# Near-parabolic Renormalization and Rigidity

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# Irrationally indifferent fixed points

We consider holomorphic functions of one variable with fixed point z=0.

$$f(z) = \lambda z + a_2 z^2 + \dots$$

If $|\lambda| = 1$, z=0 is called *indifferent fixed point*.

If $\lambda$ is a root of unity, *parabolic;* otherwise *irrationally indifferent*.

$$\lambda = e^{2\pi i \alpha} \qquad \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

If conjugate to a rotation (linearizable), then it has a Siegel disk.

Otherwise, very complicated invariant sets (hedgehogs).

Earlier works: Siegel, Bruno, Herman, Yoccoz, Perez Marco, Petersen, McMullen, Buff, Chéritat, ...

Consider

$$f_0(z) = z + a_2 z^2 + O(z^3) \qquad a_2 \neq 0$$

and its perturbation $f(z) = e^{2\pi i \alpha} z + a_2 z^2 + \dots$

Linearizability of irrationally indifferent fixed points

Siegel (under Diophantine cond.), Bruno (under Bruno condition),
Yoccoz (a new proof using renormalization and converse);
Cremer (nonlinearizable ex.)

Boundary of Siegel disks (Jordan curve in known cases)

Herman (quadratic polynomial, bouded type rotation number => J. curve)
Petersen (quad. poly., bouded type => locally connected J, measure 0)
Herman-Yoccoz, Petersen-Zackeri (weaker cond. for J. curve w. crit. pt.)
Herman (quadratic polynomial, no critical point on bdry)
Buff-Cheritat (various smoothness)

Universality/Rigidity at the boundary

Manton-Nauenberg (experiments, heuristic argument)
McMullen (quadratic-like map => rigidity and differentiability)
This talk (a new class , high type rotation number
=> rigidity and differentiability)

# Physicists' motivation

KAM torus ———·········——→ Chaos

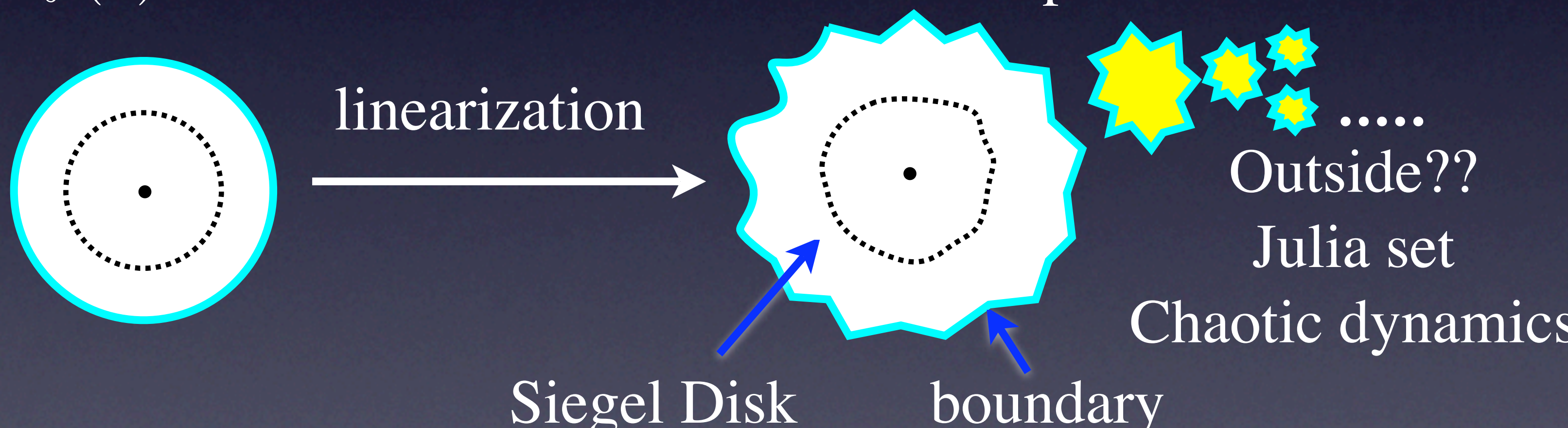
↑ last KAM torus destroyed?

Physicists expect a "universal phenomenon" at critical parameter

## Simpler model (no parameter, only in the phase space)

Irrationally indifferent fixed point

$f(z) = e^{2\pi i\alpha} z + a_2 z^2 + \ldots$ holomorphic near 0




Boundary of Siegel Disk is the closure of critical orbit (for polynomials)

Physicists expect a "universal phenomenon" at the boundary of SD

Manton-Nauenberg (physicists), McMullen (for bounded type)

**Theorem** (McMullen). *Let $f$ and $\hat{f}$ be quadratic-like maps with Siegel disks of period one with the same rotation number $\alpha$ of bounded type. Then $f$ and $\hat{f}$ are conjugate by a quasiconformal mapping $\varphi$ which is $C^{1+\gamma}$-conformal on the boundary of the Siegel disk, i.e.*

$$\varphi(z) = \varphi(z_0) + A(z - z_0) + O(|z - z_0|^{1+\gamma}) \text{ as } z \to z_0$$

*where $z_0$ is on the boundary of the Siegel disk and $A$ is a non-zero constant.*

**Theorem.** *Let $f = e^{2\pi i\alpha}h$ and $\hat{f} = e^{2\pi i\alpha}\hat{h}$ where $h$ and $\hat{h}$ are in the class $\mathcal{F}_1$ which will be defined later, and the rotation number $\alpha$ is of high type $(N)$ with sufficiently large $N$ (also defined later). Then $f$ and $\hat{f}$ are asymptotically conformally conjugate on the closure of critical orbit. Moreover the conjugacy is $C^{1+\gamma}$-conformal on the critical orbit. Furthermore there exists $0 < \lambda < 1$ such that if the contunued fraction coefficients of $\alpha$ satisfies $a_n \leq C\lambda^n$ with some $C > 0$ then the conjugacy is $C^{1+\gamma'}$-conformal on the closure of the critical orbit.*

*Remark.* The closure of critical orbit contains boundary of Siegel disk. The above theorem follows from Rigidity result (Theorem 5) via a differentiability result on quasiconformal mappings.

# Differentiable functions


In small scale...

homeomorphism: can do anything

quasi-symmetric, quasi-conformal: bounded ratio

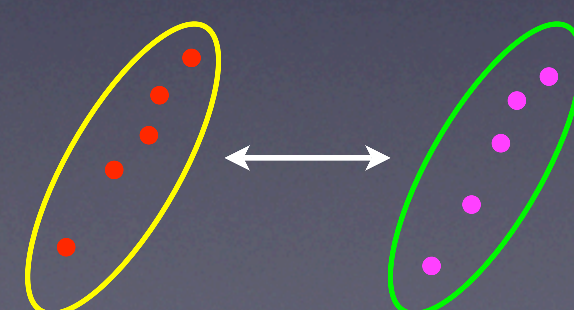
asymptotically conformal: ratio -> 1

$C^{1+\alpha}$ : ratio -> 1 "*fast*"

