Dynamics on character varieties

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Complex Dynamics and Related Topics

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September 3–6, 2007

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GOAL

- Study an action of the group
$$\Gamma_2^* = \left\{ M \in PGL(2,\mathbb{Z}) \; ; \; M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod (2) \right\}$$
on the family of surfaces
$$(S_{A,B,C,D}) \qquad x^2+y^2+z^2+xyz = Ax+By+Cz+D$$
by polynomial diffeomorphisms.

- Painlevé Equations #VI, monodromy of PVI.

  Iwasaki and Uehara, Inaba, Iwasaki, Saito, ...

- Quasi-Fuchsian Groups, Character Varieties

  Goldman, Benedetto, Brown, Neumann, Stantchev, Pickrell, Previte, Xia, Souto, Storm, Tan, Wong, Zhang, Yamashita, ...

- Holomorphic Dynamics.

  Bedford, Diller, Dinh, Dujardin, Fornaess, Lyubich, Sibony, Smillie, ...

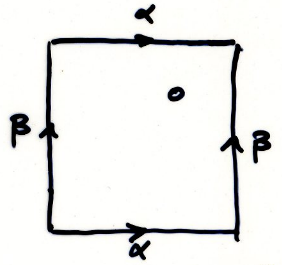
- Certain kind of "discrete Schrödinger Operators"

  Bellissard, Roberts, Casdagli, MacKay, ...

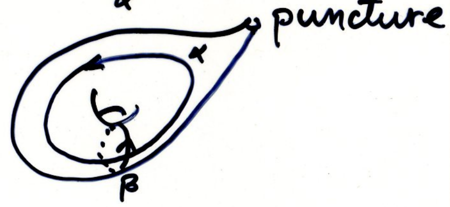
Thanks to Frank Loray (partly a joint work with him)

The Torus and The Sphere.

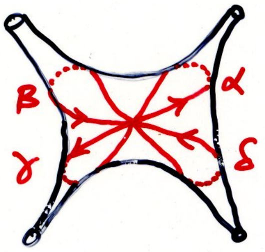
- $\mathbb{T}_1$: the once punctured torus.



$\pi_1(\mathbb{T}_1) = \langle \alpha, \beta \mid \emptyset \rangle \simeq F_2$
(free group of rank 2)
$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ makes one turn around the puncture.



- $\mathbb{S}_4$: the four punctured sphere



$\pi_1(\mathbb{S}_4) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle$
$\simeq F_3$ (free group of rank 3)

- If $X = \mathbb{T}_1$ or $\mathbb{S}_4$ then $\mathrm{euler}(X) = -1$ or $-2 < 0$.

$\Rightarrow$ $\exists\, \rho : \pi_1(X) \longrightarrow PSL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{D})$

such that $\rho(\pi_1(X))$ is a discrete subgroup of $PSL(2,\mathbb{R})$ and $\mathbb{D}/_{\rho(\pi_1(X))} \simeq X$.

Moreover, the Teichmüller space of X has real dimension 2.

- Since $\pi_1(X)$ is free, representations $\rho : \pi_1(X) \longrightarrow PSL(2,\mathbb{R})$ can be lifted to $SL(2,\mathbb{R})$.

- The Mapping Class Group of X coincides with $\mathrm{Aut}(\pi_1(X))/\mathrm{Inn}$ where Inn = inner automorphisms (= conjugations). It acts on the space of representations $\{\rho : \pi_1(X) \to SL(2,\mathbb{C})\}$ modulo $SL(2,\mathbb{C})$-conjugations.

GOAL : STUDY THIS ACTION !

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Character Varieties.

- $\mathrm{Rep}(\pi_1(X), SL(2,\mathbb{C})) = \{ \rho : \pi_1(X) \to SL(2,\mathbb{C}) ; \rho \text{ morphism} \}$

$$= \begin{cases} \{(\rho(\alpha), \rho(\beta)) \in SL(2,\mathbb{C})^2\} = SL(2,\mathbb{C})^2 \\ \text{or} \\ SL(2,\mathbb{C})^3 \text{ if } X \text{ is } \mathbb{S}_4. \end{cases}$$

- $\mathcal{X}(X) = \mathrm{Rep}(\pi_1(X), SL(2,\mathbb{C})) /\!/ \underbrace{SL(2,\mathbb{C})}$

Quotient in the sense of Geometric Invariant Theory

$SL(2,\mathbb{C})$ acts by conjugation : $(\rho, A) \mapsto A \circ \rho \circ A^{-1}$.

- **The Torus $\mathbb{T}_1$:**
  - $tr(\rho(\alpha))$, $tr(\rho(\beta))$, $tr(\rho(\alpha\beta))$ are invariant functions
  - they generate the algebra of invariant functions
  - there are no relations between these functions.

$\Rightarrow$ $\mathcal{X}(\mathbb{T}_1) = \mathbb{C}^3$, $(x,y,z) = (tr\,\rho(\alpha), \dots)$

Remark : $tr(\rho[\alpha,\beta]) = x^2 + y^2 + z^2 - xyz - 2$

- **The Sphere $\mathbb{S}_4$:**
  - $a = tr(\alpha)$ $\quad b = tr(\beta)$ $\quad c = tr\,\gamma$ $\quad d = tr(\delta)$
    $x = tr(\alpha\beta)$ $\quad y = tr(\beta\gamma)$ $\quad z = tr(\gamma\alpha)$
    generate the algebra of invariant functions.
  - They satisfy the equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$$

with $A = ab + cd$ $\quad B = bc + ad$

$C = ac + bd$ and $D = 4 - a^2 - b^2 - c^2 - d^2 - abcd$

$\Rightarrow$ $\mathcal{X}(\mathbb{S}_4^2)$ is a 6-dimensional complex quartic hypersurface in $\mathbb{C}^7$.

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Action of the Mapping Class Group

- The group $\mathrm{Aut}(\pi_1(X))$ acts on $\mathrm{Rep}(\pi_1(X), SL(2,\mathbb{C}))$ by composition :

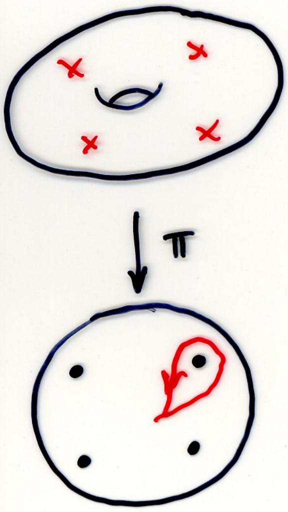
$$\rho \in \mathrm{Rep}(\pi_1(X), SL(2,\mathbb{C})),\ \Phi \in \mathrm{Aut}(\pi_1(X)) \mapsto \rho \circ \Phi .$$

- $\mathrm{Inn}(\pi_1(X)) =$ Inner automorphisms $= \{\gamma \mapsto \alpha\gamma\alpha^{-1},\ \alpha \in \pi_1(X)\}$
The group $\mathrm{Inn}(\pi_1(X))$ does not act on $\chi(X)$.

$\Rightarrow$ $\mathrm{Out}(\pi_1(X)) := \mathrm{Aut}(\pi_1(X)) / \mathrm{Inn}(\pi_1(X))$ acts on $\chi(X)$.

- The group $\mathrm{Out}(\pi_1(X))$ coincides with the mapping class group of X.

Example: The 4-punctured sphere $\mathbb{S}_4$.



$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

$\downarrow$

$\mathbb{S} = \mathbb{T}/\sigma$

where $\sigma(x,y) = (-x,-y)$

$GL(2,\mathbb{Z})$ acts on $\mathbb{T}$ and commutes with σ

$\downarrow$

$PGL(2,\mathbb{Z})$ acts on the sphere.

$H = \{(0,0), (0,\tfrac{1}{2}), (\tfrac{1}{2},0), (\tfrac{1}{2},\tfrac{1}{2})\} =$ 2-torsion of $\mathbb{T}$
also acts $\Rightarrow$ $PGL(2,\mathbb{Z}) \ltimes H$ acts on $\mathbb{S}_4$

<u>Fact</u> : This is $MCG^*(\mathbb{S}_4)$.

Remark: $\Gamma_2^* = \{ M \in PGL(2,\mathbb{Z}) \ /\ M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod (2) \}$

This group acts on $\mathbb{S}_4$ and preserves the punctures.

$\Rightarrow$ Acts on $\chi(\mathbb{S}_4)$ and preserves a,b,c,d, i.e $A, B, C,$ and D.

Automorphisms of $S_{A,B,C,D}$

- **Summary** :

The group Γ_2^* acts on the family of cubic surfaces

$(S_{A,B,C,D})$ $\qquad x^2+y^2+z^2+xyz = Ax+By+Cz+D$

where $A, B, C,$ and D are parameters (complex or real).

One wants to describe this dynamical system.

→ Tools from holomorphic dynamics are useful for that !!

Automorphisms (= polynomial diffeomorphisms)

- $s_x : (x,y,z) \in S_{A,B,C,D} \longmapsto (-x-yz+A\ ,\ y,\ z)$
- $s_y : (x,y,z) \in S_{A,B,C,D} \longmapsto (x, -y-zx+B, z)$
- $s_z : (x,y,z) \in S_{A,B,C,D} \longmapsto (x, y, -z-xy+C)$

THM (Ěl'-Huti, 1974)

- There are no relations between s_x, s_y, s_z :

$$\langle s_x, s_y, s_z\rangle \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \subset \mathrm{Aut}(S_{A,B,C,D})$$

- The index of $\langle s_x, s_y, s_z\rangle$ in $\mathrm{Aut}(X)$ is ≤ 24
- For generic A,B,C,D , $\mathrm{Aut}(X) = \langle s_x, s_y, s_z\rangle$.

