On the Dynamical Meaning of Picard-Vessiot Theory

David Blázquez-Sanz*

Abstract

The aim of these notes is to introduce the classical Picard-Vessiot from the standpoint of complex geometry. These notes were written as materials for a lecture entitled *"On the dynamical meaning of Picard-Vessiot theory"* given in the Kyoto Dynamic Days 9 at Kyoto University and the first part of a lecture entitled *"Liouville invariant tori of completely integrable linear Hamiltonian systems from the standpoint of differential Galois theory"* at Kanazawa University. These notes only cover the core of the theory and the problem of integration by quadratures. Therefore so many interesting applications and related topics could be added to cover the ambitious title. The author hopes these notes are suitable for undergraduate students with some knowledge on Riemann surfaces, homotopy, analytic functions and group theory.

Contents

*Department of Information Engineering, Niigata University, Niigata, Japan.
Permanent Address: IMA - Universidad Sergio Arboleda, Bogotá, Colombia.
e-mail: david@ima.usergioarboleda.edu.co

1 Introduction

These notes are devoted to non-autonomous systems of homogeneous linear differential equations on the complex domain. Those systems are differential equations of the form,

$$\frac{dy}{dx} = A(x)y \tag{1}$$

where x is a complex independent variable, y is a vector of n unknown functions of x and A is an $n \times n$ matrix which depends analytically on the independent variable x.

The main property of such systems is that a $\mathbb{C}$-linear combination of solutions is also a solution. Hence, the set of its solutions form a $\mathbb{C}$-vector space of dimension at most n.

We are going to present here the classical Picard-Vessiot theory, also known as differential Galois theory for linear differential equations. The Picard-Vessiot theory deals with the possibility of obtaining an algebraic formula for the general solution of the system (1) or, at least, reduce the system (1) to a its simplest form. This theory belongs to the field of differential algebra.

There is a number of good references on differential Galois theory from a differential algebraic stand point, including [2, 3, 5, 8]. Recently, a connection between Picard-Vessiot theory and dynamical systems has been fruitfully used, for instance in [7]. However, there is not available in the literature a presentation of the Picard-Vessiot theory standing on the theory of complex differential equations and complex geometry. The beautiful book [4] covers an important part of this topic, with a nice an intuitive exposition of monodromy of differential equations, however this book is focused on Fuch-Frobenius theory and overrides the Picard-Vessiot theory. The objective of this note is to develop the fundamentals of differential Galois theory using the language and intuitions of complex differential equations. Our purpose is to point out relations between algebraic invariants of the linear differential equations, such as the differential Galois group, and some dynamical and geometric aspects. I hope it will encourage some people to apply and extend the beautiful ideas due to E. Galois. This presentation is not due to myself, but just the simple application to the particular case of linear equations of the recently developed non-linear differential Galois theory [6]. A similar exposition, slightly more general and unfortunately rather more complicated can be found in [1].

1.1 Singularities

Notation. From now on, let Γ be the domain of the independent variable x. Then Γ is an open subset of the complex projective line $\overline{\mathbb{C}}$, which is endowed with the meromorphic form dx that allows us to write down the equations (1). More generally, we should consider that Γ is any Riemann surface in which we select a meromorphic 1-form that we denote dx.[1] This meromorphic form is not so important, but we need it in order to write down our differential equations in coordinates.[2] We will denote by $\mathcal{M}(\Gamma)$ the field of meromorphic functions in Γ, that becomes a differential field.[3] For any $x \in \Gamma$ we will write $\mathcal{O}_x$ for the ring of convergent power series at x, and by $\mathcal{M}(\Gamma)_x$ the field of convergent Laurent series at x. In what follows, we consider that $A(x)$ is an $n \times n$ matrix whose entries are meromorphic functions on Γ. In other words,[4]

$$A(x) \in \mathfrak{gl}(n, \mathcal{M}(\Gamma)),$$

i.e., it is an element of the Lie algebra of the general linear group of rank n with coefficients in $\mathcal{M}(\Gamma)$.

Definition 1 Assume that dx is regular at $x_0 \in \Gamma$. We say that x_0 is a singularity of (1) if x_0 is a pole of the matrix of coefficients $A(x)$.

At the zeroes and poles of dx we just take a different local coordinate [5] in order to decide if they are singularities of (1).

Example 1 Let us consider the Airy equation,

$$\frac{d^2y}{dx^2} = xy,$$

or matrix form,

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

[1] Let us recall that a meromorphic function is a function that is locally the quotient of two analytic function. And a meromorphic 1-form in $\overline{\mathbb{C}}$ is of the form $f(x)dx$. For any Riemann surface Γ the space of meromorphic forms is a 1-dimensional vector space over the field $\mathcal{M}(\Gamma)$ of meromorphic functions. It also mean that there exist quotient of two forms $\frac{\omega_1}{\omega_2}$ as a meromorphic function. In such meaning we can speak about the derivative of a meromorphic function f with respect to another meromorphic function g, it is just the quotient $\frac{df}{dg}$ that should be seen as a quotient meromorphic 1-forms in Γ. Meromorphic forms are closed, but they do not need to be exact, for instance $\frac{dx}{x}$ is not exact in $\mathbb{C}^*$ since it is $d\log(x)$ and the logarithmic function can not be defined in the whole $\mathbb{C}^*$.

[2] The meromorphic form dx determines a derivation ∂ of the field $\mathcal{M}(\Gamma)$, we just define $\partial f = \frac{df}{dx}$. Reciprocally, any derivation $\partial\colon \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma)$ determines the 1-form $\partial f df$, which does not depend on the election of the non-constant function f.

[3] Let us recall that a differential field is a field K endowed with a derivation, that is a $\mathbb{C}$-additive map $\partial\colon K \to K$ satisfying Leibniz rule $\partial(fg) = f\partial(g) + g\partial(f)$. All the theory of this notes can be done in the language of connections which is independent of the election of the derivation. However, it is the actual belief of the author (maybe wrong) that it is convenient to speak about differential equations using differential equations.

[4] The notation $\mathrm{GL}(n, K)$ and $\mathfrak{gl}(n, K)$ is used in this text for the general linear group of invertible $n \times n$ matrices with coefficients in a field K and its Lie algebra respectively. Let us recall that in this case the Lie algebra is just the space of $n \times n$ matrices, endowed with the commutator bracket. For an abstract K-vector space E we will write $\mathrm{Aut}_K(E)$ for its group of automorphisms, which is isomorphic to the general linear group.

[5] If t is a local coordinate near a singular point of dx, then we just apply that $\frac{d}{dt} = \frac{dt}{dx}\frac{d}{dx}$ for rewriting the equation (1) near the singularity.

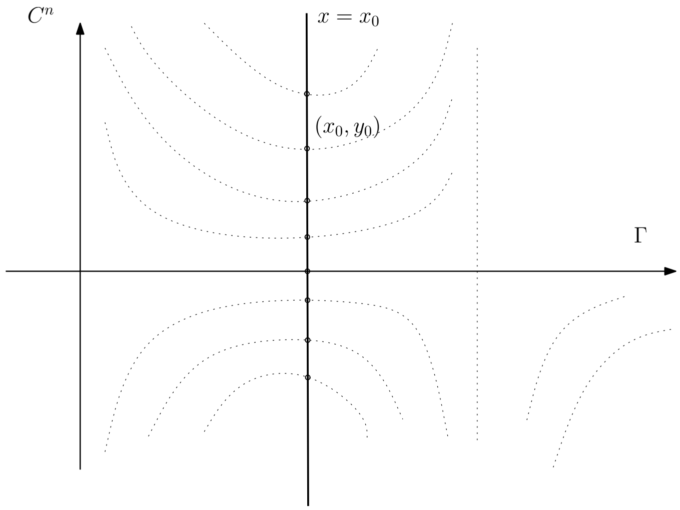

Figure 1: Near a regular point we can identify the space of germs of solutions with the space of initial conditions, $\mathbb{C}^n$.

which is defined for x varying in the complex projective line. The 1-form dx is holomorphic and it has a zero in the point at infinity. If we take $z = x^{-1}$ the point at infinity has the coordinate $z = 0$, and $dx = -z^{-2}dz$ has a pole at this point. Instead of dx we can take dz which is a regular non-vanishing 1-form at the point of infinity. We have, $\frac{d}{dx} = -z^2\frac{d}{dz}$ and hence,

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{z^2} \\ -\frac{1}{z^3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

revealing that the Airy equation has a singularity at the point of infinity. In case we want to obtain the equation written as a second order differential equation, we just eliminate the variable y_2 to obtain,

$$\frac{d^2y}{dz} = -\frac{2}{z}\frac{dy}{dz} + \frac{1}{z^5}y,$$

where $y = y_1$.