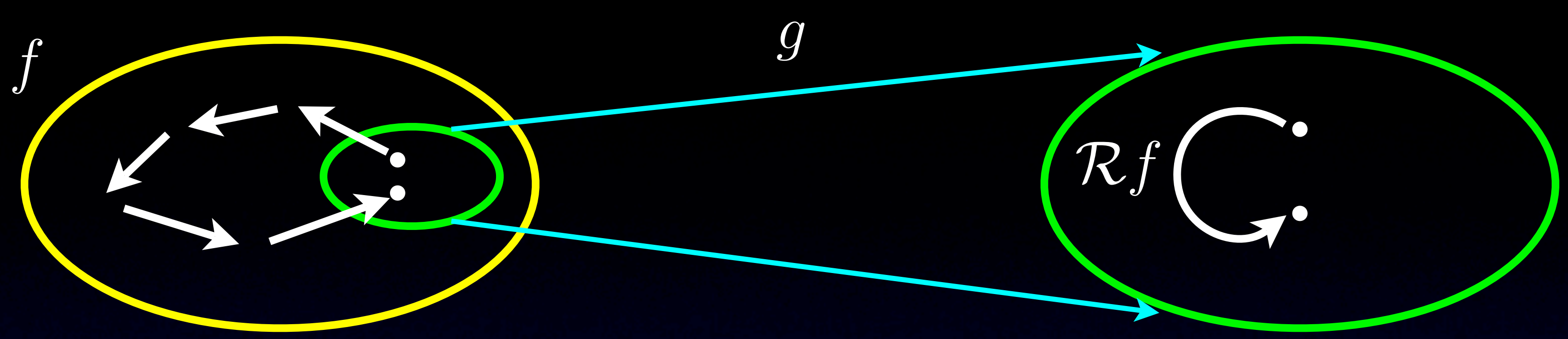
For conjugacies between dynamical systems...

compare orbits

to see details, need to iterate many times



# Return map




$$\mathcal{R}f = (\text{first return map of } f) \text{ after rescaling}$$
$$= g \circ f^k \circ g^{-1} \quad (\text{if return time} \equiv k)$$

*Renormalization*

high iterates of $f$ ⟷ fewer iterates of $\mathcal{R}f$

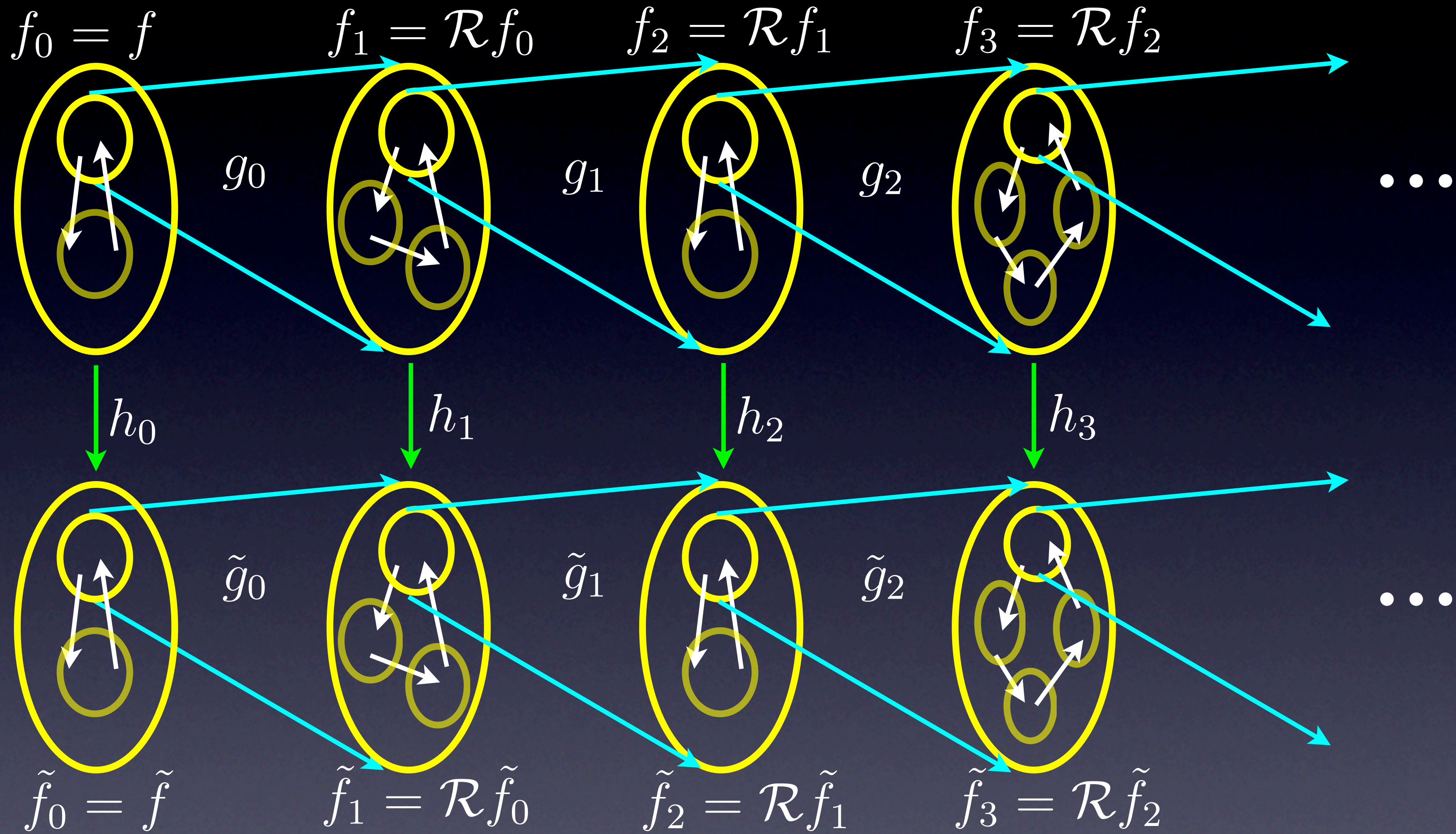
fine orbit structure for $f$ ⟷ large scale orbit structure for $\mathcal{R}f$

Successive construction of $\mathcal{R}f$, $\mathcal{R}^2 f$, $\ldots$, helps to understand the dynamics of $f$ (orbits, invariant sets, rigidity, bifurcation, $\ldots$)

If $\mathcal{R}f = f$ (fixed point of renormalization),
then $f = g \circ f^k \circ g^{-1}$ (fixed point equation)

# Renormalization and Rigidity (an oversimplified view)

Suppose $f$ and $\tilde{f}$ have "the same combinatorial type" and admit successive construction of renormalizations.




$\{h_n\}$ "bounded" $\longrightarrow$ $f$ and $\tilde{f}$ quasi-conformally conjugate

$d(f_n, \tilde{f}_n) \to 0$ $\longrightarrow$ $h_n \to$ linear $\longrightarrow$ conjugacy is asymptotically conformal or smooth, etc.

