Discontinuity of straightening maps

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Douady and Hubbard introduced the notion of polynomial-like mappings to describe renormalizations of quadratic polynomials and to study the structure of the Mandelbrot set.

Definition. A *polynomial-like map* is a proper holomorphic map $f : U' \to U$ such that $U,\ U'$ are topological disks, and $U' \Subset U$. Define the *filled Julia set* $K(f) = K(f; U', U)$ and the *Julia set* $J(f) = J(f; U', U)$ as follows:

$$K(f) = \bigcup_{n \geq 0} f^{-n}(U'), \qquad\qquad J(f) = \partial K(f).$$

We say two polynomial-like maps $f : U' \to U$ and $g : V' \to V$ are *hybrid equivalent* if there exists a quasiconformal map $\psi : U'' \to V''$ such that U'' (resp. V'') is a neighborhood of $K(f)$ (resp. $K(g)$), $\bar{\partial}\psi \equiv 0$ a.e. on $K(f)$, and $\psi \circ f = g \circ \psi$.

The most basic result on polynomial-like maps is the following.

Theorem (Straightening theorem)**.** *For any polynomial-like map $f : U' \to U$ of degree $d \geq 2$, there exists a polynomial g of the same degree hybrid equivalent to it. Furthermore, if $K(f)$ is connected, then g is unique up to affine conjugacy.*

By this theorem, we can construct a correspondence between families of polynomial-like maps and polynomials, when restricted to the connectedness loci. For $d \geq 2$, let Poly_d be the set of affine conjugacy classes of polynomials of degree d and $\mathcal{C}_d = \{P \in \mathrm{Poly}_d\ \ K(P)$ is connected$\}$ be its *connectedness locus*. We also denote

$$\mathcal{CK}_d = \{(g, z) \in \mathcal{C}_d \times \mathbb{C};\ z \in K(g)\}.$$

Definition. Let $(f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be an analytic family of polynomial-like maps of degree $d \geq 2$. Let us denote the *connectedness locus* by

$$\mathcal{C}_\Lambda = \mathcal{C}_{(f_\lambda)} = \{\lambda \in \Lambda; K(f_\lambda) \text{ is connected}\},$$

and define the *straightening map* $\mathcal{S}_\Lambda : \mathcal{C}_\Lambda \to \mathcal{C}_d$ by $\mathcal{S}_\Lambda(\lambda) = g$ if f_λ is hybrid equivalent to g.

An *analytic family of polynomial-like maps with marked point* is a family of pairs $(f_\lambda, x_\lambda)_{\lambda \in \Lambda}$ such that $(f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ is an analytic family of polynomial-like maps of degree d and $x : \Lambda \to \mathbb{C}$ is a holomorphic map with $x_\lambda \in U'_\lambda$. Let

$$\mathcal{C}_\Lambda = \mathcal{C}_{(f_\lambda, x_\lambda)} = \{\lambda \in \Lambda;\ K(f_\lambda) \text{ is connected and } x_\lambda \in K(f_\lambda)\}.$$

The straightening map $\mathcal{S}_\Lambda = \mathcal{S}_{(f_\lambda, x_\lambda)} : \mathcal{C}_\Lambda \to \mathcal{CK}_d$ for a family $(f_\lambda, x_\lambda)_{\lambda \in \Lambda}$ of polynomial-like maps with marked point is defined as follows: We have $\mathcal{S}_\Lambda(\lambda) = (g, z)$ if $f_\lambda : U'_\lambda \to U_\lambda$ is hybrid equivalent to g with conjugacy ψ_λ and $z = \psi_\lambda(x_\lambda)$ (note that $\psi|_{K(f)}$ is uniquely determined).

In the case of quadratic-like maps (i.e., when the degree $d = 2$), Douady and Hubbard proved that the straightening map is always continuous. Furthermore, in the Mandelbrot set $\mathcal{M} = \mathcal{C}_2$, there exist many homeomorphic copies of itself (called *baby Mandelbrot sets*) and the homeomorphism is given by the straightening map of renormalizations.

However, they also construct a discontinuous straightening map of a cubic-like family. Hence it is natural to ask what happens in the case of renormalizations of higher degree polynomials.

Let us denote the set of critical points of $f \in \mathrm{Poly}_d$ by $\mathrm{Crit}(f)$.

