# The Hilbert scheme of two point of Enriques surface

Hayasi Taro*

department of mathematics of Osaka University

## Introduction

Throughout this paper, we work over $\mathbb{C}$. We denote the Hilbert scheme of $n$ points of a surface $S$ by $S^{[n]} = \mathrm{Hilb}^n S$. Let $E$ be an Enriques surface. $E^{[n]}$ has a Calabi-Yau manifold $X_n$ as the universal covering space of degree 2. In [8, Theorem1.3], the author showed that for Enriques surfaces $S$, $T$, and $n \geq 3$, if the universal covering spaces of $S^{[n]}$ and $T^{[n]}$ are isomorphic, then $S^{[n]}$ and $T^{[n]}$ are isomorphic by checking the action to cohomology ring of the automorphisms of them. However, in general, $S \not\cong T$ even if their universal covering spaces are isomorphic by a result of Ohashi [10]. For $n = 2$, since the second cohmology of $X_2$ is bigger than that of $E^{[2]}$ [8, Theorem5.1], the automorphisms of $E^{[2]}$ and $X_2$ were not stduied enough in [8]. The author does not kwon $E^{[2]}$ is uniquley determined by $X_2$. In this paper, we study the automorphisms of the the Hilbert scheme of two points of Enriques surfaces by using that the second cohmology of $X_2$ is bigger than that of $E^{[2]}$. There are two main results (Theorem 0.3 and 0.5).

Let $S$ be a smooth projective surface. First we study whether $S$ could be restored from $S^{[2]}$, i.e. for an two projective surfaces $S$ and $S'$, if $S^{[2]} \cong S'^{[2]}$, then are $S$ and $S'$ isomorphic ? For $K3$ surfaces, this problem is fully studied. In [11, Example 7.2], Yoshioka showed that there exist two $K3$ surfaces $K$ and $K'$ such that $K \not\cong K'$ and $K^{[2]} \cong K'^{[2]}$. The following two theorems is very useful:

**Theorem 0.1.** *For a smooth projective surface $S$, we put*

$$h^{p,q}(S) := \dim_{\mathbb{C}} \mathrm{H}^q(S, \Omega_S^p) \text{ and}$$

$$h(S, x, y) := \sum_{p,q} h^{p,q}(S) x^p y^q.$$

*tarou-hayashi@cr.math.sci.osaka-u.ac.jp

*By* [7, Theorem 2] *and* [6, page 204]*, we have the equation* (1)*:*

$$\sum_{n=0}^{\infty}\sum_{p,q} h^{p,q}(S^{[n]})x^p y^q t^n = \prod_{k=1}^{\infty}\prod_{p,q=0}^{2}\Big(\frac{1}{1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k)}\Big)^{(-1)^{p+q}h^{p,q}(S)}.$$

**Theorem 0.2.** [7] *Let $X$ be smooth projective variety with $n = \dim X \geq 1$. Then there is an isomorphism $S_d\mathrm{H}^0(X, \omega_X^{\otimes m}) \cong \mathrm{H}^0(X^{[n]}, \omega_{X^{[n]}}^{\otimes m})$ and the Kodaira dimension $\kappa(X^{[n]} = d\kappa(X)$. whenever $mn$ is even.*

Therefore, for a smooth projective surface $S$ the Hodge number and Kodaira dimension can be restored from $S^{[n]}$. However, we may not necessarily restore $S$ from $S^{[n]}$ because there is the example of Yoshioka. In addition, the relationship of the deformation of $S$ and $S^{[n]}$ is known. Our first main result (Theorem 0.3) shows that this never happened to Enriques surfaces:

**Theorem 0.3.** *Let $E$ be an Enriques surface and $S$ a smooth projective surface. If there is an isomorphism $\varphi : E^{[2]} \xrightarrow{\sim} S^{[2]}$, then $S$ is an Enriques surface, and there is an isomorphism $\psi : E \xrightarrow{\sim} S$ such that $\varphi$ is induced by $\psi$.*

We also notice that for the universal covering $K3$ surfaces $X$ and $Y$ of Enriques surfaces $E$ and $F$, Sosna [4] showed if $X^{[n]} \cong Y^{[n]}$ for some $n \geqq 2$, then $X \cong Y$.

Our second main result (Theorem 1.3) is on the naturality problem of automorphisms of $S^{[2]}$. First we recall the definition of the natural automorphism [3].

**Definition.** Let $S$ be a smooth compact surface. For $n \geq 2$, an automorphism $g \in \mathrm{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \mathrm{Aut}(S)$ such that $g = f^{[n]}$. Here $f^{[n]}$ is the automorphism of $S^{[n]}$ that is naturaly induced by $f \in \mathrm{Aut}(S)$.

**Theorem 0.4.** *For $n \geq 2$, let $S$ be a $K3$ surface or an Enriques surface, and $D$ the exceptional divisor of the Hilbert-Chow morphism $\pi : S^{[n]} \to S^{(n)}$. An automorphism $f$ of $S^{[n]}$ is natural if and only if $f(D) = D$.*

When $S$ is not a $K3$ surface and an Enriques surface, this theorem does not hold good, i.e. there exist a smooth projective surface $S$ which has an automorphism $f$ of $S^{[n]}$ such that $f(D) = D$ but $f$ is not natural. Our second main result is the following theorem:

**Theorem 0.5.** *Let $E$ be an Enriques surface. Then* $\mathrm{Aut}(E^{[2]}) \cong \mathrm{Aut}(E)$*, i.e. all automorphisms of* $\mathrm{Aut}(E^{[2]})$ *are natural.*

For a smoooth quartic surface $Z$ of $\mathbb{P}^3$ which is a $K3$ surface, generic line $L$ on $\mathbb{P}^3$ meets $Z$ along 4 distinct points. By alternating them, Beauville showed $Z^{[2]}$ has an automorphism which is not natural [1]. Further, Oguiso showed the fact there exists a $K3$ surface $Y$ such that $[\mathrm{Aut}(Y^{[2]}) : \mathrm{Aut}(Y)] = \infty$ under the natural inclusion [9, Theroem 1.2 (1)], which is completely different from Theorem 1.3.