# Yoccoz renormalization for Siegel-Bruno Theorem

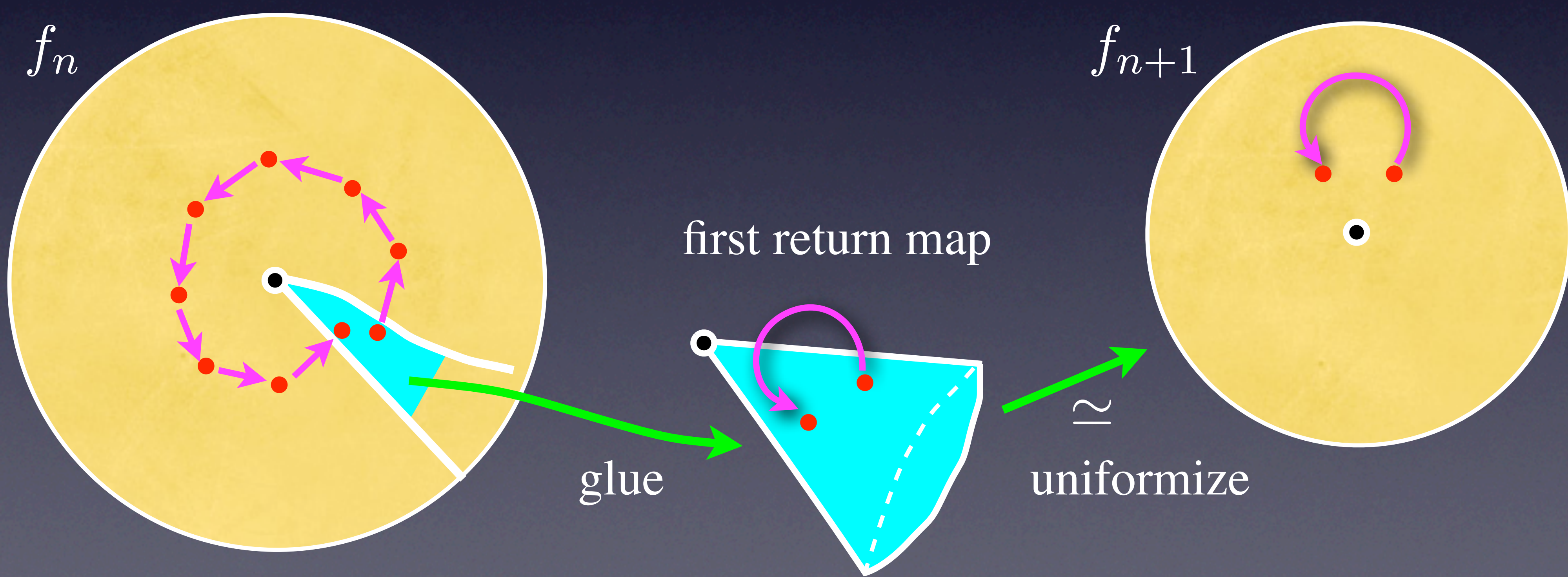
$f(z) = e^{2\pi i\alpha} z + \ldots,\ \sum_n \frac{\log q_{n+1}}{q_n} < \infty$ where $\frac{p_n}{q_n} \to \alpha$ (convergents)

$\implies f$ is conjugate to $z \mapsto e^{2\pi i\alpha} z$

Yoccoz's proof: construct the sequence of renormalizations $f_n$

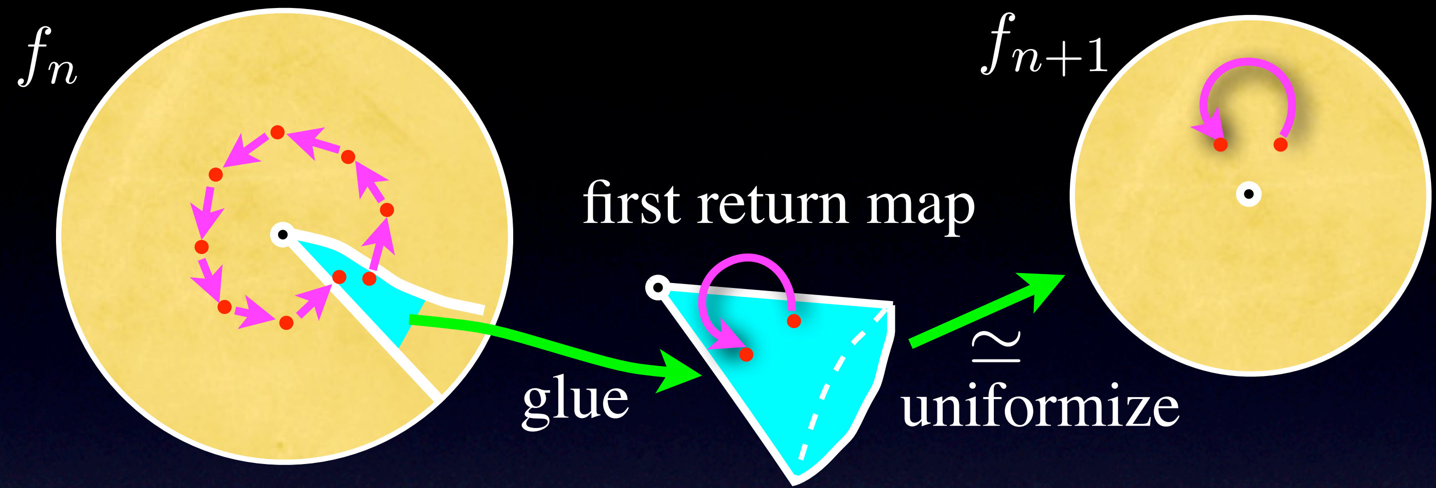
$$f_0 = f, \quad \alpha_0 = \alpha, \quad \alpha_{n+1} = dist(\tfrac{1}{\alpha_n}, \mathbb{Z})$$

$$f_n(z) = e^{2\pi i\alpha_n} z + \ldots \rightsquigarrow f_{n+1}(z) = e^{2\pi i\alpha_{n+1}} z + \ldots$$

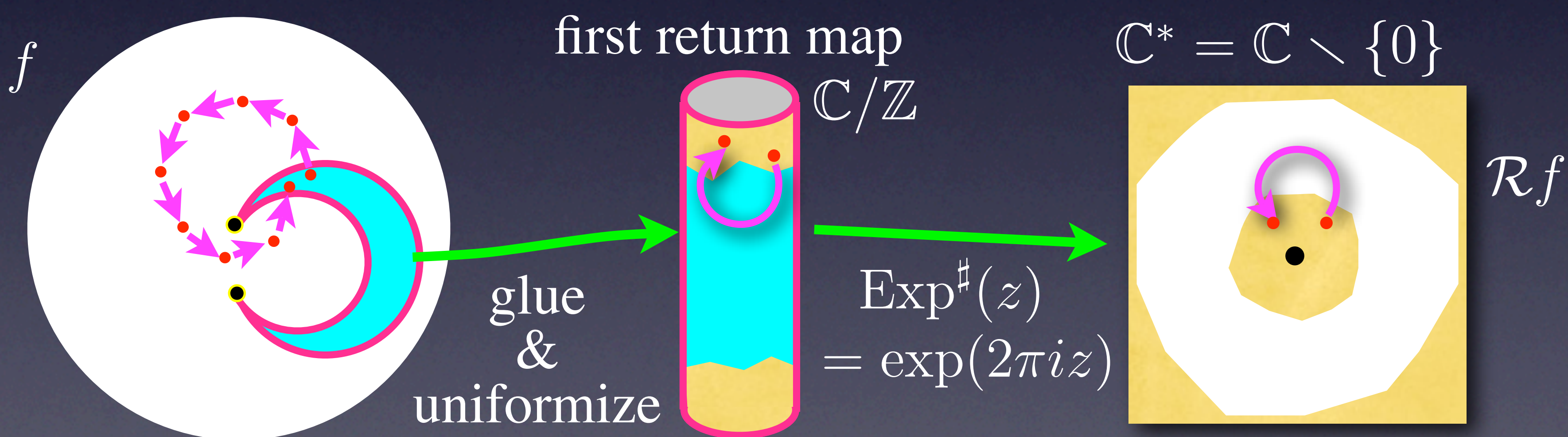


# Yoccoz renormalization for Siegel-Bruno Theorem

$$f_n(z) = e^{2\pi i \alpha_n} z + \dots \rightsquigarrow f_{n+1}(z) = e^{2\pi i \alpha_{n+1}} z + \dots$$




# Cylinder/Near-parabolic renormalization




$\mathcal{R}f$ can be defined when $f(z) = e^{2\pi i \alpha} z + \dots$ is a small perturbation of $z + a_2 z^2 + \dots$ $(a_2 \neq 0)$ and $|\arg \alpha| < \pi/4$.

