ON THE PARABOLIC BIFURCATION OF HOLOMORPHIC MAPS

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ABSTRACT. A holomorphic map having a parabolic fixed point changes its dynamics drastically under a small perturbation. In order to study such a bifurcation, a special coordinate transformation is introduced. An invariant "residue" for fixed points is defined.

§0. INTRODUCTION

The dynamics of holomorphic maps of one complex variable was studied extensively during 80's. For quadratic polynomials, Douady and Hubbard have obtained the combinatrial description of the Mandelbrot set, in terms of external rays and external angles[DH]. One of their main tools is the analysis of the parabolic bifurcation (the bifurcation of rationally indifferent periodic points) using "Ecalle cylinders". This Ecalle cylinder is also used to study the structure of the Mandelbrot set near $c = 1/4$ the cusp of the main cardioid, the non local connectivity of the cubic connectedness locus, etc, [L],[DD].

We are concerned with fixed points of holomorphic maps. We call a fixed point *parabolic* (or *rationally indifferent*) if the multiplier (=the derivative at the fixed point) is a root of unity. In other word, if the origin 0 is a parabolic fixed point of f, f has the form:

$$f(z) = \lambda z + a_2 z^2 + \ldots$$

where $\lambda = exp(2\pi ip/q)$, $p, q \in Z$, $q \geq 1$, $(p,q) = 1$. It is well known that under a small perturbation, a periodic cycle of period q can bifurcate from such a fixed point. Furthermore the behaviour of orbits depends sensitively on the perturbation, hence the global dynamics may change drastically. So our goal is to analyze the bifurcation of parabolic points in such a way that we can investigate the change of the global dynamics.

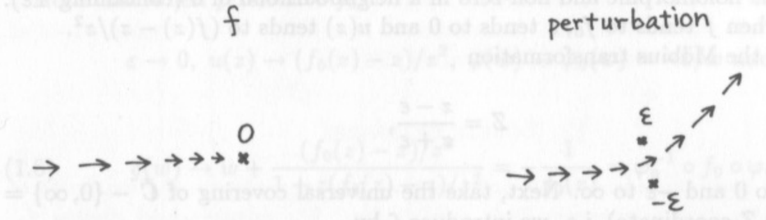

FIGURE 1. Dynamics of f and its perturbation

In the first part of this paper, we try to study the parabolic bifurcation using some special (and singular) coordinate transformation and reinterpret the method of Ecalle cylinders. In the second part, we introduce an invariant, the residue or the holomorphic index, for fixed points and study its properties specially for parabolic fixed points. In this paper we announce only the results, and the precise statements and the proofs should be published elsewhere. For simplicity, we treat only the case where the multiplier $\lambda = 1$.

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§1. COORDINATE CHANGE

Let f_0 be a holomorphic map defined in a neighbourhood of 0 and suppose that 0 is a parabolic fixed point of f_0 with multiplier 1, i.e.

$$f_0(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{1.1}$$

We need to put :

(1.2) **Non-degeneracy Assumption.** $a_2 \neq 0$.

In this case, by the linear scaling of the coordinate $z \to a_2 z$, we may assume that $a_2 = 1$. Now we perturb f_0 to f which is supposed to be defined in a neighbourhood of 0 and close to f_0 in the sense that $|f - f_0|$ is small in certain neighbourhood of 0. Since 0 is a fixed point of f_0 with multiplicity 2 (as a solution of $f_0(z) - z = 0$), f has two fixed points near 0 counted with multiplicity. Suppose f has two distinct fixed points near 0. By a coordinate change of the form $z \to z + c$ with c small, we may assume that these two fixed points are ε and $-\varepsilon$ (symmetric with respect to 0). Then we can write :

$$f(z) = z + (z^2 - \varepsilon^2)u(z) \tag{1.3}$$

where $u(z)$ is holomorphic and non-zero in a neighbourhood of 0 (containing $\pm\varepsilon$). Moreover when f tends to f_0 ε tends to 0 and $u(z)$ tends to $(f(z)-z)/z^2$.