1.2 Solutions and Initial Conditions

At regular points we can solve the Cauchy problem just by series expansions, and then we have a canonical identification of the space of solutions with the space of initial conditions (see Figure 1).

Lemma 1 (Linear Superposition Principle) *Let $x \in \Gamma$ be a regular point of* (1). *Let us consider $\mathcal{S}_x \subset \mathcal{M}(\Gamma)^n_x$ the set solutions of* (1) *in $\mathcal{O}^n_x$, that is,* *germs of solutions defined around x. Then $\mathcal{S}_x$ is a $\mathbb{C}$-vector space of dimension n. The map that assigns to any initial condition $y_0 \in \mathbb{C}^n$ the germ of the solution $\bar{y}$ of (1) such that $\bar{y}(x) = y_0$, is $\mathbb{C}$-linear.*

From now of we systematically will use the notation $\mathcal{S}_x$ for the vector space of germs of solutions defined around x.

Another interesting thing about solutions is that we can prolong them analytically along any continuous path whenever we avoid singularities. This is a direct consequence of the superposition principle.

Lemma 2 *Let $\bar{y}$ be a germ of a solution of (1) defined around $x_0 \in \Gamma$. Then y can be prolonged analytically along any path that does not contain singularities of (1). Equivalently, $\bar{y}$ extends to a uniquely defined analytic solution $y(x)$ in any simply connected domain of x_0 that does not contain singularities of (1).*

Proof. Let us consider a continuous path $\gamma\colon [0,1] \to \Gamma$ such that $\gamma(0) = x_0$. For each $t \in [0,1]$ we have that there is n linear independent solutions of (1) defined in certain neighbourhood U_t of $\gamma(t)$. Being $[0,1]$ compact, and taking a refinement of our covering if necessary, we can select a finite family $t_0 = 0 < t_1 < \ldots < t_m = 1$ in such way that the m open sets $\gamma^{-1}(U_{t_i})$ are connected and cover the interval $[0,1]$. If follows easily that the analytic continuation exist. $\square$

1.3 Non-linearity near Singularities

We may think that linear differential equations do not show interesting dynamical properties. This may be true if we avoid singularities, and restrict the variation of the independent variable to simply connected (or at least with abelian fundamental group) domains.

Near a singularity, the variable dependent can be scaled in such a way that we obtain a new differential equation which is regular in such point. Take for instance, the system,

$$\frac{dy}{dx} = \frac{A(x)}{x^\alpha} y \tag{2}$$

where $A(x)$ is analytic near zero, and α is a positive integer. We can regularize such singularity by taking a new independent variable t such that $\frac{dx}{dt}x^\alpha$. Then, we obtain a regularized equation,

$$\begin{aligned} \frac{dx}{dt} &= x^\alpha \\ \frac{dy}{dt} &= A(x)y, \end{aligned}$$

which is an autonomous differential equation in x and y and has a zero at the origin. However, this equation is no more linear: the matrix $A(x)$ may contain non-linearities in the x-variable. In general, it may be non-integrable.

1.4 Global Geometry of Solutions

Let us remind that when we allow complex values for the independent variable x, the graphs of solutions of the differential equation (1) are complex curves in $\Gamma \times \mathbb{C}^n$, which can also be seen as real surfaces. Those surfaces define a foliation of equations,

$$dy_i - \sum_{j=1}^{n} a_{ij}(x) y_j dx = 0, \quad i = 1, \dots, n \tag{3}$$

in $\Gamma \times \mathbb{C}^n$. We are going to build up the Picard-Vessiot theory by studying the geometry of the leaves of such foliation.

Definition 2 A leaf of the foliation (3) is called *exceptional* if it is contained in fiber of some $x_0 \in \Gamma$ by the natural projection $\Gamma \times \mathbb{C}^n \to \Gamma$. In such a case, x_0 is a singularity of (1).

Then, we should distinguish two classes of leaves. *Regular* ones, which are locally represented by graphs of solutions of (1), and those which are exceptional and project onto singularities of (1).

It is very interesting to consider the global geometry of the leaves, and also the geometry of the leaves near singularities. We should notice that near singularities linear differential equations with meromorphic coefficients may not behave like linear differential equations with constant coefficients.

Example 2 Let us consider the equation,

$$\frac{dy}{dx} = \frac{(a+b+1)x^2 + (a-b)x - 1}{x^3 - x}$$

Its solution is easily computed in terms of an elementary integration. It yields:

$$y(x) = \lambda_1 \left(\log(x) + a\log(x-1) + b\log(x-1)\right) + \lambda$$

We have a nice formula for the general solution, which can be defined in any *simply connected* domain in $\mathbb{C} \setminus \{-1, 0, 1\}$. If we continue our solution along a closed path that goes around zero clockwise, then we obtain a different solution that correspond to a different sheet of the same complex curve,

$$y^+(x) = y(x) + 2\pi i;$$

if we choose a path that goes around 1, we get also a different solution,

$$y^\times(x) = y(x) + 2a\pi i;$$

finally if we choose a path that goes around -1, we get the solution,

$$y^-(x) = y(x) + 2b\pi i.$$

Thus, the leaf of the foliation defined by $y(x)$ contains also the graphs of all functions of the form

$$y^*(x) = y(x) + 2\pi i(k + an + bm), \quad k, n, m \in \mathbb{Z}.$$

If we assume that $1, a, b$ are linearly independent over the rational numbers, and mutually linearly independent over the real numbers, it is clear that the leaf of containing the graph of $y(x)$ is dense in $\mathbb{C}^2$.

Example 3 Let us consider the following equation:

$$\frac{dy}{dx} = \frac{(a+b)x - b}{x^2 - x} y.$$

This equation can be integrated by elementary procedures, and its solution is given by,

$$y(x) = \lambda x^a (x-1)^b. \tag{4}$$

As in the example above, the solution defined by formula (4) can be defined for x varying simply connected subset of $\mathbb{C}$ that does not contain 0 and 1. The graph of this solution is an open subset of a leaf of the foliation (3). Let us prolong y along a loop that goes around 0 counter-clockwise. We obtain a new solution whose graph lies on the same leaf of the foliation:

$$y^+(x) = e^{2\pi i a} y(x).$$

If we prolong y along a loop that goes around 1 counter-clockwise we get,

$$y^\times(x) = e^{2\pi i b} y(x).$$

It follows that the single leaf defined by y contains all the graphs of functions of the form,

$$y^*(x) = e^{2\pi i(ma+nb)} y(x)$$

with m and n integers. Assume that a and b are complex numbers such that $a, b, 1$ are linearly independent over the rational numbers and $a, b, 1$ are mutually independent over the real numbers. In such case, it automatically follows that the leaf defined by y is dense in $\mathbb{C}^2$.

2 Monodromy Representation

The set of poles of a meromorphic function in Γ is always a discrete set, and then so is the set of singularities of (1). Let us consider $\Gamma^\times$ the Riemann surface that we obtain by removing the singularities of (1) from Γ.

Given any continuous path $\gamma\colon [0,1] \to \Gamma^\times$ we can prolong any germ of solution defined around $\gamma(0)$ along γ so that we obtain a germ of solution defined around $\gamma(1)$ (see Lemma 2 and Figure 2). From the uniqueness of the analytic prolongation in simply connected domains it is clear that this germ at the final point $\gamma(1)$ depends only on the homotopy class $[\gamma]$ of γ. The analytic continuation is also $\mathbb{C}$-linear. Therefore, $[\gamma]$ induces, through analytic continuation, a $\mathbb{C}$-linear map:[6]

$$m_{[\gamma]}\colon \mathcal{S}_{\gamma(0)} \simeq \mathbb{C}^n \to \mathcal{S}_{\gamma(1)} \simeq \mathbb{C}^n.$$

Definition 3 The *homotopy grupoid* $\Pi_1(\Gamma^\times)$ is the set of the homotopy classes of continuous paths in $\gamma\colon [0,1] \to \Gamma^\times$.

The set $\Pi_1(\Gamma^\times)$ is endowed with a double natural projection, s that sends each homotopy class to its starting point, and e that send each homotopy class to its end point:

$$(s,e)\colon \Pi_1(\Gamma^\times) \to \Gamma^\times \times \Gamma^\times, \quad [\gamma] \mapsto (s([\gamma]), e([\gamma])) = (\gamma(0), \gamma(1)).$$

For each x_0 in $\Gamma^\times$ we have that $s^{-1}(x_0) \subset \Pi_1(\Gamma^\times)$ is the set of all homotopy classes of paths with starting point ant x_0. Therefore, the following projection

$$e\colon s^{-1}(x_0) \to \Gamma^\times,$$

is the standard construction of the universal covering.

