{1-r^2}\right)$ is a disk with diameter $\left[\frac{1-r}{1+r}, \frac{1+r}{1-r}\right]$ and mapped onto $\mathbb{D}(0, r)$ by $z \mapsto \frac{z-1}{z+1}$, which is an isomorphism from H onto $\mathbb{D}$. We obtain the equality by the invariance of Poincaré metric. The general case follows immediately via a similarity. □

The following theorem gives a sharp bound on the Fatou coordinate. It gave a substantial improvement for the estimate in Proposition 5.6 compared to earlier methods the authors had tried.

Theorem 5.12 (A general estimate on Fatou coordinate)**.** *Let Ω be a disk or a half plane and $f:\Omega\to\mathbb{C}$ a holomorphic function with $f(z)\neq z$. Suppose f has a univalent Fatou coordinate $\Phi:\Omega\to\mathbb{C}$, i.e., $\Phi(f(z))=\Phi(z)+1$ when $z,f(z)\in\Omega$. If $z\in\Omega$ and $f(z)\in\Omega$, then*

$$\left|\log\Phi'(z)+\log(f(z)-z)-\frac{1}{2}\log f'(z)\right|\le\log\cosh\frac{d_\Omega(z,f(z))}{2}=\frac{1}{2}\log\frac{1}{1-r^2},$$

where r is a real number such that $0\le r<1$ and $d_{\mathbb{D}}(0,r)=d_\Omega(z,f(z))$.

Proof. Set $g=\Phi$ and $\zeta=f(z)$ in Theorem A.3 in Appendix and use $\Phi(f(z))=\Phi(z)+1$ and $\Phi'(z)=\Phi'(f(z))f'(z)$. Use (A.2) for the equality on the right hand side. $\square$

Computer Checked Inequalities. In the following, the inequalities checked with computer are denoted by $\underset{*}{<}$ and $\underset{*}{>}$ with * in the equation numbers. This was not applied to some simple inequalities which only involve π or square roots such as $\sqrt{3}$, $\sqrt{6}$, because those values are well known. For the convenience, approximate values are indicated as $x\doteqdot 1.2345\ldots$, which means $x\in[1.2345,1.2346]$ (we do not round up the next digit).

List of constants. $cp=cp_Q=5+2\sqrt{6}$ ($\doteqdot 9.899\ldots$), $cv=cv_Q=27, \eta=2, x_E=e_0=-0.18, a_E=1.24, b_E=1.04, e_1=1.14, e_{-1}=0.1, R=266, \rho=0.05, u_0=\frac{25}{\sqrt{3}}$ ($\doteqdot 14.43\ldots$), $R_1=167, \varepsilon_1=0.057, \varepsilon_2=0.406, \varepsilon_3=\frac{2}{3}, \varepsilon_4=1.13, r_1=1.25, r_2=1.4, r_3=1.54, \theta_2=\frac{\pi}{4}, \theta_3=0.4\pi, u_1=12.5, u_2=cp, u_3=\frac{27\sqrt{3}}{2}$ ($\doteqdot 23.38\ldots$), $u_4=20.8, u_5=u_3-u_1$.

5.C Covering property of $f\in\mathcal{F}_0$ and P as "subcover"

Let $f\in\mathcal{F}_0$. After a linear conjugacy, we may suppose that its critical value $cv=cv_f$ is contained in $\mathbb{R}_-$. A traditional way to consider $f:Dom(f)\to\mathbb{C}$ is to regard $Dom(f)$ as a Riemann surface spread over $\mathbb{C}$, consisting of "sheets" which are copies of the plane $\mathbb{C}$, cut along several slits and then glued together along pairs of slits, with f acting as the projection onto $\mathbb{C}$. This view helps us to understand the structure of $Dom(f)$.

Definition. Denote $\Gamma_a=(cv,0), \Gamma_b=(-\infty,cv], \Gamma_c=(0,+\infty)\subset\mathbb{R}$. Define $\mathbb{C}_{slit}=\mathbb{C}\smallsetminus(\{0\}\cup\Gamma_b\cup\Gamma_c)$, and $\mathbb{H}^+=\{z:\operatorname{Im}z>0\}$, $\mathbb{H}^-=\{z:\operatorname{Im}z<0\}$.

Description of covering properties of $f\in\mathcal{F}_0$: Since $\mathbb{C}_{slit}$ is simply connected and does not contain 0 and the critical value, $f^{-1}(\mathbb{C}_{slit})$ consists of connected components $\mathcal{U}_i$ ($i\in I$, where I is an index set, say $I=\mathbb{N}$ or $I=\{1,\ldots,n\}$), each of which is mapped by f isomorphically onto $\mathbb{C}_{slit}$. Denote $\mathcal{U}_{i\pm}=f^{-1}(\mathbb{H}^\pm)\cap\mathcal{U}_i$, $\gamma_{ai}=f^{-1}(\Gamma_a)\cap\mathcal{U}_i$, $\gamma_{bi\pm}=f^{-1}(\Gamma_b)\cap\overline{\mathcal{U}}_{i\pm}$, $\gamma_{ci\pm}=f^{-1}(\Gamma_c)\cap\overline{\mathcal{U}}_{i\pm}$ ($i\in I$), where the closures are taken within $Dom(f)$.

See Figure 10 (left).

The domain $Dom(f)$ of f can be described as the union of $\overline{\mathcal{U}}_i$'s, which are glued along boundary curves $\gamma_{bi\pm}$ and $\gamma_{ci\pm}$; each γ_{ci+} is glued with some γ_{cj-} and vice versa, the same is ture for $\gamma_{bi\pm}$. For $\gamma_{bi\pm}$, if γ_{bi+} is glued with γ_{bj-}, then γ_{bj+} must be glued with γ_{bi-}, because the critical points are simple. Since f is homeomorphic near 0, there must be a component, say $\mathcal{U}_1$, such that $0\in\overline{\mathcal{U}}_1$ and $\gamma_{c1+}=\gamma_{c1-}$.

Next consider boundary curves γ_{b1+} and γ_{b1-}. If they were glued together, then $\overline{\mathcal{U}}_1$ would be already isomorphic to $\mathbb{C}$ and $Dom(f)=\overline{\mathcal{U}}_1$. This would imply that f is isomorphic and has no critical value (since $Dom(f)$ is connected). This contradicts with the assumption that $f\in\mathcal{F}_0$. So there must be another component, say $\mathcal{U}_2$, such that $\gamma_{b1+}=\gamma_{b2-}$ and $\gamma_{b2+}=\gamma_{b1-}$.

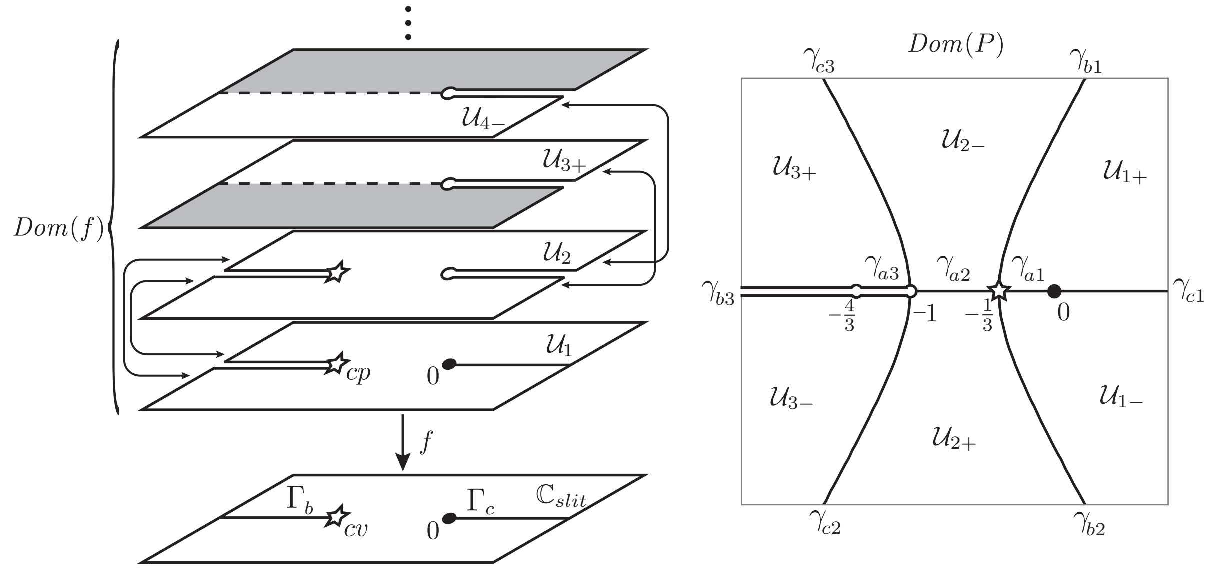

Figure 10: $Dom(f)$ as a Riemann surface spread over $\mathbb{C}$ (left) and $Dom(P)$ (right)

Note that $f^{-1}(cv) \cap \gamma_{b1+} \cap \gamma_{b2+}$ is a critical point, which we call *the closest critical point* and denote by $cp = cp_f$.

Denote $\mathcal{U}_{12} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \gamma_{b1+} \cup \gamma_{b2+}$. Then $f|_{\mathcal{U}_{12}} : \mathcal{U}_{12} \to \mathbb{C}_{slit} \cup \Gamma_b = \mathbb{C} \smallsetminus \{0\} \cup \Gamma_c$ is a branched covering of degree 2 branched over cv_f.

Example 1. Let $p(z) = z + z^2$, $Dom(p) = \mathbb{C} \smallsetminus \{-1\}$. Then the critical point is $cp = -\frac{1}{2}$ and the critical value is $cv = -\frac{1}{4}$. $\mathcal{U}_1 = \{z : \text{Re} > -\frac{1}{2}\} \smallsetminus [0, +\infty)$, $\mathcal{U}_2 = \{z : \text{Re} < -\frac{1}{2}\} \smallsetminus (-\infty, -1]$.

Example 2. Let $P(z) = z(1+z)^2$, and restrict to $Dom(P) = \mathbb{C} \smallsetminus \{-1\}$. The critical points are $cp_P = -\frac{1}{3}$ and -1, and the critical values are $cv_P = P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$. It is easy to see that $\gamma_{a1} = (-\frac{1}{3}, 0)$, $\gamma_{a2} = (-1, -\frac{1}{3})$, $\gamma_{a3} = (-\frac{4}{3}, -1)$, $\gamma_{c1+} = \gamma_{c1-} = (0, +\infty)$, $\gamma_{b3+} = \gamma_{b3-} = (-\infty, -\frac{4}{3}]$. Since other inverse images of Γ_b and Γ_c must branch from $-\frac{1}{3}$ and -1 and extend to ∞ within upper or lower half planes, it can be checked that $\gamma_{b1+} = \gamma_{b2-}$ and $\gamma_{c3+} = \gamma_{c2-}$ divide the upper half plane into $\mathcal{U}_{1+}, \mathcal{U}_{2-}, \mathcal{U}_{3+}$; $\gamma_{b2+} = \gamma_{b1-}$ and $\gamma_{c2+} = \gamma_{c3-}$ divide the lower half plane into $\mathcal{U}_{1-}, \mathcal{U}_{2+}, \mathcal{U}_{3-}$.

Figure 10 (right) illustrates the domains and curves for P. From now on, we denote $\gamma_{bi} = \gamma_{bi+}$ and $\gamma_{ci} = \gamma_{ci+}$ for simplicity.

Proof of Proposition 5.1. Now we further assume that $cv_f = \frac{4}{27} = cv_P$. We continue with the above description of $Dom(f)$ as the union of $\overline{\mathcal{U}}_i$ $(i \in I)$. We already have two special components $\mathcal{U}_1$ and $\mathcal{U}_2$ as before. Now consider γ_{c2+} and γ_{c2-}. If they were glued together, after adding an inverse image of 0 to $\mathcal{U}_2$, we would have a degree two branched cover onto $\mathbb{C}$ and this leads to the case of a quadratic polynomial.

So if f is not a quadratic polynomial, there must be components $\mathcal{U}_3$ and $\mathcal{U}_4$ such that $\gamma_{c2-} = \gamma_{c3+}$ and $\gamma_{c2+} = \gamma_{c4-}$. Note here that $\mathcal{U}_3$ and $\mathcal{U}_4$ may or may not be distinct. Further gluings for γ_{c3-} or $\gamma_{b3\pm}$ etc. depend on particular f. So we have common structure up to the half components $\mathcal{U}_{3+}$ and $\mathcal{U}_{4-}$, no matter whether $\mathcal{U}_3 = \mathcal{U}_4$ or not. Let us denote the components and curves for P by $\mathcal{U}_i^P$, γ_{ai}^P etc. as in Figure 10 (right). We can now define $\varphi : \mathbb{C} \smallsetminus (-\infty, -1] = \mathbb{C} \smallsetminus (\gamma_{b3}^P \cup \gamma_{a3}^P) \to Dom(f)$ by $\varphi(z) = \left(f|_{\mathcal{U}_{i\pm}}\right)^{-1} \circ P$ on $\mathcal{U}_{i\pm}^P$ for $i = 1, 2, 3,$

except on $\mathcal{U}^P_{3-}$, where $\left(f|_{\mathcal{U}_{4-}}\right)^{-1}\circ P$ is used. This definition extends continuously to the boundary curves γ^P_{b1}, γ^P_{b2}, γ^P_{c1}, γ^P_{c2}, γ^P_{c3}, since the gluing relation is the same (if $\mathcal{U}_{3-}$ is replaced by $\mathcal{U}_{4-}$). The origin is mapped onto the origin and $-\frac{1}{3}$ is mapped to the closest critical point of f. It is easy to see that φ is a homeomorphism from $\mathbb{C}\smallsetminus(-\infty,-1]$ onto its image. At the points other than 0 and the critical point, the map f is locally conformal, so φ is holomorphic there. By the removable singularity theorem, φ is conformal from $\mathbb{C}\smallsetminus(-\infty,-1]$ onto its image. It follows from the definition that $f = P\circ\varphi^{-1}$ and $\varphi(0)=0$. By differentiation, we also have $\varphi'(0)=1$. □

Corollary 5.13. *If $f\in\mathcal{F}_0$ and f is not a quadratic polynomial, then*

$$\left|f''(0)-5\right|\le 1 \quad \text{if } cv_f = -\frac{4}{27}, \quad \text{or} \quad \left|f''(0)\cdot cv_f + \frac{20}{27}\right| \le \frac{4}{27} \text{ in general.}$$

Remark. For the quadratic polynomial $q(z)=z+z^2$, we have $q''(0)\cdot cv_q = -\frac{1}{2}$, which does not satisfy the inequality.

Proof. Since $f''(0)\cdot cv_f$ is invariant under the linear conjugacy, we only need to deal with the case $cv_f = -\frac{4}{27}$. Therefore we may suppose that $f = P\circ\varphi^{-1}$ as in Proposition 5.1, where $\varphi:\mathbb{C}\smallsetminus(-\infty,-1]\to U$ is a conformal map with $\varphi(0)=0$, $\varphi'(0)=1$. Let $f_{Koebe}(z) = \frac{z}{(1-z)^2}$ which is a conformal map from the unit disk onto $\mathbb{C}\smallsetminus(-\infty,-1/4]$. Then $\hat\varphi(z)=\frac{1}{4}\varphi(4f_{Koebe}(z))$ is a univalent function in the class $\mathcal{S}$. Then by Theorem A.1 (a) in Appendix, $|\hat\varphi''(0)|\le 4$. On the other hand, $\hat\varphi''(0) = 4\varphi''(0)\left(f'_{Koebe}(0)\right)^2 + \varphi'(0)f''_{Koebe}(0) = 4\varphi''(0)+4$ and $\varphi''(0) = P''(0)-f''(0) = 4-f''(0)$. Therefore we have $|\varphi''(0)+1|\le 1$ and $|f''(0)-5|\le 1$, which was the assertion. □

5.D Passing from P to Q

For various estimates, it is easier to work with a parabolic fixed point and with arbitrary univalent functions defined in the complement of a disk (or an ellipse). This is why we introduced Q (and ψ_0, ψ_1) on §5.A.

Lemma 5.14. (a) *The P and Q are related by*

$$Q = \psi_0^{-1}\circ P\circ\psi_1.$$

The ψ_1 maps $\widehat{\mathbb{C}}\smallsetminus\overline{\mathbb{D}}$ (and also $\mathbb{D}$) conformally onto $\mathbb{C}\smallsetminus(-\infty,-1]$ and $\psi_1(\infty)=0$.
(b) *The map Q has four critical points $cp := 5+2\sqrt{6}(\doteqdot 9.8989\ldots)$, $cp' := 5-2\sqrt{6}(\doteqdot 0.1010\ldots)$ and ±1; the critical values are $cv := Q(cp) = Q(cp') = 27$, $Q(1)=\infty$ and $Q(-1)=0$; cp and cp' are simple critical points, whereas the local degree is 4 at $z=1$ and 6 at $z=-1$.*

Proof. (a) $P(\psi_1(z)) = -\frac{4z}{(1+z)^2}\left(1-\frac{4z}{(1+z)^2}\right)^2 = -\frac{4z(1-z)^4}{(1+z)^6} = \psi_0(Q(z))$.
The map ψ_1 can be written as $\psi_1 = \psi_{1,2}\circ\psi_{1,1}$, where $\psi_{1,1}: z\mapsto\frac{z-1}{z+1}$ and $\psi_{1,2}: w\mapsto w^2-1$. $\psi_{1,1}$ maps $\widehat{\mathbb{C}}\smallsetminus\overline{\mathbb{D}}$ (resp. $\mathbb{D}$) to the right half plane (resp. the left half plane), then $\psi_{1,2}$ maps the right half plane (or the left half plane) onto $\mathbb{C}\smallsetminus(-\infty,-1]$.
(b) Left to the reader. See also Lemma 5.21 (a). □

Definition. Define $\mathcal{U}^Q_{i\pm} = \psi_1^{-1}(\mathcal{U}^P_{i\pm})\smallsetminus\overline{\mathbb{D}}$, $\Gamma^Q_a = \psi_0^{-1}(\Gamma^P_a)$, $\gamma^Q_{ai} = \psi_1^{-1}(\gamma^P_{ai})\smallsetminus\mathbb{D}$ etc. Then $\Gamma^Q_a = (cv,+\infty) = (27,+\infty)$, $\Gamma^Q_b = (0,cv]$, $\Gamma^Q_c = (-\infty,0)$, $\gamma^Q_{a1} = (cp,+\infty)$, $\gamma^Q_{a2} = (1,cp)$, $\gamma^Q_{c1} = (-\infty,-1)$. (Here $\mathcal{U}^Q_{3-}$ is not connected with $\mathcal{U}^Q_{3+}$ and may rather be called $\mathcal{U}^Q_{4-}$ as in the previous subsection, but we name it to be consistent with P.) Note that ψ_1^{-1} split γ^P_{a3} and γ^P_{c3} into arcs on $\partial\mathbb{D}$, $\gamma^Q_{a3+} = [1,\omega]_{\partial\mathbb{D}}$, $\gamma^Q_{b3+} = [\omega,-1]_{\partial\mathbb{D}}$, $\gamma^Q_{a3-} = [1,\bar\omega]_{\partial\mathbb{D}}$, $\gamma^Q_{b3-} = [\bar\omega,-1]_{\partial\mathbb{D}}$, where $[\zeta,\zeta']_{\partial\mathbb{D}}$ denotes the arc between ζ and ζ' on $\partial\mathbb{D}$ and $\omega = \frac{1+\sqrt{3}i}{2}$. See Figure 11.

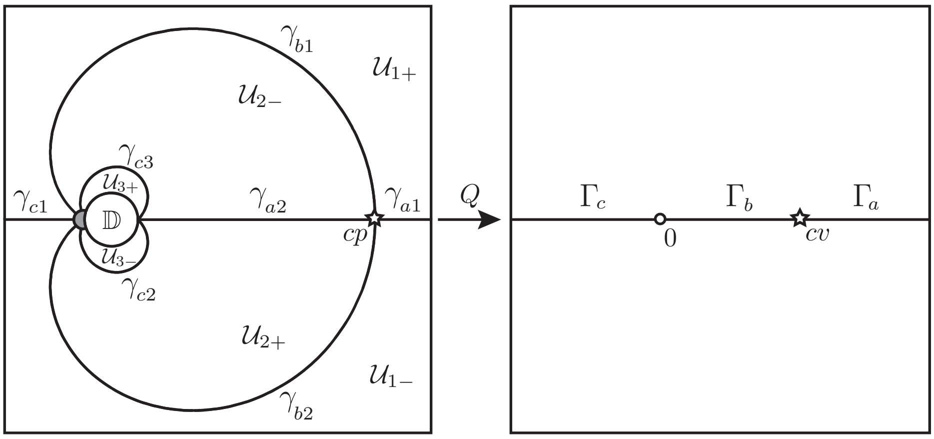

Figure 11: Domain of Q with partition by curves; $\widehat{\mathbb{C}} \smallsetminus U^Q_\eta$ consists of $\overline{\mathbb{D}}$ and two shaded regions near $+1$ and -1, however the one near $+1$ is invisible.

It is clear that Q maps each $\mathcal{U}_{i\pm}$ isomorphically onto $\{z : \pm \operatorname{Im} z > 0\} = \psi_0^{-1}(\mathbb{H}^\pm)$ and γ_{ai} homeomorphically onto Γ_a etc. Denote $\mathcal{U}^Q_{12} = \mathcal{U}^Q_1 \cup \mathcal{U}^Q_2 \cup \gamma^Q_{b1+} \cup \gamma^Q_{b2+} = \psi_1^{-1}(\mathcal{U}^P_{12})$. Then $Q|_{\mathcal{U}^Q_{12}} : \mathcal{U}^Q_{12} \to \mathbb{C} \smallsetminus \{0\} \cup \Gamma^Q_c$ is a branched covering of degree 2 branched over cv_Q.

Now we prove Proposition 5.3 assuming Proposition 5.2.

Proof of Proposition 5.3. (a) Suppose $f \in \mathcal{F}_0$. Then by Proposition 5.1, it can be expressed as $f = P \circ \varphi^{-1}$ on U, where $\varphi : \mathbb{C} \smallsetminus (-\infty, -] \to U (\subset \mathit{Dom}(f))$ is a conformal map with $\varphi(0) = 0$, $\varphi'(0) = 1$. Since $V' = U^P_\eta \subset \mathbb{C} \smallsetminus (-\infty, -1]$, we can further restrict $f = P \circ \varphi^{-1}$ to $\varphi(V')$ and obtain an element of $\mathcal{F}^P_2$. This is obviously injective because we are restricting holomorphic functions.
(b) By Proposition 5.2, we have $\overline{V} \subset V'$. Given $f = P \circ \varphi^{-1} \in \mathcal{F}^P_2$, where φ is defined on V', we can restrict φ to V. Since $\partial E \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}$, the boundary of V is non-singular real-analytic Jordan curve, hence $\varphi|_V$ has a quasiconformal extension to $\mathbb{C} \smallsetminus V$. Thus we obtain $f = P \circ (\varphi|_V)^{-1} \in \mathcal{F}^P_1$.
(c) The statement on the one to one correspondence is easy to check. Note that $\psi_1 : \widehat{\mathbb{C}} \smallsetminus E \to V$ is conformal and φ is normalized at 0 if and only if $\hat\varphi = \psi_0^{-1} \circ \varphi \circ \psi_1$ is normalized at ∞. The statement on $f''(0)$ is immediate from calculation: $F(z) = z + (10 - c_0) + O(\frac{1}{z})$ and $f(z) = \psi_0^{-1} \circ F \circ \psi_0(z) = -4/(-4/z + (10 - c_0) + O(z)) = z + \frac{10-c_0}{4} z^2 + O(z^3)$. □

The following lemma (used in Lemmas 5.17 and 5.26) shows that γ^Q_{c2} and γ^Q_{c3} go outside $\overline{\mathbb{D}}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \cup \overline{\mathbb{D}}(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}})$.

Lemma 5.15. (a) $\{z \in \mathbb{C} : z \neq -1,\ \frac{2\pi}{3} \le \pm\arg(z+1) < \pi\} \subset \mathcal{U}^P_{3\pm}$.
(b) $\overline{\mathbb{D}}(\pm\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \overline{\mathbb{D}} \subset \mathcal{U}^Q_{3\pm}$. *Hence* $\mathcal{U}^Q_{12} \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \cup \overline{\mathbb{D}}(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}})$.

