RIMS Workshop
Dynamical Systems and Applications: Recent Progress

Abstracts

Eusebius Doedel (Concordia University, Canada)

Title: Numerical methods for bifurcation problems

In the first lecture I will review some basic concepts and techniques used in the numerical computation of families of solutions to systems of nonlinear equations. These include parameter continuation, and, more importantly, pseudo-arclength continuation. Elementary examples will include a predator-prey model and the Gelfand-Bratu Problem. An introduction to the numerical continuation of singular points (e.g., folds and Hopf bifurcations) will also be given.

In the second lecture I will consider continuation of solutions of general boundary value problems in ordinary differential equations, with emphasis on discretization, using high order accurate finite element collocation methods with adaptive meshes. As an important application I will consider the specific case of computing families of periodic solutions of systems of nonlinear ordinary differential equations. The special case of periodic solutions of conservative systems will also be mentioned briefly; this case will be considered in more detail in the lectures of Andre Vanderbauwhede.

The third lecture will deal with some recent applications of numerical continuation to the computation of 2D manifolds that arise in dynamical systems; in particular, stable and unstable manifolds of stationary points and of periodic orbits. The specific approach considered is very effective for problems having, for example, stable eigenvalues of greatly different magnitude associated with a stationary point. Applications to a pacemaker model and to the Lorenz equations will be considered.

Marty Golubitsky (University of Houston, USA)

Title: Coupled cell systems

These lectures discuss a recently developed theory of patterns of synchrony and synchrony-breaking bifurcation in coupled cell systems based on network arechitecture.

A coupled cell system is a collection of interacting dynamical systems. Coupled cell models assume that the output from each cell is important and that signals from two or more cells can be compared so that patterns of synchrony can emerge. We ask: How much of the qualitative dynamics observed in coupled cells is the product of network architecture and how much depends on the specific equations? The ideas will be illustrated through a series of examples and theorems.

Network architecture is a graph that indicates which cells have the same phase space, which cells are coupled to which, and which couplings are the same. Coupled cell systems with a given architecture are called admissible. In this theory, local network symmetries (which form a groupoid) generalize symmetry as a way of organizing network dynamics, and synchrony-breaking bifurcations in admissible systems replace symmetry-breaking bifurcations in equivariant systems as a basic way in which transitions to complicated dynamics occur.

One theorem gives necessary and sufficient conditions for synchrony in terms of network architecture; a second shows that synchronous dynamics may itself be viewed as dynamics in a quotient coupled cell system, and a third discusses the possible spatio-temporal symmetries that periodic solutions can have. Examples of steady-state and Hopf bifurcations from synchronous equilibria will be given. These examples show that quite complicated normal forms can occur in codimension one in bifurcations in admissible systems.

These lectures are based on joint research with Ian Stewart and a number of co-authors: Marcus Pivato, Andrew Torok, Matthew Nicol, Yunjiao Wang, Maria Leite, Toby Elmhirst, Ana Dias, and Fernando Antoneli.

John Mallet-Paret (Brown University, USA)

Title: Functional differential equations

<u>Singularly Perturbed State-Dependent Delay Equations</u> (2 lectures)

We discuss recent results on differential-delay equations of the form

$$\varepsilon \dot{x}(t) = f(x(t), x(t-r)).$$

Here the delay $r \geq 0$ is either a constant, or else can vary in a state-dependent fashion $r = r(x(t))$. Results on asymptotic behavior of solutions (particularly periodic solutions) for small ε are obtained. A variety of mathematical techniques are used, including global bifurcation and degree theory, a priori estimates, geometric singular perturbation theory, and the theory of max-plus operators.

Intriguing numerical results suggest a very rich structure, particularly in the case of multiple delay problems. Although the theory here is in its infancy, some new techniques seem very promising.

Dynamics of Lattice Differential Equations (1 lecture)

Lattice differential equations are dynamical systems which are discrete in space and continuous in time. Of particular interest are sponaneously generated patterns (for example stripes or checks), spatial chaos, and traveling front solutions between equilibria which may either be spatially homogeneous or which exhibit regular patterns. Also of interest are the effects of anisotropy of the lattice, in particular propagation failure of fronts, and the effect of random imperfections in the lattice.