[6] Let us remind that $\mathcal{S}_x$ is the notation we already introduced for the space of germs of solutions of (1) defined around $x \in \Gamma$.

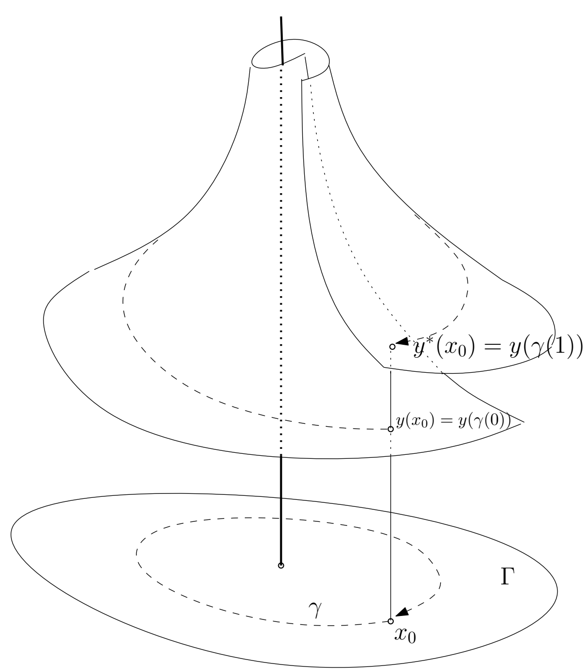

Figure 2: The monodromy matrix m_γ sends $y(x_0)$ to $y^*(x_0)$.

The natural *grupoid* structure of $\Pi_1(\Gamma^\times)$ is defined as follows. If the starting point of γ_1 coincides with the end point of γ_2 we consider the concatenation $\gamma_1 * \gamma_2$:

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_2(2t) & \text{if} \quad t \leq 1/2, \\ \gamma_1(2t-1) & \text{if} \quad t > 1/2. \end{cases}$$

We define $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$. The analytic prolongation is compatible with this procedure, so that we have:

$$m_{[\gamma_1]*[\gamma_2]} = m_{[\gamma_1]} \circ m_{[\gamma_2]}.$$

It is also clear that for any x_0 in $\Gamma^\times$ the first fundamental group $\pi_1(\Gamma^\times, x_0)$ of homotopy classes of loops with origin at x_0 is the fiber $(s,e)^{-1}(x_0, x_0)$, naturally embedded in $\Pi_1(\Gamma^\times)$.

For $[\gamma] \in \pi_1(\Gamma^\times, x_0)$ we have $m_{[\gamma]} \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{S}_{x_0})$ and using the natural identification between germs, $\mathcal{S}_{x_0}$, and initial conditions $\mathbb{C}^n$, we get a group morphism,

$$\mathrm{mon}(\Gamma^\times, x_0)\colon \pi_1(\Gamma^\times, x_0) \to \mathrm{GL}(n, \mathbb{C}),$$

which we call the *monodromy representation* of (1) based on x_0.

Definition 4 We call Monodromy group of equation (1) at $x_0 \in \Gamma^\times$, $\mathrm{Mon}(x_0, \Gamma^\times)$, to the image of the above map $\mathrm{mon}(x_0, \Gamma^\times)$.

Remark 1 The monodromy representation depends on a point x_0 of $\Gamma^\times$. We can override this dependency if we consider the monodromy representation of the homotopy groupoid into the grupoid of transversal 1-jets of linear transformations which is isomorphic to $\mathrm{GL}(n,\mathbb{C}) \times \Gamma^\times \times \Gamma^\times$. Such technical point is is some case more elegant to the eyes of the geometrist, but not necessary for the development of Picard-Vessiot theory.

Example 4 In the differential equation of Example 3, our Riemann surface $\Gamma^\times$ is $\mathbb{C} \setminus \{0,1\}$. Its fundamental homotopy group, based on any x_0 is spanned by a loop γ_0 around 0 and a loop γ_1 around 1. We consider both loops counterclockwise. Thus, the homotopy group $\pi_1(\mathbb{C} \setminus \{0,1\}, x_0) = \langle \gamma_0, \gamma_1 \rangle$, is the free non-abelian group generated by two symbols. The monodromy representation is:

$$\mathrm{mon}(\mathbb{C} \setminus \{0,1\}, x_0)\colon \pi_1(\Gamma^\times, x_0) \to \mathbb{C}^*, \quad \gamma_0 \mapsto e^{2\pi i a}, \quad \gamma_1 \mapsto e^{2\pi i b}.$$

(1) If a, b are rational, then $\mathrm{Mon}(\mathbb{C} \setminus \{0,1\}, x_0)$ is a finite group of roots of unit.

(2) If a, b are real numbers then $\mathrm{Mon}(\mathbb{C} \setminus \{0,1\}, x_0)$ is a relative compact group included in the unit circle. This group is isomorphic to $\mathbb{Z}$ if a/b is rational, otherwise it will be $\mathbb{Z}^2$.

(3) If the quotient a/b is rational then $\mathrm{Mon}(\mathbb{C}\setminus\{0,1\},x_0)$ is a discrete group isomorphic to $\mathbb{Z}$ embedded inside a logarithmic spiral.

(4) If the quotient a/b is a real irrational number then $\mathrm{Mon}(\mathbb{C}\setminus\{0,1\},x_0)$ is a dense group embedded in a logarithmic spiral.

(5) In the generic case, we have a group isomorphic to $\mathbb{Z}^2$ which is dense in $\mathbb{C}^*$.

Proposition 3 *Let γ be a path in $\Gamma^\times$, then*

$$\mathrm{Mon}(\Gamma^\times,\gamma(0)) = m_{[\gamma]}\mathrm{Mon}(\Gamma^\times,\gamma(1))m_{[\gamma]}^{-1}.$$

Proof. Just notice that the map defined by,

$$\pi_1(\Gamma^\times,\gamma(0)) \to \pi_1(\Gamma^\times,\gamma(1)), \quad [\sigma]\mapsto[\gamma*\sigma*\gamma^{-1}]$$

is an isomorphism of groups. By the compatibility between the concatenation of paths and composition of monodromy matrices we complete the proof. □

3 Univalued Solutions

Let us consider the universal covering $u\colon\widetilde{\Gamma}\to\Gamma^\times$. Let $\mathcal{M}(\widetilde{\Gamma})$ be the field of meromorphic functions in $\widetilde{\Gamma}$. Meromorphic functions in $\Gamma^\times$ lift to meromorphic functions in $\widetilde{\Gamma}$ so that we have a natural inclusion[7] $\mathcal{M}(\Gamma)\subset\mathcal{M}(\widetilde{\Gamma})$. In this way we can consider the differential equation (1) as a differential equation in $\widetilde{\Gamma}\times\mathbb{C}^n$. This has two main advantages: the domain $\widetilde{\Gamma}$ is simply connected and it contains no singularities of the differential equation.

Proposition 4 *Any leaf of the foliation* (3) *in $\widetilde{\Gamma}\times\mathbb{C}^n$ is the graph of an analytic function defined in the whole $\Gamma^\times$. Equivalently, the space $\tilde{\mathcal{S}}$ of solutions of the equation* (1) *in $\mathcal{O}(\widetilde{\Gamma})^n\subset\mathcal{M}(\widetilde{\Gamma})^n$ has complex dimension n.*

Example 5 Let us consider the differential equation

$$\frac{d}{dy} = \frac{A}{x}y,$$

with A a constant matrix. Its general solution is[8] $y=\lambda x^A y_0$ where y_0 is an arbitrary constant vector. This is in general a multivalued function. In this case the Riemann surface $\Gamma^\times$ is the pointed plane $\mathbb{C}^*$. Its universal covering is given by the exponential function,

$$u\colon\mathbb{C}\to\mathbb{C}^*, \quad z\mapsto x(z)=e^z.$$

[7] Also 1-forms lift up, so that this inclussion is in fact an extension of differential fields.

[8] In fact, the expression x^A means $\exp(\log(x)A)$. Here it happens that the example itself is the equation that motivated the notation.

By the change of variable $x = e^z$ we obtain the differential equation:

$$\frac{dy}{dz} = Ay,$$

whose general solution, $y = \exp(zA)y_0$, is univalued in $\mathbb{C}$.

Remark 2 In the general case, to give an explicit construction of the universal covering is rather complicated. For instance, the construction of the universal covering of the plane without two points is the starting point of the theory of modular functions. However, the universal covering is a very useful tool for theoretical discussions on linear differential equations.