First, by the Möbius transformation

$$Z = \frac{z-\varepsilon}{z+\varepsilon}, \tag{1.4}$$

we send ε to 0 and $-\varepsilon$ to ∞. Next, take the universal covering of $\bar{\mathbb{C}} - \{0,\infty\} = \mathbb{C} - \{0\}$ (in Z-coordinate), i.e. we introduce ζ by

$$Z = exp(2\pi i\zeta). \tag{1.5}$$

Let us denote $z_1 = f(z)$ and use Z_1, ζ_1 to denote corresponding coordinates, i.e.

$$exp(2\pi i\zeta_1) = Z_1 = \frac{z_1-\varepsilon}{z_1+\varepsilon}$$

where ζ_1 is determined up to an ambiguity by modulo $\mathbf{Z}$. In this ζ coordinate, f can be lifted to a map (depending on the choice of lift) as follows:

$$Z_1 = \frac{z_1-\varepsilon}{z_1+\varepsilon} = \frac{z-\varepsilon+(z^2-\varepsilon^2)u(z)}{z+\varepsilon+(z^2-\varepsilon^2)u(z)} = \frac{z-\varepsilon}{z+\varepsilon}\cdot\frac{1+(z+\varepsilon)u(z)}{1+(z-\varepsilon)u(z)} = Z\frac{1+\varepsilon u(z)/(1+zu(z))}{1-\varepsilon u(z)/(1+zu(z))}.$$

Hence

$$\zeta \to \zeta_1 = \frac{1}{2\pi i}\log(\frac{z_1-\varepsilon}{z_1+\varepsilon}) = \zeta + \frac{1}{2\pi i}\log\left(\frac{1+\varepsilon u(z)/(1+zu(z))}{1-\varepsilon u(z)/(1+zu(z))}\right)$$

where we choose a branch of logarithm so that $\log 1 = 0$, and we keep the original coordinate function z in the expression. Then

$$\zeta_1 = \zeta + \frac{1}{\pi i}\frac{\varepsilon u(z)}{1+zu(z)} + O(\varepsilon^2).$$

So we introduce a new coordinate by $w = \pi i\zeta/\varepsilon$, then the corresponding map $g(w)$ is:

$$w \to g(w) = w + \frac{1}{2\varepsilon}\log\left(\frac{1+\varepsilon u(z)/(1+zu(z))}{1-\varepsilon u(z)/(1+zu(z))}\right) = w + \frac{u(z)}{1+zu(z)} + O(\varepsilon). \tag{1.6}$$

The original coordinate z can be expressed under the form:

$$z = \varphi(w) = \varepsilon\frac{1+exp(2\varepsilon w)}{1-exp(2\varepsilon w)}. \tag{1.7}$$

Since we took the universal covering, w and $w + \frac{\pi i}{\varepsilon}$ should be identified.

Note that when f tends to f_0,

$$\varepsilon \to 0,\ u(z) \to (f_0(z)-z)/z^2,\ \varphi(w) \to \varphi_0(w) = -1/w \text{ and}$$

$$g(w) \to w + \frac{(f_0(z)-z)/z^2}{1+z(f_0(z)-z)/z^2} = -\frac{1}{f_0(z)} = \varphi_0^{-1}\circ f_0\circ\varphi_0(w). \tag{1.8}$$

