Holomorphic maps on $\mathbb{P}^k$ with sparse critical orbits

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$\mathbb{P}^k$ complex projective space of $\mathbb{C}$-dim $k \geq 1$

ω Fubini-Study form s.t. $\int_{\mathbb{P}^k} \omega^k = 1$

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic

Assume $\deg(f) := \int_{\mathbb{P}^k} f^* \omega \wedge \omega^{k-1}$ is at least 2.

$f^n := \overbrace{f \circ \cdots \circ f}^{n \text{ times}}$ n-th iterate

C critical set

$D := \overline{\bigcup_{n \geq 1} f^n(C)}$ closure of post-critical set

$E := \bigcap_{n \geq 1} \overline{\bigcup_{i \geq n} f^i(C)}$ critical limit set

$|\cdot|$ Fubini-Study metric

$\mathbf{T}_p$ holomorphic tangent space at $p \in \mathbb{P}^k$

A point $p \in \mathbb{P}^k$ is repelling for $f \iff$

$$\min_{v \in \mathbf{T}_p, |v|=1} |Df^j(v)| \to +\infty$$

as $j \to +\infty$.

A compact set $K (\subset \mathbb{P}^k)$ is a repeller for $f \iff$

$f(K) = K$ and there are constants $c > 0, \ \lambda > 1$

such that

$$|Df^n(v)| \geq c\lambda^n |v|$$

for $\forall v \in \bigcup_{p \in K} \mathbf{T}_p$ and $\forall n \geq 1$.

$\mathbb{D}$ unit disk $(\subset \mathbb{C})$

A holomorphic embedding $\varphi : \mathbb{D} \to \mathbb{P}^k$ is a Fatou disk

$\iff$ $\{f^n \circ \varphi\}_{n\geq 1}$ is a normal family in $\mathbb{D}$.

A Fatou disk $\varphi : \mathbb{D} \to \mathbb{P}^k$ is noncontracting $\iff$

every limit map of $\{f^n \circ \varphi\}_{n\geq 1}$ is nonconstant.

Theorem A

Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree ≥ 2. Let K be a compact set in $\mathbb{P}^k$ such that $f(K) \subset K$ and $K \cap D = \emptyset$. Then, there are subsets $K^u, K^c \subset K$ which satisfy the following properties:

(i) $K^u \cup K^c = E,\ K^u \cap K^c = \emptyset$;

(ii) $f(K^u) \subset K^u,\ f(K^c) \subset K^c$;

(iii) Each point in K^u is repelling ;

(iv) For each $p \in E^c$, there is a noncontracting Fatou disk through p.

Moreover, if $f(K) = K$ and $K^c = \emptyset$, then K is a repeller.

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic, $d = \deg(f) \geq 2$

F Fatou set (i.e. domain of normality of $\{f^n\}_{n\geq 1}$)

$J := \mathbb{P}^k \setminus F$ Julia set

$T := \lim_{n\to+\infty} \frac{1}{d^n}(f^*)^n\omega$ Green (1,1) current

$T^p := \underbrace{T \wedge \cdots \wedge T}_{p \text{ times}}$ Green (p,p) current

$J_p := \mathrm{Supp}(T^p)$ p-th Julia set

<u>Known facts</u>

$J_1 = J$ (Fornæss-Sibony, Ueda)

$J_k \subset \overline{\{\text{repelling periodic points}\}}$ (Briend-Duval)

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic, $\deg(f) \geq 2$

f is <u>critically finite</u> $\iff$ D is algebraic.

f is <u>critically sparse</u> $\iff$ D is pluripolar.

Theorem B

Suppose f is critically sparse. Then, all repellers for f is contained in J_k. In particular,

$$J_k = \overline{\{\text{repelling periodic points}\}}.$$

Example

$P : \mathbb{C}^k \to \mathbb{C}^k$ regular polynomial map, $\deg(P) \geq 2$

$K(P) := \{w \in \mathbb{C}^k \mid \{P^n(w)\}_{n\geq 0} \text{ bounded}\}$

Suppose $K(P) \cap C = \emptyset$.

Because $K(P)$ is a repeller and P is critically sparse in $\mathbb{P}^k$, we can apply Theorem B.

Hence we obtain $K(P) = J_k$.