4 Algebraic Groups and Zariski Topology

4.1 Zariski Topology in $\mathbb{C}^n$.

Let us remind that the Zariski topology in $\mathbb{C}^n$ is the topology in which the closed subsets are defined by systems of polynomial equations in the coordinates $y_1, \ldots, y_m$. A subset is closed if and only if it is an algebraic subset. The Zariski topology in a subset of $\mathbb{C}^n$ is just the restriction of the Zariski topology in $\mathbb{C}^n$.

In the space $\mathrm{GL}(n,\mathbb{C})$ we consider the matrix elements u_{ij} as coordinates. Since matrices of the general linear group are non-degenerated, the inverse of the determinant $\frac{1}{\det(u_{ij})}$ is a well-defined function. Therefore, we will consider as Zariski closed subset to the solutions of systems of polynomial equations in the coordinates u_{ij} and the inverse of the determinant $\frac{1}{\det(u_{ij})}$. This point is somehow artificial, because we can always eliminate this extra variable $\frac{1}{\det(u_{ij})}$ of our equations, but it appears easily in examples, and it is worthy to clarify that the inverse of the determinant is allowed to appear in the equations defining closed subsets of $\mathrm{GL}(n,\mathbb{C})$.[9]

Definition 5 A subgroup G of $\mathrm{GL}(n,\mathbb{C})$ is called a *linear algebraic group* if it is closed in the Zariski topology.

Linear algebraic groups are complex analytic Lie subgroups of $\mathrm{GL}(n,\mathbb{C})$. Let G be a linear algebraic group. Then G has a finite number of connected components. The connected component which contains the identity is a normal subgroup G^0 of G. It is also the smallest normal subgroup of finite index.

Example 6 Some examples of algebraic groups:

(1) Any finite group of matrices is algebraic.

[9] In fact, the ring $\mathbb{C}\left[u_{ij}, \frac{1}{\det(u_{ij})}\right]$ is the ring of *regular functions* in $\mathrm{GL}(n,\mathbb{C})$. The composition and inversion of matrices are then defined by regular functions. If we would not consider $\frac{1}{\det(u_{ij})}$, the the inversion of matrices could not be seen as an algebraic procedure.

(2) The special linear group $\mathrm{SL}(n,\mathbb{C})$ of matrices with determinant 1 is algebraic.

(3) The group of upper (or lower) triangular matrices, $\mathrm{Tr}(n,\mathbb{C})$ is algebraic.

(4) The group of orthogonal matrices,

$$O(n,\mathbb{C}) = \{A | A^t A = \mathrm{Id}\},$$

is algebraic.

(5) Any group defined by the invariance of some tensor quantities is algebraic. Conversely, any algebraic group is defined as the group of matrices that preserve some mixed tensor (Chevalley's theorem).

(6) The unimodular group,

$$\mathrm{U}(1,\mathbb{C}) = \{z \in C \,|\, z\bar{z} = 1\},$$

is *not* algebraic.

4.2 $\mathcal{M}(\Gamma)$-Zariski Topology

In a general setting, for the exposition of the Picard-Vessiot theory it is necessary some systematic study of prime ideals of the ring of polynomials with coefficients in a differential field. In the particular case of our interest, in which our field of coefficients is a field of meromorphic functions in a Riemann surface Γ, we can avoid such technical point and deal directly with the solutions of polynomial equations in the analytic space $\Gamma \times \mathbb{C}^n$.

Let $\mathcal{K} \subset \mathcal{M}(\Gamma)$ be a field of meromorphic functions in Γ.[10]

Definition 6 Let Γ^* be an open subset of Γ and $Z \subset \Gamma^* \times \mathbb{C}^n$ be a subset. We say that Z is $\mathcal{K}$-algebraic in $Z \subset \Gamma^* \times \mathbb{C}^n$ if there is a system of polynomial equations:

$$P_j(x,y) = 0, \quad j = 1,\ldots,k, \quad P_j(x,y) \in \mathcal{K}[y_1,\ldots,y_n], \tag{5}$$

such that:

(a) the poles of the coefficients of $P_j(x,y)$ are outside Γ^*;

(b) Z is the set of solutions of (5).

Definition 7 We say that $Z \subset \Gamma \times \mathbb{C}^n$ is $\mathcal{K}$-Zariski closed if there exist a covering $\{\Gamma_\alpha\}_{\alpha \in A}$ of Γ such that for all $\alpha \in A$ the set $Z_\alpha = Z \cap (\Gamma_\alpha \times \mathbb{C}^n)$ is $\mathcal{K}$-algebraic in $\Gamma_\alpha \times \mathbb{C}^n$.

[10] That is, any subfield of $\mathcal{M}(\Gamma)$.

The collection of $\mathcal{K}$-Zariski closed sets define a topology in $\Gamma\times\mathbb{C}^n$. When Γ is a compact Riemann surface then the $\mathcal{K}$-topology in $\Gamma\times\mathbb{C}^n$ is the usual Zariski topology of the product of algebraic manifolds. However, the situation is richer when we consider different field of functions in an open Riemann surface.

Remark 3 Points are closed in the $\mathcal{K}$-Zariski topology if and only if the field $\mathcal{K}$ separates points in Γ.

The $\mathcal{K}$-Zariski topology is defined in $\Gamma\times \mathrm{GL}(n,\mathbb{C})$ analogously. By restriction of this topology we also have a $\mathcal{K}$-Zariski topology in $\Gamma\times G$ for any linear algebraic group G.

5 The Automorphic Equation

5.1 Brief Review of G-Spaces

Let G be a group and X be a set. Let us recall that a G-space structure on X is just an action of G in X, [11]

$$G\times X\to X,\quad (\sigma,x)\mapsto \sigma\cdot x$$

satisfying the standard axioms of group actions. Analogously we define the notion of G-space structure by the right side.[12]

Given a point x of X, the orbit of x, denoted by $G\cdot x$, is the set of elements of the form $\sigma\cdot x$ for some σ in G. The isotropy group of x is the subgroup $H_x\subset G$ of elements $\sigma\in G$ such that $\sigma\cdot x = x$.

When G is a linear algebraic group and the action of G in a Zariski closed subset $X\subset\mathbb{C}^n$ is given by polynomials in the coordinates y_i, the matrix elements u_{ij} and the inverse of the determinant $\frac{1}{\det(u_{ij})}$, then we say that this action is algebraic. In such a case, orbits are Zariski closed and isotropy groups are also algebraic subgroups of G.

Let us recall that a G-space is called *homogeneous* if it consist of a single orbit and *free* if the isotropy groups H_x are reduced to the identity. A homogeneous and free G-space is called a *principal homogeneous G-space.*

Definition 8 Let $G\subset \mathrm{GL}(n,\mathbb{C})$ be a linear algebraic group. A *principal meromorphic G-bundle* is a subset $P\subset\Gamma\times\mathrm{GL}(n,\mathbb{C})$ such that:

(1) the image of the projection is the complement of some discrete set in Γ. If we denote this image by $\Gamma^\times$ we have that the natural projection $P\to\Gamma^\times$ is a bundle.[13]

[11] It means that the map $G\to\mathrm{End}(X)$, $\sigma\to\phi_\sigma$ defined by $\phi_\sigma(x)=\sigma\cdot x$ is a group morphism from G into $\mathrm{Aut}(X)\subset\mathrm{End}(X)$.

[12] An action by the right side is of the form $X\times G\to X$ satisfying analogous properties.

[13] It means that it is *onto* and the fibers are smooth submanifolds of constant dimension.

(2) P is $\mathcal{M}(\Gamma)$-Zariski closed in $\Gamma^\times \times \mathrm{GL}(n, \mathbb{C})$.

(3) For each $x_0 \in \Gamma^\times$ the fiber P_{x_0} of x_0 by the natural projection is a principal homogeneous G-space.

5.2 Fundamental Matrices of Solutions

A fundamental matrix of solutions for equation (1) is a $n \times n$ matrix $U(x)$ whose columns are linearly independent solutions of (1). [14]

It is clear that $U(x)$ is a fundamental matrix of solutions if and only if it is non-degenerate and it satisfy the differential equation:

$$\frac{dU}{dx} = A(x)U \tag{6}$$

which we call the *automorphic equation* associated to (1).

This differential equation can be seen as a foliation in $\Gamma \times \mathrm{GL}(n, \mathbb{C})$ defined by the equations,

$$du_{ij} - \sum_{k=1}^{n} a_{ik} y_k dx = 0, \quad i, j = 1, \ldots, n. \tag{7}$$

The advantage of considering equation (6) instead of (1) is that every single solution of the automorphic equation resumes the general solution of the original system (1). Moreover, solutions of (6) are all similar between then.