Proof. (a) If $z \in \mathbb{C}$ with $z \neq -1$ and $\frac{2\pi}{3} \le \arg(z+1) < \pi$, then it is easy to see that $\frac{2\pi}{3} < \arg z < \pi$ and therefore $2\pi < \arg P(z) = \arg z + 2\arg(z+1) < 3\pi$ and $\operatorname{Im} P(z) > 0$. This implies that

$\{z \in \mathbb{C} : z \neq -1,\ \frac{2\pi}{3} \leq \arg(z+1) < \pi\}$ is contained in a connected component of $P^{-1}(\mathbb{H}^+)$. This component must be $\mathcal{U}_{3+}$, since points near $(-\infty, -1)$ are contained in $\mathcal{U}_3$. It can be proved similarly for $\mathcal{U}_{3-}$.
(b) First we conseider the image $\psi_1(\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \overline{\mathbb{D}})$. Write $\psi_1 = \psi_{1,2} \circ \psi_{1,1}$ as in the proof of the previous lemma. Note that $\partial\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ is a circle intersecting the unit circle at 1, -1 with angle $\frac{\pi}{6}$. The Möbius transformation $\psi_{1,1}(z) = \frac{z-1}{z+1}$ maps the unit circle to the imaginary axis, 1 to 0, -1 to ∞, hence it must map $\partial\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \mathbb{D}$ onto a half line from 0 to ∞ that intersects the imaginary axis at 0 and ∞ with angle $\frac{\pi}{6}$, and contains $\psi_{1,1}(i\sqrt{3}) = \frac{1+\sqrt{3}i}{2}$. So we conclude that $\psi_{1,1}(\partial\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \mathbb{D}) = \{w : w = 0, \infty \text{ or } \arg w = \frac{\pi}{3}\}$ and $\psi_{1,1}(\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \overline{\mathbb{D}}) = \{w : w \neq 0,\ \frac{\pi}{3} < \arg w < \frac{\pi}{2}\}$. Then the latter is mapped to $\{z : z \neq -1,\ \frac{2\pi}{3} < \arg(z+1) < \pi\}$ by $\psi_{1,2}(w) = w^2 - 1$. Hence we proved $\psi_1(\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \overline{\mathbb{D}}) = \{z \in \mathbb{C} : z \neq -1,\ \frac{2\pi}{3} \leq \arg(z+1) < \pi\} \subset \mathcal{U}^P_{3+}$. This implies $\mathbb{D}(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \smallsetminus \overline{\mathbb{D}} \subset \mathcal{U}^Q_{3+}$. The same conclusion holds for $\mathbb{D}(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}})$. It follows that $\mathcal{U}^Q_{12} \cap \overline{\mathbb{D}}(\pm\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = \emptyset$. $\square$

From the following subsection, when there is no confusion, we will drop Q in the notation $\mathcal{U}^Q_i$, γ^Q_{ai} etc and denote $\mathcal{U}_i$, γ_{ai} etc.

5.E Estimates on Q: Part 1

Now we embark on the estimates which are needed for Main Theorem 1(c). From now on, throughout this section, we assume that $F = Q \circ \varphi^{-1} \in \mathcal{F}^Q_1$. Therefore $\varphi : \widehat{\mathbb{C}} \smallsetminus E \to \widehat{\mathbb{C}} \smallsetminus \{0\}$ is a normalized univalent mapping. For convenience, we usually use variable z for the ranges of Q and φ (which are the domain and range of F), whereas variable ζ is used for their domains.

Lemma 5.16. *Let $\eta = 2$, $\varepsilon_1 = 0.057$, $\varepsilon_2 = 0.406$.*
(a) *$\widehat{\mathbb{C}} \smallsetminus U^Q_\eta \cup \overline{\mathbb{D}}$ is covered by the disks $\mathbb{D}(1, \varepsilon_1)$ and $\mathbb{D}(-1, \varepsilon_2)$.*
(b) *The disks $\overline{\mathbb{D}}(1, \varepsilon_1)$, $\overline{\mathbb{D}}(-1, \varepsilon_2)$ and $\overline{\mathbb{D}}$ are contained in the interior of the ellipse E.*

See Figure 12 (left).

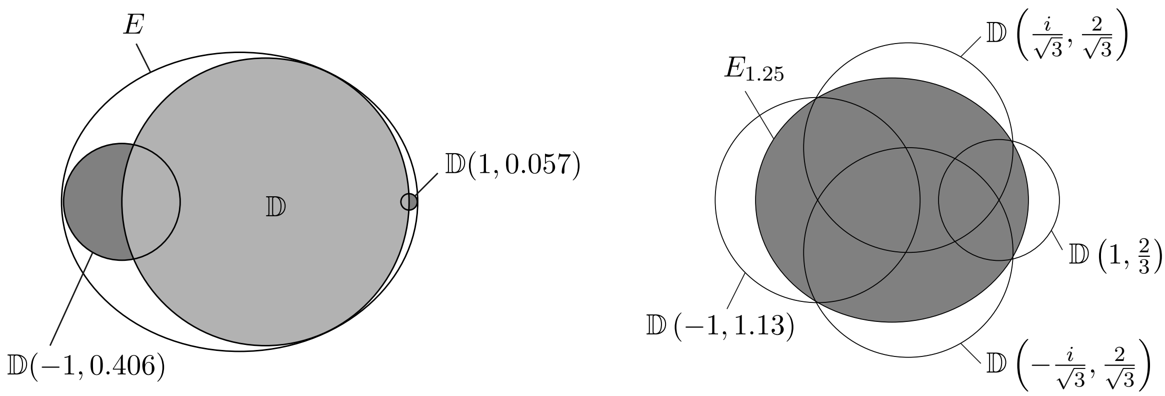

Figure 12: Ellipses E, $E_{1.25}$ and related disks for Lemmas 5.16 and 5.17

Proof. (a) By the description of U^P_η in previous subsection and the relation between P and Q, it is easy to see that $\widehat{\mathbb{C}} \smallsetminus U^Q_\eta \cup \overline{\mathbb{D}}$ consists of two connected components W and W' such that W (resp.

W') contains 1 (resp. -1) in its boundary and $|Q(\zeta)| \geq cv\, e^{2\pi\eta}$ in W (resp. $|Q(\zeta)| \leq cv\, e^{-2\pi\eta}$ in W'). If we know that $|Q(\zeta)| < cv\, e^{2\pi\eta}$ on $\partial\mathbb{D}(1, \varepsilon_1)$ (resp. $|Q(\zeta)| > cv\, e^{-2\pi\eta}$) on $\partial\mathbb{D}(-1, \varepsilon_2)$), this will mean that $W \subset \mathbb{D}(1, \varepsilon_1)$ (resp. $W' \subset \mathbb{D}(-1, \varepsilon_2)$), since W (resp. W') is connected.

Since $Q(\zeta) = \frac{(\zeta+1)^6}{\zeta(\zeta-1)^4}$, if $|\zeta - 1| = \varepsilon_1$, then we have a numerical estimate

$$|Q(\zeta)| \leq \frac{(2+\varepsilon_1)^6}{(1-\varepsilon_1)\varepsilon_1^4} (\risingdotseq 7.61 \cdots \times 10^6) \underset{*}{<} 27e^{2\pi\eta} (\risingdotseq 7.74 \cdots \times 10^6). \tag{5.2*}$$

Similarly if $|\zeta + 1| = \varepsilon_2$, then

$$|Q(\zeta)| \geq \frac{\varepsilon_2^6}{(1+\varepsilon_2)(2+\varepsilon_2)^4} (\risingdotseq 9.50 \cdots \times 10^{-5}) \underset{*}{>} 27e^{-2\pi\eta} (\risingdotseq 9.41 \cdots \times 10^{-5}). \tag{5.3*}$$

Thus it follows that $\widehat{\mathbb{C}} \smallsetminus U_\eta^Q \cup \overline{\mathbb{D}} \subset \mathbb{D}(1, \varepsilon_1) \cup \mathbb{D}(-1, \varepsilon_2)$.

(b) In order to prove $\overline{\mathbb{D}}$, $\overline{\mathbb{D}}(1, \varepsilon_1)$, $\overline{\mathbb{D}}(-1, \varepsilon_2) \subset intE$, parameterize ∂E by $x = -0.18 + 1.24\,t, y = \pm 1.04\sqrt{1-t^2}$ $(-1 \leq t \leq 1)$. Let

$$h_1(t) := x^2 + y^2 - 1 = 0.456\,t^2 - 0.4464\,t + 0.114, \tag{5.4}$$

$$h_2(t) := (x-1)^2 + y^2 - \varepsilon_1^2 = 0.456\,t^2 - 2.9264\,t + 2.470751, \tag{5.5}$$

$$h_3(t) := (x+1)^2 + y^2 - \varepsilon_2^2 = 0.456\,t^2 + 2.0336\,t + 1.589164. \tag{5.6}$$

The quadratic polynomial h_1 has discriminant

$$(0.4464)^2 - 4 \times 0.456 \times 0.114 = -0.00866304 < 0. \tag{5.7}$$

Therefore $h_1(t) > 0$ for all t and this implies $\overline{\mathbb{D}} \subset intE$. Next, $h_2(t)$ has minimum at $t = \frac{2.9264}{2\times 0.456} > 1$, and the minimum within $[-1, 1]$ will be attained by

$$h_2(1) = 0.000351 > 0. \tag{5.8}$$

Hence $h_2(t) > 0$ $(t \in [-1, 1])$, which implies $\overline{\mathbb{D}}(1, \varepsilon_1) \subset intE$. Finally, $h_3(t)$ has minimum at $t = -\frac{2.0336}{2\times 0.456} < -1$, and the minimum within $[-1, 1]$ will be attained by

$$h_3(-1) = 0.011564 > 0. \tag{5.9}$$

Hence $h_3(t) > 0$ $(t \in [-1, 1])$ and $\overline{\mathbb{D}}(-1, \varepsilon_2) \subset intE$. □

Proof of Proposition 5.2. By Lemma 5.16, we have

$$U_\eta^Q \supset \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}} \cup \overline{\mathbb{D}}(1, \varepsilon_1) \cup \overline{\mathbb{D}}(-1, \varepsilon_2) \supset \widehat{\mathbb{C}} \smallsetminus intE,$$

and also

$$\overline{V} = \psi_1(\widehat{\mathbb{C}} \smallsetminus intE) \subset \psi_1(U_\eta^Q) = U_\eta^P = V'.$$

□

In order to determine the shape of Y for Proposition 5.4 (b), we will need the following lemma.

Lemma 5.17. *Let $R = 266$, $\rho = 0.05$, $\varepsilon_3 = \frac{2}{3}$, $\varepsilon_4 = 1.13$ and $r_1 = 1.25$.*

(a) *If $\zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $|\zeta - 1| \le \varepsilon_3$, then $|Q(\zeta)| > R = 266$.*

(b) *If $\zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $|\zeta + 1| \le \varepsilon_4$, then $|Q(\zeta)| < \rho = 0.05$.*

(c) *E_{r_1} is covered by $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\mathbb{D}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\mathbb{D}(1, \varepsilon_3)$ and $\mathbb{D}(-1, \varepsilon_4)$. Hence*

$$\mathbb{C} \setminus \overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}(1, \varepsilon_3) \cup \overline{\mathbb{D}}(-1, \varepsilon_4) \subset \mathbb{C} \setminus E_{r_1}.$$

(d) *If $\zeta \in \mathcal{U}_{12}$ and $\rho \le |Q(\zeta)| \le R$, then $\zeta \in \mathbb{C} \setminus E_{r_1}$. Moreover if $\zeta \in \overline{\mathcal{U}}_1$ and $|Q(\zeta)| > R$, then ζ is also in $\mathbb{C} \setminus E_{r_1}$.*

See Figure 12 (right).

Proof. (a) It is easy to see that $\overline{\mathbb{D}}\left(1, \frac{2}{\sqrt{3}}\right) \cap \{\zeta : \operatorname{Re}\zeta \le 1\}$ is covered by $\overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Hence under the assumption of (a), we have $\operatorname{Re}\zeta > 1$ and $|\zeta + 1| \ge \sqrt{4 + r^2}$, where $r = |\zeta - 1| \le \varepsilon_3$. So

$$|Q(\zeta)| \ge h_4(r) := \frac{\left(\sqrt{4 + r^2}\right)^6}{(1 + r)r^4} = \frac{(4 + r^2)^3}{(1 + r)r^4} \tag{5.10}$$

Since $(\log h_4(r))' = \frac{6r}{4+r^2} - \frac{1}{1+r} - \frac{4}{r} \le \frac{6}{4} - 0 - 4 < 0$ for $0 < r < 1$,

$$|Q(\zeta)| \ge h_4(r) \ge h_4(\varepsilon_3) = \frac{(4 + \varepsilon_3^2)^3}{(1 + \varepsilon_3)\varepsilon_3^4} = \frac{800}{3} > R. \tag{5.11}$$

(b) Similarly, under the assumption of (b), since $\varepsilon_4 < \frac{2}{\sqrt{3}} (\doteqdot 1.154\ldots)$, we have $\operatorname{Re}\zeta < -1$, hence $|\zeta| \ge \sqrt{1 + r^2}$, $|\zeta - 1| \ge \sqrt{4 + r^2}$, where $r = |\zeta + 1| \le \varepsilon_4$. Therefore

$$|Q(\zeta)| \le \frac{r^6}{\sqrt{1 + r^2}\left(\sqrt{4 + r^2}\right)^4}. \tag{5.12}$$

Take function $h_5(s) := \frac{s^3}{\sqrt{1+s}\,(4+s)^2}$ for $s > 0$, then $(\log h_5(s))' = \frac{3}{s} - \frac{1}{2(1+s)} - \frac{2}{4+s} \ge \frac{3}{s} - \frac{1}{2s} - \frac{2}{s} = \frac{1}{2s} > 0$. Hence (5.12) is bounded by

$$|Q(\zeta)| \le h_5(r^2) \le h_5(\varepsilon_4^2) = \frac{\varepsilon_4^6}{\sqrt{1 + \varepsilon_4^2}\,(4 + \varepsilon_4^2)^2} \ (\doteqdot 0.0495\ldots) \underset{*}{<} \rho. \tag{5.13*}$$

(c) It is enough to show that the upper part of E_{r_1} is covered by $\overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\overline{\mathbb{D}}(1, \varepsilon_3)$ and $\overline{\mathbb{D}}(-1, \varepsilon_4)$. We prepare an elementary lemma:

Sublemma 5.18. *Let $\Gamma = \left\{x + iy : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, y \ge 0\right\}$ with $a > b > 0$. If two points $z_1, z_2 \in \Gamma$ are contained in a disk $\mathbb{D}(\zeta_0, r)$ with $\operatorname{Im}\zeta_0 \ge 0$, then so is the subarc of Γ between z_1 and z_2.*

Proof. The Γ is the graph of $y(x) = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$. Define $h(x) = (x - \xi_0)^2 + (y(x) - \eta_0)^2 - r^2$, where $\zeta_0 = \xi_0 + i\eta_0$ with $\eta_0 \ge 0$. If $z_j = x_j + iy(x_j) \in \Gamma$ $(j = 1, 2)$ are contained in $\mathbb{D}(\zeta_0)$, then $h(x_j) < 0$. It follows that $h(x) < 0$ for x between x_1 and x_2, since $h(x) = \left(1 - \left(\frac{b}{a}\right)^2\right) x^2 - 2b\eta_0\sqrt{1 - \left(\frac{x}{a}\right)^2} + cx + d$ is obviously a convex function. $\square$

Now we continue the proof of (c) of Lemma 5.17. After shifting the origin, we will apply this lemma to $\Gamma = \partial E_{r_1} \cap \{\zeta : \operatorname{Im}\zeta \geq 0\}$ and $y(x) = b_{_E}(1.25)\sqrt{1-\left(\frac{x-x_E}{a_{_E}(1.25)}\right)^2} = 1.345\sqrt{1-\left(\frac{x+0.18}{1.505}\right)^2}$. Let $z_1 = -1.01 + iy(-1.01)$ and $z_2 = 1.145 + iy(1.145)$ and these points divide Γ into three subsrcs Γ_1, Γ_2 and Γ_3, from left to right. The end points of Γ_1, $x_{_E} - a_{_E}(1.25) = -1.685$ and z_1, are contained in $\mathbb{D}(-1,\varepsilon_4)$, since

$$|-1.685+1| = 0.685 < \varepsilon_4 \text{ and } (-1.01+1)^2 + y(-1.01)^2 - \varepsilon_4^2 \ (\doteqdot -0.01798\ldots) \underset{*}{<} 0. \quad (5.14^*)$$

The end points of Γ_2, z_1 and z_2, are contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, since

$$(-1.01)^2 + \left(y(-1.01) - \tfrac{1}{\sqrt{3}}\right)^2 - \left(\tfrac{2}{\sqrt{3}}\right)^2 \ (\doteqdot -0.0166\ldots) \underset{*}{<} 0 \ \text{ and} \quad (5.15^*)$$

$$(1.145)^2 + \left(y(1.145) - \tfrac{1}{\sqrt{3}}\right)^2 - \left(\tfrac{2}{\sqrt{3}}\right)^2 \ (\doteqdot -0.0186\ldots) \underset{*}{<} 0. \quad (5.16^*)$$

The end points of Γ_3, z_2 and $x_{_E} + a_{_E}(1.25) = 1.325$, are contained in $\mathbb{D}(1,\varepsilon_3)$, since

$$|1.325-1| = 0.325 < \varepsilon_3 \text{ and } (1.145-1)^2 + y(1.145)^2 - \varepsilon_3^2 \ (\doteqdot -0.016\ldots) \underset{*}{<} 0. \quad (5.17^*)$$

Therefore we conclude that the convex hull of $\Gamma_1 \cup \{-1\}$ is contained in $\mathbb{D}(-1,\varepsilon_4)$, the convex hull of $\Gamma_2 \cup [-1,1]$ is contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \{\pm 1\}$ and the convex hull of $\Gamma_3 \cup \{1\}$ is contained in $\mathbb{D}(1,\varepsilon_3)$. Since the upper half of E_{r_1} is the union of these three convex hulls, we have proved (c).

(d) Let $\zeta \in \mathcal{U}_{12}$ and suppose $\rho \leq |Q(\zeta)| \leq R$. By Lemma 5.15 (b), $\zeta \in \mathbb{C} \smallsetminus \overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. By (a) and (b), ζ cannot be in $\overline{\mathbb{D}}\left(1,\varepsilon_3\right) \cup \overline{\mathbb{D}}\left(-1,\varepsilon_4\right)$. It follows from (c) that $\zeta \in \mathbb{C} \smallsetminus E_{r_1}$.

For the last statement, consider the inverse image of $\mathbb{C} \smallsetminus \overline{\mathbb{D}}(0,R)$ by $Q|_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}$. Form the relation between P and Q (Lemma 5.14, considering the inverse image of a neighborhood of 0 by P), one can show that $(Q|_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}})^{-1}(\mathbb{C} \smallsetminus \overline{\mathbb{D}}(0,R)) = U \cup U'$, where U and U' are connected components contained in $\mathcal{U}_1 \cup \gamma_{c1}$ and $\mathcal{U}_2 \cup \mathcal{U}_3 \cup \gamma_{c2} \cup \gamma_{c3}$, respectively. Moreover $\infty \in \overline{U}$ and $-1 \in \overline{U}'$. It follows from (a) that $W = \overline{\mathbb{D}}(1,\varepsilon_3) \smallsetminus \overline{\mathbb{D}}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \overline{\mathbb{D}}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ must be contained in the component U'. Therefore we conclude that $W \cap \overline{\mathcal{U}}_1 = \emptyset$. The rest is similar to the previous case. This ends the proof of Lemma 5.17. □

5.F Estimates on Q: Part 2

Lemma 5.19. *One can write*

$$Q(\zeta) = \zeta + 10 + \frac{49}{\zeta} + Q_2(\zeta), \quad \textit{where} \quad Q_2(\zeta) = \frac{160}{(\zeta-1)^2} + \frac{80\,\zeta + 32 - \frac{48}{\zeta}}{(\zeta-1)^4}$$

and

$$|Q_2(\zeta)| \leq Q_{2,max}(r) := \frac{160}{(r-1)^2} + \frac{80\,r + 32 + \frac{48}{r}}{(r-1)^4} \quad \textit{for} \quad |\zeta| \geq r > 1.$$

Proof. This is immediate by a calculation and left to the reader. □

Lemma 5.20. $Q\left(\overline{\mathbb{V}}\left(21, \frac{\pi}{6}\right)\right) \subset \mathbb{V}\left(30, \frac{\pi}{6}\right) \subset \mathbb{V}\left(cv, \frac{\pi}{6}\right)$.

Proof. Suppose $\zeta \in \overline{\mathbb{V}}\left(21, \frac{\pi}{6}\right)$ and let $\zeta' = \zeta + 9$. Since $\zeta' \in \overline{\mathbb{V}}\left(30, \frac{\pi}{6}\right)$, it suffices to show that $|\arg(Q(\zeta) - \zeta')| = \left|\arg\left(\frac{49}{\zeta} + (1 + Q_2(\zeta))\right)\right| < \frac{\pi}{6}$. If $\zeta \in \overline{\mathbb{V}}\left(21, \frac{\pi}{6}\right)$, then $|\arg \zeta| < \frac{\pi}{6}$ and $\left|\arg \frac{49}{\zeta}\right| < \frac{\pi}{6}$. On the other hand, by Lemma 5.9 (a) and Lemma 5.19, $|\arg(1 + Q_2(\zeta))| \leq \arcsin Q_{2,max}(21) = \arcsin \frac{23}{56} < \arcsin \frac{1}{2} = \frac{\pi}{6}$. Since both $\frac{49}{\zeta}$ and $1 + Q_2(\zeta)$ are in $\mathbb{V}(0, \frac{\pi}{6})$, so is their sum. Therefore $Q(\zeta) \in \mathbb{V}\left(30, \frac{\pi}{6}\right) \subset \mathbb{V}\left(cv, \frac{\pi}{6}\right)$. □

Lemma 5.21. (a)

$$Q'(\zeta) = \left(1 - \tfrac{10}{\zeta} + \tfrac{1}{\zeta^2}\right)\left(\frac{1 + \frac{1}{\zeta}}{1 - \frac{1}{\zeta}}\right)^5 = \left(1 - \tfrac{5+2\sqrt{6}}{\zeta}\right)\left(1 - \tfrac{5-2\sqrt{6}}{\zeta}\right)\left(\frac{1 + \frac{1}{\zeta}}{1 - \frac{1}{\zeta}}\right)^5.$$

(b) *If* $|\zeta| \geq r > cp_Q = 5 + 2\sqrt{6}$ $(\doteqdot 9.899\ldots)$, *then*

$$\left|\log Q'(\zeta)\right| \leq LogDQ_{max}(r) := \frac{49}{r^2} + \frac{320}{r^3} + \frac{1}{4}\left(\frac{\left(\frac{5+2\sqrt{6}}{r}\right)^4}{1 - \frac{5+2\sqrt{6}}{r}} + \frac{\left(\frac{5-2\sqrt{6}}{r}\right)^4}{1 - \frac{5-2\sqrt{6}}{r}}\right) + \frac{\frac{2}{r^5}}{1 - \frac{1}{r^2}}.$$

(c) *If* $|\zeta| > 5 + 2\sqrt{6}$, *then* $\operatorname{Re} Q'(\zeta) > 0$. *For any* $\theta \in \mathbb{R}$, Q *is injective in* $\{\zeta : \operatorname{Re}(\zeta\, e^{-i\theta}) > 5 + 2\sqrt{6}\}$.