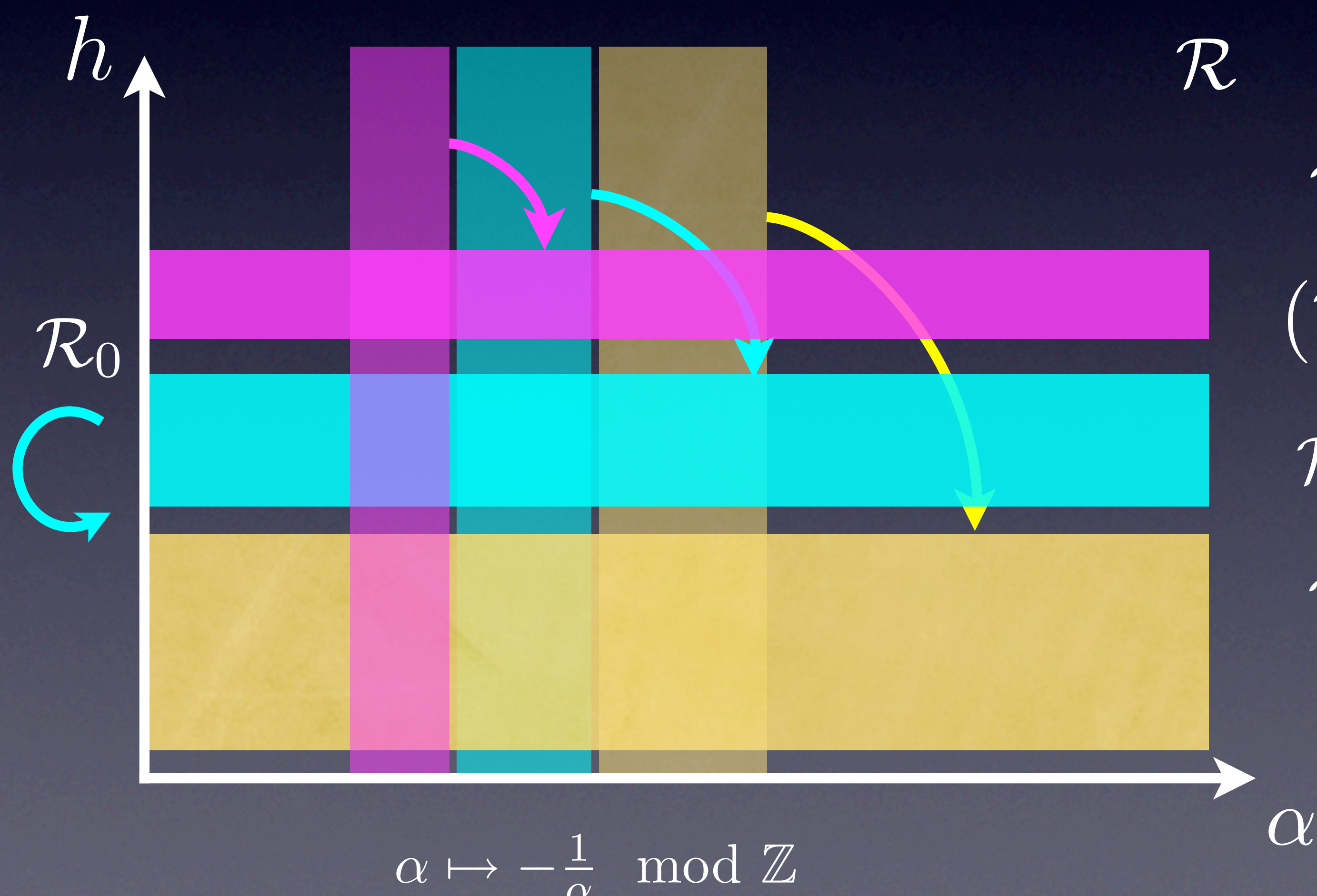
# Renormalization: The Picture

Write $f(z) = e^{2\pi i\alpha}z + O(z^2) = e^{2\pi i\alpha}h(z)$ where $h(z) = z + O(z^2)$.

$f \longleftrightarrow (\alpha, h)$

Then $\mathcal{R}f(z) = e^{-2\pi i\frac{1}{\alpha}}\mathcal{R}_\alpha h(z)$ where $\mathcal{R}_\alpha h = \mathrm{Exp}^\sharp \circ E_{(e^{2\pi i\alpha}h)} \circ (\mathrm{Exp}^\sharp)^{-1}$.

Hence $\mathcal{R} : (\alpha, h) \mapsto (-\frac{1}{\alpha}, \mathcal{R}_\alpha h)$ (skew product)




$\mathcal{R}$ hyperbolic?

($\mathcal{R}_\alpha$ contracting?)

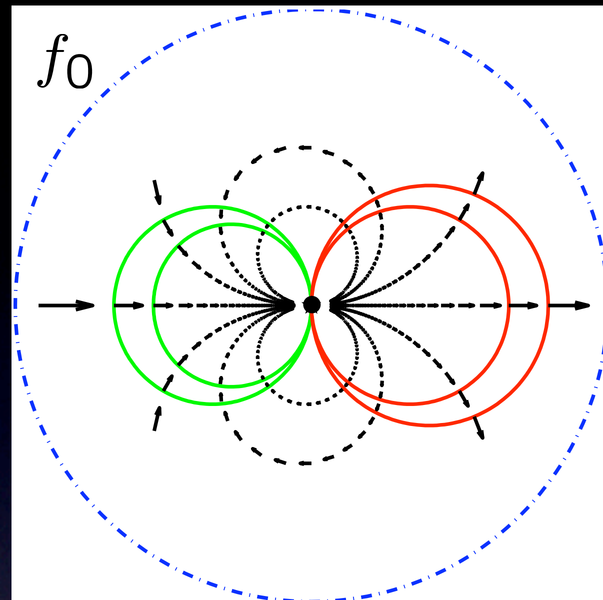
$\mathcal{R}_\alpha h \to \mathcal{R}_0 h$ ($\alpha \to 0$)

$\mathcal{R}_0$ contracting?

