Lyapounov exponents and meromorphic maps

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Complex Dynamics and Related Topics

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Lyapounov exponents and meromorphic maps

(X, ω) compact Kähler manifold of dimension k.

$f : X \to X$ dominating meromorphic map

I_f = the indeterminacy set of f.

C_f = the critical set of f.

I) Dynamical quantities:

1°) Dynamical degrees:
(Russakovskii - Schiffman for $X = \mathbb{C}\mathbb{P}^k$)

$\omega^\ell = \underbrace{\omega \wedge \dots \wedge \omega}_{\ell \text{ times}}$ $\quad 0 \leq \ell \leq k = \dim X$

$f^*(\omega^\ell)$ form with L^1_{loc} coefficients

$\delta_\ell(f) = \int f^*(\omega^\ell) \wedge \omega^{k-\ell}$

$d_\ell := \lim_{n \to +\infty} (\delta_\ell(f^n))^{1/n}$ ← this limit exists (Dinh - Sibony)

ℓ^{th} dynamical degree

Th (Khovanskii – Teissier – Gromov)

$q \to \log d_q$ is concave.

It implies that the dynamical degrees look like:

$$d_0 = 1 \leq d_1 \leq \cdots \leq d_s \geq d_{s+1} \geq \cdots \geq d_k$$

Examples:

- f holomorphic endomorphism of $\mathbb{C}\mathbb{P}^k$ of degree $d \geq 2$

$$d_0 = 1 < d_1 = d < d_2 = d^2 < \cdots < d_k = d^k$$

- f birational map of $\mathbb{C}\mathbb{P}^2$ of degree $d \geq 2$
  f algebrically stable

$$d_0 = 1 < d_1 = d > d_2 = 1$$

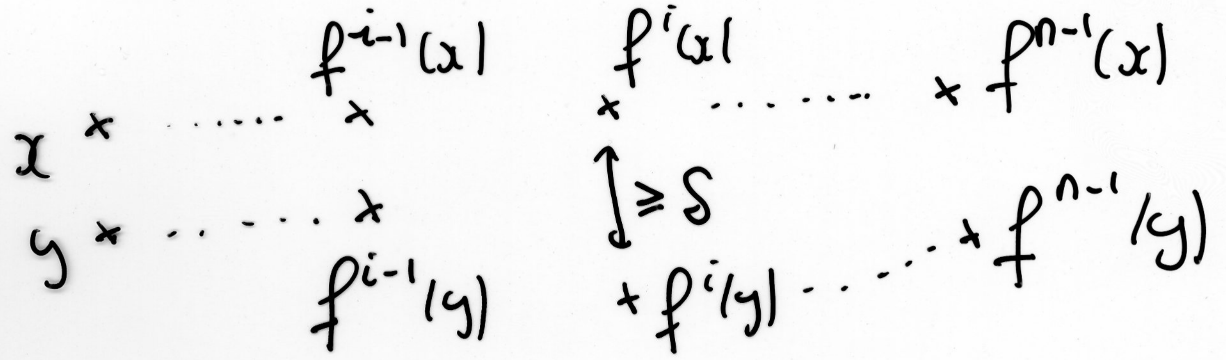
2°) Entropy:

a) Topological entropy:

$$d_n(x,y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$$

F is a (n,δ)-separated set if

$\forall x,y \in F \quad x \neq y \Rightarrow d_n(x,y) \geq \delta.$



$$h_{top}(f) := \sup_{\delta > 0} \overline{\lim} \frac{1}{n} \log \max \{ \mathrm{Card}\, F,\ F \ (n,\delta)\text{-separated set} \}.$$

h_{top} ← topological entropy

Th (Gromov: holomorphic case
Dinh-Sibony: meromorphic case)

$$h_{top}(f) \leq \max_{0 \le p \le k} \log d_p$$

d_p ← dynamical degrees

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b) Metric entropy:

μ measure $\quad \mu(I_f)=0$

$f_*\mu=\mu$ (μ is invariant)

$B_n(x,\delta)$ = ball with center x and radius δ for the metric d_n

$$h_\mu(f) := \sup_{\delta>0} \varliminf -\frac{1}{n} \log \mu B_n(x,\delta)$$

h_μ: metric entropy

x generic for μ

(Brin-Katok)

<u>Fact</u>: $h_\mu(f) \leq h_{top}(f)$.