- **Fact (easy computation)** : The group Γ_2^* acts on $S_{A,B,C,D}$.
  Its image in $\mathrm{Aut}(X)$ coincides with $\langle s_x, s_y, s_z\rangle$.

$$\left.\begin{array}{lll} \cdot\ s_x & \text{corresponds to} & \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ \cdot\ s_y & \quad " \qquad\quad " & \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \\ \cdot\ s_z & \quad " \qquad\quad " & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}\right\} \text{Those 3 matrices generate } \Gamma_2^*.$$

Example : $s_x \circ s_y \circ s_z$ corresponds to $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and is given by

$$(x,y,z) \mapsto \big(-x - (-y+xz+x^2y-Cx+B)(-z-xy+C)+A,$$
$$-y+xz+x^2y-Cx+B,\ -z-xy+C\big)$$

The Cayley Cubic.

- Choose $A, B, C, D = 0, 0, 0, 4$, then S is given by
$$x^2 + y^2 + z^2 + xyz = 4$$

- Consider $\eta : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$, $\eta(u,v) = \left(\frac{1}{u}, \frac{1}{v}\right)$

Then the map
$$\mathbb{C}^* \times \mathbb{C}^* \longrightarrow S_{0,0,0,4}$$
$$(u,v) \mapsto \left(-u - \frac{1}{u}, -v - \frac{1}{v}, -uv - \frac{1}{uv}\right)$$
provides an isomorphism between $S_{0,0,0,4}$ and $\mathbb{C}^* \times \mathbb{C}^* / \eta$

- $S_{0,0,0,4}$ has 4 singularities corresponding to the 4 fixed points of η : $(1,-1) \in \mathbb{C}^* \times \mathbb{C}^* \rightsquigarrow (-2, 2, 2) \in \mathrm{Sing}(S)$.

THM (Cayley, $\sim$ 1880)

$S_{0,0,0,4}$ is the unique surface in the family $S_{A,B,C,D}$ with 4 singularities
We shall call it the Cayley cubic and denote it S_C

- The group $GL(2, \mathbb{Z})$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ by monomial transformations:
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ (u,v) \in \mathbb{C}^* \times \mathbb{C}^* \longmapsto (u^a v^b, u^c v^d)$$

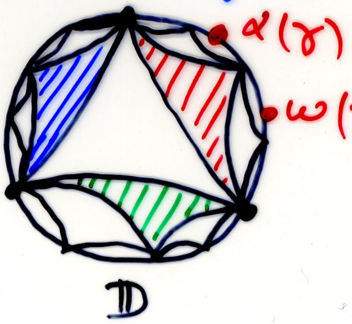
$\Rightarrow$ $PGL(2, \mathbb{Z})$ acts on S_C by polynomial diffeomorphisms

$\Rightarrow$ Γ_2^* acts on S_C : this is the same action!

- Consequence: When $A, B, C, D = 0, 0, 0, 4$,
the dynamics of Γ_2^* is "uniformized" by its usual linear action on $\mathbb{C} \times \mathbb{C}$:
$$\begin{array}{ccccc} \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & S_C \\ s, t & \longmapsto & \exp s, \exp t & \longmapsto & \left(-\frac{1}{u} - u, -v - \frac{1}{v}, -uv - \frac{1}{uv}\right) \\ \text{Linear} & & \text{Monomial} & & \end{array}$$

Action of Γ_2^* at infinity (I)

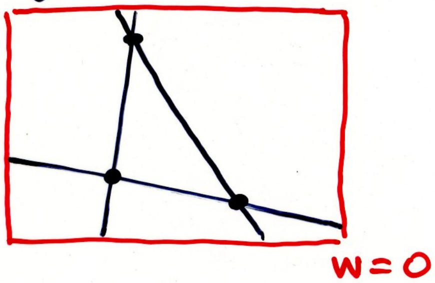
- **Description of Γ_2^*.** $\Gamma_2^* \subset PGL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{D})$ is the group of symetries of the tesselation of $\mathbb{D}$ by ideal triangles.



- **Compactification of S :** consider $\overline{S} \subset \mathbb{P}^3(\mathbb{C})$.

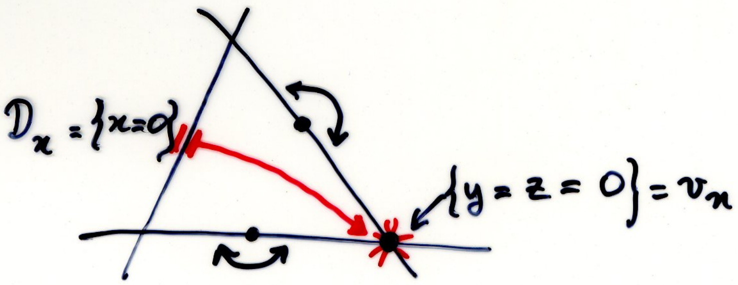
$$\overline{S} \ : \quad (x^2+y^2+z^2)w + xyz = (Ax+By+Cz)w^2 + Dw^3$$

At infinity : $xyz = 0$, $w=0$:



The group Γ_2^* acts on $\overline{S}$ by birational transformations.

- **Action of s_x at infinity :**



$\mathrm{Ind}(s_x) = \{v_x\}$

D_x is blown down on v_x

D_y and D_z are invariant.

- **Action of $s_z \circ s_y = g_x$**

$\mathrm{Ind}(g_x) = \{v_y\}$

$\mathrm{Ind}(g_x^{-1}) = \{v_z\}$

D_y and $D_z \curvearrowright v_z$

D_x is invariant.

Action of Γ_2^* at infinity (II)

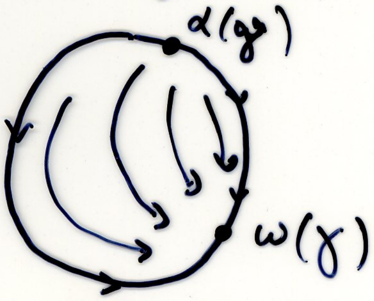
- Let $\gamma \in \Gamma_2^*$: γ corresponds to an isometry of $\mathbb{D}$
  γ corresponds to a 2×2 real matrix.

  $\lambda(\gamma) :=$ Largest $|$eigenvalue$|$ of γ.

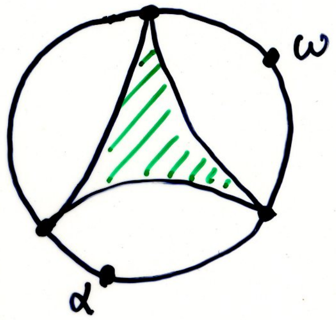
  γ is said to be hyperbolic if $\lambda(\gamma) > 1$
  γ is said to be parabolic if $\lambda(\gamma) = 1$ and $\gamma \approx \begin{pmatrix} 1 & *\neq 0 \\ 0 & 1 \end{pmatrix}$
  γ is said to be elliptic otherwise.

Fact: elliptic $\Leftrightarrow$ conjugated to s_x, s_y or s_z
parabolic $\Leftrightarrow$ " " an iterate of $s_z \circ s_y$ or $s_y \circ s_x$ or $s_x \circ s_z$.

- If γ is hyperbolic then γ has two fixed points on $\partial\mathbb{D}$ and the dynamics is:



hyperbolic isometry



up to conjugacy $\alpha(\gamma)$ and $\omega(\gamma)$ are in 2 different segments