For this convergence, we should restrict w to a suitable region, for example

$$\{w \in \mathbb{C}\mid |Im(\varepsilon w)| \le \frac{\pi}{2}\} - \bigcup_{n\in\mathbb{Z}}\{w\mid |w - n\frac{\pi i}{\varepsilon}| < R\}$$

for some constant $R > 0$. Hence one can consider that $g(w)$ is a perturbation of $g_0(w) = \varphi_0^{-1}\circ f_0\circ\varphi_0(w)$. The main difference is that for the perturbed system g, there is an additional identification:

$$w \sim w + \frac{\pi i}{\varepsilon}. \tag{1.9}$$

In fact, this identification gives the principal dependence on the perturbation (or on the parameter if f is parametrized). If we consider an orbit for f which comes into a certain neighbourhood of 0, it corresponds to an orbit for g which comes out of a set like $\{w\mid |w| < R\}$. Moreover if the f-orbit comes out of the neighbourhood, the g-orbit should arrive at another set $B_n = \{w\mid |w - n\frac{\pi i}{\varepsilon}| < R\}$, probably for n=1 or -1. Wheather this happens or not depends on the relative position of B_1 or B_{-1} with respect to B_0, hence on the argument of $\frac{\pi i}{\varepsilon}$. So if we pretend that ε is the parameter of the system, (1.9) shows a sensitive dependence on the parameter.

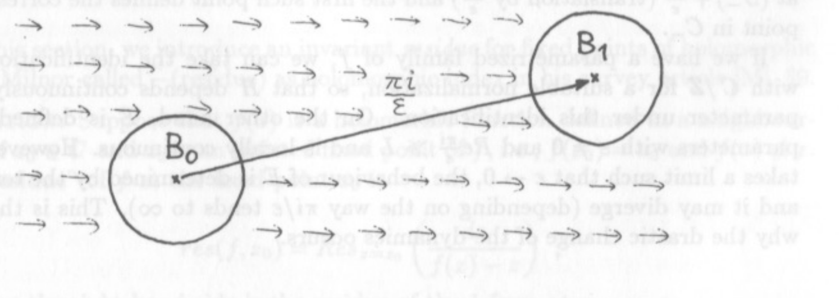

FIGURE 2. Dynamics of g

One can use this analysis, for example, to prove the landing property of external rays of rational angles with odd denominators (see [DH]). The original proof of this fact was given by the method of Ecalle cylinders.

§2. Ecalle cylinders

Using the coordinate in the previous section, we can re-prove the existence of Ecale cylinders and re-interpret their application. For unperturbed system f_0, the Ecalle cylinders $C_{0,+}$ and $C_{0,-}$ are defined as follows. In the coordinate $w = -1/z = \varphi_0^{-1}(z)$, $g_0(w) = \varphi_0^{-1} \circ f_0 \circ \varphi_0(w)$ has the expansion:

$$g_0(w) = w + 1 + O(\frac{1}{w}).$$

Let $\ell_+ = \{w |\ Re w = L\}$ and $\ell_- = \{w |\ Re\ w = -L\}$ for $L > 0$. If L is large enough, g_0 is injective and $g_0(w) > w + \frac{1}{2}$ in $\{w |\ |w| \geq L\}$. Then ℓ_+ (resp. ℓ_-) is mapped to its right hand side, and ℓ_+ and $g_0(\ell_+)$ (resp. ℓ_- and $g_0(\ell_-)$) bounds a simply connected region S_+ (resp. S_-). The *positive Ecalle cylinder* $C_{0,+}$ is

$$C_{0,+} = \overline{S}_+ / \sim$$

where $\overline{S}_+ = S_+ \cup \ell_+ \cup g_0(\ell_+)$ is a closed region in $\mathbb{C}$ and the equivalence relation $\sim$ is defined by $w \sim g_0(w)$ for $w \in \ell_+$. It is easy to see that $C_{0,+}$ has naturally the structure of a Riemann surface and is conformally isomorphic to the bi-infinite cylinder $\mathbb{C} - \{0\} \simeq \mathbb{C}/\mathbb{Z}$. The *negative Ecalle cylinder* $C_{0,-}$ is defined similarly from $\overline{S}_-$.