II) Meromorphic maps:

$f : X \to X$ dominating meromorphic map

(X, ω) compact Kähler manifold dimension k

$$d_0 \leq d_1 \leq \cdots \leq d_{s-1} < d_s > d_{s+1} \geq \cdots \geq d_k$$

Example: holomorphic endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$

$d_p = d^p$

$s = k$ in this case.

We consider a measure μ invariant, ergodic

with $\log d(\cdot, \underbrace{I_f \cup C_f}_{A}) \in L^1(\mu)$

necessary condition for the existence of the Lyapounov exponents

$$+\infty > \chi_1 \geq \cdots \geq \chi_k > -\infty$$

Rk: $h_\mu(f) \leq h_{top}(f) \leq \log d_s$

Dinh-Sibony

Th (D.1)

If $h_\mu(f) > \max(\log d_{s-1}, \log d_{s+1})$

(or $h_\mu(f) > \cancel{\max} \log d_{k-1}$ if $s = k$)

Then:

$$\chi_1 \geq \dots \geq \chi_s \geq \frac{1}{2}\left(h_\mu(f) - \log d_{s-1}\right) > 0$$

$$0 > \frac{1}{2}\left(\log d_{s+1} - h_\mu(f)\right) \geq \chi_{s+1} \geq \dots \geq \chi_k$$

In particular $\underline{\mu \text{ is hyperbolic}}$.

$\chi_1 \geq \dots \geq \chi_s > 0 \rightsquigarrow s$ directions with expansion

$0 > \chi_{s+1} \geq \dots \geq \chi_k \rightsquigarrow k-s$ directions with contraction.

Cor:

If $h_\mu(f) = \log d_s$

$$\chi_1 \geq \dots \geq \chi_s \geq \frac{1}{2} \log \frac{d_s}{d_{s-1}} > 0$$

$$0 > \frac{1}{2} \log \frac{d_{s+1}}{d_s} \geq \chi_{s+1} \geq \dots \geq \chi_k$$

Rk: These bounds are sharp.

Examples of dynamical systems and measure μ which satisfy the hypothesis of the corollary:

- Holomorphic endomorphisms of $\mathbb{C}\mathbb{P}^k$ of degree $d \geq 2$

$$d_0 = 1 < d_1 = d < \dots < d_k = d^k$$

$\mu =$ Green measure (Fornaess - Sibony)

μ is invariant and ergodic, $\log d(x, C_f) \in L^1(\mu)$

$a =$ a generic point

$$\frac{1}{d^{kn}} \sum_{f^n(a_i) = a} \delta_{a_i} \to \mu$$

(Dinh-Sibony, Briend - Duval)

$d^k =$ topological degree of f

<u>Th</u>: Gromov / Misiurewicz - Przytycki

$$h_\mu(f) = k \log d = \log d^k = \log d_k .$$

we can apply the corollary

$$\Rightarrow \chi_1 \geq \dots \geq \chi_k \geq \frac{1}{2} \log \frac{d^k}{d^{k-1}} = \frac{1}{2} \log d$$

$\to$ we found the Briend-Duval's inequality.

- X projective [redacted] manifold with dimension k

$f: X \to X$ dominating meromorphic map

with $d_k > d_{k-1} \geq \dots \geq d_0 = 1$

Guedj constructed an invariant ergodic measure μ with $\log d(x, \mathcal{I}) \in L^1(\mu)$ and $h_\mu(f) = \log d_k$.

corollary $(s=k) \rightsquigarrow \chi_1 \geq \dots \geq \chi_k \geq \frac{1}{2} \log \frac{d_k}{d_{k-1}} > 0$

$\rightsquigarrow$ this is the Guedj's inequality.