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic, $\deg(f) \geq 2$

$\rho(\cdot,\cdot)$ distance in $\mathbb{P}^k$

K compact set in $\mathbb{P}^k$ s.t. $f(K) = K$

$$\widehat{K} := \{\hat{p} = (p_i)_{i\leq 0} \;:\; p_i \in K,\; f(p_{i-1}) = p_i \text{ for all } i \leq 0\}$$

$$\pi :\; (p_i)_{i\leq 0} \in \widehat{K} \mapsto p_0 \in K$$

$$\hat{\rho}(\hat{p},\hat{q}) := \sum_{i\leq 0} 2^i \rho(p_i, q_i) \quad \hat{p},\hat{q} \in \widehat{K}$$

$$\hat{f}((p_0, p_{-1}, p_{-2}, \cdots)) := (f(p_0), p_0, p_{-1}, \cdots)$$

$$\widehat{\mathbf{T}K} := \{(\hat{p}, v) \;:\; \hat{p} \in \widehat{K},\; v \in \mathbf{T}_{\pi(\hat{p})}\mathbb{P}^k\}$$

$$\pi' : (\hat{p}, v) \in \widehat{\mathbf{T}K} \mapsto v \in \mathbf{T}\mathbb{P}^k$$

$$\widehat{Df} : (\hat{p}, v) \in \widehat{\mathbf{T}K} \mapsto (\hat{f}(\hat{p}), Df(v)) \in \widehat{\mathbf{T}K}$$

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic, $\deg(f) \geq 2$

K compact set in $\mathbb{P}^k$ s.t. $f(K) = K$

K is a hyperbolic set for f $\iff$ there exist constants $c > 0$, $\lambda > 1$ and a continuous splitting $\widehat{\mathbf{T}K} = E^u \oplus E^s$ by complex subbundles E^u, E^s such that

$$\widehat{Df}(E^u) = E^u,\ \widehat{Df}(E^s) \subset E^s,$$

$$\|\widehat{Df}^n((\hat{p}, v))\| \geq c\lambda^n \|(\hat{p}, v)\|,\ \forall(\hat{p}, v) \in E^u,$$

$$\|\widehat{Df}^n((\hat{p}, v))\| \leq c^{-1}\lambda^{-n} \|(\hat{p}, v)\|,\ \forall(\hat{p}, v) \in E^s$$

for $\forall n \geq 1$, where $\| \cdot \|$ denotes the metric in $\widehat{\mathbf{T}K}$ induced from Fubini-Study metric.

K hyperbolic set for f

δ small positive constant

Local/global stable manifold for $p \in K$

$$W^s_\delta(p) := \{y \in \mathbb{P}^k \mid \rho(f^i(y), f^i(p)) < \delta \ \ \forall i \geq 0\}$$

$$W^s(p) := \{y \in \mathbb{P}^k \mid \rho(f^i(y), f^i(p)) \to 0 \text{ as } i \to +\infty\}$$

Local/global unstable manifold for $\hat{q} \in \widehat{K}$

$$W^u_\delta(\hat{q}) := \{y \in \mathbb{P}^k \mid \exists (y_i)_{i \leq 0} \in \widehat{\mathbb{P}^k} \text{ s.t.}$$

$$y_0 = y,\ \rho(y_i, q_i) < \delta \ \ \forall i \leq 0\}$$

$$W^u(\hat{q}) := \{y \in \mathbb{P}^k \mid \exists (y_i)_{i \leq 0} \in \widehat{\mathbb{P}^k} \text{ s.t.}$$

$$y_0 = y,\ \rho(y_i, q_i) \to 0 \text{ as } i \to -\infty\ \}$$

$f : \mathbb{P}^k \to \mathbb{P}^k$ holomorphic, $\deg(f) \geq 2$

$$\Omega := \{x \in \mathbb{P}^k \,|\, \forall \text{ nbd } U \ni x, \; \exists n \geq 1 \text{ s.t. } f^n(U) \cap U \neq \emptyset\}$$

f is Axiom A $\iff$ Ω hyperbolic set

$$\Omega = \overline{\{\text{periodic points}\}}$$

When f is Axiom A, we consider decomposition

$$\Omega = \Omega_0 \sqcup \cdots \sqcup \Omega_k$$

where Ω_i is the part of unstable dimension i.

Theorem C

Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree ≥ 2. Suppose $J \cap E$ is a hyperbolic set. Then, the Fatou set F consists of the attractive basins for finitely many attracting cycles. Moreover, if the unstable dimension of $J \cap E$ is 1, then

$$E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in \widehat{J \cap E}} W^u(\hat{p}).$$

Theorem D

Let f be a critically finite map on $\mathbb{P}^2$. Then,

f is Axiom A if and only if $J \cap E$ is a hyperbolic set of unstable dimension 1.

Lemma (criticality)

f critically finite map on $\mathbb{P}^2$

X irreducible component of E

p saddle periodic point s.t. $p \in X \cap \mathrm{Reg}(D)$

Then, X is in a critical cycle of curves.

Theorem E

Let f be a Axiom A critically finite map on $\mathbb{P}^2$.

Then,

(1) each irreducible component of E is a rational curve ;

(2) J_2 is connected ;

(3) $\Omega_2 = J_2$;

(4) $\Omega_1 = J \cap E$;

(5) $\Omega_0 = \{\text{attracting periodic points}\} \neq \emptyset$;

(6) $E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in \widehat{J \cap E}} W^u(\hat{p})$;

(7) $J = J_2 \sqcup \bigcup_{p \in J \cap E} W^s(p)$;