Proof. (a) This can be checked by a calculation.
(b) Using $-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \sum_{n=4}^{\infty} \frac{x^n}{n}$, we have

$$\begin{aligned}\log Q'(\zeta) &= \log\left(1 - \tfrac{5+2\sqrt{6}}{\zeta}\right) + \log\left(1 - \tfrac{5-2\sqrt{6}}{\zeta}\right) + 5\log\left(1 + \tfrac{1}{\zeta}\right) - 5\log\left(1 - \tfrac{1}{\zeta}\right) \\ &= -\tfrac{49}{\zeta^2} - \tfrac{320}{\zeta^3} - \sum_{n=4}^{\infty}\left(\tfrac{(5+2\sqrt{6})^n}{n\zeta^n} + \tfrac{(5-2\sqrt{6})^n}{n\zeta^n}\right) + \sum_{m=2}^{\infty} \tfrac{10}{(2m+1)\zeta^{2m+1}}.\end{aligned}$$

The inequality follows easily.
(c) Consider $\arg Q'(\zeta) = \operatorname{Im} \log Q'(\zeta)$ in $|\zeta| > 5 + 2\sqrt{6}$. First note that Q' has no zeroes there. Suppose now that $\operatorname{Im} \zeta \geq 0$. Since $\operatorname{Im} \frac{1}{\zeta} \leq 0$ and $\left|\frac{5+2\sqrt{6}}{\zeta}\right| < 1$, it is easy to see that

$$\arg\left(1 + \tfrac{1}{\zeta}\right) \leq 0 \leq \arg\left(1 - \tfrac{5-2\sqrt{6}}{\zeta}\right) \leq \arg\left(1 - \tfrac{1}{\zeta}\right) \leq \arg\left(1 - \tfrac{5+2\sqrt{6}}{\zeta}\right) < \frac{\pi}{2}.$$

Therefore

$$\arg Q'(\zeta) \leq \arg\left(1 - \tfrac{5+2\sqrt{6}}{\zeta}\right) + \left(\arg\left(1 - \tfrac{5-2\sqrt{6}}{\zeta}\right) - \arg\left(1 - \tfrac{1}{\zeta}\right)\right) < \frac{\pi}{2}.$$

On the other hand, by Lemma 5.9,

$$\arg Q'(\zeta) \geq 5\arg\left(1 + \tfrac{1}{\zeta}\right) - 5\arg\left(1 - \tfrac{1}{\zeta}\right) \geq -10 \arcsin \frac{1}{5 + 2\sqrt{6}} \geq -\frac{\pi}{3} \cdot \frac{10}{5 + 2\sqrt{6}} > -\frac{\pi}{2}. \quad (5.18)$$

Thus we have $\operatorname{Re} Q'(\zeta) > 0$. The same conclusion holds when $\operatorname{Im} \zeta < 0$.

If two distinct points ζ_0 and ζ_1 can be joined by a segment within $\{\zeta : |\zeta| > 5 + 2\sqrt{6}\}$, then by

$$\frac{Q(\zeta_1) - Q(\zeta_0)}{\zeta_1 - \zeta_0} = \frac{1}{\zeta_1 - \zeta_0}\int_0^1 \frac{d}{dt} Q(\zeta_0 + t(\zeta_1 - \zeta_0))dt = \int_0^1 Q'(\zeta_0 + t(\zeta_1 - \zeta_0))dt, \quad (5.19)$$

we have $\operatorname{Re} \frac{Q(\zeta_1) - Q(\zeta_0)}{\zeta_1 - \zeta_0} > 0$. Hence $Q(\zeta_0) \neq Q(\zeta_1)$. This proves that Q is injective in $\{\zeta : \operatorname{Re}(\zeta e^{-i\theta}) > r\}$. □

5.G Estimates on φ

Let $\zeta(w)$ be as in Lemma 5.10.

Lemma 5.22. *Suppose $\varphi : \widehat{\mathbb{C}} \smallsetminus E \to \widehat{\mathbb{C}} \smallsetminus \{0\}$ is a normalized univalent map. It can be written as*

$$\varphi(\zeta) = \zeta + c_0 + \varphi_1(\zeta)$$

with $c_0 \in \mathbb{C}$ and $\lim_{\zeta\to\infty} \varphi_1(\zeta) = 0$. Then we have the following estimates:

(a) $|c_0 - c_{00}| \le c_{01,max}$, *where* $c_{00} := 0.18 = -x_E$, $c_{01,max} := 2.28 = 2e_1$.

(b) *Image*$(\varphi) \supset \{z : |z - (c_0 + x_E)| > 2e_1\} \supset \{z : |z| > 4e_1 = 4.56\}$.

(c) $e_1|w|\left(1 - \frac{1}{|w|}\right)^2 \le |\varphi(\zeta(w))| \le e_1|w|\left(1 + \frac{1}{|w|}\right)^2$ *for* $|w| > 1$.

(d) $\left|\arg \frac{\varphi(\zeta(w))}{w}\right| \le \log \frac{|w|+1}{|w|-1}$ *for* $|w| > 1$.

(e) $|\varphi_1(\zeta)| \le \varphi_{1,max}(r) := a_E \sqrt{-\log\left(1 - \left(\frac{a_E}{r-|x_E|}\right)^2\right)}$ *for* $|\zeta| \ge r > a_E + |x_E| = 1.42$.

(f) $|\log \varphi'(\zeta)| \le LogD\varphi_{max}(r) := -\log\left(1 - \left(\frac{a_E}{r-|x_E|}\right)^2\right)$ *for* $|\zeta| \ge r > a_E + |x_E| = 1.42$.

Proof. Let $\hat\varphi(w) = \frac{1}{e_1}\varphi(\zeta(w))$. Then it can be checked that $\hat\varphi$ belongs to Σ_*. Since

$$\hat\varphi(w) = w + \frac{c_0 + x_E}{e_1} + \frac{1}{e_1}\left(\varphi_1(\zeta(w)) + \frac{e_{-1}}{w}\right) = w + \frac{c_0 + x_E}{e_1} + O\left(\frac{1}{w}\right),$$

it follows from Theorem A.2 (a) that for $\hat c_0 = \frac{c_0+x_E}{e_1}$, $|\hat c_0| \le 2$ and $\{z : |z| > 4\} \subset \{z : |z - \hat c_0| > 2\} \subset$ *Image*$(\hat\varphi)$. They imply (a) and (b). Applying Theorem A.2 (d) to $\hat\varphi$, we also obtain (c) and (d).

Let $\tilde\zeta = \frac{\zeta - x_E}{a_E}$. If $|\tilde\zeta| > 1$ then $\zeta = x_E + a_E\tilde\zeta \in \mathbb{C} \smallsetminus E$ and $\tilde\varphi(\tilde\zeta) = \frac{1}{a_E}\varphi(x_E + a_E\tilde\zeta)$ is defined. Applying Theorem A.2 (b) and (c) to $\tilde\varphi$ which belongs to Σ_*, we obtain (e) and (f). □

Lemma 5.23. *If $\zeta \in \mathbb{C} \smallsetminus intE_{r_1}$, then $|\varphi(\zeta)| > \rho$ and $|\arg \frac{\varphi(\zeta)}{\zeta}| < \pi$.*

Proof. Suppose $\zeta \in \mathbb{C} \smallsetminus intE_{r_1}$, then we can write $\zeta = \zeta(w)$ with $|w| \ge r_1 = 1.25$. By Lemma 5.22 (c), using the fact that $r(1 - \frac{1}{r})^2$ is increasing in $r > 1$, we have

$$|\varphi(\zeta)| = |\varphi(\zeta(w))| \ge e_1|w|\left(1 - \frac{1}{|w|}\right)^2 \ge 1.14 \times 1.25\left(1 - \frac{1}{1.25}\right)^2 = 0.057 > \rho = 0.05.$$

Also by Lemma 5.22 (d),

$$\left|\arg \frac{\varphi(\zeta(w))}{w}\right| \le \log \frac{1.25 + 1}{1.25 - 1} = 2\log 3 \ (\fallingdotseq 2.1972\dots) \underset{*}{<} 0.7\pi \ (\fallingdotseq 2.1991\dots). \tag{5.20*}$$

On the other hand, by Lemma 5.9,

$$\left|\arg \frac{\zeta(w)}{w}\right| = \left|\arg\left(1 + \frac{x_E}{e_1 w} + \frac{e_{-1}}{e_1 w^2}\right)\right| \le \arcsin\left(|x_E| + |e_{-1}|\right) = \arcsin(0.28) \le \frac{\pi}{3} \cdot 0.28 < 0.1\pi.$$

Therefore we have $\left|\arg \frac{\varphi(\zeta)}{\zeta}\right| \le \left|\arg \frac{\varphi(\zeta(w))}{w}\right| + \left|\arg \frac{\zeta(w)}{w}\right| \le 0.7\pi + 0.1\pi < \pi$. □

We will need the following for Lemma 5.33 in §5.L.

Lemma 5.24. *If* $\zeta \in \mathbb{C} \setminus \mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \mathbb{D}\left(-\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup int E_{r_1}$ *and* $\operatorname{Re}\zeta \geq x_{_E}$*, then* $\varphi(\zeta) \notin \mathbb{R}_-$.

Proof. By Lemma 5.22 (d), we have for $|w| > 1$,

$$|\arg(\varphi(\zeta(w))| \leq |\arg w| + \left|\arg \frac{\varphi(\zeta(w))}{w}\right| \leq |\arg w| + \log \frac{|w|+1}{|w|-1}.$$

Suppose $\zeta \in \mathbb{C} \setminus E_{r_1}$ and $\operatorname{Re}\zeta \geq x_{_E}$. Then we can write as $\zeta = \zeta(w)$ with $r = |w| > r_1 = 1.25$ and $\theta = \arg w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. So in order to prove the lemma, it suffices to show that

$$\text{if } r \geq r_1 \text{ and } \ 0 \leq \theta \leq \frac{\pi}{2}, \text{ then either } \theta + \log\frac{r+1}{r-1} < \pi \text{ or } \zeta(re^{i\theta}) \in \mathbb{D}\left(\tfrac{i}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}\right). \tag{5.21}$$

We cover by 5 cases:
(a) $r > r_1 = 1.25$ and $0 \leq \theta \leq 0.3\pi$; (b) $r \geq r_3 = 1.54$ and $0.3\pi \leq \theta \leq \frac{\pi}{2}$; (c) $r_2 = 1.4 \leq r \leq r_3 = 1.54$ and $0.3\pi \leq \theta \leq 0.4\pi$; (d) $r_2 = 1.4 \leq r \leq r_3 = 1.54$ and $0.4\pi \leq \theta \leq \frac{\pi}{2}$; (e) $r_1 = 1.25 \leq r \leq r_2 = 1.4$ and $0.3\pi \leq \theta \leq \frac{\pi}{2}$.
In case (a), we have $\theta + \log\frac{r+1}{r-1} < 0.3\pi + 0.7\pi = \pi$ by (5.20*). We also have $\theta + \log\frac{r+1}{r-1} < \pi$ in cases (b) and (c) by

$$\log\frac{1.54+1}{1.54-1}\ (\doteqdot 1.548\ldots) \underset{*}{<} \frac{\pi}{2}\ (\doteqdot 1.570\ldots), \tag{5.22*}$$

$$\log\frac{1.4+1}{1.4-1}\ (\doteqdot 1.791\ldots) \underset{*}{<} 0.6\pi\ (\doteqdot 1.884\ldots). \tag{5.23*}$$

In order to show $\zeta(re^{i\theta}) \in \mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ for cases (d) and (e), we need the following:

Sublemma 5.25. *Let* $1 \leq s_1 < s_2$ *and* $0 < \theta_1 < \frac{\pi}{2}$*. If* $\zeta(s_2 i)$ *and* $\zeta(s_2 e^{i\theta_1})$ *are contained in* $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$*, then*

$$Z(s_1, s_2, \theta_1) := \{\zeta(w) : s_1 \leq |w| \leq s_2 \text{ and } \theta_1 \leq \theta \leq \frac{\pi}{2}\}.$$

is also contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

Proof. By the assumption and Lemma 5.18, the subarc $\partial E_{s_2} \cap Z(s_1, s_2, \theta_1)$ is contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Since $Z(s_1, s_2, \theta_1)$ is the region bounded by $\{\zeta : \operatorname{Re}\zeta = x_{_E}\}$, ∂E_{s_1}, ∂E_{s_2} and the upper right part of a hyperbola

$$\left(\frac{x - x_{_E}}{\cos\theta_1}\right)^2 - \left(\frac{y}{\sin\theta_1}\right)^2 = 4e_{_1}e_{_{-1}}, \quad x \geq x_{_E} + 2\sqrt{e_{_1}e_{_{-1}}}\cos\theta_1 \text{ and } y \geq 0,$$

which is concave, it is easy to see that the region $Z(s_1, s_2, \theta_1)$ is contained in the convex hull of $(\partial E_{s_2} \cap Z(s_1, s_2, \theta_1)) \cup [x_{_E}, x_{_E} + 2\sqrt{e_{_1}e_{_{-1}}}\cos\theta_1]$. Since $x_{_E} + 2\sqrt{e_{_1}e_{_{-1}}}\cos\theta_1 < 2\sqrt{0.114} < 1$, this convex hull is contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. □

We apply this to $r_2 = 1.4$, $\theta_2 = \frac{\pi}{4} < 0.3\pi$ and $r_3 = 1.54$ and $\theta_3 = 0.4\pi$. It can be checked that

$$\left|\zeta(r_2 e^{i\theta_2}) - \tfrac{i}{\sqrt{3}}\right|^2 = \left(1.14\, r_2 \cos\theta_2 - 0.18 + \tfrac{0.1\cos\theta_2}{r_2}\right)^2 + \left(1.14\, r_2 \sin\theta_2 - \tfrac{0.1\sin\theta_2}{r_2} - \tfrac{1}{\sqrt{3}}\right)^2$$
$$(\doteqdot 1.248\dots) \underset{*}{<} \left(\tfrac{2}{\sqrt{3}}\right)^2 \ (\doteqdot 1.333\dots), \tag{5.24*}$$
$$\left|\zeta(r_3 e^{i\theta_3}) - \tfrac{i}{\sqrt{3}}\right|^2 = \left(1.14\, r_3 \cos\theta_3 - 0.18 + \tfrac{0.1\cos\theta_3}{r_3}\right)^2 + \left(1.14\, r_3 \sin\theta_3 - \tfrac{0.1\sin\theta_3}{r_3} - \tfrac{1}{\sqrt{3}}\right)^2$$
$$(\doteqdot 1.208\dots) \underset{*}{<} \left(\tfrac{2}{\sqrt{3}}\right)^2, \tag{5.25*}$$
$$\left|\zeta(r_3 i) - \tfrac{i}{\sqrt{3}}\right|^2 = (-0.18)^2 + \left(1.14\, r_3 - \tfrac{0.1}{r_3} - \tfrac{1}{\sqrt{3}}\right)^2 \ (\doteqdot 1.27\dots) \underset{*}{<} \left(\tfrac{2}{\sqrt{3}}\right)^2. \tag{5.26*}$$

From (5.26*), $\zeta(r_2 i)$ is also in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Hence, by the above lemma, $Z(r_1, r_2, \theta_2)$ and $Z(r_1, r_3, \theta_3)$ are contained in $\mathbb{D}\left(\frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Therefore (5.21) is proved for cases (d) and (e). This completes the proof of Lemma 5.24. $\square$

5.H Lifting Q and φ to X

Definition. Denote $Y_{j\pm} = \left(Q|_{\overline{\mathcal{U}}_{j\pm}}\right)^{-1}(\pi_X(X_{j\pm}))$ $(j = 1, 2)$. Let

$$Y = Y_{1+} \cup Y_{1-} \cup Y_{2+} \cup Y_{2-}.$$

which is a subset of $\mathcal{U}_{12} \cup \mathbb{R}_- \subset \mathbb{C}$. Define $\widetilde{Q} : Y \to X$ (whose well-definedness is to be verified) by

$$\widetilde{Q}(\zeta) = \left(\pi|_{X_{j\pm}}\right)^{-1}(Q(\zeta)) \in X_{j\pm} \quad \text{for } \zeta \in Y_{j\pm}.$$

Also define

$$\widetilde{Y} = \mathbb{C} \smallsetminus \left(E_{r_1} \cup \mathbb{R}_+ \cup \overline{\mathbb{V}}\left(21, \tfrac{\pi}{6}\right)\right).$$

Proof of Prop 5.4 (a). Since Q maps $\mathcal{U}_{j\pm}$ isomorphically onto $\{z : \pm \operatorname{Im} z > 0\}$, $\widetilde{Q}$ maps $Y_{j\pm}$ homeorphically onto $X_{j\pm}$ $(j = 1, 2)$. Hence, in order to see that $\widetilde{Q}$ is well-defined and isomorphic, it suffices to check its consistency along their boundaries.

First note that among $\overline{\mathcal{U}}_{1+}, \overline{\mathcal{U}}_{1-}, \overline{\mathcal{U}}_{2+}$ and $\overline{\mathcal{U}}_{2-}$, the pairs whose intersections are more than $\{cp, -1\}$ are: $\overline{\mathcal{U}}_{1+} \cap \overline{\mathcal{U}}_{1-} = \overline{\gamma}_{a1} \cup \overline{\gamma}_{c1}$, $\overline{\mathcal{U}}_{1+} \cap \overline{\mathcal{U}}_{2-} = \overline{\gamma}_{b1}$, $\overline{\mathcal{U}}_{1-} \cap \overline{\mathcal{U}}_{2+} = \overline{\gamma}_{b2}$, $\overline{\mathcal{U}}_{2+} \cap \overline{\mathcal{U}}_{2-} = \overline{\gamma}_{a2}$. Moreover $[cv, +\infty) = \{cv\} \cup \Gamma_a$ does not intersect with $X_{i\pm}$ $(i = 1, 2)$, so γ_{ai}'s do not affect the intersection of $Y_{i\pm} (\subset \mathcal{U}_{i\pm})$. Neither does -1, since $Q(-1) = 0 \notin X_{i\pm}$. Hence among $\overline{Y}_{i\pm}$'s, the pairs having intersections are: $Y_{1+} \cap Y_{1-} \subset \gamma_{c1} \subset \mathbb{R}_-$, $Y_{1+} \cap Y_{2-} \subset \gamma_{b1}$, $Y_{1-} \cap Y_{2+} \subset \gamma_{b2}$.

First consider the pair Y_{1+} and Y_{1-}. In the construction of X, X_{1+} and X_{1-} are glued along the negative real axis Γ_c, but on the positive side of real axis, they are disjoint, i.e. they are considered to be on different sheets. Accordingly, Y_{1+} and Y_{1-} intersect only along $\gamma_{c1} \subset Q^{-1}(\Gamma_c)$. So this gluing is consistent for Y_{1+} and Y_{1-}, and defines a continuous map $\widetilde{Q}$ there. As for X_{1+} and X_{2-}, they are glued along $(\rho, cv) \subset \Gamma_b$, but not along negative real axis. On the other hand, Y_{1+} and Y_{2-} intersect along γ_{b1}. So the gluing is also consistent here. The same is true for the pair X_{1-} and X_{2+}. Thus all the gluings along the boundaries are consistent and $\widetilde{Q} : Y \to X$ is an isomorphisim.

The construction implies that $\pi_X \circ \widetilde{Q} = Q$ on Y. If $z \in X$ and $|\pi_X(z)| > R$, z must be on $X_{1+} \cup X_{1-}$, therefore $\widetilde{Q}^{-1}(z) \in Y_{1+} \cup Y_{1-} \subset \mathcal{U}_{1+} \cup \mathcal{U}_{1-}$. When $\pi_X(z) \to \infty$, $\widetilde{Q}^{-1}(z)$ corresponds to the inverse branch of Q near ∞, hence it has asymptotic expansion $\widetilde{Q}^{-1}(z) = \pi_X(z) - 10 + o(1)$. $\square$

Lemma 5.26. $Y \subset \widetilde{Y}$.

Proof. Suppose $\zeta \in Y$. If $\zeta \in Y_{1\pm}(\subset \overline{\mathcal{U}}_1)$, then $|Q(\zeta)| > \rho$. If $\zeta \in Y_{2\pm}(\subset \mathcal{U}_{12})$, then $\rho < |Q(\zeta)| < R$. Therefore in either case, by Lemma 5.17 (d), we have $\zeta \in \mathbb{C} \smallsetminus E_{r_1}$. Since $\pi_X(X) \cap \overline{\mathbb{V}}\left(cv, \frac{\pi}{6}\right) = \emptyset$, we have $Q(\zeta) \notin \overline{\mathbb{V}}\left(cv, \frac{\pi}{6}\right)$. It follows from Lemma 5.20 that $\zeta \notin \overline{\mathbb{V}}\left(21, \frac{\pi}{6}\right)$. Finally since $Q((1, +\infty)) = [cv, +\infty) \subset \overline{\mathbb{V}}\left(cv, \frac{\pi}{6}\right)$ and $Y \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}$, we have $\zeta \notin \mathbb{R}_+$. Thus we proved that $\zeta \in \widetilde{Y}$. □

Proof of Proposition 5.4 (b). We prove that $\varphi|_{\widetilde{Y}}$ can be lifted to $\tilde{\varphi} : \widetilde{Y} \to X$ which is well-defined and holomorphic. Then by Lemma 5.26, $Y \subset \widetilde{Y}$, so the assertion will follow.

First note that

$$\text{if } |\zeta| \geq 7,\ |\varphi(\zeta) - \zeta| \leq c_{00} + c_{01,max} + \varphi_{1,max}(7)\ (\doteqdot 2.687\ldots) \underset{*}{<} 3; \tag{5.27*}$$

$$\text{if } \zeta \in \mathbb{C} \smallsetminus E \text{ and } |\zeta| \leq 7,\ |\varphi(\zeta)| \leq 7 + c_{00} + c_{01,max} + \varphi_{1,max}(7) < 7 + 3 = 10. \tag{5.28}$$

The latter holds because the image $\varphi(\{\zeta \in \mathbb{C} \smallsetminus E : |\zeta| < 7\})$ is surrounded by the Jordan curve $\varphi(\{\zeta : |\zeta| = 7\})$. Therefore if $\zeta \in \mathbb{C} \smallsetminus \overline{\mathbb{V}}(21, \frac{\pi}{6})$ (in particular if $\zeta \in \widetilde{Y}$), then $\varphi(\zeta)$ cannot be in $\overline{\mathbb{V}}(cv, \frac{\pi}{6})$, since the distance between $\partial\mathbb{V}(21, \frac{\pi}{6})$ and $\overline{\mathbb{V}}(cv, \frac{\pi}{6})$ is 3.

Take $\zeta \in \widetilde{Y}$. By Lemma 5.23, we have $\left|\arg \frac{\varphi(\zeta)}{\zeta}\right| < \pi$ and $|\varphi(\zeta)| > \rho$ for $\zeta \in \widetilde{Y}$. Define $\tilde{\varphi}(\zeta) \in X$ so that $\pi(\tilde{\varphi}(\zeta)) = \varphi(\zeta)$ and

$$\begin{aligned}
&\tilde{\varphi}(\zeta) \in X_{1+} \cup X_{2-} && \text{if } \operatorname{Im}\zeta \geq 0 \text{ and } -\pi < \arg\tfrac{\varphi(\zeta)}{\zeta} \leq 0;\\
&\tilde{\varphi}(\zeta) \in X_{1-} \cup X_{2+} && \text{if } \operatorname{Im}\zeta \leq 0 \text{ and } 0 \leq \arg\tfrac{\varphi(\zeta)}{\zeta} < \pi;\\
&\tilde{\varphi}(\zeta) \in X_{1+} \cup X_{1-} && \text{otherwise.}
\end{aligned}$$

A possible problem with this definition is that when $\tilde{\varphi}(\zeta)$ was defined to be in $X_{2\pm}$ (first and second case), it might happen that $|\varphi(\zeta)| \geq R$. But this cannot happen because, for example, for the first case of the definition, $\varphi(\zeta)$ lies in the half plane $H = \{w : \arg\zeta - \pi < \arg w < \arg\zeta\}$ and not in $\overline{\mathbb{V}}(21, \frac{\pi}{6})$, and the distance between ζ and $H \smallsetminus \overline{\mathbb{D}}(0, R) \cup \overline{\mathbb{V}}(21, \frac{\pi}{6})$ (if not empty) is large (bounded below by the distance between $\partial\mathbb{D}(0, R) \cup \partial\mathbb{V}(21, \frac{\pi}{6})$ and the real axis, which is greater than $(R - 21)\sin\frac{\pi}{6} > 3$). This concludes that $\tilde{\varphi} : \widetilde{Y} \to X$ is well-defined.