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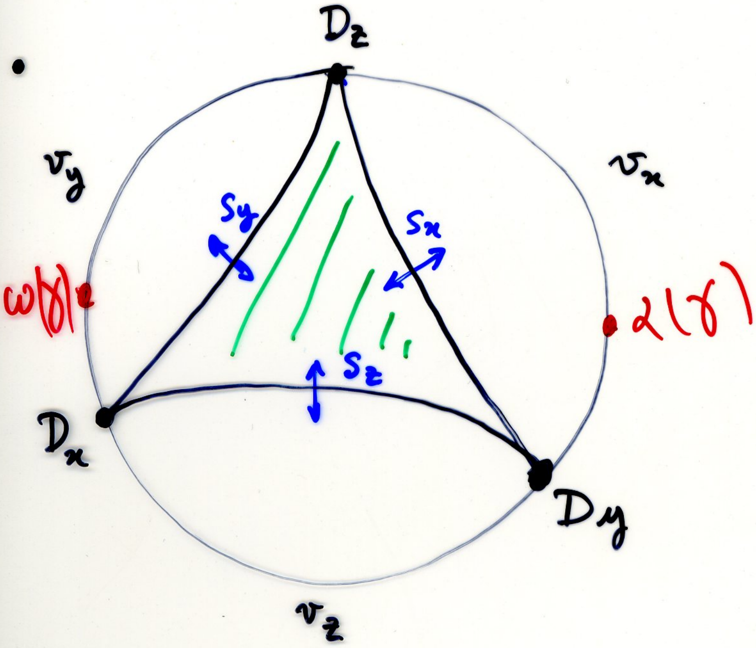
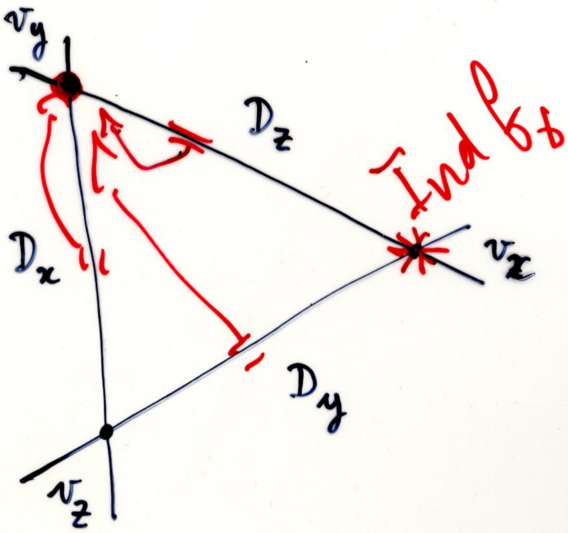
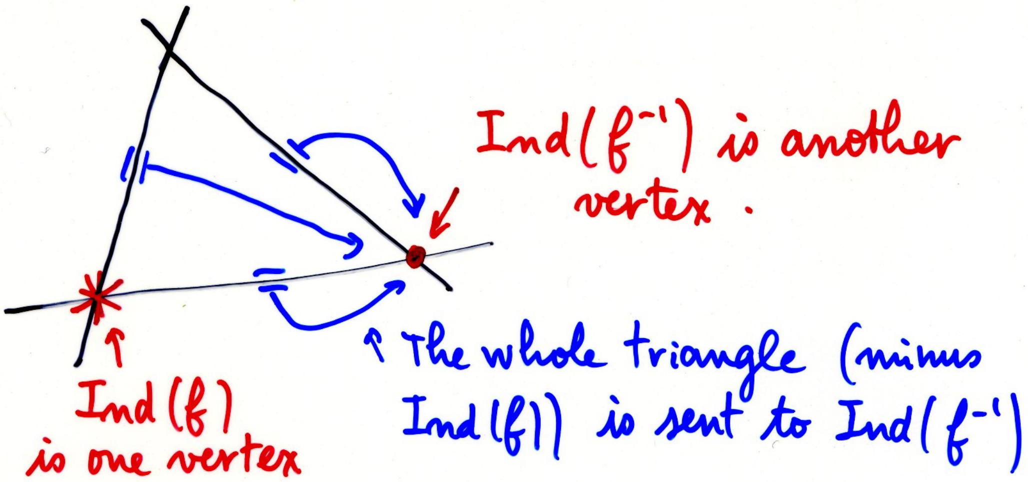




Topological Entropy.

- **Summary:** Let f be an automorphism of $S_{A,B,C,D}$. Assume that f is determined by an hyperbolic element of Γ_2^*. Then, after conjugacy in $\mathrm{Aut}(S_{A,B,C,D})$ we have:



- **Consequence:** Up to conjugacy in $\mathrm{Aut}(S_{A,B,C,D})$, f is algebraically stable.

THM (a new version of Iwasaki & Uehara)

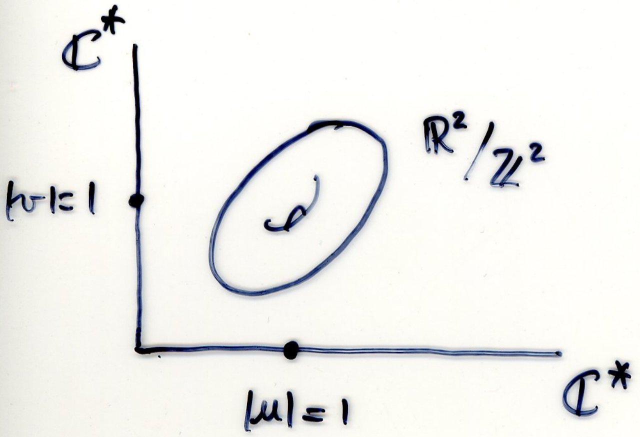
For any set of parameters $A, B, C, D \in \mathbb{C}$
For any ~~[illegible]~~ element f in $\mathrm{Aut}(S_{A,B,C,D})$,
The topological entropy of $f : S_{A,B,C,D}(\mathbb{C}) \to S_{A,B,C,D}(\mathbb{C})$ is given by

$$h_{top}(f) = \log(\lambda(f))$$

Remark: $\lambda(f) := \lambda(\gamma)^{\frac{1}{k}}$ for any $k \geqslant 1$ such that f^k is induced by $\gamma \in \Gamma_2^*$.

- **proof 1** (Smillie, Bedford & Diller, Dujardin ; Dinh & Sibony)
  - $f : S \to S$ a birational transformation of a complex projective surface.
  - $\mathrm{Ind}(f^{-1}) \cap \mathrm{Ind}(f) = \phi$, $\quad f^{-1}(\mathrm{Ind}\, f) = \mathrm{Ind}(f)$
    $f(\mathrm{Ind}\, f^{-1}) = \mathrm{Ind}(f^{-1})$
  - $f^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$
    $$\lambda(f^*) = \limsup_{n \to +\infty} \|(f^n)^*\|^{1/n}$$
    Then $h_{top}(f) = \log(\lambda(f^*))$.
  - Moreover : $H \subset S$ a hyperplane section, then
    $$h_{top}(f) = \log\left(\limsup_{n \to +\infty} \|(f^n)^*[H]\|^{1/n} \right)$$

- **proof 2** : Assume that f is induced by $\gamma \in \Gamma_2^*$.
  - The triangle at infinity is a hyperplane section of $\bar{S}_{A,B,C,D}$.
  - The action of f^* on the triangle at infinity does not depend on A, B, C, D : $f^* : \mathrm{Vect}([D_x], [D_y], [D_z]) \circlearrowleft$.
  - We compute $\lambda(f^*)$ in a specific case : The Cayley cubic case S_C.
  - In this case, the dynamics is linear :



$$h_{top}(f) = \log(\lambda(f))$$

Normal forms at infinity (I)

- **Germs of contracting holomorphic transformations (Dloussky, Favre).**

$f : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ a germ of holomorphic map near the origin.

Assume that f contracts both axes on $(0,0)$:

$$f(\{x=0\}) = f(\{y=0\}) = (0,0).$$

Let
$$f_* : \underset{\mathbb{Z}^2}{\underset{\shortparallel}{\pi_1(\mathbb{C}^* \times \mathbb{C}^*)}} \to \underset{\mathbb{Z}^2}{\pi_1(\mathbb{C}^* \times \mathbb{C}^*)}$$
be the linear map induced by f:

$$f_* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$$

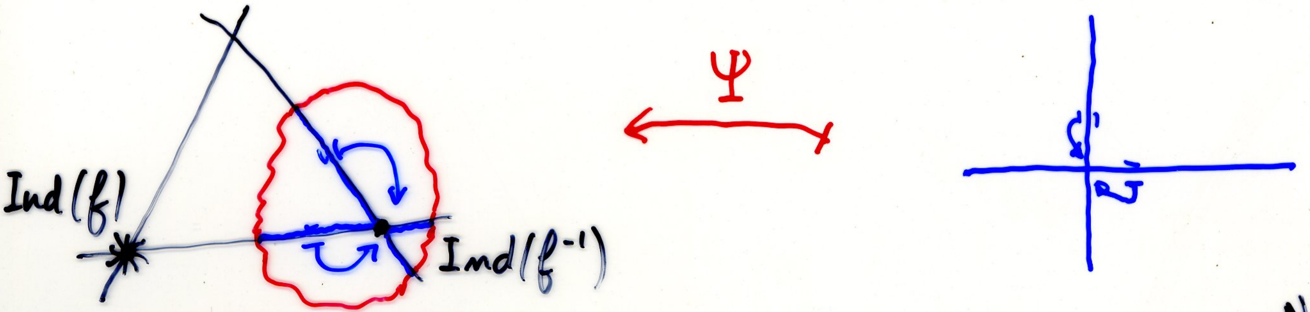
THM (Dloussky, Favre): $\exists$ a germ of holomorphic diffeomorphism $\Psi : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ such that

$$\Psi\left((x,y)^{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \right) = f(\Psi(x,y))$$

i.e. Ψ conjugates f to $(x,y) \mapsto (x^a y^b, x^c y^d)$

- **Consequence** (for $f \in \mathrm{Aut}(S_{A,B,C,D})$)

$\exists\, N_f \in GL(2,\mathbb{Z})$



f hyperbolic (after a good conjugacy in $\mathrm{Aut}(S)$)

$(u,v) \mapsto (u,v)^{N_f}$

Normal forms at infinity (II)

- **Proposition.** Let $A, B, C, D \in \mathbb{C}$.
  Let M be an element of Γ_2^*.
  Let $f : S_{A,B,C,D} \to S_{A,B,C,D}$ be the automorphism corresponding to M.
  Assume that M is hyperbolic and $\mathrm{Ind} f \neq \mathrm{Ind} f^{-1}$.
  Then
  (i) $\exists N_f$ a 2×2 integer matrix with ≥ 0 entries which is conjugate to $\pm M$.

  (ii) $\exists \Psi : (\mathbb{C}^2, 0) \longrightarrow (\overline{S}_{A,B,C,D}, \mathrm{Ind} f^{-1})$ a germ of holomorphic diffeomorphism such that
$$f(\Psi(u,v)) = \Psi\left((u,v)^{N_f}\right)$$

Remark: $\forall M \in PSL(2,\mathbb{Z})$ $\exists N$ with ≥ 0 entries such that M is conjugate to N in $PSL(2,\mathbb{Z})$.