YES for $\alpha$ small

# Horn map and Parabolic Renormalization

$$f_0(z) = z + a_2 z^2 + \dots$$
$$a_2 \neq 0$$




Horn map

$$E_{f_0} = \Phi_{attr} \circ \Phi_{rep}^{-1}$$

Parabolic Renormalization

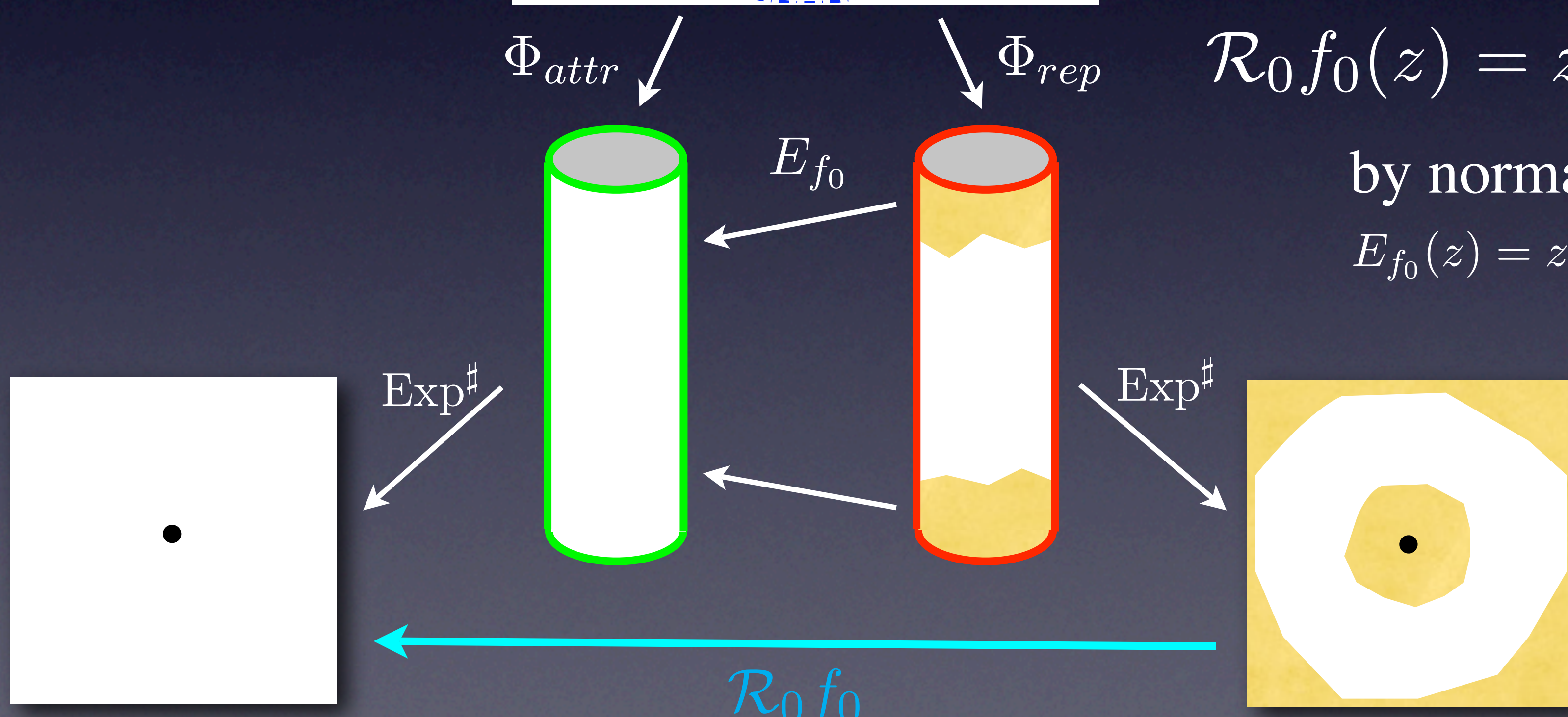
$$\mathcal{R}_0 f_0 = \mathrm{Exp}^\sharp \circ E_{f_0} \circ (\mathrm{Exp}^\sharp)^{-1}$$