Now we check the continuity. Possible discontinuities occur when the definition above switches the cases, i.e., when $\operatorname{Im}\zeta = 0$ or $\arg\frac{\varphi(\zeta)}{\zeta} = 0$. If $\zeta \in \widetilde{Y}$ and $\operatorname{Im}\zeta = 0$, then $\zeta \in \mathbb{R}_-$ hence $\tilde{\varphi}(\zeta)$ is in $X_{1+} \cup X_{1-}$ even when the first or second case of the definition is applied. If $\operatorname{Im}\zeta \neq 0$ and $\arg\frac{\varphi(\zeta)}{\zeta} = 0$, then $\tilde{\varphi}(\zeta)$ is also in $X_{1+} \cup X_{1-}$. Therefore around the switching, $\tilde{\varphi}(\zeta)$ should be in $X_{1+} \cup X_{1-}$ and this does not cause a discontinuity. Once the continuity is obtained, it is obviously holomorphic. □

5.I Estimates on F

Lemma 5.27. *Suppose $r > cp = 5 + 2\sqrt{6}$, $\theta \in \mathbb{R}$ and $\operatorname{Re}(\zeta\, e^{-i\theta}) > r$. Then the following estimates hold for $z = \varphi(\zeta)$:*

(a) $F(z) - z \in \mathbb{D}\left(10 - c_{00} + \frac{49e^{-i\theta}}{2r},\ \beta_{max}(r)\right)$, *where*

$$\beta_{max}(r) := c_{01,max} + \frac{49}{2r} + Q_{2,max}(r) + \varphi_{1,max}(r);$$

(b) $Arg\Delta F_{min}(r,\theta) \le \arg(F(z)-z) \le Arg\Delta F_{max}(r,\theta)$, *where*

$$Arg\Delta F_{\left\{ \substack{max \\ min} \right\}}(r,\theta) := -\arctan\left(\frac{\frac{49\sin\theta}{2r}}{10-c_{00}+\frac{49\cos\theta}{2r}}\right)$$
$$\pm\arcsin\left(\frac{\beta_{max}(r)}{\sqrt{\left(10-c_{00}\right)^2+\left(\frac{49}{2r}\right)^2+2\left(10-c_{00}\right)\left(\frac{49}{2r}\right)\cos\theta}}\right);$$

(c) $Abs\Delta F_{min}(r,\theta) \le |F(z)-z| \le Abs\Delta F_{max}(r,\theta)$, *where*

$$Abs\Delta F_{\left\{ \substack{max \\ min} \right\}}(r,\theta) := \sqrt{\left(10-c_{00}\right)^2+\left(\frac{49}{2r}\right)^2+2\left(10-c_{00}\right)\left(\frac{49}{2r}\right)\cos\theta} \pm \beta_{max}(r);$$

(d) $|\log F'(z)| \le LogDF_{max}(r) := LogDQ_{max}(r) + LogD\varphi_{max}(r)$.

Proof. (a) For $z=\varphi(\zeta)$, we can write $\varphi(\zeta)=\zeta+(c_{00}+c_{01})+\varphi_1(\zeta)$ and

$$F(z)-z = Q(\zeta)-\varphi(\zeta) = 10+\frac{49}{\zeta}+Q_2(\zeta)-(c_{00}+c_{01})-\varphi_1(\zeta) = \alpha+\beta = \alpha\left(1+\frac{\beta}{\alpha}\right),$$

where $\alpha = 10-c_{00}+\frac{49e^{-i\theta}}{2r}$ and $\beta = -c_{01}+\left(\frac{49}{\zeta}-\frac{49e^{-i\theta}}{2r}\right)+Q_2(\zeta)-\varphi_1(\zeta)$. Note that $\left|\frac{49}{\zeta}-\frac{49e^{-i\theta}}{2r}\right| \le \frac{49}{2r}$ by Lemma 5.11 (a). Therefore we have $|\beta| \le c_{01,max}+\frac{49}{2r}+Q_{2,max}(r)+\varphi_{1,max}(r)=\beta_{max}(r)$, for $r>1.42$. This implies (a).

When $r>cp$, α and β can be estimated as

$$|\alpha| \ge 10-c_{00}-\frac{49}{2cp}\ (\doteqdot 7.34\ldots) \underset{*}{>} \beta_{max}(cp)\ (\doteqdot 7.06\ldots) \ge |\beta|. \tag{5.29*}$$

(The estimates (b) and (c) hold whenever $|\alpha|>|\beta|$.)
(b) It follows that

$$|\arg(F(z)-z)-\arg\alpha| \le \left|\arg\left(1+\frac{\beta}{\alpha}\right)\right| \le \arcsin\left|\frac{\beta}{\alpha}\right|.$$

Since

$$\arg\alpha = -\arctan\left(\frac{\frac{49\sin\theta}{2r}}{10-c_{00}+\frac{49\cos\theta}{2r}}\right) \text{ and } |\alpha| = \sqrt{\left(10-c_{00}\right)^2+\left(\frac{49}{2r}\right)^2+2\left(10-c_{00}\right)\left(\frac{49}{2r}\right)\cos\theta},$$

we have the inequality.
(c) Similarly we have $|\alpha|-|\beta| \le |F(z)-z| \le |\alpha|+|\beta|$.
(d) This is immediate from definitions in Lemmas 5.21 (b) and 5.22 (f). □

Lemma 5.28. $F\left(\overline{\mathbb{V}}(cv,\frac{\pi}{6})\right) \subset \mathbb{V}(30,\frac{\pi}{6}) \subset V(cv,\frac{\pi}{6})$.

Proof. In the proof of Proposition 5.4 (b), we showed that $\varphi(\mathbb{C}\setminus\mathbb{V}(21,\frac{\pi}{6}))\cap\overline{\mathbb{V}}(cv,\frac{\pi}{6})=\emptyset$. Therefore $\varphi^{-1}(\overline{\mathbb{V}}(cv,\frac{\pi}{6}))\subset\mathbb{V}(21,\frac{\pi}{6})$. By Lemma 5.20, we have $F(\overline{\mathbb{V}}(cv,\frac{\pi}{6}))\subset Q(\mathbb{V}(21,\frac{\pi}{6}))\subset\mathbb{V}(30,\frac{\pi}{6})$. □

5.J Repelling Fatou coordinate $\widetilde{\Phi}_{rep}$ on X

Proof of Proposition 5.5. First it is easy to see that on X, $F \circ \pi_X \circ g = Q \circ \varphi^{-1} \circ \pi_X \circ \tilde{\varphi} \circ \widetilde{Q}^{-1} = Q \circ \widetilde{Q}^{-1} = \pi_X$.

Near ∞, F has an inverse branch $\bar{g}(z) = z - (10 - c_0) + o(1)$ as $z \to \infty$. By Lemma 5.9, $\left|\arg(10 - c_0)\right| \le \arcsin\left(\frac{c_{01,max}}{10-c_{00}}\right) \le \frac{\pi}{3} \cdot \frac{2.28}{9.82} < \frac{\pi}{10}$. If we take a large $L > 0$, then $\bar{g}$ exists and injective in $W = \mathbb{C} \smallsetminus \overline{\mathbb{V}}(-L, \frac{\pi}{10})$ and satisfies $|\arg(\bar{g}(z) - z) - \pi| < \frac{\pi}{10}$, hence $\bar{g}(W) \subset W$, and also $\operatorname{Re} \bar{g}(z) < \operatorname{Re} z - (10 - c_{00}) + c_{01,max} + 1 < \operatorname{Re} z - 6$. By the behavior of $\widetilde{Q}^{-1}$ near ∞ (Proposition 5.4 (a)), we have $\pi_X(g(\pi_X^{-1}(z))) \to \infty$ as $z \in \pi_X(X)$ and $z \to \infty$. Therefore it must coincide with $g(z)$ as the only inverse of z by F near ∞, hence $\pi_X(g(z)) = \bar{g}(\pi_X(z))$ if $\pi_X(z)$ is large.

By a general theory of Fatou coordinates (see Theorem 1.1), there exists a Fatou coordinate $\Phi_{rep}(z)$ holomorphic and injective in $\{z : \operatorname{Re} z < -L'\}$ for large $L' > L$ and satisfies $\Phi_{rep}(\bar{g}(z)) = \Phi_{rep}(z) - 1$. Then it can be extended to W, and the extension is still injective, because of the injectivity of the original Φ_{rep} and $\bar{g}|_W$. This is a repelling Fatou coordinate for F.

Let $W' = \pi_X^{-1}(W \cap \pi_X(X))$, then $\pi_X|_{W'}$ is injective if L is large. Define $\widetilde{\Phi}_{rep} = \Phi_{rep} \circ \pi_X$ on W'. It naturally satisfies $\widetilde{\Phi}_{rep}(g(z)) = \widetilde{\Phi}_{rep}(z) - 1$ in W'. Now we want to extend this function to the whole X via the functional equation. We need the following:

Lemma 5.29. *For any point $z \in X$, there exists an $n \in \mathbb{N}$ such that $g^n(z) \in W'$.*

Proof. Pick a point $z_0 \in W'$. Let $\partial W'$ be the boundary of W' within X. Then $\pi_X(\partial W')$ is a union of two finite segments. Note that X is hyperbolic as a Riemann surface, since it is isomorphic to Y which is a proper subdomain of $\mathbb{C}$. Since $\partial W'$ is relatively compact within $\mathbb{C} \smallsetminus \overline{\mathbb{D}}(0, \rho)$, in which $g^n(z_0)$ tend to the boundary, the Poincaré distance $d_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}(0,\rho)}(g^n(z_0), \partial W') \to \infty$ as $n \to \infty$. The same holds with respect to the Poincaré distance d_X of X, since by Schwarz-Pick theorem (see [A2]), the projection $\pi_X : X \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}(0, \rho)$ does not expand the Poincaré distance. It follows that for any other point $z \in X$,

$$d_X(g^n(z), g^n(z_0)) \le d_X(z, z_0) < d_X(g^n(z_0), \partial W')$$

for sufficiently large n, where the left inequality is also given by Schwarz-Pick theorem applied to g^n. Hence $g^n(z) \in W'$ for these n. $\square$

Thus the Fatou coordinate $\widetilde{\Phi}_{rep}$ can be extended to X by $\widetilde{\Phi}_{rep}(z) = \widetilde{\Phi}_{rep}(g^n(z)) + n$, where n is chosen so that $g^n(z) \in W'$. It is well defined and satisfies the functional equation. Moreover it is injective on X, because of the injectivity of the original Φ_{rep} and g. We also have $\operatorname{Re} \pi_X(g^n(z)) \to -\infty$ as $n \to \infty$ for any point $z \in X$. Proposition 5.5 is proved. $\square$

5.K Attracting Fatou coordinate Φ_{attr} and domains D_1, $D_1^\sharp$

Definition. Denote $pr_+(z) = \operatorname{Re}(z\, e^{-i\pi/6})$ and $pr_-(z) = \operatorname{Re}(z\, e^{+i\pi/6})$, which correspond to the orthogonal projection to the line with angle $\pm\frac{\pi}{6}$ to the real axis. Let

$$H_1^\pm = \{z : pr_\pm(z) > u_1 := 12.5\}, \qquad H_2^\pm = \{\zeta : pr_\pm(\zeta) > u_2 := cp\},$$
$$H_3^\pm = \{z : pr_\pm(z) \ge u_3 := pr_+(cv) = \tfrac{27\sqrt{3}}{2} (\doteqdot 23.38\dots)\}, \quad H_4^\pm = \{\zeta : pr_\pm(\zeta) \ge u_4 := 20.8\}.$$

Lemma 5.30 (Attracting Fatou coordinate Φ_{attr})**.** (a) $\varphi(H_2^\pm) \supset H_1^\pm$, $\varphi(H_4^\pm) \supset H_3^\pm$. *Hence F is defined on $H_1^+ \cup H_1^-$.*
(b) *Q is injective in $H_2^\pm$. Therefore F is injective in $H_1^\pm$.*

(c) *If* $z \in H_1^\pm$, *then* $|\arg(F(z)-z)| < \frac{\pi}{3}$, *hence* $F(H_1^\pm) \subset H_1^\pm$. *Therefore the sector* $H_1^+ \cup H_1^- = \mathbb{V}\left(u_0, \frac{2\pi}{3}\right)$ *is forward invariant under* F *and contained in* $Basin(\infty)$, *where* $u_0 = \frac{25}{\sqrt{3}}$.
(d) *An attracting Fatou coordinate* Φ_{attr} *for* F *exists in* $\mathbb{V}\left(u_0, \frac{2\pi}{3}\right)$ *and is injective in each of* $H_1^\pm$.

We normalize the Fatou coordinate Φ_{attr} so that $\Phi_{attr}(cv) = 1$.

Proof. (a) By Lemma 5.22 (b), $H_1^\pm$ is contained in $Image(\varphi)$. If $\zeta \in \partial H_2^\pm$, then By Lemma 5.22 (e),

$$\begin{aligned} pr_\pm(\varphi(\zeta)) &= pr_\pm(\zeta) + pr_\pm(c_{00}) + pr_\pm(c_0 - c_{00}) + pr_\pm(\varphi_1(\zeta)) \\ &\le cp + \frac{c_{00}\sqrt{3}}{2} + c_{01,max} + \varphi_{1,max}(cp) \ (\doteqdot 12.493\dots) \underset{*}{<} 12.5. \end{aligned} \tag{5.30*}$$

Hence $\varphi(\zeta) \notin H_1^\pm$. Thus $\varphi^{-1}(H_1^\pm)$ must be contained in one side of $\partial H_2^\pm$. However if we take a point ζ in $H_2^\pm$ far from $\partial H_2^\pm$, then $\varphi(\zeta) \in H_1^\pm$, therefore $\varphi^{-1}(H_1^\pm)$ must be contained in $H_2^\pm$, i.e., $\varphi(H_2^\pm) \supset H_1^\pm$.

If $\zeta \in \partial H_4^\pm$, then

$$\begin{aligned} pr_\pm(\varphi(\zeta)) &= pr_\pm(\zeta) + pr_\pm(c_{00}) + pr_\pm(c_0 - c_{00}) + pr_\pm(\varphi_1(\zeta)) \\ &\le 20.8 + \frac{c_{00}\sqrt{3}}{2} + c_{01,max} + \varphi_{1,max}(20.8) \ (\doteqdot 23.31\dots) \\ &\underset{*}{<} pr_\pm(cv) \ (\doteqdot 23.38\dots). \end{aligned} \tag{5.31*}$$

As before, we conclude that $\varphi(H_4^\pm) \supset H_3^\pm$.
(b) The injectivity of Q in $H_2^\pm$ follows from Lemma 5.21 (c). The injectivity of F in $H_1^\pm$ follows immediately.
(c) If $z \in H_1^\pm$, then $\zeta = \varphi^{-1}(z) \in H_2^\pm$ by (a). By Lemma 5.27 (b),

$$\begin{aligned} |\arg(F(z)-z)| &\le \max\{Arg\Delta F_{max}(cp, \pm\tfrac{\pi}{6}),\ -Arg\Delta F_{min}(cp, \pm\tfrac{\pi}{6})\} \\ &(\doteqdot \max\{0.524\dots,\ 0.731\dots\}) \underset{*}{<} 1 < \frac{\pi}{3}. \end{aligned} \tag{5.32*}$$

This implies the forward invariance of $H_1^\pm$ and also $H_1^+ \cup H_1^-$, which can be shown to coincide with $\mathbb{V}\left(u_0, \frac{2\pi}{3}\right)$. The fact that $H_1^\pm$ is contained in $Basin(\infty)$ and (d) can be proven as in the proof of Proposition 5.5. □

Lemma 5.31 (Estimates on Φ_{attr}). (a) *The attracting Fatou coordinate* Φ_{attr} *satisfies the following inequalities:*

$$-\frac{\pi}{6} < \arg \Phi'_{attr}(z) < \frac{\pi}{5} \quad \textit{for} \quad z \in H_3^+ \quad \textit{and} \quad -\frac{\pi}{5} < \arg \Phi'_{attr}(z) < \frac{\pi}{6} \quad \textit{for} \quad z \in H_3^-; \tag{5.33}$$

$$0.055 < |\Phi'_{attr}(z)| < 0.176 \quad \textit{for} \quad z \in H_3^+ \cup H_3^- = \overline{\mathbb{V}}\left(cv, \tfrac{2\pi}{3}\right). \tag{5.34}$$

(b) Φ_{attr} *is injective in* $H_3^+ \cup H_3^- = \overline{\mathbb{V}}\left(cv, \frac{2\pi}{3}\right)$. *There exists a domain* $\mathcal{H}_1$ *such that* Φ_{attr} *is a homeomorphism from* $\overline{\mathcal{H}_1}$ *onto* $\{z : \operatorname{Re} z \ge 1\}$, *and* $\mathcal{H}_1$ *satisfies* $\overline{\mathbb{V}}\left(cv, \frac{\pi}{3}\right) \subset \mathcal{H}_1 \cup \{cv\} \subset \overline{\mathcal{H}_1} \subset \mathbb{V}\left(cv, \frac{2\pi}{3}\right) \cup \{cv\}$ *and* $cv \in \partial\mathcal{H}_1$.

Proof. (a) Suppose $z \in H_3^+$. Then $\zeta = \varphi^{-1}(z) \in H_4^+$, i.e., $\mathrm{Re}(\zeta e^{-i\pi/6}) \geq u_4 = 20.8$. We will derive the estimates from Theorem 5.12. First we claim that

$$F(z) \in \mathbb{D}_{H_1^+}(z, s(r_4)), \tag{5.35}$$

with $r_4 = 0.43$, where $s(\cdot)$ is defined in Lemma 5.11 (b). According to Lemma 5.11 (b) with $H = H_1^+$, $t = u_1$, $u = pr_+(z) - u_1$, $r = r_4$, $\theta = \frac{\pi}{6}$, this is equivalent to

$$F(z) - z \in \mathbb{D}\left(\frac{2ur_4^2 e^{i\pi/6}}{1-r_4^2}, \frac{2ur_4}{1-r_4^2}\right). \tag{5.36}$$

Note that this disk contains 0, so it is increasing with u. Therefore we only need to check when u is the smallest, i.e. $u_5 = u_3 - u_1 = pr_+(cv) - 12.5$. According to Lemma 5.27 (a), we can write $F(z) - z = \alpha + \beta$ with $\alpha = 10 - c_{00} + \dfrac{49e^{-i\pi/6}}{2u_4}$, $u_4 = 20.8$ and $|\beta| \leq \beta_{max}(u_4)$. By a numerical estimate, we have

$$\begin{aligned}
&\left|\alpha - \frac{2u_5 r_4^2 e^{i\pi/6}}{1-r_4^2}\right| + \beta_{max} - \frac{2u_5 r_4}{1-r_4^2} \\
&\quad = \sqrt{\left(10 - c_{00} + \frac{49\sqrt{3}}{4u_4} - \frac{\sqrt{3}u_5 r_4^2}{1-r_4^2}\right)^2 + \left(\frac{49}{4u_4} + \frac{u_5 r_4^2}{1-r_4^2}\right)^2} + \beta_{max}(u_4) - \frac{2u_5 r_4}{1-r_4^2} \\
&\quad (\doteqdot -0.289\ldots) \underset{*}{<} 0,
\end{aligned} \tag{5.37*}$$

which implies (5.36) and (5.35).

Applying Theorem 5.12 to Φ_{attr} with $\Omega = H_1^+$, $r = r_4$ and using Lemma 5.27, we obtain

$$\begin{aligned}
\arg \Phi'_{attr}(z) &\leq -\arg(F(z) - z) + \frac{1}{2}\left|\log F'(z)\right| + \frac{1}{2}\log\frac{1}{1-r_4^2} \\
&\leq -Arg\Delta F_{min}(u_4, \tfrac{\pi}{6}) + \frac{1}{2}LogDF_{max}(u_4) - \frac{1}{2}\log(1-r_4^2) \\
&(\doteqdot 0.6175\ldots) < \frac{\pi}{5}\ (\doteqdot 0.6283\ldots),
\end{aligned} \tag{5.38*}$$

$$\begin{aligned}
\arg \Phi'_{attr}(z) &\geq -\arg(F(z) - z) + \frac{1}{2}\left|\log F'(z)\right| - \frac{1}{2}\log\frac{1}{1-r_4^2} \\
&\geq -Arg\Delta F_{max}(u_4, \tfrac{\pi}{6}) - \frac{1}{2}LogDF_{max}(u_4) + \frac{1}{2}\log(1-r_4^2) \\
&(\doteqdot -0.5089\ldots) > -\frac{\pi}{6}\ (\doteqdot -0.5235\ldots).
\end{aligned} \tag{5.39*}$$

A similar estimate can be given for $z \in H_3^-$.

As for $|\Phi'_{attr}(z)|$ on H_3^+ or H_3^-, again by Theorem 5.12 and Lemma 5.27, we have

$$\begin{aligned}
|\Phi'_{attr}(z)| &\leq \exp\left(-\log|F(z) - z| + \frac{1}{2}|\log F'(z)| + \frac{1}{2}\log\frac{1}{1-r_4^2}\right) \\
&\leq \frac{\exp\left(\frac{1}{2}LogDF_{max}(u_4)\right)}{Abs\Delta F_{min}(u_4, \frac{\pi}{6})\sqrt{1-r_4^2}}\ (\doteqdot 0.1752\ldots) \underset{*}{<} 0.176,
\end{aligned} \tag{5.40*}$$

$$\begin{aligned}
|\Phi'_{attr}(z)| &\geq \exp\left(-\log|F(z) - z| - \frac{1}{2}|\log F'(z)| - \frac{1}{2}\log\frac{1}{1-r_4^2}\right) \\
&\geq \frac{\sqrt{1-r_4^2}}{Abs\Delta F_{max}(u_4, \frac{\pi}{6})\exp\left(\frac{1}{2}LogDF_{max}(u_4)\right)}\ (\doteqdot 0.0558\ldots) \underset{*}{>} 0.055.
\end{aligned} \tag{5.41*}$$

It is easy to check that $H_3^+ \cup H_3^- = \overline{\mathbb{V}}(cv, \frac{2\pi}{3})$.

(b) Suppose that $[z_1, z_2]$ is a non-trivial segment within $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$. It is easy to see that

$$\text{if } \theta < \arg \Phi'_{attr}(z) < \theta' \leq \theta + \pi \text{ on } [z_1, z_2], \text{ then } \theta < \arg \frac{\Phi_{attr}(z_2) - \Phi_{attr}(z_1)}{z_2 - z_1} < \theta'. \tag{5.42}$$

(Apply (5.19) to $e^{-i\theta - i\pi/2}\Phi_{attr}(z)$ and $e^{-i\theta' + i\pi/2}\Phi_{attr}(z)$ and consider the real part.) In particular, taking $\theta = -\frac{\pi}{5}$ and $\theta' = \frac{\pi}{5}$, we have $\operatorname{Re} \frac{\Phi_{attr}(z_2) - \Phi_{attr}(z_1)}{z_2 - z_1} > 0$ and $\Phi_{attr}(z_1) \neq \Phi_{attr}(z_2)$. If two points $z_1, z_2 \in \overline{\mathbb{V}}(cv, \frac{2\pi}{3})$ cannot be joined by one segment in $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$, then one can choose z_3 so that $[z_1, z_3]$ and $[z_3, z_2]$ are contained in $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$ and $\frac{\pi}{3} \leq \arg(z_3 - z_1) \leq \frac{2\pi}{3}$ and $\frac{\pi}{3} \leq \arg(z_2 - z_3) \leq \frac{2\pi}{3}$ (interchanging z_1 and z_2 if necessary). By (5.42), $0 < \frac{\pi}{3} - \frac{\pi}{5} < \arg(\Phi_{attr}(z_2) - \Phi_{attr}(z_1)) < \frac{2\pi}{3} + \frac{\pi}{5} < \pi$. The same estimates holds for $z_2 - z_3$ and therefore $\operatorname{Im}(\Phi_{attr}(z_2) - \Phi_{attr}(z_1)) > 0$. Thus Φ_{attr} is injective in $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$.

Similarly if $z_1, z_2 \in H_3^+$ and $z_1 \neq z_2$, then

$$\arg(z_2 - z_1) - \frac{\pi}{6} < \arg(\Phi_{attr}(z_2) - \Phi_{attr}(z_1)) < \arg(z_2 - z_1) + \frac{\pi}{5}. \tag{5.43}$$

In particular, if $\arg(z - cv) = \frac{2\pi}{3}$ (z is on the upper boundary of $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$), $\frac{\pi}{2} = \frac{2\pi}{3} - \frac{\pi}{6} < \arg(\Phi_{attr}(z) - 1) < \frac{2\pi}{3} + \frac{\pi}{5} < \pi$ (note here that $\Phi_{attr}(cv) = 1$), i.e., $\operatorname{Re}(\Phi_{attr}(z) - 1) < 0$ and $\operatorname{Im}(\Phi_{attr}(z) - 1) > 0$. A similar result holds for H_3^-. By (5.19) and (a), we also have

$$\left| \frac{\Phi_{attr}(z) - 1}{z - cv} \right| \geq \int_0^1 \operatorname{Re} \Phi'_{attr}(cv + t(z - cv))dt \geq 0.055 \cos(\tfrac{\pi}{5}) > 0.$$

So as $z \to \infty$ in $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$, $\Phi_{attr}(z) \to \infty$.