- **Unbounded orbits:**

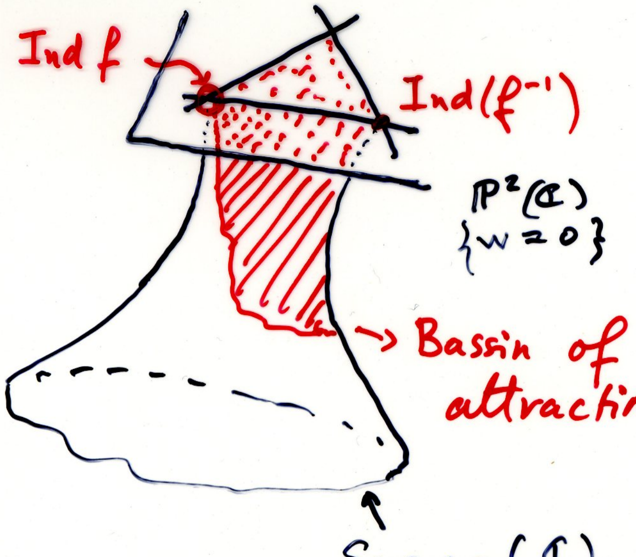
  Let $(x,y,z) \in S_{A,B,C,D}(\mathbb{C})$. Assume that the forward orbit of (x,y,z) is not bounded, then
$$f^n(x,y,z) \xrightarrow[n\to+\infty]{} \mathrm{Ind}(f^{-1})$$
  and the following limit is well defined:

Green $$G_f^+(x,y,z) = \lim_{n\to+\infty} \frac{1}{\lambda(f)^n} \log \| f^n(x,y,z) \|$$

(Here $\|(x,y,z)\| = |x|^2 + |y|^2 + |z|^2$.

Basin of attraction of $\mathrm{Ind}(f^{-1})$

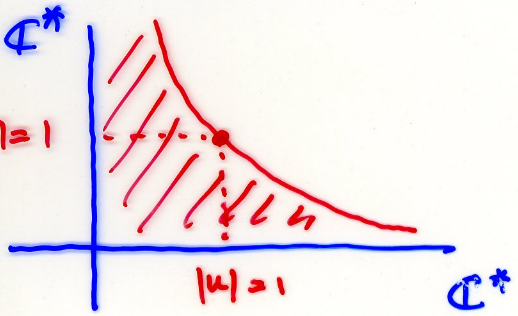
- **Basin of attraction of $\mathrm{Ind}(f^{-1})$:**



$$\Omega^*(\mathrm{Ind}(f^{-1})) = \left\{ m \in S_{A,B,C,D}(\mathbb{C}) \,;\ f^n(m) \underset{n\to+\infty}{\longrightarrow} \mathrm{Ind}(f^{-1}) \right\}$$

$$\Omega(\mathrm{Ind}(f^{-1})) = \left\{ m \in \overline{S}_{A,B,C,D}(\mathbb{C}) \,;\ f^n(m) \underset{n\to+\infty}{\longrightarrow} \mathrm{Ind}(f^{-1}) \right\}$$

- **Monomial Model:**



$$\Omega^*(N_f) = \left\{ (u,v) \in \mathbb{C}^* \times \mathbb{C}^*,\ |v| < |u|^{s(f)} \right\}$$

where $N_f \begin{pmatrix} 1 \\ s(f) \end{pmatrix} = \lambda(f) \begin{pmatrix} 1 \\ s(f) \end{pmatrix}$

(i.e. $s(f)$ is the slope of the eigenline of N_f corresponding to the eigenvalue $\lambda(f)$)

Proposition:

The conjugacy Ψ extends to a holomorphic <u>diffeomorphism</u> between $\Omega^*(N_f)$ and $\Omega^*(\mathrm{Ind}(f^{-1}))$.

Julia Sets and Currents.

- If the orbit of a point $m \in S_{A,B,C,D}(\mathbb{C})$ is unbounded, then

either $f^n(m) \underset{n \to +\infty}{\longrightarrow} \mathrm{Ind}(f^{-1})$ and $m \in \Omega^*(\mathrm{Ind}\, f^{-1})$

or $f^n(m) \underset{n \to -\infty}{\longrightarrow} \mathrm{Ind}(f)$ and $m \in \Omega^*(\mathrm{Ind}\, f)$

- Notations.

— Interesting sets —

- $K^+(f) = \{ m \mid$ the forward orbit of m is bounded$\}$
$=$ complement of $\Omega^*(\mathrm{Ind}\, f^{-1})$

$K^-(f) = \{ m \mid$ the backward orbit is bounded$\}$

$K(f) = K^+(f) \cap K^-(f)$

- $J^+(f) = \partial K^+(f)$ $\qquad$ $J^-(f) = \partial K^-(f)$

$J(f) = J^+(f) \cap J^-(f) \underset{(\neq)}{\subset} \partial K(f)$

- $J^*(f) =$ closure of the set a saddle periodic points of f.

— Eigen currents —

- $T_f^+ = dd^c G_f^+$ where $G_f^+(m) = \lim\limits_{n \to +\infty} \dfrac{1}{\lambda(f)^n} \log \| f^n(m) \|$

$T_f^- = dd^c G_f^-$ where $G_f^-(m) = \lim\limits_{n \to -\infty} \dfrac{1}{\lambda(f)^{-n}} \log \| f^n(m) \|$

- $\mu_f = T_f^+ \wedge T_f^-$

If T_f^+ and T_f^- are normalized correctly, then

μ_f is an f-invariant probality measurea.

Results from holomorphic dynamics

(Bedford, Diller, Dinh, Dujardin, Fornaess, Lyubich, Sibony, Smillie, ...)

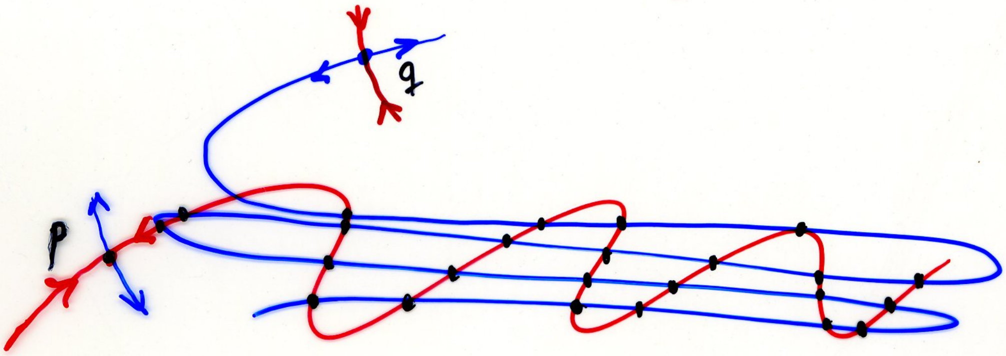
- G_f^+ and G_f^- are Hölder continuous.
  $\Rightarrow$ μ_f is well defined.

- μ_f is the unique f-invariant probability measure with maximal entropy:
$$h_\mu(f) = h_{top}(f) = \log \lambda(f)$$

- The number of periodic points of f of period N is finite (Iwasaki-Uehara: explicit formula) $\approx \lambda(f)^N$.
  Most of them are hyperbolic saddle points.
$$\frac{1}{\lambda(f)^N} \sum_{m \in Per(f,N)} \delta irac_m \xrightarrow[N \to +\infty]{} \mu_f$$
  where $Per(f,N) = \begin{cases} \text{periodic points of period } N \\ \text{or saddle periodic points } \text{———} \end{cases}$.

- $J^*(f)$ coincides with the support of μ_f.
  Any periodic saddle point is in the support of μ_f.
  If p, q are periodic saddle points then
$$\overline{W^s(p) \cap W^u(q)} = J^*(f)$$

stable manifold of p — unstable manifold of q



- If p is a saddle periodic point of f, then $W^u(p)$ is parametrized by $\mathbb{C}$:
$$\exists\, \xi : \mathbb{C} \xrightarrow[\text{holo}]{} S_{A,B,C,D}(\mathbb{C})$$
with ξ injective, $\xi(0) = p$ and $\xi(\mathbb{C}) = W^u(p)$

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, let χ be a smooth non negative function on $\xi(\mathbb{D})$ with $\chi(m) > 0$ and $\chi \equiv 0$ along $\partial\mathbb{D}$.
Let $[\xi(\mathbb{D})]$ be the current of integration on $\xi(\mathbb{D})$:
$$\langle [\xi(\mathbb{D})] \mid \alpha \text{ a 2-form} \rangle = \int_{\mathbb{D}} \xi^*\alpha .$$
Then
$$\frac{1}{\lambda(f)^n} f_*^{+n}\left(\chi\cdot[\xi(\mathbb{D})]\right) \xrightarrow[n\to+\infty]{} c^{*} T_f^-$$

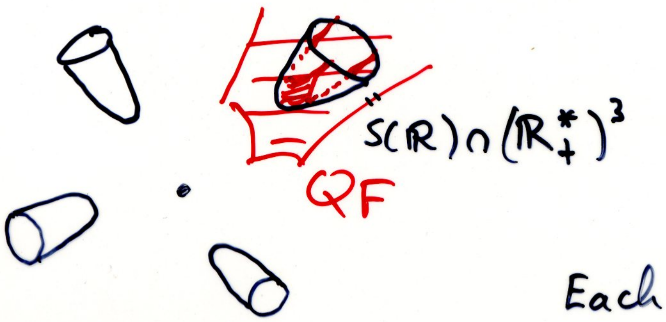
- Since f is area preserving, we have
$$\begin{aligned}\text{Interior}(K(f)) &= \text{Interior}(K^+(f))\\ &= \text{Interior}(K^-(f))\\ &= \text{bounded open subset}\\ &\quad\ \text{of } S_{A,B,C,D}(\mathbb{C}).\end{aligned}$$

The Quasi-Fuchsian Space.