Let $U(x)$ be a solution of the automorphic system (6). By elementary computation, it is clear that given any constant y_0 in $\mathbb{C}^n$ the product $U(x)y_0$ is a solution of (1). It also follows easily that any solution of (1) can be obtained in this way from a unique fundamental matrix. Applying the same argument, if σ is a non-degenerate matrix then $U(x)\sigma$ is also a fundamental matrix of solutions. Thus, any fundamental matrix of solutions can be obtained from $U(x)$ in this way.

Notation. For $x \in \Gamma^\times$ let us denote by $\mathcal{P}_x$ the set of germs of solutions of (6) in $\mathrm{GL}(n, \mathcal{O}_x)$. We also consider the lifting of equation (6) to the universal covering $\widetilde{\Gamma}$ of $\Gamma^\times$. By $\widetilde{\mathcal{P}}$ we mean the set of solutions of (6) in $\mathrm{GL}(n, \mathcal{O}(\widetilde{\Gamma}))$. [15] The above discussion is summarized in the following result:

Theorem 5 *The group* $\mathrm{GL}(n, \mathbb{C})$ *acts on* $\mathcal{P}_x$ *for any* $x \in \Gamma^\times$ *and on* $\widetilde{\mathcal{P}}$ *by the right side. These actions,*

$$\mathcal{P}_x \times \mathrm{GL}(n, \mathbb{C}) \to \mathcal{P}_x, \quad (U(x), \sigma) \mapsto U(x)\sigma,$$

$$\widetilde{\mathcal{P}} \times \mathrm{GL}(n, \mathbb{C}) \to \widetilde{\mathcal{P}}, \quad (U(x), \sigma) \mapsto U(x)\sigma,$$

give to $\mathcal{P}_x$ *and to* $\widetilde{\mathcal{P}}$ *structures of principal homogeneous* $\mathrm{GL}(n, \mathbb{C})$*-spaces by the right side.*

[14] And therefore, such a matrix is non-degenerate.

[15] By $\mathcal{O}(\widetilde{\Gamma})$ we mean the ring of analytic functions defined in the whole surface $\widetilde{\Gamma}$.

5.3 Monodromy of Fundamental Solutions

Let us consider a leaf L of the foliation (7) in $\Gamma^{\times} \times \mathrm{GL}(n, \mathbb{C})$, and a constant non-degenerate matrix σ. The set [16]

$$L \cdot \sigma = \{(x, U\sigma) \,|\, (x, U) \in L\}$$

is also a leaf of the foliation (7). This means that the group $\mathrm{GL}(n, \mathbb{C})$ also acts on the set of leaves. Let us denote this set by $\mathcal{L}$. Since any fundamental matrix can be obtained from another one by composition with a constant matrix, it is also clear that $\mathcal{L}$ is an homogeneous $\mathrm{GL}(n, \mathbb{C})$-space.

Notation. From now on, for $x_0 \in \Gamma^{\times}$, let us denote by L_{x_0} to the leaf of the foliation (7) that passes through the point (x_0, Id) where Id represents the identity matrix.

Proposition 6 *The monodromy group* $\mathrm{Mon}(\Gamma^{\times}, x_0)$ *is the group of isotropy of the leaf* L_{x_0}.

Proof. Let γ be a continuous loop with origin at x_0. The monodromy matrix $m_{[\gamma]}$ acts on the initial conditions,

$$m_{[\gamma]} \colon \mathbb{C}^n \to \mathbb{C}^n,$$

so that the analytic prolongation along γ of the solution $y(x)$ with initial condition $y(x_0) = y_0$ will lead to the solution $y^*(x)$ with initial condition $y^*(x_0) = m_{[\gamma]} y_0$. Let $U(x)$ be a germ fundamental matrix of solutions defined around x_0 with the initial condition $U(x_0) = \mathrm{Id}$. Then, the solution $y(x)$ with initial condition $y(x_0) = y_0$ is given by $y(x) = U(x)y_0$, and the solution with initial condition $y(x_0) = m_{[\gamma]}(y_0)$ is $y^*(x) = U(x)m_{[\gamma]}(y_0)$.

So that it is clear that if we prolong analytically the germ $U(x)$ along γ we obtain the germ $U(x)m_{[\gamma]}$ near the end point. This means that the graphs of $U(x)$ and $U(x)m_{[\gamma]}$ lie on the same leaf of the foliation (7), and then the monodromy matrix $m_{[\gamma]}$ is in the isotropy group of L_{x_0}. $\square$

Being the universal covering $\widetilde{\Gamma}$ simply connected, the following result becomes apparent.

Proposition 7 *Any leaf* L *of the foliation* (7) *in* $\widetilde{\Gamma} \times \mathrm{GL}(n, \mathbb{C})$ *is the graph of an analytic solution of* (1) *defined in* $\widetilde{\Gamma}$. *The set* $\widetilde{\mathcal{L}}$ *of leaves of* (7) *in* $\widetilde{\Gamma} \times \mathrm{GL}(n, \mathbb{C})$ *is then a principal homogeneous* $\mathrm{GL}(n, \mathbb{C})$*-space.*

6 The Galois group

6.1 Galois Bundle

Let L be a regular leaf of the foliation (7) in $\Gamma^{\times} \times \mathrm{GL}(n, \mathbb{C})$. We denote its $\mathcal{M}(\Gamma)$-Zariski closure by $\overline{L}^{\mathrm{zar}}$ (see Figure 3).

Lemma 8 *The set* $\overline{L}^{\mathrm{zar}}$ *is union of leaves of the foliation* (7).

Proof. In an adequate subset $\Gamma^{*} \times \mathrm{GL}(n, \mathbb{C})$ the set $\overline{L}^{\mathrm{zar}}$ is an $\mathcal{M}(\Gamma)$-algebraic set so that it is the solution set of some algebraic equations,

$$P_k(x, U) = 0, \quad k = 1, ..., l, \quad P_k = \mathcal{M}(\Gamma)[u_{ij}],$$

which span the ideal of all polynomial equations satisfied by L in $\Gamma^{*} \times \mathrm{GL}(n, \mathbb{C})$. Let $U(x) = (u_{ij}(x))$ be a fundamental matrix of solution, defined in a sufficiently small open subset of $\Gamma^{\times}$, and such that the graph of $U(x)$ is part of the leaf L. We differentiate the above algebraic equations implicitly with respect to x to obtain:

$$0 = \frac{d}{dx} P_k(x, U(x)) = \frac{\partial P_k}{\partial x}(x, U(x)) + \sum_{i,j=1}^{n} \frac{\partial P_k}{\partial u_{ij}}(x, U(x)) \frac{du_{ij}(x)}{dx}.$$

[16] From now on, we shall use the dot "$\cdot$" for representing the product of matrices done "point by point" in subsets of $\Gamma^{\times} \times \mathrm{GL}(n, \mathbb{C})$ and $\Gamma^{\times} \times \mathbb{C}^n$. That means that, for instance, if X is a subset of $\Gamma^{\times} \times \mathrm{GL}(n, \mathbb{C})$ and $y_0 \in \mathbb{C}^n$ is a constant vector, then $X \cdot y_0$ is the set of elements of the form $(x, \sigma y_0)$ where $(x, \sigma) \in X$.

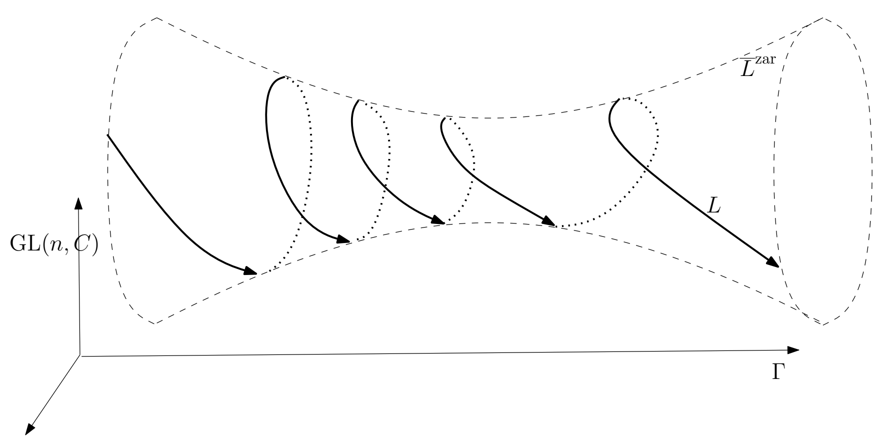

Figure 3: In order to compute the Galois group we take the $\mathcal{M}(\Gamma)$-Zariski closure $\overline{L}^{\mathrm{zar}}$ of any regular leaf L of the foliation (7).