- f regular birational map of $\mathbb{C}\mathbb{P}^k$ (Dinh - Sibony)

They constructed an invariant ergodic measure μ with $\log d(x, \mathcal{I}) \in L^1(\mu)$ and with maximal entropy

$\rightsquigarrow$ μ is hyperbolic.

cor.

- other examples: holomorphic automorphisms in Kähler manifolds (Dinh - Sibony) ....

III) A general inequality:

($d_0 = 1 \leq d_1 \leq \dots \leq d_s \geq \dots \geq d_k$)

Th (D.)

Let μ be an invariant ergodic measure

$\log d(x, \mathcal{A}) \in L^1(\mu)$.

$\chi_1 \geq \dots \geq \chi_k$ the Lyapounov exponents

Fix s $1 \leq s \leq k$ and we define

$\ell = \ell(s)$ with:
$\ell' = \ell'(s)$

$$\chi_1 \geq \dots \geq \chi_{s-\ell-1} > \chi_{s-\ell} = \dots = \chi_s = \dots = \chi_{s+\ell'} > \chi_{s+\ell'+1} \geq \dots \geq \chi_k$$

(with $s-\ell = 1$ if $\chi_1 = \dots = \chi_s$
$s+\ell' = k$ if $\chi_s = \dots = \chi_k$)

Then

$$h_\mu(f) \leq \max_{0 \leq q \leq s-\ell-1} \log d_q + 2\chi_{s-\ell}^+ + \dots + 2\chi_k^+$$

$$\chi_i^+ = \max(\chi_i, 0)$$

$$h_\mu(f) \leq \max_{s+\ell' \leq q \leq k} \log d_q - 2\chi_1^- - \dots - 2\chi_{s+\ell'}^- .$$

$$\chi_i^- = \min(\chi_i, 0).$$

Cor 1:

Let μ be an invariant ergodic measure

$\log d(x, \mathcal{R}) \in L^1(\mu)$

$$h_\mu(f) \leq 2\chi_1^+ + \dots + 2\chi_k^+$$

(Ruelle's inequality)

Proof: take $s = 1$ in the first inequality

$d_0 = 1$.

Cor 2:

Same hypothesis

$$h_\mu(f) \leq \log \underset{\substack{\uparrow \\ \text{topological degree}}}{d_k} - 2\chi_1^- - \dots - 2\chi_k^-.$$

"inverse Ruelle's inequality"

Proof: $s = k$ in the second inequality.

Proof of the previous theorem

$$d_0 = 1 \leq d_1 \leq \dots \leq d_{s-1} < d_s > d_{s+1} \geq \dots \geq d_k$$

measure μ invariant ergodic
with $\log d(x, \mathcal{R}) \in L^1(\mu)$
$h_\mu(f) > \max(\log d_{s-1}, \log d_{s+1})$
$\Rightarrow \mu$ is hyperbolic $\quad \chi_1 \geq \dots \geq \chi_s > 0$
$0 > \chi_{s+1} \geq \dots \geq \chi_k$

<u>Proof</u>:

. Suppose $\chi_s \leq 0$

1^{st} formula:

$$\chi_{s-l}^+ = \dots = \chi_s^+ = 0 = \dots = \chi_k^+$$

because $\quad 0 \geq \chi_{s-l} = \dots = \chi_s \geq \dots \geq \chi_k$

$$\log d_{s-1} < h_\mu(f) \leq \log d_{s-l-1} \leq \log d_{s-1}$$

$\rightarrow$ contradiction.

$\Rightarrow \chi_s > 0$

By using the second formula with $s = s+1$

$\rightsquigarrow \chi_{s+1} < 0$.