Given any $R' > 0$, take $R'' > 0.055 \cos(\frac{\pi}{5}) \times R'$ and denote $G = \mathbb{V}(cv, \frac{2\pi}{3}) \cap \mathbb{D}(cv, R'')$. The above results imply that $\Phi_{attr}(\partial G)$ does not intersect $\{z : \operatorname{Re} z \geq 1\} \cap \overline{\mathbb{D}}(1, R')$ except cv. Since $\{z : \operatorname{Re} z \geq 0\} \cap \overline{\mathbb{D}}(1, R')$ contains at least one point of $\Phi_{attr}(G)$ (such as $cv + t$ with small $t > 0$ by (5.43)), the Jordan curve $\Phi_{attr}(\partial G)$ has winding number 1 around this point. Therefore this is true around any point in $\{z : \operatorname{Re} z \geq 0\} \cap \overline{\mathbb{D}}(1, R')$ except cv. Hence by Argument Principle, $\{z : \operatorname{Re} z \geq 1\} \cap \overline{\mathbb{D}}(1, R') \subset \Phi_{attr}(G) \cup \{cv\}$. Since $R' > 0$ was arbitrary, $\{z : \operatorname{Re} z \geq 1\}$ is contained in the image of $\mathbb{V}(cv, \frac{2\pi}{3}) \cup \{cv\}$ by Φ_{attr}. Define $\mathcal{H}_1 = \Phi_{attr}^{-1}(\{z : \operatorname{Re} z > 1\})$. If $z \in \overline{\mathbb{V}}(cv, \frac{\pi}{3}) = z \in H_3^+ \cap H_3^-$, again by (5.43), where $\frac{\pi}{5}$ can be replaced by $\frac{\pi}{6}$ in this case, we have $|\arg(\Phi_{attr}(z) - 1)| < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$. Hence $\Phi_{attr}(\overline{\mathbb{V}}(cv, \frac{\pi}{3}))$ should be contained in $\{z : \operatorname{Re} z > 1\} \cup \{cv\}$. Therefore we have $\overline{\mathbb{V}}\left(cv, \frac{\pi}{3}\right) \subset \mathcal{H}_1 \cup \{cv\} \subset \overline{\mathcal{H}_1} \subset \mathbb{V}\left(cv, \frac{2\pi}{3}\right) \cup \{cv\}$. □

Proof of Proposition 5.6. Lemmas 5.30 and 5.31 already proved (a). For (b), simply define $D_1 = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2,\ -\eta < \operatorname{Im} z < \eta\})$, $D_1^\sharp = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z > \eta\})$, $D_1^\flat = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z < -\eta\})$, where the inverse image is taken only within $\overline{\mathbb{V}}(cv, \frac{2\pi}{3})$. Suppose $|\arg(z - F(cv))| \leq \frac{\pi}{3}$ (on the right of W_1). Then $z \in \overline{\mathbb{V}}(cv, \frac{\pi}{3})$, since by Lemma 5.28, $F(cv) \in \mathbb{V}(cv, \frac{\pi}{6})$. So as before we obtain $|\arg(\Phi_{attr}(z) - \Phi_{attr}(F(cv)))| < \frac{\pi}{2}$. Hence $\operatorname{Re} \Phi_{attr}(z) > \operatorname{Re} \Phi_{attr}(F(cv)) = 2$. This shows that D_1, $D_1^\sharp$, $D_1^\flat$ must be contained in W_1. Similarly if $|\arg(z - cv)| \leq \frac{\pi}{6}$, then $|\arg(\Phi_{attr}(z) - 1)| < \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$, and $\Phi_{attr}(z)$ cannot be in $\{z : 1 < \operatorname{Re} z < 2,\ |\operatorname{Im} z| > \eta\}$ because $\tan \frac{\pi}{3} = \frac{\sqrt{3}}{2} < \eta = 2$. This implies that $D_1^\sharp$ and $D_1^\flat$ are contained in $\{z : \frac{\pi}{6} < \pm \arg(z - cv) < \frac{2\pi}{3}\}$. Finally it remains to show $D_1 \subset \mathbb{D}(cv, R_1)$. Since the derivative of Φ_{attr}^{-1} is bounded by $\frac{1}{0.055}$ by (a) and $\{z : 1 < \operatorname{Re} z < 2,\ -\eta < \operatorname{Im} z < \eta\} \subset \mathbb{D}(1, \sqrt{1 + \eta^2})$, we have $D_1 \subset \mathbb{D}(cv, \sqrt{1 + \eta^2}/0.055)$. We only need to check that $\sqrt{1 + \eta^2}/0.055 < R_1 = 239$. In fact, this inequality is true even for a much bigger η such as $\eta = 13.0$ because

$$\sqrt{1 + 13.0^2}/0.055\ (\doteqdot 237.06\ldots) \underset{*}{<} 239. \tag{5.44*}$$

□

5.L Locating domains D_0, D'_0, D_{-1} and D''_{-1}

Lemma 5.32. (a) *Let* $\widetilde{W}_0 := \{\zeta : \operatorname{Re}\zeta > cp$ *or* $pr_+(\zeta) > \frac{2cp}{\sqrt{3}}$ *or* $pr_-(\zeta) > \frac{2cp}{\sqrt{3}}\}$. *Then* $\mathbb{V}(cv, \frac{2\pi}{3}) \subset Q(\widetilde{W}_0) \subset \mathbb{C} \smallsetminus (-\infty, cv]$ *and* $\widetilde{W}_0 \subset \mathcal{U}_1$.
(b) $\varphi(\widetilde{W}_0) \subset W_0 := \{z : \operatorname{Re} z > 7.6$ *or* $pr_+(z) > 9.1$ *or* $pr_-(z) > 9.1\}$.
(c) $Q^{-1}(W_0) \smallsetminus \overline{\mathbb{D}} \subset \widetilde{W}_{-1} := \mathbb{V}(0, \frac{2\pi}{3}) \smallsetminus (\overline{\mathbb{D}} \cup \{\zeta : \operatorname{Re}\zeta \le 0$ *and* $|\zeta| \le 7\})$.

We postpone the proof until later in this subsection.

Definition/Construction. Note that Q maps both $\mathcal{U}_1$ and $\mathcal{U}_2$ homeomorphically onto $\mathbb{C} \smallsetminus (-\infty, cv]$. Define

$$\widetilde{\mathcal{H}}_0 = (Q|_{\mathcal{U}_1})^{-1}(\mathcal{H}_1),\ \widetilde{D}_0 = (Q|_{\mathcal{U}_1})^{-1}(D_1),\ \widetilde{D}_0^\sharp = (Q|_{\mathcal{U}_1})^{-1}(D_1^\sharp),\ \widetilde{D}'_0 = (Q|_{\mathcal{U}_2})^{-1}(D_1).$$

These domains are contained in $\mathbb{C} \smallsetminus E_{r_1}$, because of Lemma 5.17 (d) and $\mathcal{H}_0 \cup D_1^\sharp \subset \overline{\mathbb{V}}(cv, \frac{2\pi}{3}) \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}(0, \rho)$, $D_1 \subset \overline{\mathbb{D}}(0, R) \smallsetminus \overline{\mathbb{D}}(0, \rho)$. Hence we can define

$$\mathcal{H}_0 = \varphi(\widetilde{\mathcal{H}}_0),\ D_0 = \varphi(\widetilde{D}_0),\ D_0^\sharp = \varphi(\widetilde{D}_0^\sharp),\ D'_0 = \varphi(\widetilde{D}'_0).$$

It is easy to see that $F(\mathcal{H}_0) = \mathcal{H}_1$ and Φ_{attr} naturally extends to $\overline{\mathcal{H}}_0$ so that it is a homeomorphism onto $\{z : \operatorname{Re} z \ge 0\}$. Moreover $\Phi_{attr}(D_0) = \{z : 0 < \operatorname{Re} z < 1, |\operatorname{Im} z| < \eta\}$ and $D_0 \subset \mathcal{H}_0 \smallsetminus \overline{\mathcal{H}_1}$, in particular D_0 does not intersect $\overline{\mathbb{V}}(cv, \frac{\pi}{3}) (\subset \overline{\mathcal{H}_1})$. By Lemma 5.32 (a), (b), D_0 must be contained in W_0, since $D_1 \subset \mathbb{V}(cv, \frac{2\pi}{3})$. So D_0 must be contained in $\mathbb{C} \smallsetminus (-\infty, 0] \cup [cv, +\infty)$.

Since Q maps $(\mathcal{U}_{1+} \cup \mathcal{U}_{2-} \cup \gamma_{b1})$ and $(\mathcal{U}_{1-} \cup \mathcal{U}_{2+} \cup \gamma_{b2})$ homeomorphically onto $\mathbb{C} \smallsetminus (-\infty, 0] \cup (cv, +\infty)$, we can define

$$\widetilde{D}_{-1} = \left(Q|_{(\mathcal{U}_{1+} \cup \mathcal{U}_{2-} \cup \gamma_{b1})}\right)^{-1}(D_0) \ \text{ and } \ \widetilde{D}''_{-1} = \left(Q|_{(\mathcal{U}_{1-} \cup \mathcal{U}_{2+} \cup \gamma_{b2})}\right)^{-1}(D_0).$$

These domains are contained in $\mathbb{C} \smallsetminus E_{r_1}$ by the lemma below. So finally define

$$D_{-1} = \varphi(\widetilde{D}_{-1}) \ \text{ and } \ D''_{-1} = \varphi(\widetilde{D}''_{-1}).$$

It is clear from the construction that F maps D_0, D'_0, D_{-1} and D''_{-1} homeomorphically on to D_1, D_1, D_0 and D_0 respectively. Recall that $R = 266$, $R_1 = 239$.

Lemma 5.33. (a) $\widetilde{D}_0 \subset \widetilde{W}_0 \cap \mathbb{D}(17, R_1 + 1)$; $D_0 \subset W_0 \cap \mathbb{D}(17, R_1 + 4)$.
(b) $\widetilde{D}_0 \cup \widetilde{D}'_0 \cup \widetilde{D}_{-1} \cup \widetilde{D}''_{-1} \subset \widetilde{W}_{-1} \cap \mathbb{D}(0, R_1 + 18) \cap \mathcal{U}_{12} \cap (\mathbb{C} \smallsetminus E_{r_1})$.
(c) $D_0 \cup D'_0 \cup D_{-1} \cup D''_{-1} \subset \mathbb{D}(0, R_1 + 21)$.

Proof. (a) If $|\zeta| \ge 100$, then

$$|Q(\zeta) - (\zeta + 10)| \le \frac{49}{100} + Q_{2,max}(100) < \frac{49}{100} + \frac{160}{99^2} + \frac{80 \times 100 + 48 + 1}{99^4} < 1.$$

Hence if $|\zeta - 17| \ge R_1 + 1$, then $|\zeta| > 100$ and we have $|Q(\zeta) - 27| = |(Q(\zeta) - (\zeta + 10)) + (\zeta - 17)| \ge |\zeta - 17| - |Q(\zeta) - (\zeta + 10)| > R_1$. So

$$\widetilde{D}_0 \cup \widetilde{D}'_0 \subset Q^{-1}(\mathbb{D}(cv, R_1)) \subset \mathbb{D}(17, R_1 + 1).$$

On the other hand, if $\zeta \in \mathbb{C} \setminus E$ and $|\zeta - 17| < R_1 + 1$, then its image $\varphi(\zeta)$ is surrounded by the Jordan curve $\varphi(\{\zeta' : |\zeta' - 17| = R_1 + 1\})$ which is contained in $\mathbb{D}(17, R_1 + 4)$ by (5.27*). Hence $D_0 \cup D_0' \subset \mathbb{D}(17, R_1 + 4)$. It follows from Lemma 5.32 (a), (b) that $\widetilde{D}_0 \subset \widetilde{W}_0$ and $D_0 \subset W_0$.

(b) Proceeding similarly, we have $\widetilde{D}_{-1} \cup \widetilde{D}''_{-1} \subset Q^{-1}(\mathbb{D}(17, R_1 + 4)) \subset \mathbb{D}(7, R_1 + 5)$, and $D_{-1} \cup D''_{-1} \subset \mathbb{D}(7, R_1 + 8)$. Let $\zeta \in \widetilde{D}_0 \cup \widetilde{D}'_0 \cup \widetilde{D}_{-1} \cup \widetilde{D}''_{-1}$. By the above, we have $\zeta \in \mathbb{D}(17, R_1 + 1) \cup \mathbb{D}(7, R_1 + 5) \subset \mathbb{D}(0, R_1 + 18)$. It is also contained in $Q^{-1}(D_0 \cup D_1) \subset Q^{-1}(W_0)$. By Lemma 5.32 (c), it is in $\widetilde{W}_{-1}$. The definition shows that $\zeta \in \mathcal{U}_{12}$. Since $D_0 \cup D_1 \subset W_0 \cap (\mathbb{D}(17, R_1 + 4) \cup \mathbb{D}(27, R_2)) \subset \overline{\mathbb{D}}(0, R) \setminus \overline{\mathbb{D}}(0, \rho)$, it follows from Lemma 5.17 (d) that $\zeta \in \mathbb{C} \setminus E_{r_1}$.

(c) It was already shown that he left hand side is contained in $\mathbb{D}(17, R_1+4) \cup \mathbb{D}(7, R_1+8)$ which is in $\mathbb{D}(0, R_1 + 21)$. $\square$

Proof of Proposition 5.7. The above construction and the previous lemma show that statements (a), (b), (c) and (d) of Proposition 5.7 hold. We now need to check $\overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}_{-1} \cup \overline{D}''_{-1} \setminus \{cv\} \subset \mathbb{D}(0, R) \setminus \left(\overline{\mathbb{D}}(0, \rho) \cup \mathbb{R}_- \cup \overline{\mathbb{V}}(cv, \frac{\pi}{6})\right) = \pi_X(X_{2+}) \cup \pi_X(X_{2-})$. Lemma 5.33 (c) shows that the left hand side is contained in $\mathbb{D}(0, 27)$.

Let $\zeta \in closure(\widetilde{D}_0 \cup \widetilde{D}'_0 \cup \widetilde{D}_{-1} \cup \widetilde{D}''_{-1})$. Lemma 5.33 (b) implies that $\zeta \in int E_{r_1}$. Hence by Lemma 5.23 $|\varphi(\zeta)| > \rho$. Furthermore, by Lemma 5.24, if $\operatorname{Re} \zeta \geq 0$, $\varphi(\zeta) \notin \mathbb{R}_-$. If $\operatorname{Re} \zeta \leq 0$, then $\zeta \in closure(\widetilde{W}_{-1})$ hence $|\operatorname{Im} \zeta| \geq 7 \sin \frac{2\pi}{3} > 3$. However, since $|\zeta| \geq 7$, we have $|\varphi(\zeta) - \zeta| < 3$ by (5.27*). Therefore $\varphi(\zeta) \notin \mathbb{R}_-$.

Finally let $z \in \overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}_{-1} \cup \overline{D}''_{-1}$. Then $F(z) \in \overline{\mathcal{H}}_0$ and $0 \leq \operatorname{Re} \Phi_{attr}(F(z)) \leq 2$. On the other hand, by Lemma 5.31 (b), for $z' \in \overline{\mathbb{V}}(cv, \frac{\pi}{6})$ with $z' \neq cv$, we have $\operatorname{Re} \Phi_{attr}(z') > 1$ hence $\operatorname{Re} \Phi_{attr}(F(z')) > 2$. So z cannot be in $\overline{\mathbb{V}}(cv, \frac{\pi}{6}) \setminus \{cv\}$. Altogether, we have proved (e) of Proposition 5.7. $\square$

The rest of this subsection is devoted to the proof of Lemma 5.32.

Proof of Lemma 5.32. (a) Note that the boundary $\partial\widetilde{W}_0$ consists of $\ell_0^\pm : \zeta = cp \pm it$ $(0 \leq t \leq \frac{cp}{\sqrt{3}})$ and $\ell_1^\pm : \zeta = \left(1 \pm \frac{i}{\sqrt{3}}\right) cp + s\, e^{\pm\frac{2\pi i}{3}}$ $(s \geq 0)$. We first show that $Q(\ell_0^\pm), Q(\ell_1^\pm) \subset \{z : \frac{2\pi}{3} < \pm \arg(z - cv) < \pi\} \cup \{cv\}$.

By an easy computation, we have

$$Q(\zeta) - cv = \frac{(\zeta^2 - \zeta + 1)(\zeta - 5 - 2\sqrt{6})^2(\zeta - 5 + 2\sqrt{6})^2}{\zeta(\zeta - 1)^4}. \tag{5.45}$$

Take $\zeta = cp + it$ $(0 < t \leq \frac{cp}{\sqrt{3}})$ on $\ell_0^+ \setminus \{cv\}$. We give bounds on

$$\arg(Q(\zeta) - cv) = \pi - \arg \zeta + \arg\left(1 + \frac{\zeta}{(\zeta - 1)^2}\right) + 2 \arg\left(1 + \frac{1 - cp'}{\zeta - 1}\right).$$

Note that $0 < \arg \zeta \leq \frac{\pi}{6}$. Since $\operatorname{Re} \frac{(\zeta-1)^2}{\zeta} = \operatorname{Re}\left(\zeta - 2 + \frac{1}{\zeta}\right) > cp - 2$ and $\operatorname{Im} \frac{(\zeta-1)^2}{\zeta} = \left(1 - \frac{1}{|\zeta|^2}\right) t > 0$, we have by Lemma 5.9,

$$0 < -\arg\left(1 + \frac{\zeta}{(\zeta-1)^2}\right) \leq \arcsin\left|\frac{\zeta}{(\zeta-1)^2}\right| \leq \arcsin\frac{1}{cp - 2} \leq \frac{\pi}{3} \cdot \frac{1}{cp-2} < \frac{\pi}{18}.$$

We also have $0 < -\arg\left(1 + \frac{1-cp'}{\zeta-1}\right) \leq \arcsin \frac{1}{cp-2} < \frac{\pi}{18}$. Hence it follows that

$$\pi > \arg(Q(\zeta) - cv) > \pi - \frac{\pi}{6} - \frac{\pi}{18} - \frac{2\pi}{18} = \frac{2\pi}{3}.$$

This implies $Q(\ell_0^+) \subset \{z : \frac{2\pi}{3} < \arg(z - cv) < \pi\} \cup \{cv\}$.

Next assume $\zeta \in \ell_1^+$, i.e., $\zeta = \left(1 + \frac{i}{\sqrt{3}}\right) cp + s\, e^{+\frac{2\pi i}{3}}$ $(s \geq 0)$. We now want to show that $pr_+(Q(\zeta)) < pr_+(cv)$. We write as in Lemma 5.19,

$$Q(\zeta) = \zeta + 10 + \frac{49}{\zeta} + \frac{160}{(\zeta - 1)^2} + Q_3(\zeta)$$

with $|Q_3(\zeta)| \leq Q_{3,max}(r) := \dfrac{80r + 32 + \frac{48}{r}}{(r-1)^4}$ for $|\zeta| \geq r > 1$.

It is easy to check that $pr_+(\zeta) = \frac{2cp}{\sqrt{3}}$, $pr_+(10) = \frac{10\sqrt{3}}{2}$, $\frac{\pi}{6} \leq \arg\zeta \leq \frac{2\pi}{3}$, hence $-\frac{5\pi}{6} \leq \arg\left(\frac{e^{-i\pi/6}}{\zeta}\right) \leq -\frac{\pi}{3}$, which implies $pr_+\left(\frac{1}{\zeta}\right) \leq \frac{\cos\frac{\pi}{3}}{|\zeta|} \leq \frac{1/2}{2cp/\sqrt{3}}$. Also $\frac{\pi}{6} \leq \arg(\zeta - 1) \leq \frac{2\pi}{3}$, hence $-\frac{3\pi}{2} \leq \arg\left(\frac{e^{-i\pi/6}}{(\zeta-1)^2}\right) \leq -\frac{\pi}{2}$, and $pr_+\left(\frac{1}{(\zeta-1)^2}\right) \leq 0$. Thus we have

$$\begin{aligned} pr_+(Q(\zeta)) &\leq \frac{2cp}{\sqrt{3}} + \frac{10\sqrt{3}}{2} + \frac{49/2}{2cp/\sqrt{3}} + 0 + Q_{3,max}\Big(\frac{2cp}{\sqrt{3}}\Big) \\ &(\doteqdot 22.3\ldots) \underset{*}{<} pr_+(cv)\ (\doteqdot 23.3\ldots). \end{aligned} \tag{5.46*}$$

Finally we want to show $\operatorname{Im} Q(\zeta) > 0$. Since ℓ_1^+ is a half line which intersect orthogonally $\{\zeta' : \arg\zeta' = \frac{\pi}{6}\}$ at distance $\frac{2cp}{\sqrt{3}}$ from the origin, its image by $\zeta \mapsto \frac{1}{\zeta}$ is on the circle that passes through 0 and intersects orthogonally $\{\zeta' : \arg\zeta' = -\frac{\pi}{6}\}$ at distance $\frac{\sqrt{3}}{2cp}$ from the origin. The imaginary part on this circle is at least $-\frac{3}{4}\cdot\frac{\sqrt{3}}{2cp}$. Hence we have $\operatorname{Im}\frac{1}{\zeta} \geq -\frac{3}{4}\cdot\frac{\sqrt{3}}{2cp}$ for $\zeta \in \ell_1^+$. Hence

$$\operatorname{Im} Q(\zeta) \geq \frac{cp}{\sqrt{3}} - 49 \cdot \frac{3}{4} \cdot \frac{\sqrt{3}}{2cp} - Q_{2,max}\Big(\frac{2cp}{\sqrt{3}}\Big)\ (\doteqdot 0.94\ldots) \underset{*}{>} 0. \tag{5.47*}$$

Thus we have proved that $Q(\ell_1^+) \subset \{z : \frac{2\pi}{3} < \arg(z - cv) < \pi\}$. Similar estimates hold for for $Q(\ell_0^-)$, $Q(\ell_1^-)$.

By an argument similar to the proof of Lemma 5.31 (b), it is easy to show that $\mathbb{V}(cv, \frac{2\pi}{3}) \subset Q(\widetilde{W}_0)$. Since $Q(\partial\widetilde{W}_0)$ does not intersect $\Gamma_b^Q = (0, cv]$ except at cv, $\partial\widetilde{W}_0$ does not intersect the Jordan curve $\gamma_{b1} \cup \gamma_{b2} \cup \{-1\}$ except at cp. Since $\partial\widetilde{W}_0$ is unbounded, it (except cp) must be contained in the unbounded component of $\mathbb{C} \smallsetminus \gamma_{b1} \cup \gamma_{b2} \cup \{-1\}$, which is $\mathcal{U}_1 \cup (-\infty, 0)$. The Jordan curve must be on left hand side of $\partial\widetilde{W}_0$ and $\widetilde{W}_0$ is on the right. So it follows that $\widetilde{W}_0$ must be contained in $\mathcal{U}_1$. Therefore $Q(\widetilde{W}_0) \subset \mathbb{C} \smallsetminus (-\infty, cv]$.

(b) Suppose $\zeta \in \widetilde{W}_0$. Hence $\operatorname{Re}\zeta \geq cp$ or $pr_\pm(\zeta) \geq \frac{2cp}{\sqrt{3}}$. If $\operatorname{Re}\zeta \geq cp$, then

$$\operatorname{Re}\varphi(\zeta) \geq cp + c_{00} - c_{01,max} - \varphi_{1,max}(cp)\ (\doteqdot 7.6401\ldots) \underset{*}{>} 7.6. \tag{5.48*}$$

If $pr_\pm(\zeta) \geq \frac{2cp}{\sqrt{3}}$, then

$$pr_\pm(\varphi(\zeta)) \geq \frac{2cp}{\sqrt{3}} + \frac{\sqrt{3}c_{00}}{2} - c_{01,max} - \varphi_{1,max}\Big(\frac{2cp}{\sqrt{3}}\Big)\ (\doteqdot 9.169\ldots) \underset{*}{>} 9.1. \tag{5.49*}$$

Therefore in either case, $\varphi(\zeta) \in W_0$.