Using equation (6), we realize that this quantity is also a polynomial in the matrix elements u_{ij},

$$\frac{d}{dx}P_k(x,U) + \sum_{i,j,k=1}^{n} a_{ik}(x)u_{kj}\frac{\partial P_k}{\partial u_{ij}}(x,U).$$

Such quantity vanishes on L, by definition, and consequently it also vanish on the Zariski closure $\overline{L}^{\mathrm{zar}}$. By iteration of the same argument we conclude that the successive derivatives $\frac{d^\alpha P_k}{dx^\alpha}(x,U)$ are also polynomials in U and then they all vanish on $\overline{L}^{\mathrm{zar}}$.

Consider any point (x_0,U_0) of $\overline{L}^{\mathrm{zar}}$ in $\Gamma^* \times \mathrm{GL}(n,\mathbb{C})$. Let $U(x)$ be a fundamental matrix of solutions such that $U(x_0)=U_0$. By hypothesis,

$$\frac{d^\alpha P_k}{dx^\alpha}(x_0,U(x_0)) = 0, \quad \alpha = 0,1,2,\dots$$

Being $P_k(x,U(x))$ a holomorphic function at x_0, it follows that $P_k(x,U(x))$ vanishes and then the graph of $U(x)$ is contained in $\overline{L}^{\mathrm{zar}}$. Since any leaf of the foliation (7) is covered by the graphs of fundamental matrices of solutions, we complete the proof. □

Definition 9 We call $\mathrm{Gal}(L)$ the Galois group of L which is the set of matrices σ such that $L\cdot\sigma \subset \overline{L}^{\mathrm{zar}}$.

Proposition 9 *The following statements hold:*

(i) $\overline{L}^{\mathrm{zar}} = \bigcup_{\sigma\in\mathrm{Gal}(L)} L\cdot\sigma$

(ii) $\mathrm{Gal}(L)$ *is a linear algebraic group.*

(iii) If L and L' are two leaves of the foliation (7) *then:*

$$\mathrm{Gal}(L) = \sigma^{-1}\cdot\mathrm{Gal}(L')\cdot\sigma,$$

for any constant non-degenerate matrix σ such that $L\cdot\sigma = L'$.

Proof. (i) The inclusion $\overline{L}^{\mathrm{zar}} \supseteq \bigcup_{\sigma\in\mathrm{Gal}(L)} L\cdot\sigma$ is a direct conclusion of definition 9. For the opposite inclusion let (x_0,U_0) be in $\overline{L}^{\mathrm{zar}}$ and let L_0 be the leaf of the foliation (7) which passes through (x_0,U_0). By Lemma 8 we see that L_0 is contained in $\overline{L}^{\mathrm{zar}}$. By Proposition 7 there exists $\sigma\in\mathrm{GL}(n,\mathbb{C})$ such that $L\cdot\sigma = L_0$, i.e., σ is in $\mathrm{Gal}(L)$. It finishes the proof of the point (1).

(ii) If L and L' are two leaves of foliation (7) having the same Zariski closure then $\mathrm{Gal}(L) = \mathrm{Gal}(L')$. It follows easily that $\mathrm{Gal}(L)$ is closed under the composition and inversion of matrices, and henceforth it is a group. Let (x_0,U_0) be a point of L. In some open subset of the form $\Gamma^*\times\mathrm{GL}(n,\mathbb{C})$ the set $\overline{L}^{\mathrm{zar}}$ is given by some polynomial equations,

$$P_k(x,U) = 0, k = 1,...,l, \quad P_k \in \mathcal{M}(\Gamma)[u_{ij},\det(u_{ij})^{-1}].$$

In virtue of the part (i) the set

$$Z = \{U_0\sigma \,|\, \sigma \in \mathrm{Gal}(L)\} \subset \mathrm{GL}(n,\mathbb{C}),$$

is given by the polynomial equations,

$$P_k(x_0,U) = 0, \quad k = 1,...,l, \quad P_k(x_0,U) \in \mathbb{C}[u_{ij}, \det(u_{ij})^{-1}].$$

Applying a left translation we obtain the equations of the Galois group,

$$Q_k(U) = P_k(x_0, U_0^{-1}U), \quad k = 1,...,l, \quad Q_k(U) \in \mathcal{M}(\Gamma)[u_{ij}, \det(u_{ij})^{-1}],$$

which clearly define a Zariski closed set, and hence $\mathrm{Gal}(L)$ is a linear algebraic group.

(iii) Let us assume that $L' = L\sigma$. The right translation,

$$R_\sigma \colon \Gamma^\times \times \mathrm{GL}(n,\mathbb{C}) \to \Gamma^\times \times \mathrm{GL}(n,\mathbb{C}), \quad (x,U) \mapsto (x,U\sigma),$$

is clearly a homeomorphism for $\mathcal{M}(\Gamma)$-topology so that $\overline{L'}^{\mathrm{zar}} = \overline{L}^{\mathrm{zar}} \cdot \sigma$. The conjugation of the Galois groups follows immediately. $\square$

Remark 4 In comparison with the algebraic theory in [3, 8], the Zariski closure $\overline{L}^{\mathrm{zar}}$ is what is usually called the *torsor space* for equation (1). The ring of regular[17] functions on $\overline{L}^{\mathrm{zar}}$ is usually called the *Picard-Vessiot algebra*.

One of the problems with the above definition of the Galois group is that its embedding into $\mathrm{GL}(n,\mathbb{C})$ is not uniquely defined. It depends on the election of a leaf of the automorphic foliation. However, we can assign canonically to each $x_0 \in \Gamma^\times$ the leaf L_{x_0} that passes through the point (x_0, Id). It leads us to the definition of the Galois bundle:

Definition 10 We call *Galois bundle* $\mathrm{Gal}(\Gamma^\times)$ to the subset of $\Gamma^\times \times \mathrm{GL}(n,\mathbb{C})$ defined by:

$$\mathrm{Gal}(\Gamma^\times) = \{(x,\sigma) \,|\, \sigma \in \mathrm{Gal}(L_x)\}$$

It is clear that the projection $\mathrm{Gal}(\Gamma^\times) \to \Gamma^\times$ is a bundle, and its fibre at x_0, is an algebraic group $\mathrm{Gal}(\Gamma^\times, x_0)$. We call it the *Galois group of* (1) *based on* x_0.

Proposition 10 $\mathrm{Gal}(\Gamma^\times)$ *is* $\mathcal{M}(\Gamma)$*-Zariski closed in* $\Gamma^\times \times \mathrm{GL}(n,\mathbb{C})$.

Proof. The proof of this proposition is not so hard, but it relies on some previous results on algebraic geometry that are beyond the purpose of these notes. Here we give here a sketch of the proof. Let us consider a regular leaf L and its $\mathcal{M}(\Gamma)$-Zariski closure T. Let us denote by G the abstract Galois group, which is an algebraic group. Then $T \to \Gamma^\times$ is a meromorphic principal bundle

[17]Rational functions having no poles on its domain of definition.

modeled over G. It is known that algebraic G-bundles are *meromorphically isotrivial.* It means that we can cover $\Gamma^\times$ by surfaces $\{\Gamma_\alpha\}_{\alpha\in A}$ in such way that for any $\alpha \in A$ there exist a finite ramified covering $\hat\Gamma_\alpha \to \Gamma$ and a meromorphic section $\sigma_\alpha(\hat x)$ of T defined on $\hat\Gamma$. Let $\hat T_\alpha$ be the lift of T to $\hat\Gamma$, and consider the following commutative diagram:

$$\begin{tikzcd}
\hat\Gamma_\alpha \times \mathrm{GL}(\mathrm{n},\mathbb{C}) \arrow[r] \arrow[dr, "\pi"'] & \hat\Gamma_\alpha \times \mathrm{GL}(\mathrm{n},\mathbb{C}) \arrow[d] & (\hat x, U) \arrow[r] \arrow[dr] & (\hat x, U\hat\sigma(\hat x)^{-1}) \arrow[d] \\
 & \Gamma_\alpha \times \mathrm{GL}(\mathrm{n},\mathbb{C}) & & (x, U\hat\sigma(\hat x)^{-1})
\end{tikzcd}$$

The projection π is the composition of a birrational isomorphism and a finite projection. It means that it is closed by the Zariski topology. We complete the proof just by realizing that $\pi(\hat T_\alpha) = \mathrm{Gal}(\Gamma_\alpha)$, where $\mathrm{Gal}(\Gamma_\alpha)$ denotes the intersection of $\mathrm{Gal}(\Gamma^\times)$ with $\Gamma_\alpha \times \mathrm{GL}(n,\mathbb{C})$. □

Definition 11 The connected component of $\mathrm{Gal}(\Gamma^\times)$ containing the identity section of $\Gamma^\times \times \mathrm{GL}(n,\mathbb{C})$ will be denoted by $\mathrm{Gal}^0(\Gamma^\times)$. It is clear and consistent with our notation, that $\mathrm{Gal}^0(\Gamma^\times, x)$ is the connected component of the identity of $\mathrm{Gal}(\Gamma^\times, x)$.