The dynamics of g_0 induces a mapping H_0 from a neighbourhood of the ends of the negative cylinder C_- to a neighbourhood of the ends of the positive cylinder C_+ as follows. If $w \in \overline{S}_-$ and $|Imw|$ is large enough, its orbit eventually arrives at $\overline{S}_+$. This correspondence defines the map.

For a perturbed system f, if the perturbation is small and if $Re \frac{\pi i}{\varepsilon} \gg L$, we can define similarly the positive and negative cylinders C_+, C_- and the mapping H near the ends of C_-. Since there is the identification (1.9), we can define an isomorphism E from C_+ to C_- as follows. If $w \in \overline{S}_+$, its orbit eventually arrives at $(\overline{S}_-) + \frac{\pi i}{\varepsilon}$ (translation by $\frac{\pi i}{\varepsilon}$) and the first such point defines the corresponding point in C_-.

If we have a parametrized family of f, we can take the identification of $C_\pm$ with $\mathbb{C}/\mathbb{Z}$ for a suitable normalization, so that H depends continuously on the parameter under this identification. On the other hand, E is defined for the parameters with $\varepsilon \neq 0$ and $Re \frac{\pi i}{\varepsilon} \gg L$ and is locally continuous. However, if one takes a limit such that $\varepsilon \to 0$, the behaviour of E is determined by the term $\pi i/\varepsilon$, and it may diverge (depending on the way $\pi i/\varepsilon$ tends to ∞). This is the reason why the drastic change of the dynamics occurs.

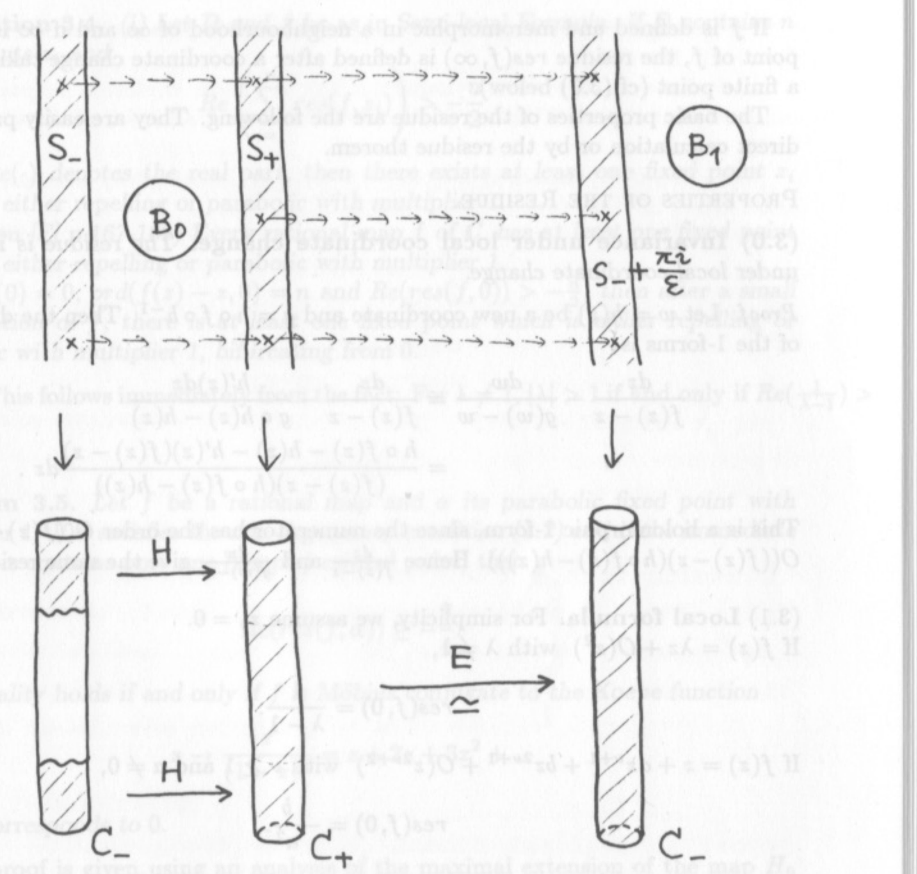

FIGURE 3. Ecalle cylinders and maps H and E

§3. Residue for fixed points

In this section, we introduce an invariant *residue* for fixed points of holomorphic maps. Milnor called −(residue) as holomorphic index in his survey article [M], §9.