- **Quasi Fuchsian Space.** (for the once punctured torus).

  - We consider $\mathcal{X}(\mathbb{T}_1) = \mathrm{Rep}(\pi_1(\mathbb{T}_1), SL(2,\mathbb{C}))/\!\!/_{SL(2,\mathbb{C})}$
  and we add the condition
  $$\mathrm{tr}(\rho[\alpha,\beta]) = -2.$$

  - The real surface $S(\mathbb{R})$: $\quad x^2+y^2+z^2 = xyz$

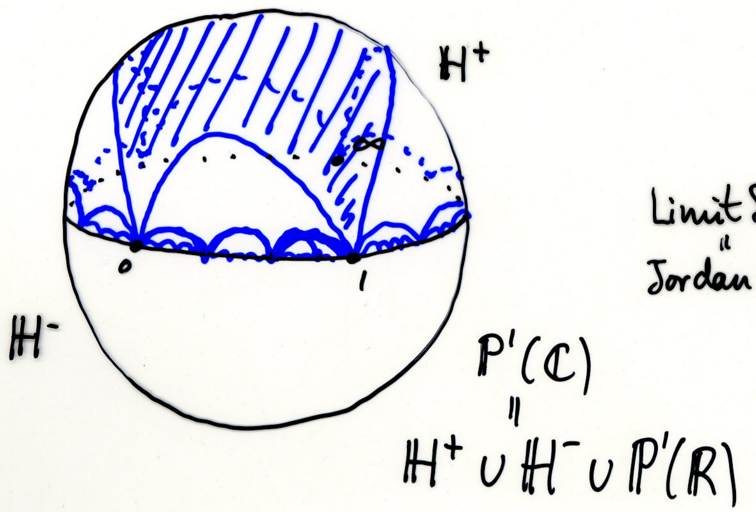


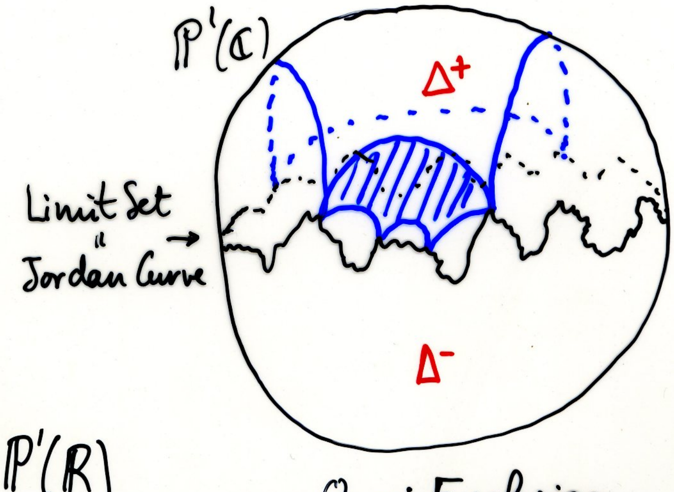
$$x = \mathrm{tr}(\rho(\alpha))$$
$$y = \mathrm{tr}(\rho(\beta))$$
$$z = \mathrm{tr}(\rho(\alpha\beta))$$

Each connected component $\neq \{(0,0,0)\}$ is homeomorphic to $\mathbb{D}$.

The action of $PGL(2,\mathbb{Z})$ ($\cong \Gamma_2^*$) on $S(\mathbb{R}) \cap (\mathbb{R}_+^*)^3$ is conjugate to the action of $MCG^*(\mathbb{T}_1)$ on $\mathrm{Teich}(\mathbb{T}_1)$, i.e. to the action of $PGL(2,\mathbb{Z})$ on $\mathbb{D}$: In particular, it is totally discontinuous.

- Quasi Fuchsian deformation.





Quasi Fuchsian.

Bers Parametrization.

- Small deformations of fuchsian representations
  → quasi fuchsian representations :

$$QF \begin{cases} \rho : F_2 = \langle \alpha, \beta \rangle \longrightarrow SL(2, \mathbb{C}) \\ \rho \text{ is faithful} \\ \rho(F_2) \text{ is discrete} \\ \rho(F_2) \text{ preserves a Jordan Curve } \Lambda \text{ and } \mathbb{P}^1(\mathbb{C}) \setminus \Lambda \\ \text{is the union of 2-invariant disks } \Delta^+ \text{ and } \Delta^- . \end{cases}$$

QF is an open subset of $S(\mathbb{C})$.

$\overline{QF} = DF := \{ [\rho] : F_2 \to SL(2, \mathbb{C}) \text{ discrete faithful} \}$

- **Bers Parametrization.**

$\mathbb{T}_1' =$ the once punctured torus, with the opposite orientation.

$\text{Teich}(\mathbb{T}_1) \simeq \mathbb{H}^+$, $\text{Teich}(\mathbb{T}_1') \simeq \mathbb{H}^-$.

$GL(2, \mathbb{Z})$ acts on $\mathbb{H}^+$ and $\mathbb{H}^-$ simultaneously.

Thm (Bers) $\exists$ Bers : $\mathbb{H}^+ \times \mathbb{H}^- \longrightarrow QF$ a holomorphic diffeomorphism such that

$$\text{Bers}(f(X), f(Y)) = f(\text{Bers}(X, Y))$$

$\forall (X, Y) \in \mathbb{H}^+ \times \mathbb{H}^- = \text{Teich}(\mathbb{T}_1) \times \text{Teich}(\mathbb{T}_1')$

$\forall f \in GL(2, \mathbb{Z}) = MCG(\mathbb{T}_1)$

This action also conjugates the action of $MCG(\mathbb{T}_1)$ on

$$\underbrace{\{(z_1, z_2) \in \mathbb{H}^+ \times \mathbb{H}^- \ / \ z_1 = \overline{z_2}\}}_{\simeq \ \text{Teich}(\mathbb{T}_1)}$$

to the action of $PGL(2, \mathbb{Z})$ on $S(\mathbb{R}) \cap (\mathbb{R}_*^+)^3$.

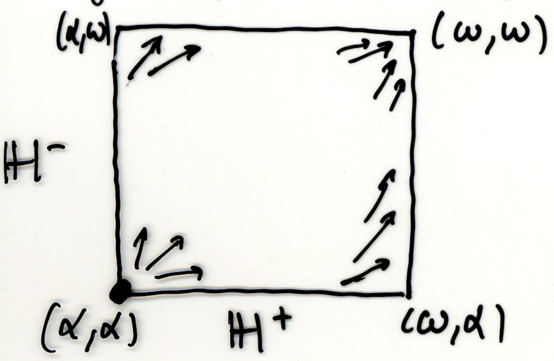
Dynamics on $\overline{QF}$

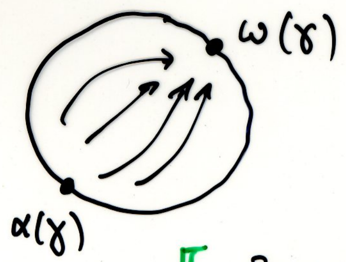
- **THM (Minsky)**

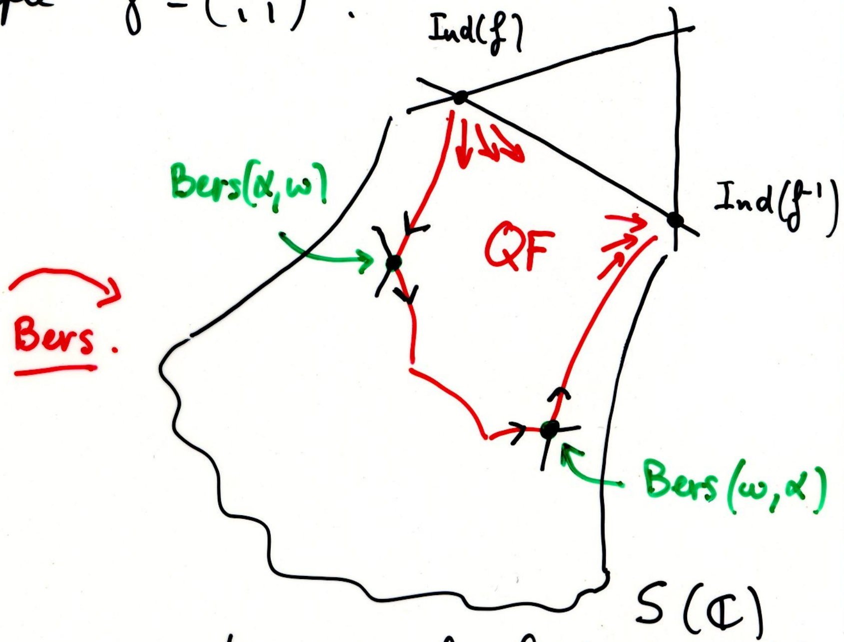
  The Bers map extends up to
  $$\partial^*(\mathbb{H}^+\times\mathbb{H}^-) = \partial(\overline{\mathbb{H}^+\times\mathbb{H}^-})\setminus\{(x,x);\, x\in\mathbb{P}^1(\mathbb{R})\}$$
  and provides a continous bijection between
  $$\overline{\mathbb{H}^+\times\mathbb{H}^-}\setminus\{(x,x)\in\mathbb{P}^1(\mathbb{R})\} \text{ and } DF.$$

- **Consequence:** Take $\gamma\in PGL(2,\mathbb{Z})$, hyperbolic.
  For example $\gamma=\begin{pmatrix}2&1\\1&1\end{pmatrix}$.