Michał Misiurewicz (IUPUI, USA)

Title: Rotation Theory

Rotation Theory has its roots in the theory of rotation numbers for circle homeomorphisms, developed by Poincare. It is particularly useful for the study and classification of periodic orbits of dynamical systems. Its main idea is to consider limits of ergodic averages not at almost all points, like in Ergodic Theory, but for all points. We present the general ideas of the Rotation Theory and its applications to some classes of dynamical systems, like continuous circle maps homotopic to the identity, torus homeomorphisms homotopic to the identity, subshifts of finite type and continuous interval maps.

Tim Sauer (George Mason University, USA)

Title: Attractor reconstruction from data

On the 25th anniversary of Takens' Theorem, we provide an introduction to the theory and application of data-driven attractor reconstruction. We will review theorems of Whitney and Takens, as well as later generalizations. Applications of attractor reconstruction such as noise filtering, time series prediction and chaos control will be discussed.

In the second lecture, recent extensions of attractor reconstruction will be surveyed. Three important examples include versions for (1) stochastic systems, (2) skew systems, and (3) interspike intervals. Applications will be discussed along with future directions.

References:

[1] F. Takens, Detecting strange attractors in turbulence, in Lecture Notes in Mathematics, **898**, Springer-Verlag (1981).

[2] T. Sauer, J. A. Yorke, M. Casdagli, Embedology. J. Statistical Physics, **65**, 579–616 (1991).

[3] E. Ott, T. Sauer, J. A. Yorke, Coping with Chaos. Wiley Interscience, New York (1994).
[4] T. Sauer, Reconstruction of dynamical systems from interspike intervals. Phys. Rev. Lett., **72**, 3811–3814 (1994).
[5] H. Kantz, T. Schreiber, Nonlinear Time Series Analysis. Cambridge Univ. Press (1999).
[6] J. Stark, D.S. Broomhead, M.E. Davies and J. Huke, Delay embeddings of forced systems II: Stochastic forcing, J. Nonlinear Science, **13**, 519–577 (2003).

Masato Tsujii (Hokkaido University, Japan)

Title: Functional-analytic methods in hyperbolic dynamical systems

In this series of lectures, we address some functional-analytic methods in the study of hyperbolic dynamical systems. In the study of chaotic dynamical systems from the view point of ergodic theory, one efficient method is to look at their actions on the spaces of functions: The pull-back operators (Koopman operators) and their formal adjoints (Perron-Frobenius operators). Indeed, many important properties, such as exponential decay of correlations, are expressed in terms of spectral properties of these operators.

To consider spectral properties of operators, it is primarily important to choose appropriate Banach spaces of functions as the operand. In classical results on expanding and piecewise expanding dynamical systems, the Banach spaces are taken as sets of function in the usual sense, such as the Hölder spaces of C^r functions and the space of functions with bounded variation. However those Banach spaces, or any other Banach spaces of functions in the usual sense, do not work in the case of hyperbolic dynamical systems with both contracting and expanding directions. This and preceding study of analytic diffeomorphism by H.H Rugh motivate us to consider Banach spaces of generalized functions (distributions).

The first major progress in this direction is achieved by C. Liverani. More recently, V. Baladi and the author introduced different kind of Banach spaces of distributions defined in terms of Fourier analysis, which are natural anisotropic versions of the usual Hölder and Sobolev spaces. In the lectures, we aim to explain the recent results obtained by using those Banach spaces.

In the first lecture, we begin with some classical results for motivation and then introduce the Banach spaces of distributions mentioned above. In the second, we give results on spectral properties and dynamical Fredholm determinant of the Ruelle transfer operators for hyperbolic diffeomorphisms. In the third, we discuss the parallel argument for hyperbolic flows.

Andre Vanderbauwhede (University of Ghent, Belgium)

Title: Continuation and bifurcation of periodic orbits in Hamiltonian and reversible systems

In these lectures we describe and discuss several aspects of the branching behaviour of periodic orbits in Hamiltonian and reversible systems. Typically in such systems (and in contrast with what happens in general systems) periodic orbits come in families which are at least one-dimensional but whose dimension may increase when additional first integrals are present. Such systems appear frequently in applications; we just mention all kinds of pendula, and the n-body problem in celestial mechanics.