IV) Ideas for the proof of the inequalities:

an easier case:

$X = \mathbb{C}\mathbb{P}^2$

$\chi_1 \geq \chi_2$ we suppose $\chi_1 > \chi_2$ $s=2$

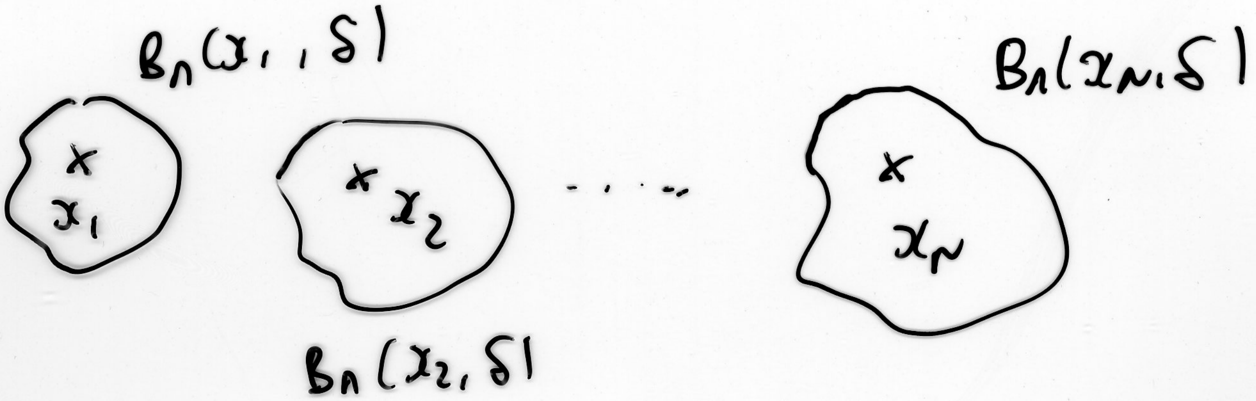
the first inequality becomes $(\ell=0)$

$$\boxed{h_\mu(f) \leq \log d_1 + 2\chi_2^+}$$

$$h_\mu(f) = \lim_{\delta\to 0} \underline{\lim} -\frac{1}{n} \log \mu\, B_n(x,\delta)$$

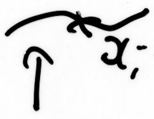
$\mu\, B_n(x,\delta) \simeq e^{-h(\mu)n}$ x generic for μ

⇝ we can find $x_1, \dots, x_N \in X$
with $N \simeq e^{h_\mu(f)n}$ $d_n(x_i, x_j) \geq \delta$ if $i \neq j$
and x_i = good points for Pesin's theory.



$\chi_1 > \chi_2$ so through each x_i we can construct an "approximate stable manifold"

$W^s(x_i)$



$W^s(x_i)$ dimension 1

diameter $f^\ell(W^s(x_i)) \le \frac{\delta}{4}$

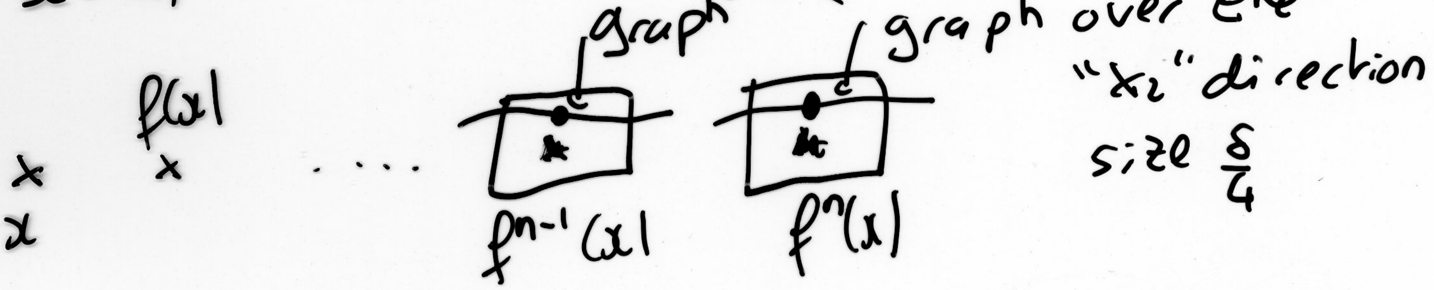
$\ell = 0, \dots, n-1$

area

$\simeq e^{-2\chi_2^+ n}$

to realize that, we use the graph trans-form:

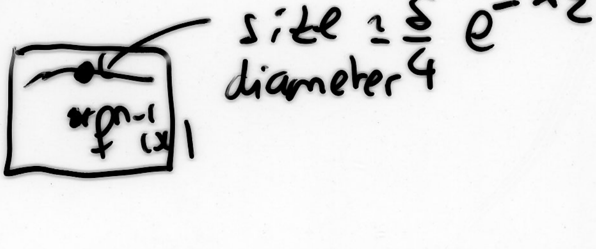
$x = x_i$



$\chi_1 > \chi_2$

if $\chi_2 \le 0 \to$ we do a cut-off
we keep the part in a box of size $\delta/4$

if $\chi_2 > 0 \to$ we keep all



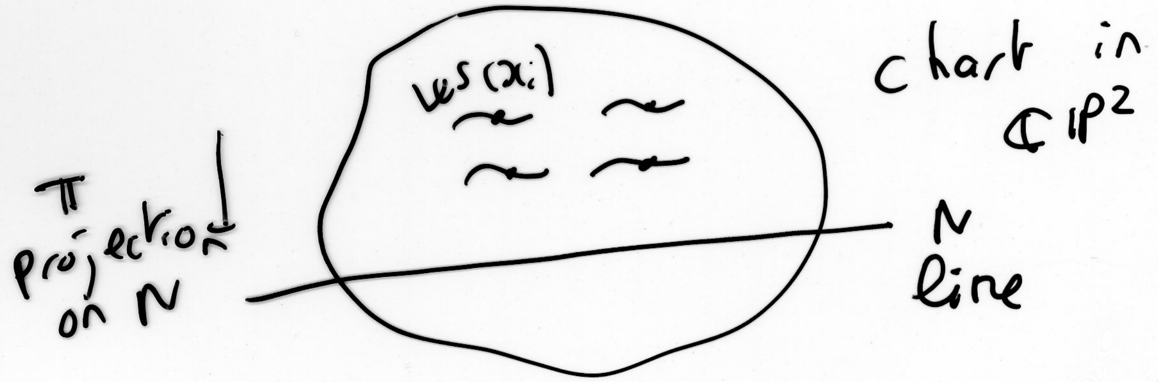
We start again this process

x, $W^s(x)$ — size diameter $\simeq \delta/4$ if $X_2 \le 0$

$\frac{\delta}{4} e^{-X_2 n}$ if $X_2 > 0$

area $\simeq e^{-2X_2^+ n} c(\delta)$ $\quad X_2^+ = \max(X_2, 0)$

So we have $\simeq e^{h_\mu(f) n}$ approximate stable manifold which have area $\simeq e^{-2X_2^+ n} c(\delta)$



a_n = The area of $\pi\left(\bigcup_{i=1}^{N} W^s(x_i)\right) \geq e^{h_\mu(f)n - 2X_2^+ n}$.

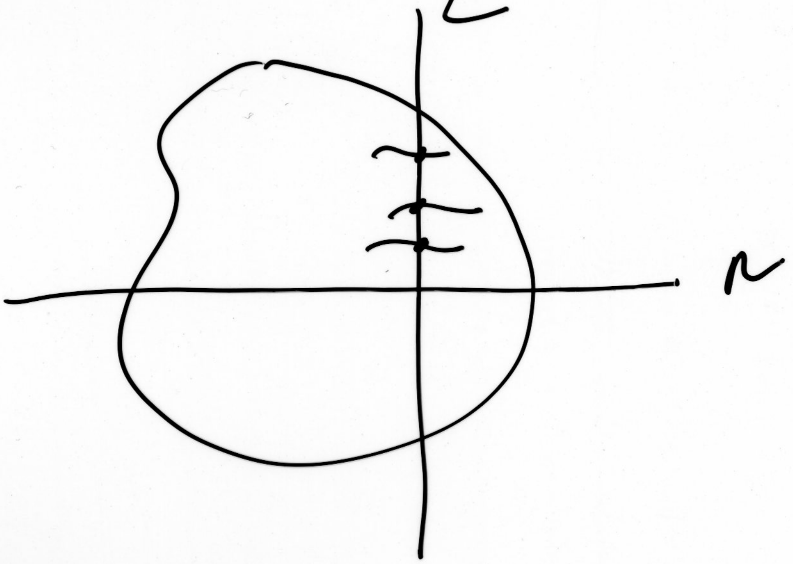
$\qquad\qquad\qquad\qquad\qquad\qquad W^s$