γ acts on $\mathbb{H}^+\times\mathbb{H}^-$







dynamics of f (corresponding to γ)

Fact:

Bers(α,ω) and Bers(ω,α) are two hyperbolic fixed points of f.

Bers$(\alpha,\mathbb{H}^-)\subset W^u(\text{Bers}(\alpha,\omega))$

Bers$(\mathbb{H}^+,\omega)\subset W^s(\text{Bers}(\omega,\alpha))$

Nice Orbits.

- **The origin (0,0,0)**

The point $(0,0,0) \in S$ is a singular point

$$(S) \qquad x^2 + y^2 + z^2 = xyz.$$

It corresponds to the finite representation $\rho_0 : F_2 \to SL(2,\mathbb{C})$ defined by:

$$\rho_0(\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_0(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

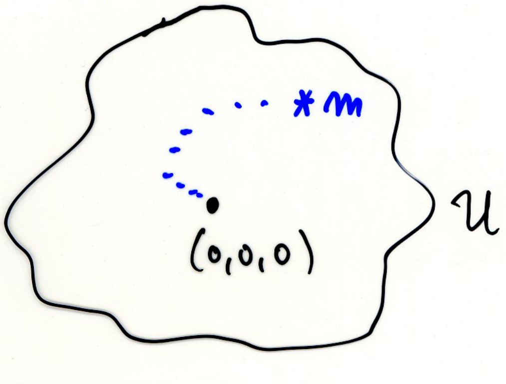
- **THM:** Let $\gamma \in PGL(2,\mathbb{Z})$ be any hyperbolic element. Let f be the automorphism of S determined by γ. Let q be one of the 2 fixed points of f on ∂QF.

  There exists $[\rho] \in S(\mathbb{C})$ such that the <u>closure</u> of the orbit $MCG(T_1)\cdot[\rho]$ contains both q and the origin $(0,0,0) = [\rho_0]$.

Proof:

Step 1 (Bowditch): $\exists$ a neighborhood $\mathcal{U}$ of the origin $((0,0,0) \in \mathcal{U} \subset S(\mathbb{C}))$ such that

$$\forall m \in \mathcal{U} \qquad \overline{MCG(T_1)\cdot m} \ni (0,0,0).$$



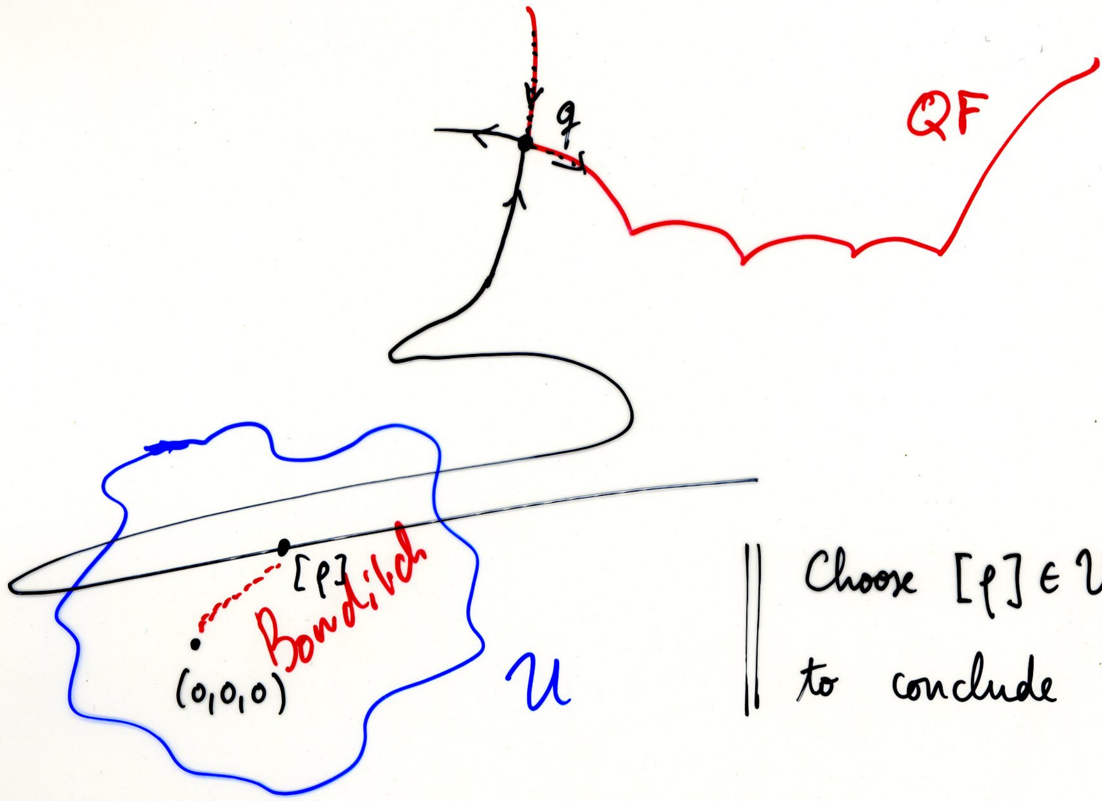
- **Step 2:**

  - $(0,0,0) \in K(f)$ because this is a fixed point.
  - If $(0,0,0) \in \mathrm{Int}(K^-(f)) = \mathrm{Int}(K(f))$, then f is linearizable at the origin
  - but $Df_{|(0,0,0)}$ has finite order and f is not periodic, so $(0,0,0) \notin \mathrm{Int}(K^-(f))$.

  $\Rightarrow (0,0,0) \in \partial K^-(f)$.

- **Conclusion:**

  Since $W^s(q)$ is dense in $\partial K^-(f)$, $W^s(q)$ intersects the open set $\mathcal{U}$.



Choose $[p] \in \mathcal{U} \cap W^s(q)$ to conclude

$\boxtimes$

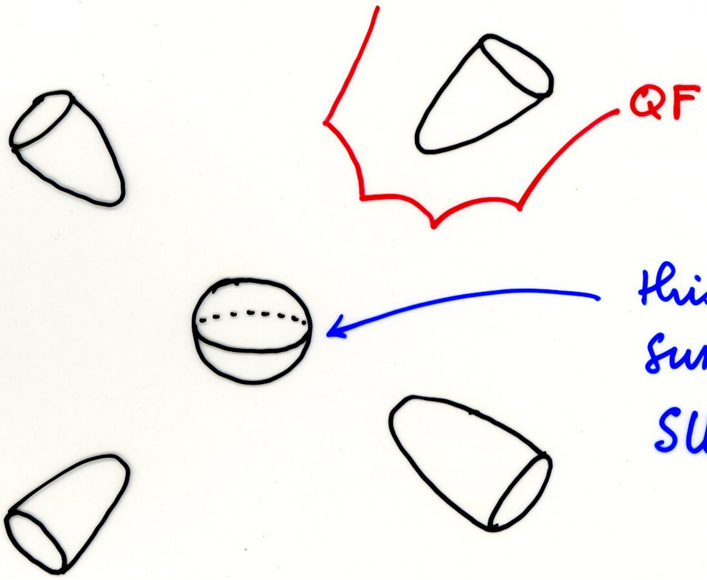
Another Example (Orbifold Structure on T_1)

- Impose the condition $\operatorname{tr}(\rho[\alpha,\beta]) = 0$.
  i.e. $\rho[\alpha,\beta]^4 = \mathrm{Id}$

The surface is now $x^2+y^2+z^2-xyz = 2$.

- We can use Teichmüller theory + quasi fuchsian deformations in the orbifold category.

- New feature: The topology of $x^2+y^2+z^2-xyz=2$.



this component of the real surface correspond to $SU(2)$ representations.

THM: $\forall \gamma \in PGL(2,\mathbb{Z})$ hyperbolic
$\forall q$ one of the 2 fixed points of f on ∂QF

If f : → has a periodic saddle point then
$\exists\ m \in \{x^2+y^2+z^2-xyz=2\}$ such that

$$f^n(m) \xrightarrow[n\to+\infty]{}$$

$$f^n(m) \xrightarrow[n\to-\infty]{} q.$$

Moreover, if $\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, this works and $\overline{MCG(T_1)\cdot m}$ contains the whole bounded component

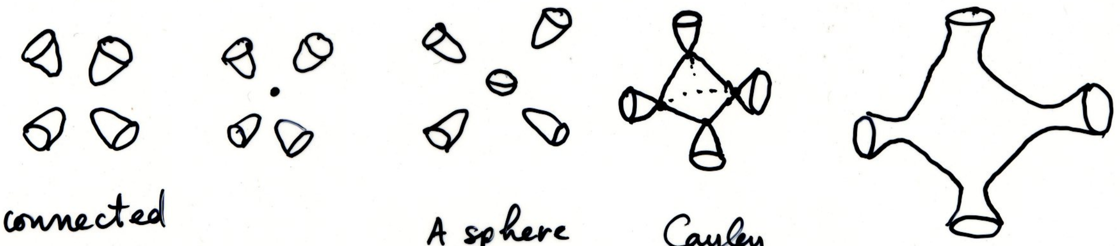
REAL versus COMPLEX Dynamics.