(c) Note that $\mathbb{C} \smallsetminus (\widetilde{W}_{-1} \cup \overline{\mathbb{D}}) = \{\zeta : \operatorname{Re}\zeta \leq 0 \text{ and } 1 < |\zeta| \leq 7\} \cup \{\zeta : |\zeta| \geq 7 \text{ and } \frac{2\pi}{3} \leq \arg\zeta \leq \frac{4\pi}{3}\}$. We need to show that if ζ is in this set, then $Q(\zeta) \notin W_0$.

First suppose that $\operatorname{Re}\zeta \le 0$ and $1 < |\zeta| \le 7$. Then $|\zeta+1| \le |\zeta-1|$. Therefore

$$|Q(\zeta)| = \left|\left(\zeta - \frac{1}{\zeta}\right)\left(\frac{\zeta+1}{\zeta-1}\right)^5\right| \le |\zeta| + \frac{1}{|\zeta|}.$$

Hence $|Q(\zeta)| \le 7 + \frac{1}{7} < 7.6$, which implies $Q(\zeta) \in \mathbb{D}(0, 7.6) \subset \mathbb{C} \smallsetminus W_0$.

Next assume that $r = |\zeta| \ge 7$ and $\frac{2\pi}{3} \le \arg\zeta \le \pi$. Note that $|\zeta - 1| \ge |\zeta| = r \ge 7$, hence $Q_2(\zeta)$ has an estimate:

$$|\zeta\, Q_2(\zeta)| \le \frac{160}{7} + \frac{80 \times 7 + 32 + \frac{48}{7}}{7^3} < \frac{160}{7} + \frac{80 \times 7 + 32 + 143}{7^3} = 25.$$

Thus we have

$$\begin{aligned} \operatorname{Re}(Q(\zeta)) &\le r\cos\left(\frac{2\pi}{3}\right) + 10 + \frac{49\cos(\frac{2\pi}{3})}{r} + \operatorname{Re}\frac{\zeta\, Q_2(\zeta)}{\zeta} \\ &\le -\frac{r}{2} + 10 - \frac{49}{2r} + \frac{25}{r} \le -\frac{7}{2} + 10 + \frac{1}{14} < 7.6. \end{aligned} \tag{5.50}$$

As for $pr_+(Q(\zeta))$, we have $pr_+(\zeta) \le 0$ and $-\frac{7\pi}{6} \le \arg\left(\frac{e^{-i\pi/6}}{\zeta}\right) \le -\frac{5\pi}{6}$. Hence we have

$$\begin{aligned} pr_+(Q(\zeta)) &\le 0 + pr_+(10) + \frac{49\cos(\frac{5\pi}{6})}{r} + pr_+\left(\frac{\zeta\, Q_2(\zeta)}{\zeta}\right) \\ &\le \frac{10\sqrt{3}}{2} + \frac{1}{r}\left(-\frac{49\sqrt{3}}{2} + 25\right) < \frac{10\sqrt{3}}{2} < \frac{10 \times 1.8}{2} < 9.1. \end{aligned} \tag{5.51}$$

Now for $pr_-(Q(\zeta))$, we have $pr_-(\zeta) \le -7\cos(\frac{\pi}{6})$ and $-\frac{5\pi}{6} \le \arg\left(\frac{e^{i\pi/6}}{\zeta}\right) \le -\frac{\pi}{2}$, so $pr_-(\frac{1}{\zeta}) \le 0$. Hence

$$\begin{aligned} pr_-(Q(\zeta)) &\le -7\cos\left(\frac{\pi}{6}\right) + pr_-(10) + 0 + pr_-\left(\frac{\zeta\, Q_2(\zeta)}{\zeta}\right) \\ &\le -\frac{7\sqrt{3}}{2} + \frac{10\sqrt{3}}{2} + \frac{25}{7} = \frac{3\sqrt{3}}{2} + \frac{25}{7} < 3 + 4 < 9.1. \end{aligned} \tag{5.52}$$

These three inequalities imply that $Q(\zeta) \notin W_0$. The same conclusion holds when $\pi \le \arg\zeta \le \frac{4\pi}{3}$. This ends the proof of Lemma 5.32. $\square$

5.M Construction of Ψ_1 – Relating D_n's to P

Proof of Proposition 5.8. The sets D_0, D_0', D_{-1}, D_{-1}'', $D_0^\sharp$, and $D_1^\sharp$ are contained in $\pi_X(X_{1+} \cup X_{2-})$, so we regard them as subsets of $X_{1+} \cup X_{2-}$. (However sometimes we will abuse the notation to mean their projection.) Define for $n = 1, 2, \dots,$

$$D_{-n-1} := g^n(D_{-1});\ D_{-n}' := g^n(D_0');\ D_{-n-1}'' = g^n(D_{-1}'');\ D_{-n}^\sharp := g^n(D_0^\sharp).$$

Note here that our definition does not automatically guarantee that $g(D_0) = D_{-1}$ and $g(D_1^\sharp) = D_0^\sharp$ (see lemma below). (For example, if we lifted D_0 etc. to $\pi_X(X_{1-} \cup X_{2+})$, then we would get $g(D_0) = D_{-1}''$.) The Fatou coordinate Φ_{attr} extends naturally to $\widetilde{\Phi}_{attr}$ on these domains together with their closure. Let

$$\mathsf{D} = \{z : 0 < \operatorname{Re} z < 1 \text{ and } |\operatorname{Im} z| < \eta\} \text{ and } \mathsf{D}^\sharp = \{z : 0 < \operatorname{Re} z < 1 \text{ and } \eta < \operatorname{Im} z\}.$$

We name their boundary segments by

$$\partial^l_+ \mathsf{D} = 0 + i[0,\eta];\ \partial^l_- \mathsf{D} = 0 + i[0,-\eta];\ \partial^r_+ \mathsf{D} = 1 + i[0,\eta];\ \partial^r_- \mathsf{D} = 1 + i[0,-\eta];$$
$$\partial^h_+ \mathsf{D} = \partial^h \mathsf{D}^\sharp = i\eta + [0,1];\ \partial^h_- \mathsf{D} = -i\eta + [0,1];\ \partial^l \mathsf{D}^\sharp = 0 + i[\eta,+\infty];\ \partial^r \mathsf{D}^\sharp = 1 + i[\eta,+\infty].$$

Here l, r and h stand for left, right and horizontal. Since $\Phi_{attr}(z) - 1$ maps homeomorphically D_1 and $D_1^\sharp$ onto D and $\mathsf{D}^\sharp$ including the boundaries, we name the boundary segments of D_1 and $D_1^\sharp$ by $\partial^l_+ D_1$, $\partial^h D_1^\sharp$, etc according to their images by $\Phi_{attr}(z) - 1$. We will apply the same naming convention to domains (such as D_n, D'_n, D''_n, $D_n^\sharp$, $\widetilde{D}_0$ etc. with $n \le 0$) which are mapped homeomorphically onto D_1 and $D_1^\sharp$ by iterates of F or by Q.

Lemma 5.34. (a) $g(D_0) = D_{-1}$ *and* $g(D_1^\sharp) = D_0^\sharp$.
(b) *Among closed domains* $\{\overline{D}_n, \overline{D}'_n, \overline{D}''_{n-1}, \overline{D}^\sharp_n \,|\, n = 0, -1, -2, \dots\}$, *intersecting pairs are exactly as follows:*

$$\begin{cases} \overline{D}_n \cap \overline{D}_{n-1} = \partial^l_+ D_n = \partial^r_+ D_{n-1}, & \overline{D}_{n-1} \cap \overline{D}'_n = \partial^r_- D_{n-1} = \partial^l_- D'_n, \\ \overline{D}'_n \cap \overline{D}''_{n-1} = \partial^l_+ D'_n = \partial^r_+ D''_{n-1}, & \overline{D}''_{n-1} \cap \overline{D}_n = \partial^r_- D''_{n-1} = \partial^l_- D_n, \\ \overline{D}_n \cap \overline{D}'_n = \overline{D}_{n-1} \cap \overline{D}''_{n-1} = \textit{a point}, & \\ \overline{D}_n \cap \overline{D}^\sharp_n = \partial^h_+ D_n = \partial^h D^\sharp_n, & \overline{D}^\sharp_n \cap \overline{D}^\sharp_{n-1} = \partial^l D^\sharp_n = \partial^r D^\sharp_{n-1}, \\ \overline{D}_n \cap \overline{D}^\sharp_{n-1} = \overline{D}_{n-1} \cap \overline{D}^\sharp_n = \textit{a point}. & \end{cases} \tag{5.53}$$

Proof. First consider four domains D_0, D'_0, D_{-1}, D''_{-1}. They are defined through $\widetilde{D}_0$, $\widetilde{D}'_0$, $\widetilde{D}_{-1}$, $\widetilde{D}''_{-1}$, which are inverse images of D_1, D_0 by two-fold branched covering $Q : \mathcal{U}_{12} \to \mathbb{C} \smallsetminus (-\infty, 0]$, branched only over cv. Since D_1 and D_0 meet at cv along $\partial^l_+ D_1 = \partial^r_+ D_0$ and $\partial^l_- D_1 = \partial^r_- D_0$, one can check the three lines of (5.53) for $n = 0$ first for $\widetilde{D}_0$ etc., then for $D_0 = \varphi(\widetilde{D}_0)$ etc.

Let us show (a) now. Since g corresponds to the unique branch of F^{-1} taking value near ∞, near ∞ we have $g(\partial^r D_1^\sharp) = \partial^l D_1^\sharp$. By the construction, we also have $\partial^l D_1^\sharp = \partial^r D_0^\sharp$. This means that g maps the left side of $\partial^r D_1^\sharp$ to the left side $\partial^r D_0^\sharp$, therefore we conclude $g(D_1^\sharp) = D_0^\sharp$. So $g(\partial^r D_0^\sharp) = g(\partial^l D_1^\sharp) = \partial^l D_0^\sharp$. Note that D_0 and $D_0^\sharp$ are defined so that $\partial^r D_0^\sharp \cup \partial^r_+ D_0$ is a single arc joining ∞ to cv. Continuing the branch g along this curve up to cv, we obtain $g(\partial^r_+ D_0) = \partial^l_+ D_0$ which coincides with $\partial^r_+ D_{-1}$ by the above. Considering the left side of the curves, we conclude $g(D_0) = D_{-1}$.

A similar lifting argument can be used to conclude the last two line of (5.53) from the intersection of $\overline{D}_0^\sharp \cup \overline{D}_1^\sharp$ with $\overline{D}_0 \cup \overline{D}_1$. Since this intersection is lifted to $\partial^h_+ D_{-1} \cup \partial^h_+ D_0$, the other lift $\overline{D}'_0 \cup \overline{D}''_{-1}$ cannot intersect with $\overline{D}^\sharp_{-1} \cup \overline{D}^\sharp_0$. Thus we conclude that among the domains with indices 0 and -1, all intersections are listed in (5.53) with $n = 0$.

By applying g (and using (a)), we obtain the intersection relations (for $n > 0$) between two domains whose indices are the same or differ by one. If the indices differ by two or more, two domains cannot intersect, because they (or their projection) will be mapped to disjoint sets by iterates of F. $\square$

Now let

$$\mathcal{U} = \mathcal{U}^P_{1+} \cup \mathcal{U}^P_{1-} \cup \gamma^P_{c1}, \quad \mathcal{U}' = \mathcal{U}^P_{2-} \cup \mathcal{U}^P_{3+} \cup \gamma^P_{c3}, \quad \mathcal{U}'' = \mathcal{U}^P_{2+} \cup \mathcal{U}^P_{3-} \cup \gamma^P_{c2}.$$

Each domain is mapped homeomorphically by P onto $\mathbb{C} \smallsetminus (-\infty, 0]$. The map $\Psi_0(z) = cv_P e^{2\pi i z} = -\frac{4}{27} e^{2\pi i z}$ defined in Proposition 5.8 maps D onto $(\mathbb{C} \smallsetminus (-\infty, 0]) \cap \{z : e^{-2\pi\eta} < |z| < e^{2\pi\eta}\}$ and

$\mathsf{D}^\sharp$ onto $(\mathbb{C} \smallsetminus (-\infty, 0]) \cap \{z : 0 < |z| < e^{-2\pi\eta}\}$. Define Ψ_1 first in the interior of the domains D_n etc by

$$\Psi_1 = \begin{cases} (P|_{\mathcal{U}})^{-1} \circ \Psi_0 \circ \widetilde{\Phi}_{attr} & \text{on } D_n \cup D_n^\sharp \\ (P|_{\mathcal{U}'})^{-1} \circ \Psi_0 \circ \widetilde{\Phi}_{attr} & \text{on } D'_n \\ (P|_{\mathcal{U}''})^{-1} \circ \Psi_0 \circ \widetilde{\Phi}_{attr} & \text{on } D''_n. \end{cases}$$

Then Ψ_1 on each domain is a homeomorphism onto its image, and extends continuously to the closure. We need to know that, on a common boundary on two domains, the two extensions are consistent. Since Ψ_1 is defined as a branch of $P^{-1} \circ \Psi_0 \circ \widetilde{\Phi}_{attr}$, as soon as these extensions match, Ψ_1 will be holomorphic. (In fact, for the points corresponding to the critical value of P, use the removable singularity theorem.)

Let us check the matching conditions according to the intersection relation (5.53). If $z \in D_n$ tends to $\partial^l_+ D_n$, then $\Psi_0 \circ \widetilde{\Phi}_{attr}(z)$ tends to $[cv_P, 0) = \Gamma^P_a$ from lower side, hence $\Psi_1(z) \in \mathcal{U}$ tends to $[cp_P, 0) = \gamma^P_{a1}$ from lower side. If $z \in D_{n-1}$ tends to the same boundary curve $\partial^l_+ D_n = \partial^r_+ D_{n-1}$ from the other side, then $\Psi_0 \circ \widetilde{\Phi}_{attr}(z)$ tends to Γ^P_a from upper side, hence $\Psi_1(z) \in \mathcal{U}$ tends to γ^P_{a1} from upper side. Since P is homeomorphic in a neighborhood of γ^P_{ai}, Ψ_1 matches completely along $\overline{D}_n \cap \overline{D}_{n-1} = \partial^l_+ D_n = \partial^r_+ D_{n-1}$, and is holomorphic there. Similarly if $z \in D_{n-1}$ tends to $\partial^r_- D_{n-1}$, then hence $\Psi_1(z) \in \mathcal{U}$ tends to $\gamma^P_{b1} = \gamma^P_{b1+}$, while if $z \in D'_n$ tends to $\partial^l_- D'_n$, then $\Psi_1(z) \in \mathcal{U}'$ tends to $\gamma^P_{b2-} = \gamma^P_{b1+}$. Hence Ψ_1 matches along $\overline{D}_{n-1} \cap \overline{D}'_n = \partial^r_- D_{n-1} = \partial^l_- D'_n$. It is easy to check the matching for the rest of (5.53), for example, $\partial^l_+ D'_n = \partial^r_+ D''_{n-1}$ corresponds to $\gamma^P_{a2-} = \gamma^P_{a2+}$ and $\partial^r_- D''_{n-1} = \partial^l_- D_n$ to $\gamma^P_{b2+} = \gamma^P_{b1-}$. Thus we obtained Ψ_1 defined on $U =$ the interior of $\bigcup_{n=-\infty}^0 \left(\overline{D}_n \cup \overline{D}'_n \cup \overline{D}''_{n-1} \cup \overline{D}^\sharp_n\right)$. It is easy to to see that $P \circ \Psi_1 = \Psi_0 \circ \widetilde{\Phi}_{attr}$ and it is surjective onto $U^P_\eta = V'$. By the description of the images $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$ and matching relations, we can conclude (b) of Proposition 5.8. By (b), $\psi = \Psi_0 \circ \widetilde{\Phi}_{rep} \circ \Psi_1^{-1} : V' \smallsetminus \{0\} \to \mathbb{C}^*$ is well-defined and injective. The relation in (c)

$$P \circ \psi^{-1} = P \circ \Psi_1 \circ \widetilde{\Phi}^{-1}_{rep} \circ \Psi_0^{-1} = \Psi_0 \circ \widetilde{\Phi}_{attr} \circ \widetilde{\Phi}^{-1}_{rep} \circ \Psi_0^{-1} = \Psi_0 \circ E_F \circ \Psi_0^{-1};$$

is self-explanatory. Here $\widetilde{\Phi}_{attr}$, $\widetilde{\Phi}_{rep}$ are the lifted versions of Φ_{attr}, Φ_{rep} hence we have $\widetilde{\Phi}_{attr} \circ \widetilde{\Phi}^{-1}_{rep} = E_F$. From this and the normalization $E_F(z) = z + o(1)$ at $\operatorname{Im} z \to +i\infty$, we conclude that ψ extends holomorphically to $z = 0$ and $\psi(0) = 0$, $\psi'(0) = 1$.

It remains to show the holomorphic dependence (e). Recall the formal expression (5.1) at the beginning of §5.A, where $\Phi_{rep} \circ F^{-n} \circ {\Phi_{attr}}^{-1}$ should be understood as follows: first take inverse image of Φ_{attr} in the right half plane $\{\operatorname{Re} z > L\}$ where we know that it is injective, next take inverse orbits along an appropriate inverse branches of F^{-1}, finally apply Φ_{rep} in the left half plane $\{\operatorname{Re} z < -L\}$ where we know it is well-defined. The choice of the inverse branches was made precise in the above construction. This involves local branching only when it is related to the critical orbit of F, which corresponds to cp_P in the domain of definition of ψ. Given a holomorphic family φ_λ, the Fatou coordinates (on the right/left half planes) and local branches of F^{-n} can be constructed so that they depend holomorphically on λ (restricting to a smaller parameter region if necessary), except along the critical orbit. Hence the resulting $\psi_\lambda(z)$ depends holomorphically on λ, except at cp_P. But the exception can be removed by the removable singularity theorem and we have the holomorphic dependence for all of V'.

The proof of Proposition 5.8 is complete. □

5.N Remarks

(a) As we commented at the end of proof of Proposition 5.6, we can take η to be 13.0 there and the rest of proof works for this η. Therefore the resulting ψ in Main Theorem 1 (c) has univalent extension to $U^P_{13.0}$.

(b) Notice that in the proof, the horn map E_F was constructed by taking inverse orbits which only go through $\varphi(\mathcal{U}_{12} \smallsetminus E_{r_1})$. So even though the class $\mathcal{F}_1$ was defined using the cubic polynomial P, we only use the "degree 2 part" of the map. The remainder $\mathcal{U}_{3\pm}$ provides a valuable "space" for estimates on univalent functions.

(c) Among the constants that appeared in the proof, important ones are η, ρ, R and r_1. It was crucial to choose appropriate values for these constants. Here is a brief account on their relation. The choice of η affects the ellipse E via Lemma 5.16 and hence the class $\mathcal{F}_1$ itself. The ρ and R are related to r_1 via Lemma 5.17 and also via Lemma 5.23. If ρ and R are given, Lemma 5.17 (c) suggests that r_1 cannot be too large while Lemma 5.23 suggests that r_1 cannot be too small (cannot be too close to 1). In fact, Lemmas 5.23 (angle estimate) and 5.24 indicate that r_1 cannot be too small in any case. The η is related to $R_1 = R - 27$ by Proposition 5.6 (b) (see (5.44*)).

6 Proof of Main Theorem 2 – Teichmüller contraction

We now make a connection between our $\mathcal{F}_1$ and the Teichmüller space of a punctured disk. Refer to [A1], [GL], [IT], [Le], [Hu] for the theory of Teichmüller spaces.

6.A Teichmüller space of a punctured disk

Definition (Teichmüller space)**.** Let W_1 be a Jordan domain in $\widehat{\mathbb{C}}$. Fix a point $p \in W_1$ and define $W = W_1 \smallsetminus \{p\}$ (which is isomorphic to $\mathbb{D} \smallsetminus \{0\}$). We say that $\varphi : \overline{W} \to \widehat{\mathbb{C}}$ is a quasiconformal map if $\varphi : \overline{W} \to \varphi(\overline{W})(\subset \widehat{\mathbb{C}})$ is a homeomorphism and $\varphi : W \to \varphi(W)$ is quasiconformal in the usual sense. The *Teichmüller space* of W is

$$\mathit{Teich}(W) = \{\varphi : \overline{W} \to \widehat{\mathbb{C}} \text{ quasiconformal map}\}/\sim,$$

where $\varphi \sim \psi$ if and only if there exists a conformal map $h : \varphi(W) \to \psi(W)$ (automatically extending homeomorphically to the closure) which coincides with $\psi \circ \varphi^{-1}$ on the boundary. Do not forget that the boundary ∂W includes the puncture p.

This definition is equivalent to the standard definition of the Teichmüller space with marked boundary. The equivalence $\sim$ for the standard one involves an isotopy between h and $\psi \circ \varphi^{-1}$. But we do not need the isotopy condition, since two homeomorphisms between Jordan domains are isotopic relative to the boundary if they agree on the boundary, and the isotopy can be adjusted so that it does not move the puncture. The Teichmüller space can also be regarded as the quotient space of measurable Beltrami differentials $\boldsymbol{\mu} = \mu(z)\frac{d\bar{z}}{dz}$ with $\|\boldsymbol{\mu}\|_\infty < 1$, where $\|\boldsymbol{\mu}\|_\infty = \operatorname{ess\,sup}|\mu(z)|$ is L^∞-norm. Two definitions are related by $\varphi \longmapsto \boldsymbol{\mu}_\varphi := \left(\frac{\partial\varphi}{\partial\bar{z}} \Big/ \frac{\partial\varphi}{\partial z}\right)\frac{d\bar{z}}{dz}$.

The *Teichmüller distance* of $[\varphi], [\psi] \in \mathit{Teich}(W)$ is defined to be

$$d_{\mathit{Teich}}([\varphi],[\psi]) = \inf\left\{\log K \;\middle|\; \begin{array}{c}\text{there is a } K\text{-quasiconformal map } h : \varphi(W) \to \psi(W) \\ \text{which coincides with } \psi \circ \varphi^{-1} \text{ on the boundary}\end{array}\right\}.$$

It is known that this is a complete metric on $\mathit{Teich}(W)$.

We have another equivalent formulation of $\mathit{Teich}(W)$.

Lemma 6.1. *Let W be as above with the puncture at $p = \infty$ and assume that $V := \mathbb{C} \setminus \overline{W}$ contains 0 and ∂W is smooth and non-singular Jordan curve. Define*

$$\mathcal{S}^{qc}(V) := \left\{ \varphi : V \to \mathbb{C} \;\middle|\; \begin{array}{l} \text{univalent with } \varphi(0) = 0, \varphi'(0) = 1 \\ \text{and has a quasiconformal extension to } \mathbb{C} \end{array} \right\}.$$

Then there exists a bijection $\rho : \mathcal{S}^{qc}(V) \to Teich(W)$ defined by $\rho(\varphi) = [\hat{\varphi}|_W]$, where $\hat{\varphi} : \mathbb{C} \to \mathbb{C}$ is a quasiconformal extension of φ. If $\varphi_n, \varphi \in \mathcal{S}^{qc}(V)$ and $d_{Teich}(\rho(\varphi_n), \rho(\varphi)) \to 0 (n \to \infty)$, then $\{\varphi_n\}$ converges to φ uniformly on compact sets in V. A mapping $\tau(\lambda)$ from a complex manifold Λ to $Teich(W)$ is holomorphic if and only if there exists a holomorphic function $\boldsymbol{\varphi} : \Lambda \times V \to \mathbb{C}$ such that $\varphi_\lambda := \boldsymbol{\varphi}(\lambda, \cdot) \in \mathcal{S}^{qc}(V)$ and $\rho(\varphi_\lambda) = \tau(\lambda)$.

Proof. The map $\rho(\varphi) = [\hat{\varphi}|_W]$ is well-defined, since the ambiguity of the extension $\hat{\varphi}$ to W is absorbed by $\sim$ for $Teich(W)$. It is surjective; given any quasiconfomal map $\psi : W \to \widehat{\mathbb{C}}$, measurable Riemann mapping theorem yields a quasiconfomal map $\varphi : \mathbb{C} \to \mathbb{C}$ such that φ is conformal (univalent) in V and $\varphi \circ \psi^{-1}$ is also conformal on $\psi(W)$, then after a proper normalization, we have $\rho(\varphi) = [\varphi|_W] = [\psi]$. This also justifies the statement on the convergence.