Proposition 11 *If* $\gamma\colon [0,1] \to \Gamma^\times$ *is a continuous path in* $\Gamma^\times$ *then,*

$$\mathrm{Gal}(\Gamma^\times, \gamma(0)) = m_{[\gamma]}^{-1}\mathrm{Gal}(\Gamma^\times, \gamma(1)) m_{[\gamma]}.$$

Proof. Let us consider the leaf $L_{\gamma(0)}$ that passes through $(\gamma(0), \mathrm{Id})$. By analytic continuation along γ it is clear that this leaf also passes through $(\gamma(1), m_{[\gamma]})$. Therefore, the leaf $L_{\gamma(0)} \cdot m_{[\gamma]}^{-1}$ passes through $(\gamma(0), \mathrm{Id})$. Thus, we have

$$\overline{L}^{\mathrm{zar}}_{\gamma(0)} = \overline{L}^{\mathrm{zar}}_{\gamma(1)} \cdot m_{[\gamma]},$$

and the result follows from part (iii) of Proposition 9. □

Proposition 12 *For all* $x \in \Gamma^\times$

$$\mathrm{Mon}(\Gamma^\times, x) \subseteq \mathrm{Gal}(\Gamma^\times, x).$$

Proof. Let us recall that the monodromy matrices at x_0 can be obtained by analytic prolongation along loops of the solution passing through (x_0, Id). Let E_{x_0} denote the fiber of x_0 by the projection $\Gamma^\times \times \mathrm{GL}(n,\mathbb{C}) \to \Gamma^\times$. We have $\mathrm{Mon}(\Gamma^\times, x) = L_{x_0} \cap E_{x_0}$. Also, by the definition of the Galois group we have $\mathrm{Gal}(\Gamma^\times, x) = \overline{L}^{\mathrm{zar}}_{x_0} \cap E_{x_0}$. It is clear that $L_{x_0} \subseteq \overline{L}^{\mathrm{zar}}_{x_0}$, which completes the proof. □

Theorem 13 *Let* Y *be a leaf of the foliation* (3) *in* $\Gamma^\times \times \mathbb{C}^n$. *Its Zariski closure is produced by the action of the Galois group:*

$$\overline{Y}^{\mathrm{zar}} = \mathrm{Gal}(\Gamma^\times) \cdot Y = \{(x, \sigma y) \,|\, (x,y) \in Y, \sigma \in \mathrm{Gal}(\Gamma^\times, x)\}.$$

Proof. Let us consider $(x_0, y_0) \in Y$. From Theorem 5 if follows that

$$Y = L_{x_0} \cdot y_0 = \{(x, \sigma \cdot y_0) \,|\, x \in \Gamma^\times,\ (x, \sigma) \in L_{x_0}\},$$

and hence $Y \subseteq \overline{L}_{x_0}^{\mathrm{zar}} \cdot y_0$. We have $\overline{L}_{x_0}^{\mathrm{zar}} \cdot y_0 = \mathrm{Gal}(\Gamma^\times) \cdot Y$ since the points of Y are of the form $(x, m_{[\gamma]} y_0)$, where γ is a path connecting x_0 and x. This set $\mathrm{Gal}(\Gamma^\times) \cdot Y$ is Zariski closed by construction, let us prove that it is the Zariski closure of Y. Let us consider a Zariski closed set Z such that $Y \subset Z \subset \mathrm{Gal}(\Gamma^\times) \cdot Y$. The following set,

$$M = \{(x, \sigma) \,|\, x \in \Gamma^\times, (x, \sigma y_0) \in Z\}$$

contains L_{x_0}, is Zariski closed and also $M \cdot y_0 = Z$. By definition of the Zariski closure we have $M \supseteq \overline{L}_{x_0}^{\mathrm{zar}}$ so that $Z \supseteq \overline{L}_{x_0}^{\mathrm{zar}} \cdot y_0 = \mathrm{Gal}(\Gamma^\times) \cdot Y$. □

6.2 Galois Group and Monodromy

Definition 12 A singularity $x_0 \in \Gamma$ of the differential equation (1) is called a regular singular point if any solution $y(x)$ defined in some sector Δ with vertex at x_0 satisfies the following property: there exists a function h holomorphic at x_0 such that,

$$\lim_{\substack{x \to x_0 \\ x \in \Delta}} h(x) y(x) = 0.$$

A second order linear differential equation has a regular singular point at x_0 if and only if it is written in the form:

$$\frac{d^2 y}{dx^2} + \frac{a_1(x)}{x - x_0} \frac{dy}{dx} + \frac{a_0(x)}{(x - x_0)^2} y = 0,$$

with a_0 and a_1 holomorphic at x_0. Such a simple characterization is not possible for linear systems of the form (1). It is true that systems of the form

$$\frac{dy}{dx} = \frac{A(x)}{(x - x_0)} y,$$

have a regular singularity ant x_0, but also system with higher order poles may have a regular singularity. The usual method for the characterization of a singularity is the reduction to a single higher order equation[18] However, it is well known that a differential equation of the form (1) with a regular singular point at x_0 admits admits a normal form,

$$\frac{dz}{dx} = \frac{C}{x - x_0} y,$$

with C a constant matrix, after an adequate change of variables,

$$z = B(x - x_0) y$$

where B is a matrix of convergent power series.

[18] This is called the method of the *cyclic vector*.

Definition 13 A linear differential equation is said to be *of fuchsian type* if all its singularities are regular.

Theorem 14 (Schlesinger) *Let us assume that* (1) *is of fuchsian type. Then the monodromy group is Zariski-dense in the Galois group.*

7 Integrability by Quadratures

7.1 Triangular Groups

Proposition 15 *There is a finite covering* $\bar{\Gamma} \to \Gamma$ *ramified over the singularities of* (1) *such that:*

(i) The group of covering automorphisms $\mathrm{Aut}(\bar{\Gamma}/\Gamma)$ *is isomorphic to the quotient,* $\mathrm{Gal}(\Gamma^\times, x_0)/\mathrm{Gal}^0(\Gamma^\times), x_0)$.

(ii) Let us consider the Riemann surface $\bar{\Gamma}^\times$ *obtained by removing the singularities of* (1) *from* $\bar{\Gamma}$*. The Galois bundle* $\mathrm{Gal}(\bar{\Gamma}^\times)$ *for the differential equation* (1)*, defined in* $\bar{\Gamma}$*, is the lifting to* $\bar{\Gamma}$ *of the bundle* $\mathrm{Gal}^0(\Gamma^\times)$*, and therefore its Galois group is connected.*

Proof. (i) Let L be a leaf of (7) and consider its Galois group, $\mathrm{Gal}(L)$, and its connected component of the identity, $\mathrm{Gal}^0(L)$. By Theorem 9 we have that $\mathrm{Gal}(L)$ acts on $\overline{L}^{\mathrm{zar}}$ in such a way that all fibers of the projection $\overline{L}^{\mathrm{zar}} \to \Gamma^\times$ are principal homogeneous $\mathrm{Gal}(L)$-spaces. Then we consider the quotient $\overline{L}^{\mathrm{zar}}/\mathrm{Gal}^0(L)$. Since $\mathrm{Gal}^0(L)$ is of finite index[19] the fibers of the projection are finite sets and isomorphic to the quotient $\mathrm{Gal}(L)/\mathrm{Gal}^0(L)$. Hence, it follows that this projection is a finite covering $\bar{\Gamma}^\times \to \Gamma^\times$. In general a finite covering can be completed adding exceptional points at ramifications so that we get a ramified covering $\bar{\Gamma} \to \Gamma$. The quotient group $\mathrm{Gal}(L)/\mathrm{Gal}^0(L)$ naturally acts in $\bar{\Gamma}$ by automorphisms.