$$\mathrm{Exp}^\sharp(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*$$

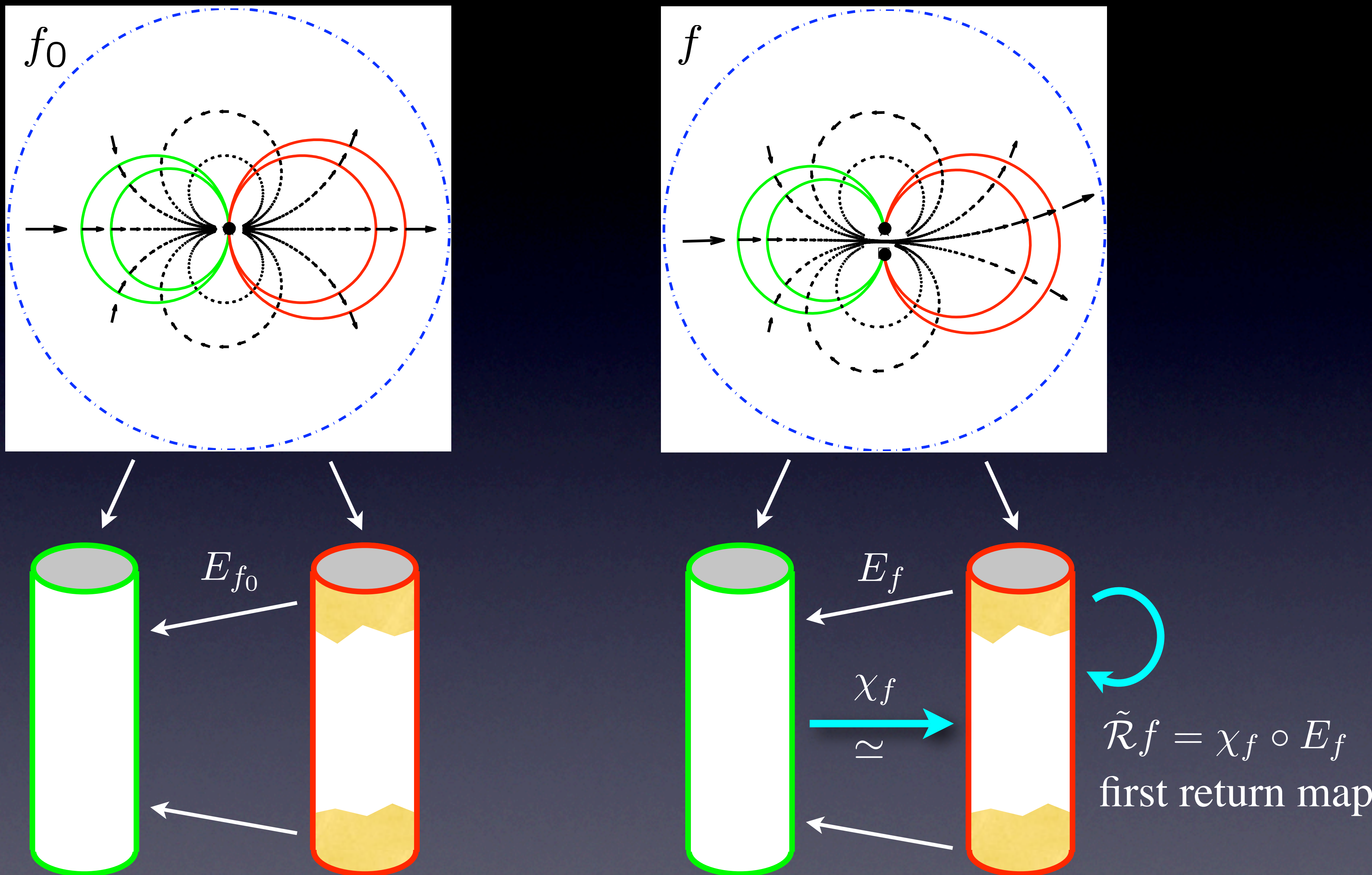
$$\mathcal{R}_0 f_0(z) = z + \dots$$

by normalization

$$E_{f_0}(z) = z + o(1) \quad (\mathrm{Im}\, z \to +\infty)$$



# Perturbation (Douady-Hubbard-Lavaurs) $f'(0) = e^{2\pi i\alpha}$, $\alpha$ small $|\arg\alpha| < \frac{\pi}{4}$




$E_f$ depends continuously on $f$

(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

# Main Theorems 1-4 (with H. Inou)

*We define a class of functions* $\mathcal{F}_1$, *(and* $\mathcal{F}_1' \subset \mathcal{F}_1$*) such that if* $f \in \mathcal{F}_1$, *then* $f$ *is holomorphic,* $f(0) = 0$, $f'(0) = 1$, $f$ *has a unique critical point* $c_f$ *in its domain of definition and the critical value* $f(c_f) = -\frac{4}{27}$ *(fixed). Moreover* $f''(0) \neq 0$.

**Theorem 1.** $\mathcal{F}_1 \xrightarrow{\mathcal{R}_0} \mathcal{F}_1' \subset \mathcal{F}_1$.
*Moreover* $\mathcal{R}_0$ *is "holomorphic" and* $\mathcal{R}_0(z + z^2) \in \mathcal{F}_1'$.

**Theorem 2.** *For small* $\alpha$ $(\in \mathbb{R})$, $\mathcal{F}_1 \xrightarrow{\mathcal{R}_\alpha} \mathcal{F}_1' \subset \mathcal{F}_1$.
*Hence there exists a large* $N$ *such that if* $f = e^{2\pi i \alpha} h$ *with* $\alpha$ *of high type* $(N)$ *and* $h \in \mathcal{F}_1$, *then the sequence of renormalizations*

$$f = f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \dots$$

*is defined so that* $f_n = e^{2\pi i \alpha_n} h_n(z)$, $h_n \in \mathcal{F}_1$. *(Here* $\alpha_{n+1} = ||\frac{1}{\alpha_n}||$ *and* $h_{n+1} = \mathcal{R}_{\alpha_n} h_n$, *possibly after complex conjugation.)*

# “Irrational numbers of high type” ($N$)

$$\alpha = \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{a_3 \pm \cfrac{1}{\ddots}}}} \qquad \text{where } a_i \geq N$$

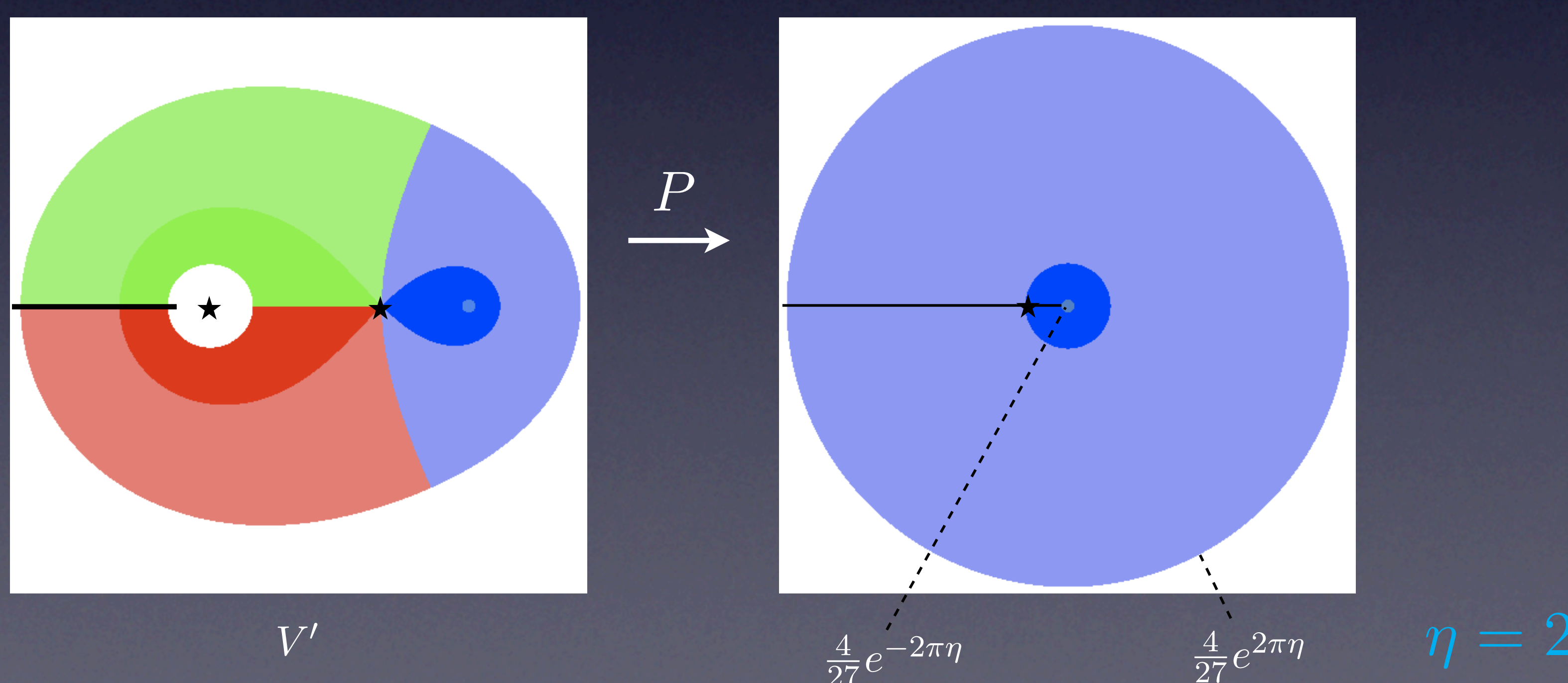
# Definition of $\mathcal{F}_1$ and $\mathcal{F}_1'$

Let $P(z) = z(1+z)^2$. We take specific simply connected open sets $V$ and $V'$ with $0 \in V \subset \overline{V} \subset V' \subset \mathbb{C}$.

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

Define $\mathcal{F}_1'$ with $V$ replaced by $V'$.

$P(0) = 0, \quad P'(0) = 1 \qquad$ critical points: $-\dfrac{1}{3}$ and $-1 \qquad$ critical values: $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$




$V$ slightly smaller domain than $V'$

**Theorem 3.** *After modifying the definition slightly, $\mathcal{F}_1$ is in one to one correspondence with the Teichmüller space of a punctured disk. With respect to the Teichmüller distance (which is complete), $\mathcal{R}_0$ is a uniform contraction.*

**Theorem 4.** *The same statement for small $\alpha$ $(\in \mathbb{R})$.*
*Hence when restricted to the subset where $|\alpha|$ is small, the renormalization $\mathcal{R}$ is hyperbolic.*

Teichmüller space is like the unit disk with Poincaré metric.
holomorphic self map does not expand the distance.