- Now we focus on the one parameter family

$$x^2 + y^2 + z^2 = xyz + D \qquad (S_D)$$

- **Topology of $S_D(\mathbb{R})$, $D \in \mathbb{R}$ (Benedetto, Goldman)**

$D<0$ — $D=0$ — $0<D<4$ — $D=4$ — $D>4$



4 connected components, all unbounded

A sphere appears

Cayley

Only one connected component.

- **Description of the real dynamics.** (for $f \in \mathrm{Aut}(S_D)$, hyper.)

THM

$D<0$	$D=0$	$0<D<4$	$D>4$
All periodic points of f are complex: $\mathrm{Per}(f) \subset S_D(\mathbb{C}) \setminus S_D(\mathbb{R})$	The origine is the unique real periodic point	There are always complex (= non real) periodic points.	All periodic points are real.
$\mathrm{Supp}(\mu_f) \cap S_D(\mathbb{R}) = \emptyset$		$\mathrm{Supp}(\mu_f)$ may intersects $S_D(\mathbb{R})$ but is not contained in $S_D(\mathbb{R})$	$\mathrm{Supp}(\mu_f)$ is contained in $S_D(\mathbb{R})$
$h_{top}(f_{\mathbb{R}}) = 0$	$h_{top}(f_{\mathbb{R}}) = 0$	$h_{top} <$? $\log(\lambda(f))$	$h_{top}(f_{\mathbb{R}}) = \log(\lambda(f))$
Totally discontinuous	"	Totally discont. on the 4 disks	$\boxed{\text{Uniformly hyperbolic on the Julia Set.}}$

Corollary:

Assume that A, B, C, D are real parameters.
Let $\gamma \in \Gamma_2^*$ be hyperbolic.
Let f be the automorphism of $S_{A,B,C,D}$ induced by γ.
If $S_{A,B,C,D}(\mathbb{R})$ is connected then the measure μ_f is singular with respect to the Lebesgue measure of $S_{A,B,C,D}(\mathbb{R})$; $\quad$ Haus-Dim $(\mathrm{Supp}\, \mu_f) < 2$.

Sketch of the proof. (When $A, B, C, D = 0, 0, 0, D$)

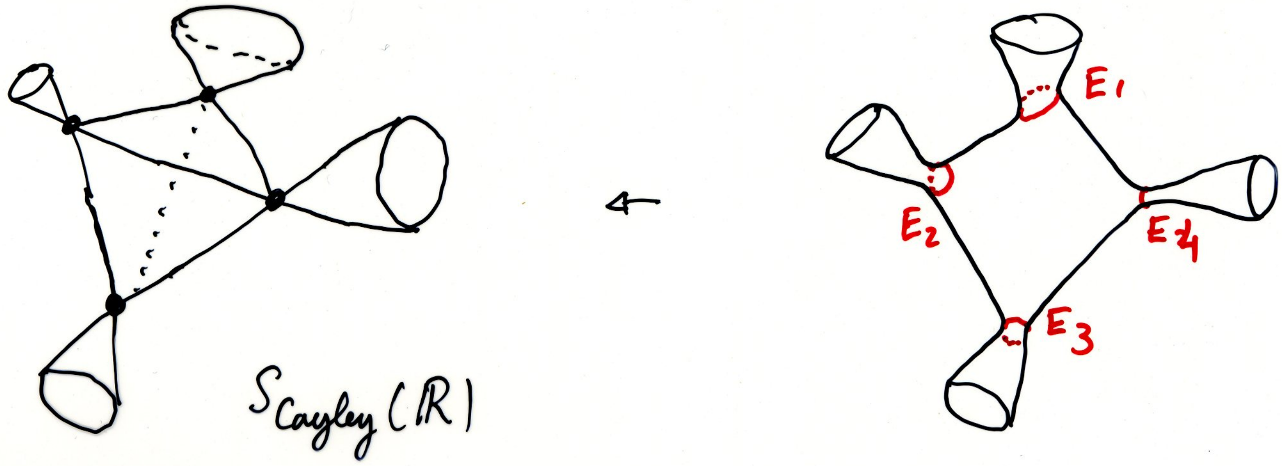
Since the surface is connected, $D \geqslant 4$ and by the previous theorem the dynamics is uniformly hyperbolic. If the Hausdorff dimension of $\mathrm{Supp}(\mu_f) = 2$, then a result of Bowen and Ruelle implies that $K(f) \cap S_D(\mathbb{R})$ is an <u>attractor</u> for $f : S_D(\mathbb{R}) \circlearrowleft$.
This contradicts the fact that $K(f)$ is compact and that f is area preserving. $\blacksquare$.

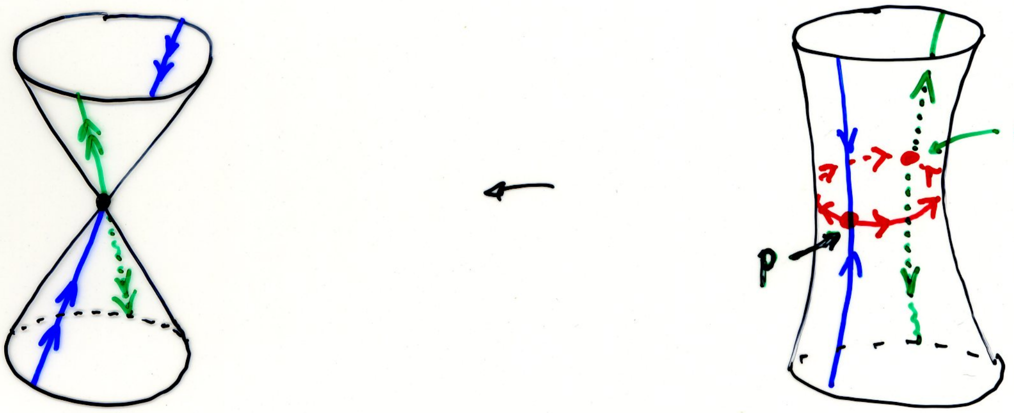
Consequence (Answer to a question by Iwasaki).

There are parameters $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ of the sixth Painlevé equation such that the monodromy along any loop with $\lambda(\gamma) > 1$ has a <u>singular</u> measure of maximal entropy.

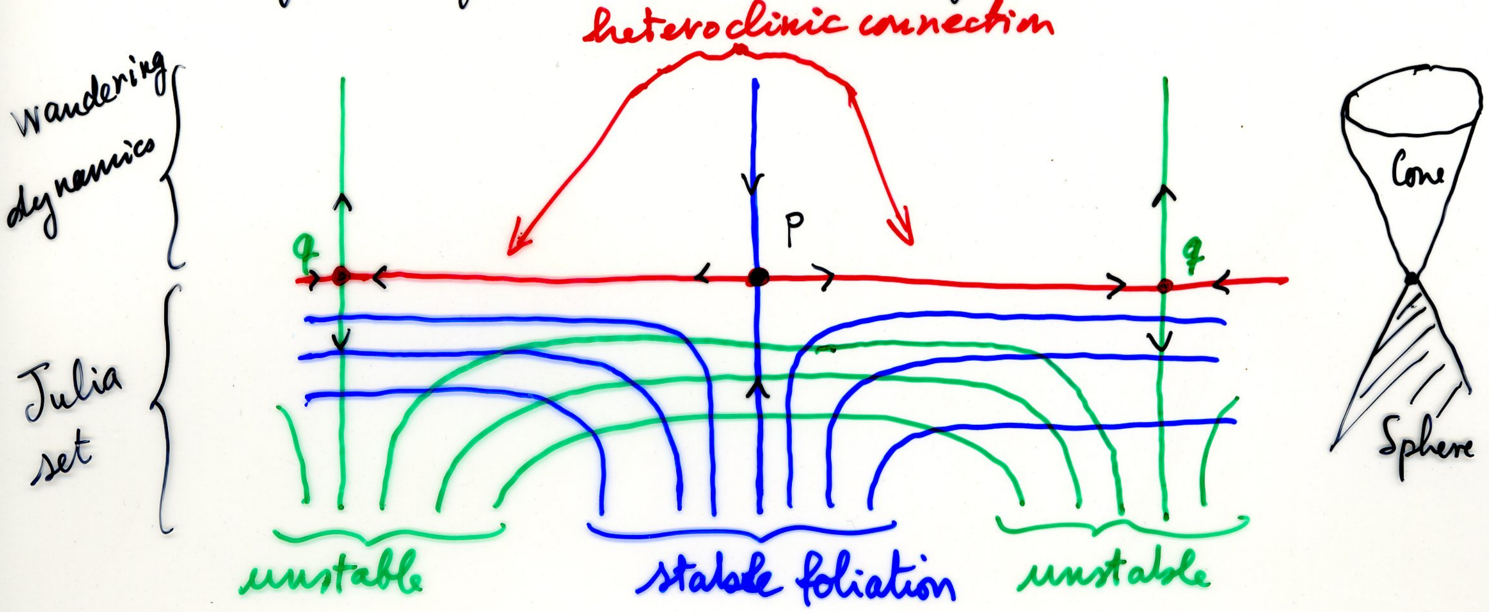
Sketch of the proof of the theorem I.