DEFINITION. Suppose that $f(z)$ is a holomorhic function defined in a neighbourhood of $z_0 \in \mathbb{C}$ and z_0 is an isolated fixed point of f, i.e., $f(z_0) = z_0$ and $f(z) \not\equiv z$. The "residue" of f at the fixed point z_0 is

$$res(f, z_0) = Res_{z=z_0}\left(\frac{dz}{f(z) - z}\right),$$

where the right hand side is the residue of the 1-form at z_0.

If f is defined and meromorphic in a neighbourhood of ∞ and if ∞ is a fixed point of f, the residue $res(f, \infty)$ is defined after a coordinate change taking ∞ to a finite point (cf (3.0) below).

The basic properties of the residue are the following. They are easily proved by direct calculation or by the residue thorem.

PROPERTIES OF THE RESIDUE.

(3.0) Invariance under local coordinate change. *The residue is invariant under local coordinate change.*

Proof. Let $w = h(z)$ be a new coordinate and $g = h \circ f \circ h^{-1}$. Then the difference of the 1-forms is

$$\frac{dz}{f(z)-z} - \frac{dw}{g(w)-w} = \frac{dz}{f(z)-z} - \frac{h'(z)dz}{g \circ h(z) - h(z)}$$
$$= \frac{h \circ f(z) - h(z) - h'(z)(f(z)-z)}{(f(z)-z)(h \circ f(z) - h(z))} dz .$$

This is a holomorphic 1-form, since the numerator has the order $O((f(z)-z)^2) = O((f(z)-z)(h \circ f(z) - h(z)))$. Hence $\frac{dz}{f(z)-z}$ and $\frac{dw}{g(w)-w}$ give the same residue. □

(3.1) **Local formula.** For simplicity, we assume $z_0 = 0$.
If $f(z) = \lambda z + O(z^2)$ with $\lambda \neq 1$,

$$res(f, 0) = \frac{1}{\lambda - 1}.$$

If $f(z) = z + az^{\nu+1} + bz^{2\nu+1} + O(z^{2\nu+2})$ with $\nu \geq 1$ and $a \neq 0$,

$$res(f, 0) = \frac{b}{a^2}.$$

REMARK. In the above formulae, the multiplier λ and the ratio b/a^2 are well-known formal and analytic invariants. The residue unifies these two invariants.

(3.2) **Semi-local formula.** Let D be a bounded region in $\mathbb{C}$ bounded by smooth Jordan curves and $f : \overline{D} \to \overline{\mathbb{C}}$ a continuous map. Suppose f is holomorphic in D and has no fixed point on ∂D. Then

$$\sum_{\substack{z_i \in D \\ \text{fixed point of } f}} res(f, z_i) = \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{f(z) - z}.$$

(3.3) **Global formula.** (Fatou [F] p.167-169) For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$,

$$\sum_{z_i \ : \ \text{fixed point of } f} res(f, z_i) = -1 .$$

Proposition 3.4. *(i) Let D and f be as in Semi-local Formula. If D contains n fixed points z_i and*

$$Re\left(\sum_{i=1}^{n} res(f, z_i)\right) > -\frac{n}{2} ,$$

where $Re(\cdot)$ denotes the real part, then there exists at least one fixed point z_i which is either repelling or parabolic with multiplier 1.
(ii) (Fatou [F] p.167-169) Every rational map f of $\overline{\mathbb{C}}$ has at least one fixed point which is either repelling or parabolic with multiplier 1.
(iii) If $f(0) = 0$, $ord(f(z) - z, 0) = n$ and $Re(res(f, 0)) > -\frac{n}{2}$, then after a small perturbation of f, there is at least one fixed point which is either repelling or parabolic with multiplier 1, bifurcating from 0.

