NEAR-PARABOLIC RENORMLAIZATION AND RIGIDITY

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In this talk, we will discuss the problems of rigidity for holomorphic maps of one variable with irrationally indifferent fixed points. There have been a lot of works (Siegel, Bruno, Herman, Yoccoz, Petersen, Buff, Chéritat, McMullen ...) toward the linearizability of irrationally fixed points, properties of the boundary of Siegel disks (maximal domain of linearization), and the properties of Julia sets. For example, if a rational map has a Siegel disk (maximal domain of linearization), then physicists (Manton-Nauenberg) asked whether its boundary has a certain kind of universality or rigidity (for example, if there are two maps of similar kinds with the same rotation number for the Siegel disk, then there is a "smooth" conjugacy on the boundary.

In this direction, McMullen obtained the following rigidity result:

Theorem (McMullen)**.** *Let f and $\hat{f}$ be quadratic-like maps with Siegel disks of period one with the same rotation number α of bounded type. Then f and $\hat{f}$ are conjugate by a quasiconformal mapping φ which is $C^{1+\gamma}$-conformal on the boundary of the Siegel disk, i.e.*

$$\varphi(z) = \varphi(z_0) + A(z - z_0) + O(|z - z_0|^{1+\gamma}) \text{ as } z \to z_0$$

where z_0 is on the boundary of the Siegel disk and A is a non-zero constant.

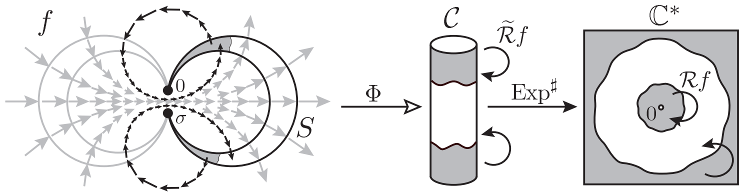
The result we present is a proof of a similar result under different (in a sense opposite) condition with a different method:

Theorem. *Let $f = e^{2\pi i\alpha}h$ and $\hat{f} = e^{2\pi i\alpha}\hat{h}$ where h and $\hat{h}$ are in the class $\mathcal{F}_1$ which will be defined later, and the rotation number α is of high type (N) with sufficiently large N (also defined later). Then f and $\hat{f}$ are asymptotically conformally conjugate on the closure of critical orbit. Moreover the conjugacy is $C^{1+\gamma}$-conformal on the critical orbit. Furthermore there exists $0 < \lambda < 1$ such that if the contunued fraction coefficients of α satisfies $a_n \leq C\lambda^n$ with some $C > 0$ then the conjugacy is $C^{1+\gamma'}$-conformal on the closure of the critical orbit.*

Remark. The closure of critical orbit contains the boundary of Siegel disk, if there is any.

The differentiability result follows from asymptotic conformality of the conjugacy with estimate on the dilatation via a differentiability result on quasiconformal mappings.

In order to study these questions, we introduce the "parabolic renormalization" which is defined as follows. For $f(z) = e^{2\pi i\alpha}z + a_2z^2 + \dots$, with $a_2 \neq 0$ and the rotation number $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ sufficiently small, there exists a "croissant-shaped" fundamental domain S, and we identify its boundary curves by the map f to obtain a quotient cylinder $\mathcal{C}$ isomorphic to $\mathbb{C}/\mathbb{Z}$, which in turn is identified with $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$. The orbits near 0 define the first



return map on S, which induces the return map $\mathcal{R}f$ near 0 of $\mathbb{C}^*$. In a joint work with Hiroyuki Inou (Kyoto University), we proved the "dynamical hyperbolicity" of $\mathcal{R}$ as a meta-dynamics on the space of maps of certain class, and as a consequence, we have

Theorem (Inou-S.). *There exist a class $\mathcal{F}_1$ of functions and a large constant N such that:*
(a) *For $f \in \mathcal{F}_1$, $f : Dom(f) \to \mathbb{C}$ holomorphic , $f(0) = 0, f'(0) = 1, f''(0) \neq 0$.*
(b) *If $h \in \mathcal{F}_1$ or $h(z) = z + z^2$ and α is an irrational number such that*

$$\alpha = \pm \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}} \qquad \text{where } a_i \in \mathbb{N} \text{ and } a_i \geq N,$$

then $f(z) = e^{2\pi i\alpha} h(z)$ is infinitely parabolic-renormalizable, that is,

$$f = f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \ldots$$

are defined and $f_n = e^{2\pi i\alpha_n} h_n(z)$, $h_n \in \mathcal{F}_1$ (possibly with complex conjugations). Moreover if $\hat{f} = e^{2\pi i\alpha}\hat{h}$ ($\hat{h} \in \mathcal{F}_1$) is another map with the same α, then for corresponding $\hat{f}_n$'s, $d(f_n, \hat{f}_n)$ converges to 0 exponentially. Here $d(\cdot,\cdot)$ is a distance on $\mathcal{F}_1$.

Definition. Let $P(z) = z(1 + z)^2$. We take specific simply connected open sets V and V' with $0 \in V \subset \overline{V} \subset V' \subset \mathbb{C}$. Define

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}.$$

[IS] H. Inou and M. Shishikura, The renormalization for parabolic fixed points and their perturbation, preprint. http://math.kyoto-u.ac.jp/~mitsu/pararenorm/