(ii) Let us consider the lift of L to $\bar{\Gamma}^\times \times \mathrm{GL}(n, \mathbb{C})$. It is not connected and the number of connected components coincides with the index of $\mathrm{Gal}^0(L)$. □

Theorem 16 (Lie-Kolchin) *Let* $H \subset \mathrm{GL}(n,\mathbb{C})$ *be a connected solvable linear algebraic group. Then there exists a non-degenerate matrix* $\sigma \in \mathrm{GL}(n,\mathbb{C})$ *such that the conjugated group* $\sigma H \sigma^{-1}$ *is a subgroup of the group of upper triangular matrices* $\mathrm{Tr}(n,\mathbb{C})$.

Definition 14 Let H be a connected linear algebraic group. A *Borel subgroup* $B \subset H$ is a maximal connected solvable group of H.

By Lie-Kolchin theorem the Borel subgroups of $\mathrm{GL}(n,\mathbb{C})$ are conjugated to the group $\mathrm{Tr}(n,\mathbb{C})$ of upper triangular matrices.

[19] It means that the quotient $\mathrm{Gal}(L)/\mathrm{Gal}^0(L)$ is a finite group.

7.2 Moving Frames

Let us consider the following change of variables in the differential equations (1) and (6),

$$z = B(x)y, \quad V = B(x)U, \tag{8}$$

where $B(x)$ is a matrix whose entries are meromorphic. Then we obtain equivalent linear and automorphic systems for z and V,

$$z = G(x)z, \quad V = G(x)V,$$

where $G(x) = \frac{dB(x)}{dx}B(x)^{-1} + B(x)A(x)B(x)^{-1}$. Such kind of transformation is called a *change of frame*, and it is clear that it does not affect the Galois bundle. However it may introduce new *apparent* singularities at the poles of $B(x)$, and eliminate some apparent singularity of matrix of coefficients $A(x)$ of the original differential equation (1).

7.3 Integration by Quadratures

We can integrate (by classical methods of separation of variables and variation of constants) differential equations of the form (1) whenever the matrix of coefficients is a triangular matrix.

Lemma 17 *Assume that the coefficient matrix $A(x)$ of (1) is a upper triangular matrix. Then for each $x \in \Gamma^\times$ the Galois group $\mathrm{Gal}(\Gamma^\times, x)$ is a subgroup of the group of upper triangular matrices.*

For a general equation of the form (1), if we can find an adequate change of frame (8) such that the new matrix of coefficients is written in triangular form, then we can integrate our differential equation by elementary methods.

In order to give the reduction algorithm for reducing a differential equation to triangular form, we need the following result whose proof will be not included here. This result fits inside the theory of Galois cohomology, and it is equivalent to the vanishing of the first Galois cohomology set $H^1(H, \mathcal{M}(\Gamma))$ for any connected solvable group.

Lemma 18 *Let $H \subset \mathrm{GL}(n, \mathbb{C})$ be a connected solvable group, and let Let $P \subset \Gamma^\times \times \mathrm{GL}(n, \mathbb{C})$ be a meromorphic H-bundle over $\Gamma^\times$. Then there is a meromorphic section of P defined on $\Gamma^\times$.*

Proposition 19 *The Galois group $\mathrm{Gal}(\Gamma^\times, x_0)$ is a subgroup of some Borel subgroup of $\mathrm{GL}(n, \mathbb{C})$ if and only if there is a change of frame $B(x) \in \mathrm{GL}(n, \mathcal{M}(\Gamma))$ such that the transformed system,*

$$z = G(x)z, \tag{9}$$

is written in triangular form, and therefore it can be integrated by quadratures.

Proof. Let L be a regular leaf of the automorphic foliation (7). By hypothesis $\mathrm{Gal}(L) \subset \mathrm{B}$ for a certain Borel subgroup. All Borel subgroups are conjugated to $\mathrm{Tr}(n,\mathbb{C})$, so that there exists $\sigma \in \mathrm{GL}(n,\mathbb{C})$ such that $\sigma \mathrm{B} \sigma^{-1} = \mathrm{Tr}(n,\mathbb{C})$. We consider leaf $L_1 = L \cdot \sigma$. By part (iii) of Proposition 9 we have that $\mathrm{Gal}(L_1) \subseteq \mathrm{Tr}(n,\mathbb{C})$.

We define the following $\mathcal{M}(\Gamma)$-Zariski closed set P which contains $\overline{L}_1^{\mathrm{zar}}$:

$$P = \overline{L}_1^{\mathrm{zar}} \cdot \mathrm{Tr}(n,\mathbb{C}) = \{(x, m\cdot\sigma) \,|\, (x,m) \in \overline{L}_1^{\mathrm{zar}},\, \sigma \in \mathrm{Tr}(n,\mathbb{C})\}.$$

It is not hard to see that $P \to \Gamma^\times$ is a meromorphic $\mathrm{Tr}(n,\mathbb{C})$-bundle over Γ. By Lemma 18 there is a meromorphic section $\sigma(x)$ of this bundle. Let us consider the change of variables (8) with $B(x) = \sigma(x)^{-1}$. We get the transformed the differential equation,

$$z = G(x)z, \quad V = G(x)V,$$

obtained after the change of frame (8). From now on we will also remove from Γ the new singularities, so that $\Gamma^\times$ does not contain singularities of the new equation.

The set $L_2 = B(x) \cdot L_1$ is now a leaf of the new automorphic equation. We apply the same transformation to the principal bundle P so that we obtain that $P \cdot B(x) = \Gamma^\times \times \mathrm{Tr}(n,\mathbb{C})$. Hence:

$$L_2 \subset \Gamma^\times \times \mathrm{Tr}(n,\mathbb{C}).$$

Let us consider any point $x_0 \in \Gamma^\times$. There is a germ $V(x)$ of a fundamental matrix of solutions of (9) such that its graph lies on the leaf L_2. Therefore, $V(x)$ is a triangular matrix for all x in its domain of definition. From equation (9) we have:

$$G(x) = \frac{dV(x)}{dx} V(x)^{-1}.$$

The matrix $\frac{dV(x)}{dx}$ is upper triangular, and so is $V(x)^{-1}$ so that $G(x)$ is upper triangular in the domain of definition of $V(x)$. It follows that $G(x)$ is a upper triangular matrix.

□

Lemma 20 *Let $\bar\Gamma \to \Gamma$ be a finite ramified covering, and let $\bar\Gamma^\times$ be the surface obtained by removing from $\bar\Gamma$ the singularities of equation (1) as a system with coefficients in $\mathcal{M}(\bar\Gamma)$. Let us consider $\bar x$ in $\bar\Gamma^\times$ and x its image in $\Gamma^\times$. Then,* $\mathrm{Gal}^0(\Gamma^\times, x) = \mathrm{Gal}^0(\Gamma^\times, x)$.

By combination of Lemma 20 Propositions 15 and 19 if follows the following final result on integration by quadratures and algebraic functions.

Theorem 21 *The connected component of the identity* $\mathrm{Gal}^0(\Gamma^\times, x_0)$ *of the Galois group is solvable if and only if there exist a finite ramified covering $\bar\Gamma$ and* *a change of frame with meromorphic coefficients in* $\bar{\Gamma}$, $B(x) \in \mathrm{GL}(n, \mathcal{M}(\bar{\Gamma}))$, *such that the transformed system,*

$$z = G(x)z,$$

is written in triangular form, and therefore it can be integrated by quadratures.

Acknowledgements

Thanks to Professor Kazuyuki Yagasaki for his valuable help during the proof-reading of these notes. My current research is partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for JSPS Fellows #21·09222, and by MICINN-FEDER grant MTM2009-06973, CUR-DIUE grant 2009SGR859 and Universidad Sergio Arboleda.

References

[1] D. Blázquez Sanz, *"Differential Galois Theory and Lie-Vessiot Systems,"* VDM publishing 2008.

[2] I. Kaplansky, *"An introduction to differential algebra",* Hermann, Paris 1957.

[3] E. R. Kolchin, *Differential Algebra and Algebraic Groups,* Academic Press, New York 1973.

[4] M. Kuga, *Galois' dream: group theory and differential equations.* Translated from the 1968 Japanese original by Susan Addington and Motohico Mulase. Birkhäuser Boston, Inc., Boston, MA, 1993.

[5] A. R. Magid, *"Lectures on differential Galois theory",* University Lecture Series, 7. American Mathematical Society, Providence, RI, 1994.

[6] B. Malgrange, *"Le grupoïde de Galois d'un feuilletage,"* Monographie **38** vol **2** de L'ensegnaiment mathématique (2001).

[7] J.J. Morales-Ruiz, *"Differential Galois Theory and Non-Integrability of hamiltonian systems,"* Progress in Mathematics **179**, Birkhäuser (1999).

[8] M. Vanderput, M. Singer, *"Galois theory of linear differential equations"*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 328. Springer-Verlag, Berlin, 2003.