Proof. This follows immediately from the fact: For $\lambda \neq 1$, $|\lambda| > 1$ if and only if $Re(\frac{1}{\lambda - 1}) > -\frac{1}{2}$.

Theorem 3.5. *Let f be a rational map and α its parabolic fixed point with multiplier 1. If α satisfies the non-degeneracy condition (1.2) and if the immediate basin of α contains only one (simple) critical point, then*

$$Re(res(f, \alpha)) \geq -\frac{3}{4}.$$

The equality holds if and only if f is Möbius conjugate to the Koebe function

$$z \to \frac{z}{(1-z)^2} = z + 2z + 3z^2 + ...$$

and α corresponds to 0.

The proof is given using an analysis of the maximal extension of the map H_0 between Ecalle cylinders in §2.

Corollary 3.6. *Under the hypothesis of Theorem 3.5, after a small perturbation of f, there is at least one fixed point which is either repelling or parabolic with multiplier 1, bifurcating from α.*

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PERIODIC ORBITS NEAR THE BOUNDARY OF A 3-DIMENSIONAL MANIFOLD

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Abstract

In this work we describe the orbit structure of one parameter families of vector fields X_λ (for real parameter values near 0), defined on a 3-dimensional smooth manifold M, such that X_o has a periodic orbit tangent to ∂M, the boundary of M.

1. INTRODUCTION

In this paper we study smooth families of vector fields X_λ depending on a real ι-dimensional parameter λ in $\mathbb{R}^\iota$, tangent to a 3-dimensional smooth manifold M, with boundary ∂M. Our main interest is to describe the orbit structure of the family, for parameter values near λ = 0, on a small neighborhood of a periodic orbit γ_o of X_o, tangent to ∂M.

Without loss of generality, we assume that M is contained in a boundaryless compact manifold M′ and that the boundary is implicitly defined by a smooth real valued function f, as follows:

$df_p \neq 0$ for p in ∂M, M = { f ≥ 0} and ∂M= { f = 0}.

Also we assume that the families X_λ of vector

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PREFACE

The papers in this volume are contributed by the lecturers of the *International Conference on Dynamical Systems and Related Topics* held at the Nagoya International Center in Nagoya, Japan on September 3–7, 1990. Most of them were presented at the conference. The number of participants from Japan is about 100, and that from outside of Japan is 41.

The conference was held under the auspices of Nagoya University, Nagoya City, and the Mathematical Society of Japan. And it was supported financially by Nagoya University, Nagoya City, Daiko Foundation, Inamori Foundation, Ishida Foundation, Japan Association for Mathematical Sciences, Kato Ryutaro Gakujutu Shinko Kikin (Kato Ryutaro Foundation for the Promotion of Science), and some private contributors participating in the conference. Without their kind help, we would not have been able to hold the conference. We express our sincere gratitude for their cooperation.

We should also mention that our conference had been planned as a satellite conference of the International Congress of Mathematicians held at Kyoto on August 21–29, 1990 (ICM 90). Our conference was made possible by the support of the organizing committee of ICM, and we also express our appreciation for their help.

For the preparation of the conference, the Scientific Committee had been organized. Their efforts toward the success of the conference are much appreciated. The collaboration of staffs of the College of General Education (especially the Department of Mathematics) of Nagoya University was also valuable.

Finally we would like to express our gratitude to all the lecturers and participants of the conference without whom there would be no conference at all. We hope the proceedings of this conference would contribute to the development of dynamical systems and related topics in the world.

June 1991

Kenichi Shiraiwa
Editor and the
Chairman of the Organizing Committee

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Dynamical Systems and Related Topics