- Goal : [Prove that the dynamics is uniformly hyperbolic if $D > 4$, and that $h_{top}(f_{\mathbb{R}}) = \log(\lambda(f))$ (if $D > 4$)
- The Cayley Cubic — Blow Up Singularities.





Cut along the green unstable manifold :



Sketch of the proof of the theorem II

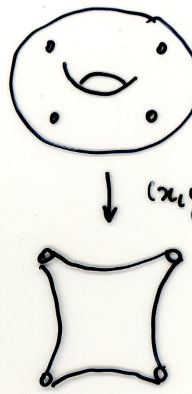
Entropy.

- To Compute the entropy we know

$$h_{top}(f_{\mathbb{R}}) \leq h_{top}(f_{\mathbb{C}}) = \log(\lambda(f))$$

New Version of Iwasaki-Uehara.

- The estimate from below comes from Bowen's inequality:



$(x,y) \sim (-x,-y)$

Sphere $\setminus$ 4 points

In the Cayley Case, we remark that if you take a generic loop $\ell \in \pi_1(\text{Sphere} \setminus 4\,\text{pts})$ then

$$\text{length } f_*^N[\ell] \sim \lambda(f)^N$$

word metric in $\pi_1(\mathbb{S}_4)$

Bowen's inequality says $h_{top}(f_{\mathbb{R}}) \geq \log(\lambda(f))$.

Since the action of f on $\pi_1(S_D(\mathbb{R}))$ does not depend on $D > 4$ and is the same as the action of $\pi_1(\mathbb{S}_4)$, we get

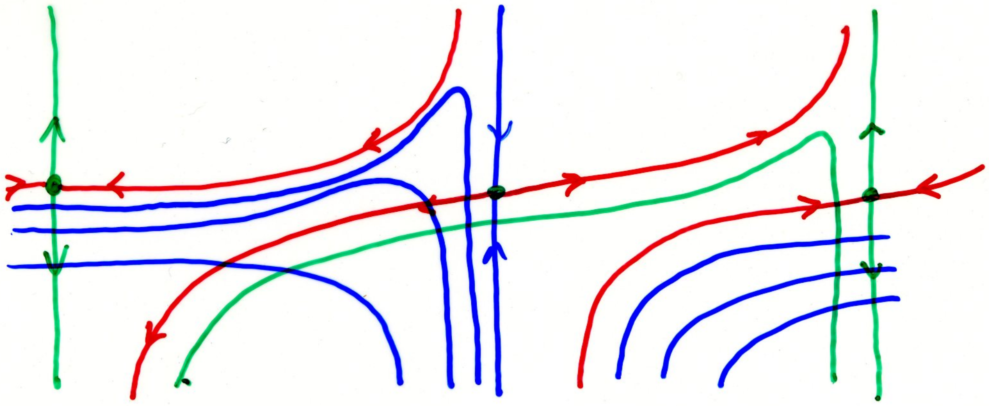
$$\forall D > 4 \qquad h_{top}(f_{\mathbb{R}}) \geq \log(\lambda(f)).$$

- In particular,

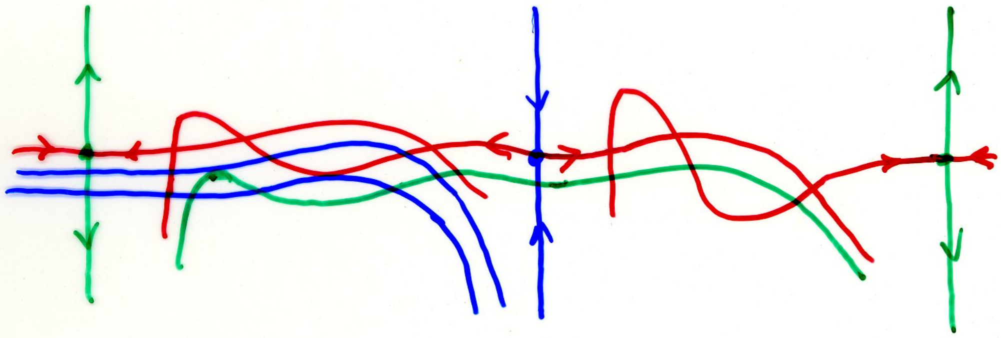
$$\left.\begin{array}{l} K(f) \subset S_D(\mathbb{R}) \\ Per(f) \subset S_D(\mathbb{R}) \\ W^s \cap W^u \subset S_D(\mathbb{R}) \end{array}\right\} \forall D \geq 4$$

Sketch of the proof of the theorem III.

- What we want to show is that the bifurcation after a small perturbation, or even a large perturbation, with $D > 4$, gives rise to the following local picture:



and not something like

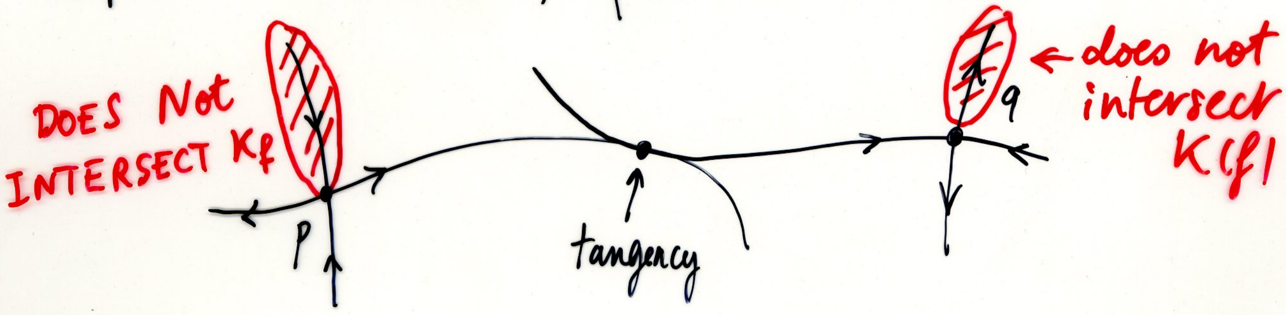


Theorem (Bedford, Smillie)

. Assume $D > 4$. If the dynamics of f on $K(f)$ is not uniformly hyperbolic then $\exists\, p, q$ saddle fixed points such that

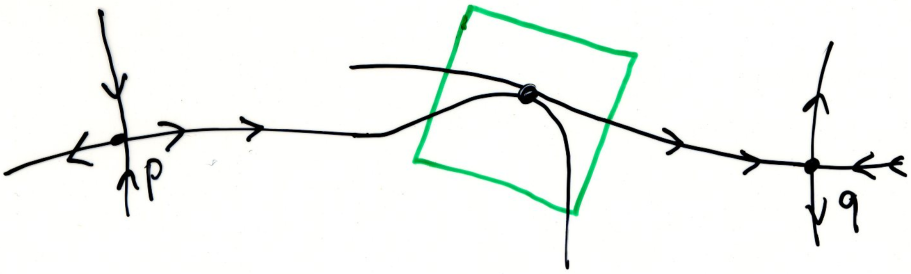
(i) $W^u(p)$ intersects $W^s(q)$ tangentially (with order 2)

(ii) p is s-one sided, q is u-one sided.

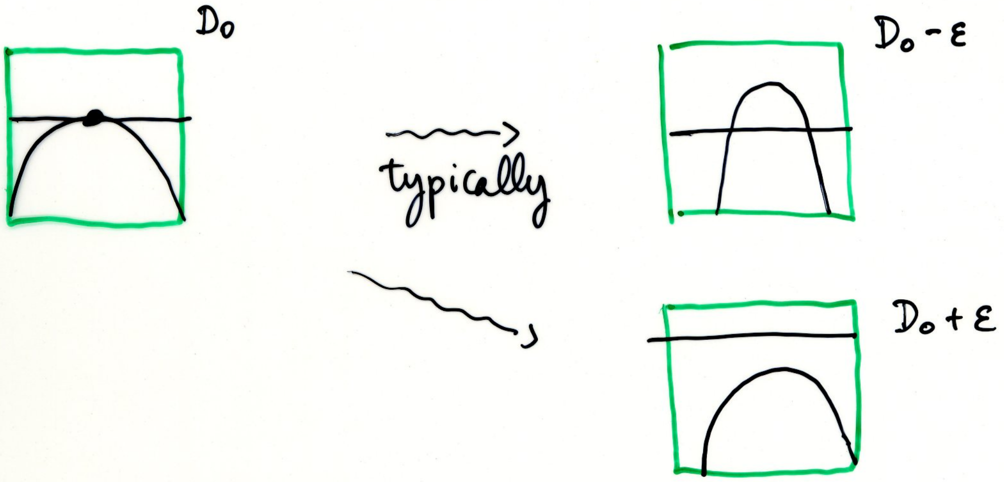


Sketch of the proof of the theorem IV.

- Assume $D_0 > 4$, not uniformly hyperbolic



- Deforme D_0 :



this "typical deformation" is not possible because for $D = D_0 + \varepsilon$, $W^u(p) \cap W^s(q) \not\subset S_D(\mathbb{R})$

- Consequence : The tangency persists when one deformes D between D_0 and 4, up to $D = 4$

- Conclusion : Get a contradiction at $D = 4$!

(Not so easy but it does work)

☒