(Royden-Gardiner Theorem: Teichmüller distance = Kobayashi distance)

$$\mathcal{F}_1 \xrightarrow{\mathcal{R}_0} \mathcal{F}_1' \subset \mathcal{F}_1$$

Estimate of contraction of $\mathcal{F}_1' \hookrightarrow \mathcal{F}_1$ via cotangent space which is the space of integrable holomorphic quadratic differentials. + modulus-area inequality

# Applications

**Theorem.** *Under the assumption of Theorem 2, the critical orbit stays in the domain of $f$ and can be iterated infinitely many times. Moreover if $f$ is (a part of) a rational map, then the critical orbit is not dense.*

**Theorem** (Buff-Chéritat)**.** *There exists an irrational number $\alpha$ such that the Julia set of the quadratic polynomial $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has positive Lebesgue measure.*

**Theorem.** *Suppose $f$ and $f'$ satisfy the assumption of Theorem 2, with the same rotation number $\alpha$. Then they have small periodic cycles $\zeta_n$ and $\zeta'_n$ around $0$ with period $q_n$. Let $\lambda(\zeta_n)$, $\lambda(\zeta'_n)$ be their multipliers. The differences*

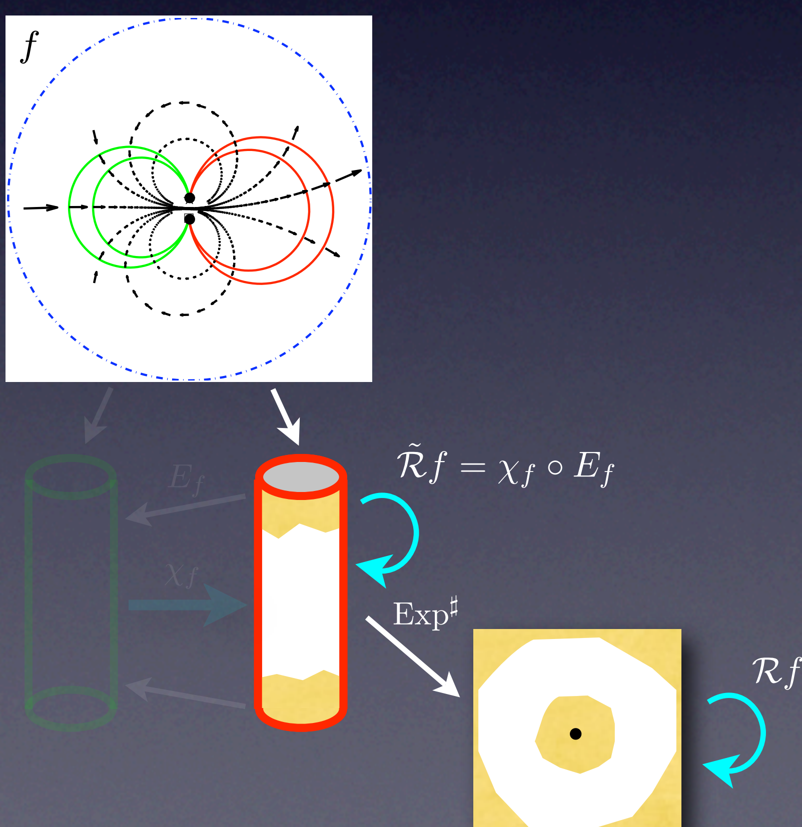
$$|\lambda(\zeta_n) - \lambda(\zeta'_n)| \quad \textit{and} \quad \left|\frac{1}{1-\lambda(\zeta_n)} - \frac{1}{1-\lambda(\zeta'_n)}\right|$$

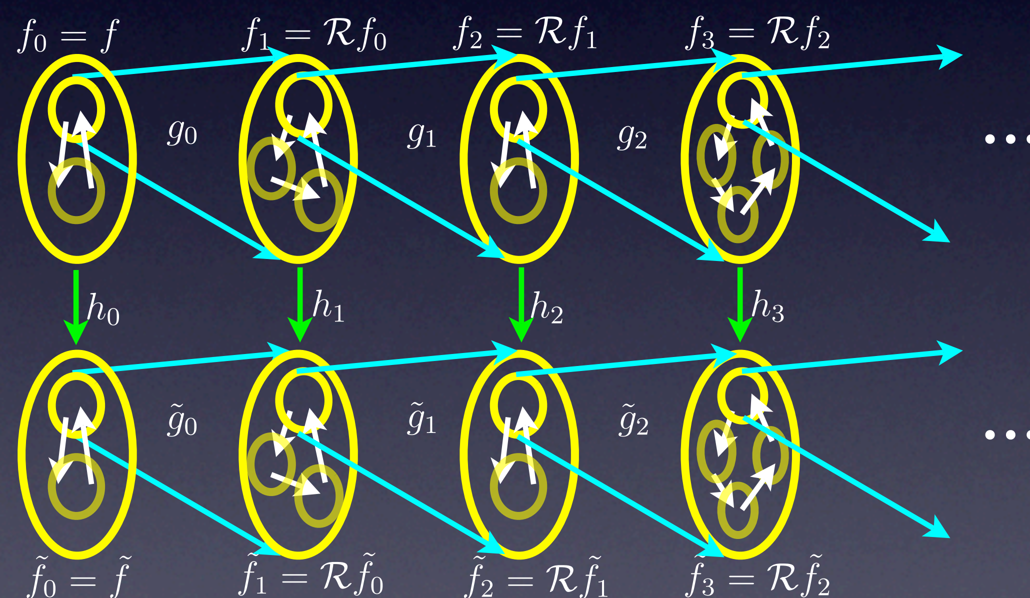
*tends to $0$ exponentially fast as $n \to \infty$ with a uniform rate.*

# Application 2: Rigidity

**Theorem 5** (Rigidity). *If $h, \tilde{h} \in \mathcal{F}_1$ and $\alpha$ satisfies the hypothesis of Theorem 2, then there exists a quasiconformal homeomorphism $\varphi$ which conjugates $f = e^{2\pi i \alpha} h$ and $\tilde{f} = e^{2\pi i \alpha} \tilde{h}$ along their critical orbits, and asymptotically conformal on the closure of critical orbits.*

Within this class of maps, the same rotation number implies a better conjugacy.

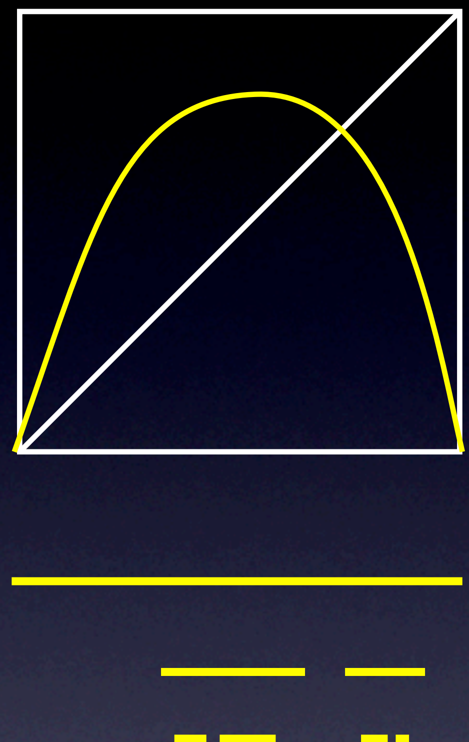



$g_n$'s, $\tilde{g}_n$'s are "exponential-like" (very expanding).