if $a_n \le d_1^n$

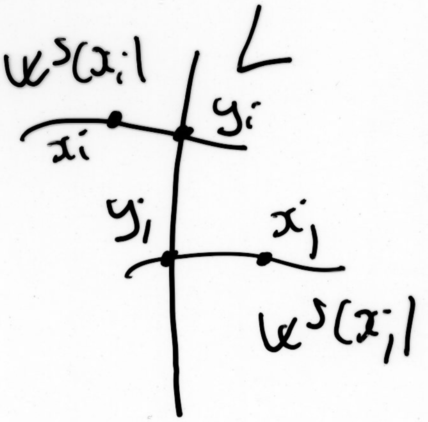
$\Rightarrow \quad \log d_1 \geq h_\mu(f) - 2X_2^+ \longrightarrow \underline{\underline{\text{OK}}}$.

We can find a line L with

$$\#(L \cap W^s) \simeq e^{h_\mu f\, n - 2\chi_2^+ n}$$



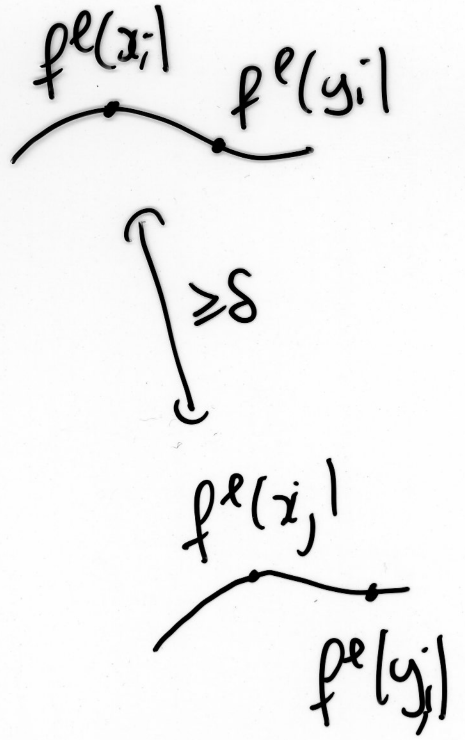
<u>Fundamental remark:</u> Buzzi / Newhouse



$$d_n(y_i, y_j) \geq \delta/2$$

the diameter of $f^\ell(W^s(x_j)) \leq \delta/4$ $\quad \ell = 0, \dots, n-1$

and $d_n(x_i, x_j) \geq \delta \to \exists \ell \in [\![0, n-1]\!]$
$d(f^\ell(x_i), f^\ell(x_j)) \geq \delta$



$f^\ell(W^s(x_i))$

$f^\ell(W^s(x_j)) \leftarrow$ diameter $\leq \delta/4$

$$\Rightarrow d(f^\ell(y_i), f^\ell(y_j)) \geq \delta/2.$$

we proved

$e^{h_\mu(f)n - 2\chi_2^+ n} \leq$ maximal cardinality of a $(n, \frac{\delta}{2})$ separated set in L.

$\leq d_1^n$ ← the idea is the same than for (Dinh-Sibony)

$$h_{top}(f) \leq \max_{0 \leq q \leq k} \log d_q$$

$$X \rightsquigarrow L$$