In the first lecture we concentrate on the continuation of periodic orbits in Hamiltonian systems having several constants of motion (symmetries). We describe some theoretical continuation results and show how these results can be implemented numerically. The main idea is to reformulate the problem in such a way that it leads to one-dimensional solution branches which are theoretically given by the implicit function theorem, and numerically by a pseudo-arclength method. To be more specific, we modify the Hamiltonian vectorfield by adding a linear combination of the gradients of the first integrals (including the Hamiltonian itself); the coefficients of these linear combinations are considered as additional parameters. One can easily show that along periodic orbits these parameters must necessarily be zero. The method works when the periodic orbits which one wants to continue are "normal", a concept which originated in earlier work of Sepulchre and MacKay and which we have extended to the situation considered here. We describe several explicit applications of the method; in particular, we will show an extensive collection of graphics illustrating the intricate branching behaviour of periodic orbits of the three-body problem.

In the second lecture we will be concerned with the continuation of so-called "doubly-symmetric solutions" in reversible systems with one or more first integrals. In many cases such doubly-symmetric solutions are automatically periodic. We first introduce the concept of a "quasi-submersive mapping" and obtain the main properties of such mappings. Next we introduce "normality conditions" under which a certain class of "constrained mappings" are quasi-submersive at their zeros; this then leads to some general continuation results for such zeros. The continuation problem for doubly-symmetric solutions can be reformulated in terms of such constrained mappings, and the normality condition can be specified in terms of an appropriate monodromy matrix. We illustrate the theoretical results with some numerical study of the continuation of the figure-eight and the supereight choreographies in the n-body problem.

The topic of the third lecture is subharmonic bifurcation in reversible systems. We first describe the generic bifurcation pattern which is as follows. In the considered time-reversible systems symmetric periodic orbits typically appear in one-parameter families; along such families simple multipliers can be locked on the unit circle, and when they pass a root of unity one sees under generic conditions two bifurcating branches of subharmonic periodic orbits, one stable, one unstable. These generic conditions are: (i) the simplicity of the multiplier (together with the non-existence of other resonant multipliers), and (ii) a transversality condition which requires that the root of unity is passed with non-zero speed (using an appropriate parametrization of the primary family). We show how these results can be obtained using the Poincaré map and a general Liapunov-Schmidt type of reduction. Next we will describe how the bifurcation scenario changes when either one of these generic conditions is not satisfied, as can happen in one- or more parameter families of reversible systems. The main emphasis will be on case (ii) where so-called "bananas" and "banana-splits" appear. Such bananas can be found for example along the short period family of periodic orbits emanating from L4 (and L5) for a certain range of the mass ratio in the restricted 3-body problem; they play an important role in the bifurcations of the long period family as L4 passes through a sequence of resonances.

The work presented in these lectures was done in collaboration with Maria-Cristina Ciocci (Imperial College), Francesco Javier Muñoz-Almaraz (Valencia), Emilio Freire and Jorge Galán (University of Sevilla), and Eusebius Doedel (Concordia University, Montreal).

References:

Lecture 1:

F.J. Muñoz-Almaraz, E. Freire, J. Galán, E.J. Doedel and A. Vanderbauwhede, Continuation of periodic orbits in conservative and Hamiltonian systems, Physica D**181** (2003), 1–38.

E.J. Doedel, R.C. Paffenroth, H.B. Keller, D.J. Dichmann, J. Galán and A. Vanderbauwhede, Continuation of periodic solutions in conservative systems with application to the 3-Body problem, Int. J. Bifurcation and Chaos, **13** (2003), 1353–1381.

Lecture 2:

F.J. Muñoz-Almaraz, E. Freire, J. Galán, E.J. Doedel and A. Vanderbauwhede, Continuation of normal doubly symmetric orbits in conservative reversible systems, Preprint 2006. Submitted to Celestial Mechanics.

Lecture 3:

M.-C. Ciocci and A. Vanderbauwhede, Bifurcation of Periodic Points in Reversible Diffeomorphisms New Progress in Difference Equations. Proceedings ICDEA2001, Augsburg, S. Elaydi, G. Ladas and B. Aulbach (Eds.), Chapman & Hall, CRC Press (2004), 75–93.