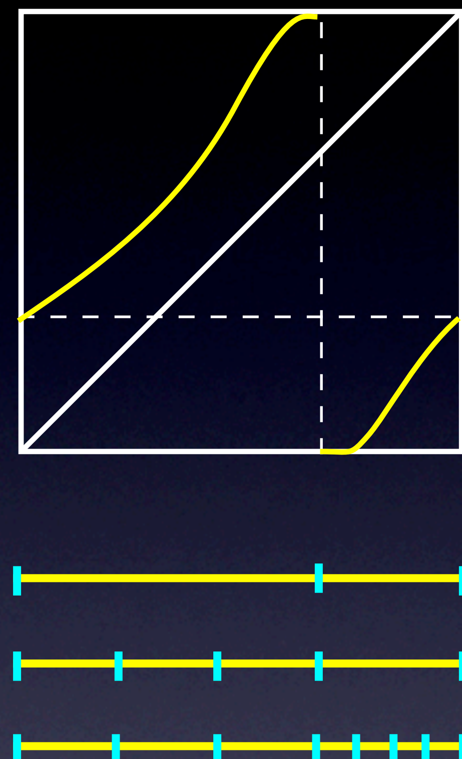
# Various Renormalizations

## Feigenbaum



proper subintervals
-> Cantor set

## Circle map



partition of interval

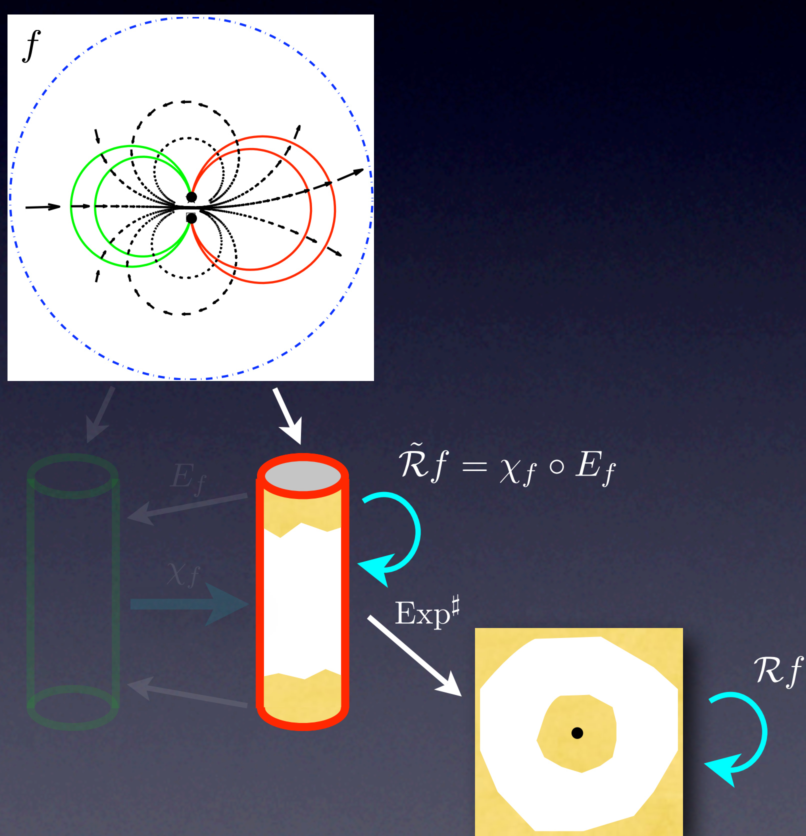
## Near-parabolic

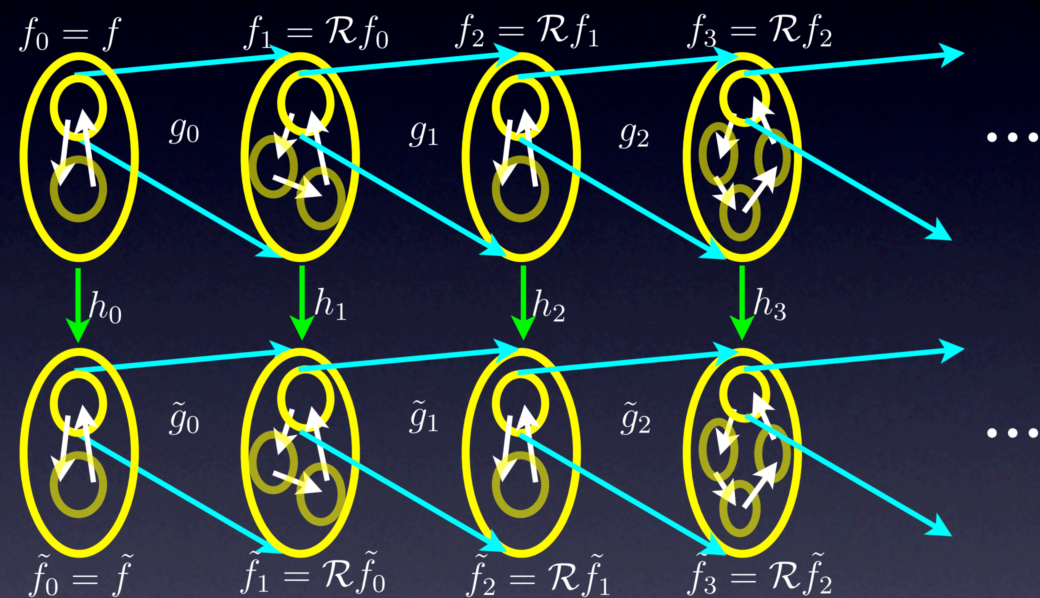
covering by sector or croissant-like domains

gluing/identification needed to define the renormalization

# Return to Theorem 5

**Theorem 5** (Rigidity). *If $h, \tilde{h} \in \mathcal{F}_1$ and $\alpha$ satisfies the hypothesis of Theorem 2, then there exists a quasiconformal homeomorphism $\varphi$ which conjugates $f = e^{2\pi i\alpha}h$ and $\tilde{f} = e^{2\pi i\alpha}\tilde{h}$ along their critical orbits, and asymptotically conformal on the closure of critical orbits.*




$g_n$'s, $\tilde{g}_n$'s are "exponential-like" (very expanding).

Need to *reconstruct* the dynamics of $f$ in subdomains (with control on geometry) from $f_n = \mathcal{R}^n f$. Because the relation between $f$ and $f_n = \mathcal{R}^n f$ is less obvious.

# Difficulty in proving rigidity for irrationally indiff. fixed pts.

Knowing $\mathcal{R}f$, what can be said about $f$?

How to transfer information (e.g. geometry) on $\mathcal{R}^n f$
to previous generations of renormalizations $\mathcal{R}^{n-1}f, \mathcal{R}^{n-2}f, \ldots, f$?

Fundamental domains (and their boundary curves) are not unique.

Need to cover previous fund. regions with next generation fund. regions
WITH OVERLAP. (not partition)

Need to reconstruct the dynamics of $f$ from that of $\mathcal{R}f$
so that one can understand $f$ better.

this is like ...

Thank you!