To injectivity, let $\varphi, \varphi_1 \in \mathcal{S}^{qc}(V)$ and suppose $\rho(\varphi) = \rho(\varphi_1)$. This means that for extensions $\hat{\varphi}$, $\hat{\varphi}_1$, there exists a conformal map $h : \hat{\varphi}(W) \to \hat{\varphi}_1(W)$ with $h = \hat{\varphi}_1 \circ \hat{\varphi}^{-1}$ on $\partial\hat{\varphi}(W)$. Extend h to $\varphi(V)$ by $h = \varphi_1 \circ \varphi^{-1}$ which is conformal there. Then h is quasiconformal either by by Rickman's theorem ([Ri], see also Lemma 2 in Chap. 1 of [DH1]) or because two conformal maps are glued along quasicircle. Since h is conformal in $\varphi(V)$ and $\hat{\varphi}(W) = \mathbb{C} \setminus \overline{\varphi(V)}$ and $\hat{\varphi}(\partial V)$ has Lebesgue measure 0, h is conformal on all $\mathbb{C}$, therefore affine. By the normalization at 0, $h(z) \equiv z$, hence $\varphi_1 = \varphi$. Thus ρ is injective.

In order to discuss the complex structure on $Teich(W)$, we review the Bers embedding in this setting (see the above references). Fix a quasicoformal map $\psi_0 : \mathbb{C} \to \mathbb{C}$ such that $\psi_0(\mathbb{C} \setminus \overline{\mathbb{D}}) = W$, $\psi_0(\mathbb{D}) = V$, $\psi_0(0) = 0$, and ψ_0 is conformal in $\mathbb{D}$. For any $\varphi \in \mathcal{S}^{qc}(V)$, $\varphi \circ \psi_0$ can be lifted to $\tilde{\varphi} : \mathbb{C} \to \mathbb{C}$ such that $\mathrm{Exp}^\sharp \circ \tilde{\varphi} = \varphi \circ \psi_0 \circ \mathrm{Exp}^\sharp$. Let $S\tilde{\varphi}$ be the Schwarzian derivative of $\tilde{\varphi}$. Then it can be checked that the map $\varphi \mapsto S\tilde{\varphi}$ corresponds to

$$\mathcal{S}^{qc}(V) \xrightarrow{\rho} Teich(W) \xrightarrow{(\psi_0)^*} Teich(\mathbb{C} \setminus \overline{\mathbb{D}}) \xrightarrow{Bers} Q^\infty_{\mathbb{Z}}(\mathbb{H}),$$

where $(\psi_0)^*$ is the isomorphism induced by ψ_0, $Bers$ is Bers embedding of $Teich(\mathbb{C} \setminus \overline{\mathbb{D}})$ into the space $Q^\infty_{\mathbb{Z}}(\mathbb{H})$ of $\mathbb{Z}$-invariant holomorphic quadratic differentials $\boldsymbol{q} = q(z)dz^2$ with norm $\|\boldsymbol{q}\|_{Q^\infty} = \sup\{(\mathrm{Im}\, z)^2|q(z)| : z \in \mathbb{H}\} < \infty$. Here $\mathbb{Z}$-invariance is required because the deck transformations of $\mathrm{Exp}^\sharp : \mathbb{H} \to \mathbb{D}^*$ are the translations by $\mathbb{Z}$. The image of Bers embedding is a bounded open set in $Q^\infty_{\mathbb{Z}}(\mathbb{H})$, and this define the structure of complex Banach manifold for $Teich(\mathbb{C} \setminus \overline{\mathbb{D}})$ and $Teich(W)$. Any holomorphic function $\Lambda \ni \lambda \mapsto \tau(\lambda) \in Teich(W)$ is represented by holomorphic family of quadratic differentials $\boldsymbol{q}_\lambda = q_\lambda(z)dz^2$ which are holomorphic in (λ, z) with $\boldsymbol{q}_\lambda \in Q^\infty_{\mathbb{Z}}(\mathbb{H})$, and vice versa. (To see the converse, we need to check that $\frac{\partial \boldsymbol{q}_\lambda}{\partial \lambda} \in Q^\infty_{\mathbb{Z}}(\mathbb{H})$, when Λ is 1-dimensional. But this follows from Cauchy formula applied to λ-variable.) From this description and the construction $\varphi \mapsto S\tilde{\varphi}$, the last statement is obvious. (Remark that the Schwarzian derivative taken directly from $\varphi \in \mathcal{S}^{qc}(V)$ does not determine the position of puncture, therefore insufficient for the embedding.) □

6.B Proof of Main Theorem 2

Now we turn to our class $\mathcal{F}_1$ and prove Main Theorem 2.

Proof of Main Theorem 2 (modulo Theorem 6.3). Let V, V' be as in Main Theorem 1. Take a domain V'' so that $\overline{V} \subset V'' \subset \overline{V''} \subset V'$ and $\partial V''$ is a non-singular real-analytic Jordan curve. We denote $W := \mathbb{C} \setminus \overline{V}$ and $U := \mathbb{C} \setminus \overline{V''}$. They have a puncture at $p = \infty$.

If $f = P \circ \varphi^{-1} \in \mathcal{F}_1$, then by definition $\varphi \in \mathcal{S}^{qc}(V)$ and $\rho(\varphi)$ defines a point in *Teich*(W). The above lemma shows that this is one to one correspondence. Let $\mathcal{R}_0^{Teich}$ denote the induced map on *Teich*(W) from the parabolic renormalization $\mathcal{R}_0$. In fact, $\mathcal{R}_0$ induces a map $\widehat{\mathcal{R}}_0^{Teich}$: *Teich*$(W) \to$ *Teich*(U), defined by $\rho(\varphi) \mapsto \rho(\psi)$ where $\mathcal{R}_0(P \circ \varphi^{-1}) = P \circ \psi^{-1}$, and this map is holomorphic by Main Theorem (e) and the above lemma. Hence it satisfies

$$d_{Teich(U)}(\widehat{\mathcal{R}}_0^{Teich}(\tau_1), \widehat{\mathcal{R}}_0^{Teich}(\tau_2)) \le d_{Teich(W)}(\tau_1, \tau_2) \quad \text{for } \tau_1, \tau_2 \in Teich(W), \tag{6.1}$$

due to Royden-Gardiner Theorem.

Theorem 6.2 (Royden-Gardiner)**.** *Any holomorphic map between Teichmüller spaces does not expand the Teichmüller distance.*

Now we can write $\mathcal{R}_0^{Teich} = \Xi \circ \hat{\mathcal{R}}_0^{Teich}$, where Ξ : *Teich*$(U) \to$ *Teich*(W) is defined as follows: if $\psi \in \mathcal{S}^{qc}(V'')$ with quasiconformal extension $\hat{\psi}$ to $\mathbb{C}$, then $\Xi(\rho(\psi)) = \rho(\psi|_V)$, or equivalently $\Xi([\hat{\psi}|_U]) = [\hat{\psi}|_W]$. It follows from Theorem 6.3 below that Ξ is well-defined with relatively compact image and satisfies (6.2). The estimate in Main Theorem 2 follows immediately, by letting V'' tend to V'. □

6.C Extension map and contraction

Theorem 6.3 (Extension map)**.** *Let W_1 and U_1 be Jordan domains in $\widehat{\mathbb{C}}$ such that $\overline{U_1} \subset W_1$. Fix a point $p \in U_1$ and define $W = W_1 \setminus \{p\}$ and $U = U_1 \setminus \{p\}$. The inclusion $U \hookrightarrow W$ induces a canonical map*

$$\Xi : Teich(U) \to Teich(W)$$

so that $\Xi(\tau) = \tau'$ if and only if there is a quasiconformal map $\psi : W \to \widehat{\mathbb{C}}$ satisfying $[\psi] = \tau'$ in Teich(W), $[\psi|_U] = \tau$ in Teich(U) and $\frac{\partial \psi}{\partial \bar{z}} = 0$ a.e. in $W \setminus U$. The image of Ξ is relatively compact (hence bounded) with respect to $d_{Teich(W)}$. Moreover it is a uniform contraction with an explicit bound:

$$d_{Teich(W)}(\Xi(\tau_1), \Xi(\tau_2)) \le \lambda\, d_{Teich(U)}(\tau_1, \tau_2) \quad \textit{for } \tau_1, \tau_2 \in Teich(U), \tag{6.2}$$

where $\lambda = e^{-2\pi \bmod (W \setminus \overline{U})} < 1$.

As for the Teichmüller spaces without removing the puncture p (universal Teichmüller space), the same conclusion holds for the map Teich$(U_1) \to$ Teich(W_1) is a contraction with the factor $e^{-4\pi \bmod (W_1 \setminus \overline{U_1})}$.

Proof. In terms of definition of *Teich*(W)'s by Beltrami differentials, Ξ is defined to be the 0-extension map $[\boldsymbol{\mu}] \mapsto [\hat{\boldsymbol{\mu}}]$, where $\boldsymbol{\mu}$ is defined on U and $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}$ on U and $\hat{\boldsymbol{\mu}} = 0$ on $W \setminus U$. In terms of quasiconformal maps, it can be expressed as follows: Let $\varphi : U \to \widehat{\mathbb{C}}$ be a quasiconformal map, then take its Beltrami differential $\boldsymbol{\mu}_\varphi$. Then by measurable Riemann mapping theorem, there exists a quasiconformal map $\psi : W \to \widehat{\mathbb{C}}$ such that $\boldsymbol{\mu}_\psi = \boldsymbol{\mu}_\varphi$ a.e. on U and $\boldsymbol{\mu}_\psi = 0$ a.e. on $W \setminus U$. Then $\psi \circ \varphi^{-1}$ is conformal in $\varphi(W)$, hence $[\psi|_U] = [\varphi]$ in *Teich*(U). Define $\Xi([\varphi]) = [\psi] \in$ *Teich*(W).

Let us check that this is well-defined. This can follow from Lemma 6.1 if ∂U is smooth or quasicircle. But we prove without this assumption. Take another representative φ_1 of $[\varphi]$ in *Teich*(U), hence there exists a conformal map $h : \varphi(U) \to \varphi_1(U)$ which coincides with $\varphi_1 \circ \varphi^{-1}$ on the boundary. Let ψ_1 be the result of the above construction for φ_1. Now define the map $\hat{h} : \psi(W) \to \psi_1(W)$ by $\hat{h} = \psi_1 \circ \varphi_1^{-1} \circ h \circ \varphi \circ \psi^{-1}$ on $\psi(U)$ and $\hat{h} = \psi_1 \circ \psi^{-1}$ on $\psi(U) \setminus \psi(U)$. Since $\varphi_1^{-1} \circ h \circ \varphi = id$ on ∂U, $\hat{h}$ is continuous on $\partial\psi(U)$, then by Rickman's theorem (quoted above), it is a quasiconformal map. Moreover, $\hat{h}$ is conformal in $\psi(U)$ because $\psi_1 \circ \varphi_1^{-1}$, h,

$\varphi \circ \psi^{-1}$ are so in corresponding domains. We also have $\frac{\partial \hat{h}}{\partial \bar{z}} = 0$ a.e. in $\psi(W) \setminus \psi(U)$ because $\boldsymbol{\mu}_\psi = \boldsymbol{\mu}_{\psi_1} = 0$ a.e. on $W \setminus U$. Hence $\hat{h}$ is a conformal map coinciding with $\psi_1 \circ \psi^{-1}$ on the boundary. Therefore $\psi_1 \sim \psi$ and Ξ is well-defined.

Next we prove the relative compactness of the image of Ξ. We may suppose that $W = \mathbb{D}^*$. Let $[\psi_n]$ be a sequence in $\Xi(\mathit{Teich}(\mathbb{D}^*))$. The representative ψ_n can be chosen so that $\frac{\partial \psi}{\partial \bar{z}} = 0$ a.e. in $W \setminus U$ and that $\psi_n(1) = 1$, $\psi_n(\mathbb{D}^*) = \mathbb{D}^*$ (which correspond to Beltrami differentials symmetric with respect to $\partial\mathbb{D}$). Even in the case of the universal Teichmüller space $\mathit{Teich}(\mathbb{D})$, ψ_n can be adjusted so that $\psi_n(0) = 0$ by composing a Möbius transformation of $\mathbb{D}$. Lift ψ_n to $\tilde{\psi}_n$ by $\mathrm{Exp}^\sharp$ so that $\psi_n \circ \mathrm{Exp}^\sharp = \mathrm{Exp}^\sharp \circ \tilde{\psi}_n$ with normalization $\tilde{\psi}_n(0) = 0$, $\tilde{\psi}_n(1) = 1$. By Schwarz reflection principle, we obtain a conformal map $\tilde{\psi}_n$ defined on Ω, where Ω is the union of $\mathbb{R}$, ${\mathrm{Exp}^\sharp}^{-1}(\mathbb{D} \setminus \overline{U})$ and its reflection. Applying Koebe distortion theorem to $\tilde{\psi}_n$ with the above normalization, we obtain a subsequence $\{\tilde{\psi}_{n_k}\}$ which converges to a limit $\tilde{\psi}$ uniformly near $\mathbb{R}$. Ahlfors-Beurling Theorem gives a new quasiconformal extension $\tilde{\psi}'_{n_k} : \mathbb{C} \to \mathbb{C}$ such that $\tilde{\psi}'_{n_k} = \tilde{\psi}_{n_k}$ on $\mathbb{R}$ and $\tilde{\psi}'_{n_k}(z+1) = \tilde{\psi}'_{n_k}(z) + 1$. Moreover their Beltrami differentials $\boldsymbol{\mu}_{\tilde{\psi}'_{n_k}}$ are uniformly bounded from 1 and converge uniformly to $\boldsymbol{\mu}_{\tilde{\psi}}$. This implies that the maximal dilatation of $\tilde{\psi}'_{n_k} \circ \tilde{\psi}^{-1}$ tends to 0. Therefore $\{[\psi_{n_k}]\}$ converges to ψ induced from $\tilde{\psi}$. This proves the relative compactness of $\Xi(\mathit{Teich}(\mathbb{D}^*))$.

Before proving the contraction, let us recall the infinitesimal definition of Teichmüller metric (see, for example, [Hu] Chap. 6). For a point $\tau = [\psi] \in \mathit{Teich}(W)$, the tangent space $T_\tau \mathit{Teich}(W)$, its "pre-dual" space $T_\tau^{\circledast} \mathit{Teich}(W)$, the pairing $(\boldsymbol{q}, \boldsymbol{\mu})$ and Teichmüller norm $\|\cdot\|_{\mathit{Teich}}$ are defined by

$$
\begin{aligned}
T_\tau \mathit{Teich}(W) &= \left\{\boldsymbol{\mu} = \mu(z)\frac{d\bar{z}}{dz} \text{ measurable Beltrami differential on } \psi(W) \text{ with } \|\boldsymbol{\mu}\|_\infty < \infty\right\}/\sim \\
T_\tau^{\circledast} \mathit{Teich}(W) &= \left\{\boldsymbol{q} = q(z)dz^2 \text{ holomorphic quadratic differential on } \psi(W) \text{ with } \|\boldsymbol{q}\|_1 < \infty\right\} \\
(\boldsymbol{q}, \boldsymbol{\mu}) &= \iint_{\psi(W)} q(z)\mu(z)dx\,dy \quad \text{for } \boldsymbol{q} \in T_\tau^{\circledast} \mathit{Teich}(W) \text{ and } \boldsymbol{\mu} \in T_\tau \mathit{Teich}(W) \\
\|\boldsymbol{\mu}\|_{\mathit{Teich}} &= \sup\left\{|(\boldsymbol{q}, \boldsymbol{\mu})| : \boldsymbol{q} \in T_\tau^{\circledast} \mathit{Teich}(W) \text{ with } \|\boldsymbol{q}\|_1 = 1\right\},
\end{aligned}
$$

where $\|\boldsymbol{q}\|_1 = \iint |q(z)|dx\,dy$ is L^1-norm on the domain of definition and the equivalence relation for $T_\tau \mathit{Teich}(W)$ is defined as $\boldsymbol{\mu} \sim \boldsymbol{\nu}$ if and only if $\|\boldsymbol{\mu} - \boldsymbol{\nu}\|_{\mathit{Teich}} = 0$. A different representative of the class $\tau = [\psi]$ will give canonically isomorphic tangent space and its pre-dual space. The Teichmüller distance coincides with the distance (a Finsler metric) defined as the infimum of the length of paths joining the two points, where the length is defined by integrating Teichmüller norm $\|\cdot\|_{\mathit{Teich}}$ along the path. Note that the finiteness of $\|\boldsymbol{q}\|_1$ forces that $\boldsymbol{q}$ can have a simple pole at the puncture.

Now according to the description of Ξ in terms of Beltrami differentials, the derivative $D_\tau\Xi$ at $\tau = [\psi] \in T_\tau \mathit{Teich}(W)$ is the 0-extension operator $[\boldsymbol{\mu}] \mapsto [\hat{\boldsymbol{\mu}}]$, where $\boldsymbol{\mu}$ is defined on $\psi(U)$ and $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}$ on $\psi(U)$ and $\hat{\boldsymbol{\mu}} = 0$ on $\psi(W) \setminus \psi(U)$. Therefore its "pre-adjoint" $D_\tau^{\circledast}\Xi : T_\tau^{\circledast} \mathit{Teich}(W) \to T_\tau^{\circledast} \mathit{Teich}(U)$ is defined by the restriction operator $\boldsymbol{q} \mapsto \boldsymbol{q}|_{\psi(U)}$, and satisfies

$$
(\boldsymbol{q}, D_\tau\Xi(\boldsymbol{\mu})) = (D_{\tau'}^{\circledast}\Xi(\boldsymbol{q}), \boldsymbol{\mu}) \quad \text{for } \boldsymbol{q} \in T_{\tau'}^{\circledast} \mathit{Teich}(W) \text{ and } \boldsymbol{\mu} \in T_\tau \mathit{Teich}(U), \tag{6.3}
$$

where $\tau \in \mathit{Teich}(U)$ and $\tau' = \Xi(\tau) \in \mathit{Teich}(W)$. In view of the definition of Teichmüller norm, in order to prove the contraction inequality, it suffices to prove the following infinitesimal contraction inequality on the pre-adjoint $D_\tau^{\circledast}\Xi$

$$
\|D_\tau^{\circledast}\Xi(\boldsymbol{q})\|_1 = \|\boldsymbol{q}|_{\psi(U)}\|_1 \leq \lambda\, \|\boldsymbol{q}\|_1 \quad \text{for } \boldsymbol{q} \in T_\tau^{\circledast} \mathit{Teich}(W). \tag{6.4}
$$

This is exactly the content of Theorem 6.6 below.

Suppose that $\|\boldsymbol{q}\|_1 = 1$ in (6.3), then

$$|(\boldsymbol{q}, D_\tau\Xi(\boldsymbol{\mu}))| = |(D^{\circledast}_{\tau'}\Xi(\boldsymbol{q}), \boldsymbol{\mu})| = \lambda|(\tfrac{1}{\lambda}D^{\circledast}_{\tau'}\Xi(\boldsymbol{q}), \boldsymbol{\mu})| \le \lambda\|\boldsymbol{\mu}\|_{Teich}$$

because $\|\frac{1}{\lambda}D^{\circledast}_{\tau'}\Xi(\boldsymbol{q})\|_1 \le 1$. Taking supremum over $\boldsymbol{q} \in T^{\circledast}_{\tau'}Teich(W)$ with $\|\boldsymbol{q}\|_1 = 1$, we have

$$\|D_\tau\Xi(\boldsymbol{\mu})\|_{Teich} \le \lambda\|\boldsymbol{\mu}\|_{Teich}. \tag{6.5}$$

By integration along paths, we obtain the claimed inequality for the Teichmüller distance. □

In order to prove (6.4), we need a preparation:

Theorem 6.4 (Isoperimetric Inequality for quadratic differential with a simple pole)**.** *If D is a Jordan domain with real-analytic boundary and $q(z)$ is meromorphic in a neighborhood of $\overline{D}$ with at most one simple pole which is in D, then*

$$\left(\int_{\partial D}\sqrt{|q(z)|}\,|dz|\right)^2 \ge 2\pi\iint_D |q(z)|dx\,dy.$$

If $q(z)$ has no pole, then 2π can be replaced by 4π.

Proof. This is a modified version of Carleman's inequality (see [Ca]). It is enough to prove the inequality when D is the unit disk $\mathbb{D}$ and the pole is at 0. In fact, if $\psi : \mathbb{D} \to D$ is a conformal map (which extends conformally to a neighborhood of $\overline{\mathbb{D}}$), the inequality for $\psi^*q(z) = q(\psi(z))(\psi'(z))^2$ on $\mathbb{D}$ yields the inequality for $q(z)$. Now we need a lemma:

Lemma 6.5. *If $\varphi_1(z)$ and $\varphi_2(z)$ are holomorphic in the neighborhood of $\overline{\mathbb{D}}$, then for $s > -2$*

$$\iint_{\mathbb{D}} |\varphi_1(z)\varphi_2(z)|^2\,|z|^s dx\,dy \le \frac{1}{2\pi}\max\left\{\frac{1}{s+2}, \frac{1}{2}\right\}\left(\int_{\partial\mathbb{D}}|\varphi_1(z)|^2\,|dz|\right)\left(\int_{\partial\mathbb{D}}|\varphi_2(z)|^2\,|dz|\right).$$

Proof. Expand $\varphi_1(z)$, $\varphi_2(z)$ and $\varphi_1(z)\varphi_2(z)$ as

$$\varphi_1(z) = \sum_{\nu=0}^{\infty}a_\nu z^\nu,\ \varphi_2(z) = \sum_{\nu=0}^{\infty}b_\nu z^\nu,\ \varphi_1(z)\varphi_2(z) = \sum_{\nu=0}^{\infty}c_\nu z^\nu,\ \text{where}\ \ c_\nu = \sum_{\mu=0}^{\nu}a_\mu b_{\nu-\mu}.$$

Then

$$\int_{\partial\mathbb{D}}|\varphi_1(z)|^2\,|dz| = \int_0^{2\pi}\left(\sum_{\nu=0}^{\infty}a_\nu e^{i\nu\theta}\right)\left(\sum_{\nu=0}^{\infty}\bar{a}_\nu e^{-i\nu\theta}\right)d\theta = 2\pi\sum_{\nu=0}^{\infty}|a_\nu|^2.$$

A similar equality holds for $\varphi_2(z)$. We also have

$$\begin{aligned}\iint_{\mathbb{D}}|\varphi_1(z)\varphi_2(z)|^2\,|z|^s dx\,dy &= \int_0^1\int_0^{2\pi}\left(\sum_{\nu=0}^{\infty}c_\nu r^\nu e^{i\nu\theta}\right)\left(\sum_{\nu=0}^{\infty}\bar{c}_\nu r^\nu e^{-i\nu\theta}\right)r^{s+1}dr\,d\theta\\ &= 2\pi\sum_{\nu=0}^{\infty}\frac{|c_\nu|^2}{2\nu+s+2}.\end{aligned}$$

It can be checked (using $2|a_0b_\nu\bar{a}_1\bar{b}_{\nu-1}| \le |a_0b_\nu|^2 + |a_1b_{\nu-1}|^2$ etc) that

$$|c_\nu|^2 \le (\nu+1)\left(|a_0b_\nu|^2 + |a_1b_{\nu-1}|^2 + \cdots + |a_\nu b_0|^2\right).$$

Hence, using $\frac{\nu+1}{2\nu+s+2} \le \frac{1}{2}$ $(s \ge 0)$, $\frac{\nu+1}{2\nu+s+2} \le \frac{1}{s+2}$ $(-2 < s < 0)$, we have

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{|c_\nu|^2}{2\nu+s+2} &\le \max\left\{\frac{1}{s+2}, \frac{1}{2}\right\} \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} |a_\mu|^2 |b_{\nu-\mu}|^2 \\ &= \max\left\{\frac{1}{s+2}, \frac{1}{2}\right\} \left(\sum_{\nu=0}^{\infty} |a_\nu|^2\right)\left(\sum_{\nu=0}^{\infty} |b_\nu|^2\right).\end{aligned}$$

The desired inequality follows. □

Now we continue the proof of Theorem 6.4. Suppose $q(z)$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ except at $z = 0$, which is at most a simple pole. Let $\alpha_1, \ldots, \alpha_m$ be zeroes of $q(z)$ within $\overline{\mathbb{D}}$. By shifting the boundary a little bit, we may suppose that they are all in $\mathbb{D}$. Factoring out Blaschke factors for the zeroes, we can write

$$q(z) = z^s q_*(z) \prod_{\nu=1}^{m} \left(\frac{z-\alpha_\nu}{1-\bar{\alpha}_\nu z}\right),$$

where $s = -1$ or 0 depending on whether 0 is a pole or not, and $q_*(z)$ has no zeroes in $\overline{\mathbb{D}}$. Hence there exists a holomorphic function $\varphi(z)$ in a neighborhood of $\overline{\mathbb{D}}$ such that $q_*(z) = (\varphi(z))^4$. Since

$$|q(z)| \le |z|^s |\varphi(z)|^4 \ \text{ in } \mathbb{D} \ \text{ and } \ \sqrt{|q(z)|} = |\varphi(z)|^2 \ \text{ on } \partial\mathbb{D},$$

Lemma 6.5 with $\varphi_1 = \varphi_2 = \varphi$ yields the isoperimetric inequality. □

Now we can prove the following, which is equivalent to (6.4) and completes the proof of Theorem 6.3.