Definition. We say a polynomial $f_0 \in C_d$ ($d \geq 3$) satisfies the condition (P) if the following hold;

1. there exists a quadratic-like restriction $f_0^p : W_0' \to W_0$ hybrid equivalent to $z + z^2$;
2. there exist critical points ω_0, ω_0' and $N > 0$ such that $\omega_0 \in W_0'$ and $f_0^N(\omega_0') = \omega_0$;
3. every critical point $\omega \in \mathrm{Crit}(f_0) \setminus \{\omega_0, \omega_0'\}$ lies in Fatou set and is eventually periodic;
4. $K(f_0^p; W_0', W_0)$ is disjoint from the closure of any attracting Fatou component.

If a polynomial f is Misiurewicz (every critical point is strictly preperiodic), then we can find such f_0 arbitrarily close to f.

For a periodic point x for f of period p, let $\mathrm{mult}_f(x) = (f^p)'(x)$ be its *multiplier*.

Theorem 1. *Let $(f_\lambda : U_\lambda' \to U_\lambda)_{\lambda \in \Lambda}$ be an analytic family of polynomial-like maps of degree $d \geq 3$. Assume*

- *$g_0 = \mathcal{S}_\Lambda(f_0)$ satisfies the condition* (P);
- *$\mathcal{S}_\Lambda : C_\Lambda \to \mathcal{S}_\Lambda(C_\Lambda) \subset C_d$ is a homeomorphism and its image $\mathcal{S}_\Lambda(C_\Lambda)$ is a neighborhood of f_0 in C_d.*

Then for any repelling periodic point $x \in J(g_0^p; W_0', W_0)$, we have

$$|\mathrm{mult}_{g_0}(x)| = |\mathrm{mult}_{f_0}(\psi_0^{-1}(x))|,$$

where ψ_0 is the hybrid conjugacy between f_0 and g_0 and $g_0^p : W_0' \to W_0$ is the quadratic-like map in the definition of (P).

Theorem 2. *Let $d < d'$ and $f_\lambda \in \mathrm{Poly}_{d'}$. Assume*

1. *$(f_\lambda : U_\lambda' \to U_\lambda)_{\lambda \in \Lambda}$ be an analytic family of polynomial-like maps of degree d;*
2. *f_0 satisfies the condition* (P) *and $\{f_\lambda; \lambda \in \Lambda\}$ is a neighborhood of f_0 in $\mathrm{Poly}_{d'}$;*
3. *there exists a marked critical point $\omega_\lambda' \in \mathrm{Crit}(f_\lambda) \setminus U_\lambda'$ for $\lambda \in \Lambda$ such that ω_0' is in the definition of* (P);
4. *$(f_\lambda : U_\lambda' \to U_\lambda, f_\lambda^N(\omega_\lambda'))$ is an analytic family of polynomial-like map with marked point and its straightening map $\mathcal{S} : C_\Lambda \to \mathcal{S}(C_\Lambda) \subset \mathcal{CK}_d$ is continuous.*

Then for any repelling periodic point $x \in J(f_0^p; W_0', W_0)$, we have

$$|\mathrm{mult}_{f_0}(x)| = |\mathrm{mult}_{g_0}(\psi_0(x))|,$$

where $\mathcal{S}(0) = (g_0, \omega)$, ψ_0 is the hybrid conjugacy between f_0 and g_0, and $f_0^p : W_0' \to W_0$ is the quadratic-like map in the definition of (P).

By applying the following theorem of Prado, Przytycki and Urbanski, we can obtain much stronger result.

Theorem 3 (Prado-Przytycki-Urbanski)**.** *Let $f : U' \to U$ and $g : V' \to V$ be polynomial-like maps. Assume f and g are hybrid equivalent via a conjugacy ψ. If $|\mathrm{mult}_f(x)| = |\mathrm{mult}_g(\psi(x))|$ for any periodic point x, then f and g are analytically conjugate.*

Proposition 4. *Let f and g are polynomials. Assume that $f : U' \to U$ and $g : V' \to V$ are polynomial-like restrictions, which are analytically conjugate. Then there exist polynomials P, φ_1 and φ_2 such that $f \circ \varphi_1 = \varphi_1 \circ P$ and $g \circ \varphi_2 = \varphi_2 \circ P$. In particular, we have $\deg f = \deg g$.*

We say that f and g are *semiconjugate up to finite cover* if the conclusion of the above proposition holds.

Corollary 5. *Under the assumption of Theorem 1 or Theorem 2, if f_0 is a polynomial* (*this is contained in the assumption of Theorem 2*), *then f_0 and g_0 are semiconjugate up to finite cover.*

This means that straightening maps for a renormalizable family of polynomials are not homeomorphisms in most cases. Furthermore, if renormalizations are of capture type, then the straightening map is discontinuous.