Theorem 6.6 (Modulus-Area Inequality for quadratic differential with a simple pole)**.** *Let A be an annulus in $\mathbb{C}$ with finite modulus* $\mathrm{mod}\, A$*, and K the bounded component of $\mathbb{C} - A$. If $q(z)$ is a meromorphic function in $A \cup K$ such that $q(z)$ has at most one simple pole, the pole (if any) is in K and $\iint_{A\cup K} |q(z)| dx\, dy < \infty$, then*

$$\iint_K |q(z)| dx\, dy \le e^{-2\pi \,\mathrm{mod}(A)} \iint_{A\cup K} |q(z)| dx\, dy.$$

If $q(z)$ has no pole, then 2π can be replaced by 4π.

Proof. This is a word-to-word translation of Modulus-Area Inequality (see Milnor [Mi] Appendix B, Corollary B.9, McMullen Inequality) with Euclidean metric replaced by the conformal metric $\sqrt{|q(z)|}\,|dz|$ induced from quadratic differential $q(z)dz^2$. We include the proof for reader's convenience.

By previous lemma, for any smooth Jordan curve γ which is not null-homotopic in A (hence surrounds K),

$$\left(\int_\gamma \sqrt{|q(z)|}\,|dz|\right)^2 \ge 2\pi \iint_K |q(z)| dx\, dy.$$

Since $\mathrm{mod}(A)$ can be defined as the inverse of extremal length of real-analytic Jordan curves that are not null-homotopic in A (see [A2]), by considering $\sqrt{|q(z)|}\,|dz|$ as a conformal metric on A, we have

$$\frac{1}{\mathrm{mod}(A)} \ge \frac{\left(\inf \int_\gamma \sqrt{|q(z)|}\,|dz|\right)^2}{\iint_A |q(z)| dx\, dy} \ge \frac{2\pi \iint_K |q(z)| dx\, dy}{\iint_A |q(z)| dx\, dy},$$

where the infimum is taken over all Jordan curve γ as above. Therefore

$$\iint_{A\cup K}|q(z)|dx\,dy \geq (1+2\pi\,\mathrm{mod}(A))\iint_K |q(z)|dx\,dy.$$

Now divide A into nested subannuli (n annuli with modulus $\frac{1}{n}\,\mathrm{mod}(A)$), and apply the above inequality repeatedly. We obtain, by letting $n\to\infty$,

$$\iint_{A\cup K}|q(z)|dx\,dy \geq \left(1+2\pi\tfrac{1}{n}\,\mathrm{mod}(A)\right)^n\iint_K |q(z)|dx\,dy \to e^{2\pi\,\mathrm{mod}(A)}\iint_K |q(z)|dx\,dy.$$

□

7 Proof of Main Theorem 3 and Corollaries

In order to prove Main Theorem 3, we first show the following:

Lemma 7.1. *Given any $f_0\in\mathcal{F}_1$, there exist a neighborhood $\mathcal{N}_{f_0}$ of f_0 and $\alpha_*(f_0)>0$ such that if $f\in\mathcal{N}_{f_0}$ satisfies $f(0)=0$, $f'(0)=e^{2\pi i\alpha}$ with $|\arg\alpha|\leq\frac{\pi}{4}$ and $0<|\alpha|<\alpha_*(f_0)$, then the horn map E_f for f is defined and $\Psi_0\circ E_f\circ\Psi_0^{-1}$ belongs to $\mathcal{F}_2^P$.*

Proof. This claim follows from the continuity of horn maps (Theorem 2.1) if we allow ourselves to take a slightly smaller V' than U_η^P. (Note here that the uniform convergence of E_{f_n} on $\{x+iy : 0\leq x\leq 1, y_0\leq y\leq y_1\}$ implies that of $\Psi_0\circ E_{f_n}\circ\Psi_0^{-1}$ on $\{z : e^{-2\pi y_1}\leq |z|\leq e^{-2\pi y_0}\}$ then they are also uniformly convergent on $\{z : |z|\leq e^{-2\pi y_0}\}$ by the maximum value principle.)

If we want to keep the same $V'=U_\eta^P$, it can be proved as follows. As in [Sh2], we can construct the "pre-Fatou coordinate" $z=\tau_f(w):=\frac{\sigma(f)}{1-e^{-2\pi i\alpha(f)w}}$ for f near f_0 with $|\arg\alpha(f)|\leq\frac{\pi}{4}$. Then $f(z)$ in z-plane lifts to $F_f(w)$ on $\mathbb{C}\smallsetminus\cup_{n\in\mathbb{Z}}\overline{\mathbb{D}}(\frac{n}{\alpha(f)},R_2)$ with some large $R_2>0$ and when f tends to f_0, F_f converges to $F_0=F_{f_0}=\tau_0^{-1}\circ f_0\circ\tau_0$ uniformly on $\{w : |\mathrm{Re}(\alpha(f)\,w)|\leq\frac{1}{2}$ and $|w|\geq R_2\}$, where $\tau_0(w)=-\frac{1}{w}$. Therefore the Fatou coordinates $\Phi_{+,f}$ and $\Phi_{-,f}$ exist in $\Omega_{+,f}=\{w : |\alpha(f)|R_2<\mathrm{Re}(\alpha(f)\,w)<\frac{1}{2}\}$ and $\Omega_{-,f}=\{w : -\frac{1}{2}<\mathrm{Re}(\alpha(f)\,w)<-|\alpha(f)|R_2\}$ and they converge to Φ_{attr,F_0} and Φ_{rep,F_0} respectively, when f tends to f_0 (taking larger R_2 if necessary).

Let $D_{n,0}$, $D'_{n,0}$, $D''_{n,0}$, $D^\sharp_{n,0}$ ($n=1,0,-1,\dots$) denote the domains for F_0 corresponding to D_n, D'_n, D''_n, $D^\sharp_n$ in §5. Define $D_{n,0}=F_0^{n-1}(D_{1,0})$ for $n=2,3,\dots$. If we take sufficiently large $\ell,m>0$, then $\overline{D}_{\ell,0}\subset\{w : |w|>R_2, |\arg w|<\frac{\pi}{4}\}$ and $\overline{D}_{-m,0},\overline{D}'_{-m,0},\overline{D}''_{-m,0},\overline{D}^\sharp_{-m,0}\subset\{w : |w|>R_2, \frac{3\pi}{4}<\arg w<\frac{5\pi}{4}\}$. If f is sufficiently close to f with $|\arg\alpha(f)|\leq\frac{\pi}{4}$, then these domains are also contained in $\Omega_{+,f}$ and $\Omega_{-,f}$. Note that $\Phi_{attr,F_0}\circ F_0^{m+\ell}$ maps $\overline{D}_{-m,0}$ homeomorphically onto $\overline{\mathsf{D}}_\ell=\{z : \ell\leq\mathrm{Re}\,z\leq\ell+1, |\mathrm{Im}\,z|\leq\eta\}$. Consider $\overline{\mathsf{D}}_\ell(r)=\overline{\mathsf{D}}_\ell\smallsetminus\mathbb{D}(\ell,r)\cup\mathbb{D}(\ell+1,r)$ for a small $r>0$ and define $\overline{D}_{-m,0}(r)=\overline{D}_{-m,0}\cap\left(\Phi_{attr,F_0}\circ F_0^{m+\ell}\right)^{-1}(\overline{\mathsf{D}}_\ell(r))$. Since $\Phi_{attr,F_0}\circ F_0^{m+\ell}$ is diffeomorphic on $\overline{D}_{-m,0}(r)$, there exists a neighborhood W of $\overline{D}_{-m,0}(r)$ such that if f is sufficiently close to f_0, then $\Phi_{+,f}\circ F_f^{m+\ell}$ is defined and diffeomorphic on W and the image contains $\overline{\mathsf{D}}_\ell(r)$ (by Rouché's theorem). This defines $\overline{D}_{-m,f}(r)=W\cap\left(\Phi_{+,f}\circ F_f^{m+\ell}\right)^{-1}(\overline{\mathsf{D}}_\ell(r))$. Similarly $\overline{D}'_{-m,f}(r)$ and $\overline{D}''_{-m,f}(r)$ are defined. Also for $\{w_{-m}\}=\overline{D}_{-m,0}\cap\overline{D}'_{-m,0}\cap\overline{D}_{-m-1,0}\cap\overline{D}''_{-m-1,0}$, there is a neighborhood W' such that $\Phi_{+,f}\circ F_f^{m+\ell}$ on W' covers $\overline{\mathbb{D}}(\ell,r)$ twice with branching over ℓ. Adding proper portion of W' to $\overline{D}_{n,f}(r)$ etc, we obtain $D_{n,f}$, $D'_{n,f}$, $D''_{n,f}$ for $n=-m,-m-1$, which are similar to D_n, D'_n, D''_n in §5.M.

The same argument works for $D^\sharp_{n,f}$ except that for the part corresponding to $\operatorname{Im}\Phi_{attr}(z) \geq R_3$ with large R_3, we already have a uniform control by the above convergence $F_f \to F_0$.

Thus we have obtained domains $D_{n,f}$, $D'_{n,f}$, $D''_{n,f}$, $D^\sharp_{n,f}$ with the same intersection relation as (5.53) for f close to f_0. This is enough to construct $\psi = \psi_f$ so that $\Psi_0 \circ E_f \circ \Psi_0^{-1} = P \circ \psi^{-1} \in \mathcal{F}_2^P$ as in §5.M. □

Proof of Main Theorem 3. By Koebe Distortion theorem (see the references cited in Appendix), the space of normalized univalent functions in V is sequentially compact with respect to the topology of uniform convergence on compact sets. Hence the above lemma implies that there must be a uniform $\alpha_* > 0$ such that if $h \in \mathcal{F}_1$, $|\arg\alpha| \leq \frac{\pi}{4}$ and $0 < |\alpha| < \alpha_*$, then $\mathcal{R}f$ is defined for $f(z) = e^{2\pi i\alpha}h(z)$ and $\mathcal{R}_\alpha h = \Psi_0 \circ E_f \circ \Psi_0^{-1} \in \mathcal{F}_2^P$. This proves the invariance part of Main theorem 3.

The statements on the holomorphic dependence and the contraction are proved exactly as in §5.M and in §6. □

Proof of Corollary 4.1. In order to clarify, let us denote by $\widetilde{\mathcal{R}}_0$ the parabolic renormalization acting on $\mathcal{F}_0$. Then for $f \in \mathcal{F}_0$, the only difference between $\mathcal{R}_0 f (\in \mathcal{F}_1)$ and $\widetilde{\mathcal{R}}_0 f (\in \mathcal{F}_0)$ is the domain of definition, and once the position of the critical value is fixed, they coincide in a neighborhood of 0.

For $\mathcal{R}_0$ acting on $\mathcal{F}_1$, the existence of the unique fixed point and the convergence are immediate from Main Theorem 2 and the completeness of the Teichmüller distance. For $f \in \mathcal{F}_0$, Theorem 3.2 guarantees that $\widetilde{\mathcal{R}}_0^n f$ are in $\mathcal{F}_0$ therefore can be represented as $\widetilde{\mathcal{R}}_0^n f = g_{Koebe} \circ \psi_n^{-1}$ with $\psi_n \in \mathcal{S}$. So by the compactness of $\mathcal{S}$, we can choose a subsequence $n_k \nearrow \infty$ such that $\{\psi_{n_k}\}$ converges uniformly on compact sets in $\mathbb{D}$. By the convergence to the fixed point in $\mathcal{F}_1$, we know that we always have the same limit function in a neighborhood of 0 for any convergent subsequence. Therefore the whole sequence $\{\psi_n\}$ must converge to a limit function. This implies that $\widetilde{\mathcal{R}}_0^n f$ converge to a fixed point and the fixed point must be in $\mathcal{F}_0$. □

Proof of Corollary 4.2. Let $f(z) = e^{2\pi i\alpha}h(z)$, where $h \in \mathcal{F}_1$ and $|\arg\alpha| \leq \frac{\pi}{4}$ and $|\alpha|$ small. Take the fundamental region $S_{attr,f}$ such that $\overline{S}_{attr,f} = \overline{D}_{1,f} \cup \overline{D}^\sharp_{1,f} \cup \overline{D}^\flat_{1,f}$ (corresponding to $1 \leq \operatorname{Re}\Phi_{attr,f}(z) \leq 2$). Consider $g(z) = \mathcal{R}_\alpha h(e^{-2\pi i\frac{1}{\alpha}}z)$, which is linear conjugate to $\mathcal{R}f(z) = e^{-2\pi i\frac{1}{\alpha}}\mathcal{R}_\alpha h(z)$. It can be shown as in [Sh1], [Sh2] that there exists $\alpha_{**} > 0$ and $C > 0$ such that if $|\alpha| < \alpha_{**}$, $z_1, z_2 \in \overline{S}_{attr,f}$, $w_i = \Psi_0(\Phi_{attr,f}(z_i))$ $(i = 1, 2)$, $w_1 \in Dom(g)$ and $g(w_1) = w_2$, then there exists an integer $m > 0$ such that $f^m(z_1) = z_2$ with $\operatorname{Re}\frac{1}{\alpha} - C \leq m \leq \operatorname{Re}\frac{1}{\alpha} + C$. So taking α_{**} small so that $\operatorname{Re}\frac{1}{\alpha_{**}} - C \geq 2$, this implies that if w_1 can be iterated n times under g, then the corresponding point z_1 (in $\overline{S}_{attr,f}$) can be iterated at least $2n$ times under f.

Let f be as in the assumption of Corollary, with $N \geq \frac{1}{\alpha_{**}} + 1$. Then the sequence $\{f_n\}$ as in (3.6) is defined so that $f_n \in (0, \alpha_{**}] * \mathcal{F}_1$. Since for each f_n, the critical value can be iterated once, by the above argument, the critical value can be iterated 2^n times for f. This works for any $n \in \mathbb{N}$, so we conclude that the critical value can be iterated infinitely many times.

The domain of definition of $\chi_f \circ E_f$ has two components, upper and lower ones. The construction of the return map $\mathcal{R}f$ in §3 uses only the upper component. Having infinite critical orbits throughout renormalization steps means that the critical orbit stays in the upper component. This implies that the critical orbit of f_n does not accumulate to the fixed point $\sigma(f_n)$, since the lower component for $\chi_{f_n} \circ E_{f_n}$ corresponds to a neighborhood of $\sigma(f_n)$. Therefore, for the original f, there exists a sequence of periodic orbits (corresponding to $\sigma(f_n)$), such that the critical orbit does not accumulate to any of these periodic orbits.

When $f(z) = e^{2\pi i\alpha}z + z^2$ is a quadratic polynomial, f itself is not in $\mathcal{F}_1$. But $\mathcal{R}_0(z+z^2) \in \mathcal{F}_2^P$, so for sufficiently small α we have $\mathcal{R}_\alpha(z + z^2) \in \mathcal{F}_1$, therefore we have the above sequence f_n

with $f_n \in \mathcal{F}_1$ for $n = 1, 2, \ldots$. The rest is similar and it follows that the critical orbit is not dense in the Julia set. $\square$

A Univalent functions

In this appendix, we prepare some estimates on univalent functions. Refer to [Po], [Du] for the theory of univalent functions.

Definition. A complex valued function is called *univalent* if it is holomorphic and injective. Important classes of univalent functions are:

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \,|\, f \text{ is univalent and } f(0) = 0, f'(0) = 1\},$$
$$\Sigma = \{g : \mathbb{C} \smallsetminus \overline{\mathbb{D}} \to \mathbb{C} \,|\, g \text{ is univalent and } \lim_{z \to \infty} \frac{g(z)}{z} = 1\}.$$

For $g \in \Sigma$, we can consider that g is a holomorphic map from $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ to $\widehat{\mathbb{C}}$ with $g(\infty) = \infty$. It can be written as $g(z) = z + c_0 + g_1(z)$, where $c_0 \in \mathbb{C}$ and g_1 is holomophic in $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ with $\lim_{z \to \infty} g_1(z) = 0$. We define subclasses of Σ by

$$\Sigma_0 = \{g \in \Sigma \,|\, c_0 = \lim_{z \to \infty} (g(z) - z) = 0\}, \quad \Sigma_* = \{g \in \Sigma \,|\, 0 \notin \mathit{Image}(g)\}.$$

Theorem A.1. *For $f \in \mathcal{S}$, we have*

(a) $|f''(0)| \le 4$.

(b) $\left|\log\left(z\,\dfrac{f'(z)}{f(z)}\right)\right| \le \log \dfrac{1 + |z|}{1 - |z|}$ *for* $|z| < 1$.

(c) $\left|\log \dfrac{f(z)}{z} + \log(1 - |z|^2)\right| \le \log \dfrac{1 + |z|}{1 - |z|}$.

Here the branches of log *on the left hand side in (a) and in (b) are (well-defined and) taken so that they have value* 0 *at* $z = 0$

Proof. (a) This is well-known. See [Po] Chap. 1, Theorem 1.5. [Du] Theorem 2.2. For (b), see [Po] Corollary 3.5, page 66, or [Du], Corollary 3, page 126.

To prove (c), fix $f \in \mathcal{S}$ and $z_1 \in \mathbb{D}$. Define $A(z) = -\frac{z - z_1}{1 - \bar{z}_1 z}$ and $f_1(z) = c(f \circ A(z) - f(z_1))$, where c is determined so that $f_1'(0) = 1$. Then $f_1 \in \mathcal{S}$. Since $f_1'(z) = -cf'(A(z))\frac{1 - |z_1|^2}{(1 - \bar{z}_1 z)^2}$, we have

$$z_1 \frac{f_1'(z_1)}{f_1(z_1)} = -z_1 \frac{cf'(0)\frac{1 - |z_1|^2}{(1 - \bar{z}_1 z_1)^2}}{c(f(0) - f(z_1))} = \frac{z_1}{f(z_1)(1 - |z_1|^2)}.$$

The assertion follows from (b) applied to f_1 at z_1. See also [Du], Exercise 2, page 141. $\square$

Theorem A.2. *Let $g(z) = z + c_0 + g_1(z) \in \Sigma$. Then the following estimates hold:*

(a) $\{z \in \mathbb{C} : |z - c_0| > 2\} \subset \mathit{Image}(g)$. *In particular, if $g \in \Sigma_*$, then $|c_0| \le 2$.*

(b) $|g_1(z)| \le \sqrt{\log \dfrac{1}{1 - |z|^{-2}}}$.

(c) $\left|\log g'(z)\right| \le \log \dfrac{1}{1 - |z|^{-2}}$.

(d) *If $g \in \Sigma_*$, then*

$$\left|\log \frac{g(z)}{z} - \log\left(1 - \frac{1}{|z|^2}\right)\right| \le \log \frac{|z|+1}{|z|-1}.$$

In particular,

$$|z|\left(1 - \frac{1}{|z|}\right)^2 \le |g(z)| \le |z|\left(1 + \frac{1}{|z|}\right)^2 \quad \text{and} \quad \left|\arg \frac{g(z)}{z}\right| \le \log \frac{|z|+1}{|z|-1}.$$

Proof. (a) See [Po], Theorem 1.4, page 19. If $\omega \notin \mathit{Image}(g)$, then let $f(z) = \frac{1}{g(\frac{1}{z})-\omega}$. We have $f(z) = z - (c_0 - \omega)z^2 + O(z^3) \in \mathcal{S}$. It follows from Theorem A.1 that $|c_0 - \omega| \le 2$.
(b) If we write $g(z) = z + c_0 + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$, the coefficients satisfy the Area Inequality ([Po] Theorem 1.3, or [Du] Theorem 2.1,)

$$\sum_{n=1}^{\infty} n|c_n|^2 \le 1.$$

By Cauchy-Schwarz inequality and the expansion $-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ ($|x| < 1$)

$$|g_1(z)| \le \sum_{n=1}^{\infty} \frac{|c_n|}{|z|^n} = \sqrt{\sum_{n=1}^{\infty} n|c_n|^2}\sqrt{\sum_{n=1}^{\infty} \frac{1}{n|z|^{2n}}} \le \sqrt{\log \frac{1}{1-|z|^{-2}}}.$$

(c) This follows from Theorem A.3 below. Or see [Po], Chap. 3.2, (5), page 65, or [Du] Chap.4, Exercise 1, page 140.
(d) Let $f(z) = \frac{1}{g(\frac{1}{z})}$. Then it is easy to see that $f \in \mathcal{S}$. The first inequality follows from Theorem A.1. The rest follows from the first. (In fact, the one for $|g(z)|$ follows from a standard estimate for $|f(z)|$.) ☐

Theorem A.3 (A consequence of Golusin inequalities)**.** *Let Ω be a disk or a half plane in $\widehat{\mathbb{C}}$ (including the case of the complement of a closed disk). If $g : \Omega \to \widehat{\mathbb{C}}$ is a univalent holomorphic mapping, then for $z, \zeta \in \Omega$ with $z, \zeta, g(z), g(\zeta) \ne \infty$ and $z \ne \zeta$,*

$$\left|\log \frac{g'(z)g'(\zeta)(z-\zeta)^2}{(g(z)-g(\zeta))^2}\right| \le 2\log\cosh \frac{d_\Omega(z,\zeta)}{2}. \tag{A.1}$$

Remark. There exists a Möbius transformation which sends Ω to $\mathbb{D}$, z to 0 and ζ to $r \in [0,1)$. In this case, $s = d_\Omega(z,\zeta) = \log \frac{1+r}{1-r}$, therefore it is easy to check that

$$2\log\cosh \frac{d_\Omega(z,\zeta)}{2} = \log\left(\frac{e^s + 2 + e^{-s}}{4}\right) = \log \frac{1}{1-r^2}. \tag{A.2}$$

Proof. Notice that the both sides of the inequality is invariant under pre- and post-composition of Möbius transformations, provided that the domain of definition Ω is transformed accordingly. In fact, for the left hand side, one can express in terms of cross ratios:

$$\frac{g'(z)g'(\zeta)(z-\zeta)^2}{(g(z)-g(\zeta))^2} = \lim_{\substack{z'\to z\\ \zeta'\to\zeta}} \frac{(g(z')-g(z))(g(\zeta')-g(\zeta))}{(g(z)-g(\zeta))(g(z')-g(\zeta'))} \cdot \frac{(z-\zeta)(z'-\zeta')}{(z'-z)(\zeta'-\zeta)}. \tag{A.3}$$

Therefore this also has a meaning even when $z, \zeta, g(z)$ or $g(\zeta)$ is equal to ∞, as long as $z \ne \zeta$.

When $\Omega = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $g(\infty) = \infty$, $\lim_{z\to\infty} \frac{g(z)}{z} = 1$, $z, \zeta \neq \infty$ and $z \neq \zeta$, the inequality (A.1) is known as a consequence of Golusin inequalities (see [Po], Chap. 3.2, (6), page 65, or [Du] Chap.4, proof of Corollary 2, page 126), where the right hand side becomes (cf. (A.2))

$$2 \log \cosh \frac{d_\Omega(z,\zeta)}{2} = \log \frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)}. \tag{A.4}$$

By the Möbius invariance, it also holds in general cases. □

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