The author does not know whether Theorem 1.1 and 1.3 are true or no for $n \geq 3$.

# Preliminaries

It is well known that $S^{[2]} \cong \mathrm{Blow}_{\Delta_S} S^2/\mathcal{S}_2$, where

$$\Delta_S := \{(x,y) \in S^2 : x = y\},$$

and $\mathcal{S}_2$ is the symmetric group of degree 2, which acts by interchanging the two factors of the product.

Let $E$ be an Enriques surface, and $\mu : K \to E$ its universal covering space. Let $\pi : X \to E^{[2]}$ be the universal covering space of $E^{[2]}$.

From $\mathrm{Blow}_{\Delta_E} E^2/\mathcal{S}_2$ and $\mu : K \to E$, we will construct $X$. Let $\sigma$ be the covering involution of $\mu$, $H$ the finite subgroup of $\mathrm{Aut}(K^2)$ which is generated by $\mathcal{S}_2$ and $\sigma \times \sigma$, and $G$ the finite subgroup of $\mathrm{Aut}(K^2)$ which is generated by $\mathcal{S}_2$ and $\mathrm{id}_K \times \sigma$. Since $K^2/G = E^2/\mathcal{S}_2$, and $H$ is a normal subgroup of $G$, the covering space $\mu^2 : K^2 \to E^2$ induces the covering spaces [8, Lemma 2.3, 2.4]:

$$K^2 \backslash {\mu^2}^{-1}(\Delta_E) \to (E^2 \backslash \Delta_E)/\mathcal{S}_2, \text{ and}$$

$$\mathrm{Blow}_{{\mu^2}^{-1}(\Delta_E)} K^2/H \to \mathrm{Blow}_{\Delta_E} E^2/\mathcal{S}_2.$$

Since $|G/H| = 2$, and $E^{[2]} \cong \mathrm{Blow}_{\Delta_E} E^2/\mathcal{S}_2$, we have $X \cong \mathrm{Blow}_{{\mu^2}^{-1}(\Delta_E)} K^2/H$, and the automorphism $\mathrm{id}_K \times \sigma$ of $K^2/H$ induces the covering involution $\rho$ of $\pi : X \to E^{[2]}$. From here, we consider $X$ as $\mathrm{Blow}_{{\mu^2}^{-1}(\Delta_E)} K^2/H$.

Let $\eta : \mathrm{Blow}_{{\mu^2}^{-1}(\Delta_E)} K^2/H \to K^2/H$ be the natural morphism. We put

$$T := \{(x,y) \in K^2 : \sigma(x) = y\}.$$

Then we have ${\mu^2}^{-1}(\Delta_E) = \Delta_K \cup T$. Furthermore, we put

$$D_1 := \eta^{-1}(T),\ D_2 := \eta^{-1}(\Delta_K),$$

and

$$h_i \text{ the first chern class of } D_i \text{ for } i = 1,2.$$

Since $\pi^{-1}(D) = D_1 \cup D_2$, we get

$$\mathrm{H}^2(X,\mathbb{C}) = \pi^*(\mathrm{H}^2(E^{[2]},\mathbb{C})) \oplus \mathbb{C}\langle h_1 - h_2 \rangle.$$

Thus $\dim \mathrm{H}^2(X,\mathbb{C}) = 12 = \dim \mathrm{H}^2(E^{[2]},\mathbb{C}) + 1$. Pay attention that for $n \geq 3$ $\dim \mathrm{H}^2(X,\mathbb{C}) = \dim \mathrm{H}^2(E^{[n]},\mathbb{C}) = 11$. Furthermore since $(\mathrm{id}_K \times \sigma)(T) = \Delta_K$, we get $\rho^* h_1 = h_2$,

$$\text{the eigenspace for the eigenvalue } -1 \text{ of } \rho^* \text{ is } \mathbb{C}\langle h_1 - h_2 \rangle,$$

and

$$\text{the eigenspace for the eigenvalue 1 of } \rho^* \text{ is } \pi^*(\mathrm{H}^2(E^{[2]},\mathbb{C})).$$

## Main theorems

Theorem 0.3 and 0.4 are followed by the following theorem:

**Theorem 0.6.** *Let $E$ and $E'$ be two Enriques surfaces. For an isomorphism $g : E^{[2]} \stackrel{\sim}{\to} E'^{[2]}$, we get $g(D) = D'$.*

**sketch 0.7.** *From the uniqueness of the universal covering space, there is an isomorphism $f : X \stackrel{\sim}{\to} X'$ such that $g \circ \pi = \pi \circ f$. From this, we have only to show $f(\pi^{-1}(D)) = \pi'^{-1}(D')$. Since the each degree of $\pi$ and $\pi'$ is 2, we have $f^{-1} \circ \rho' \circ f = \rho$ and $\rho^* = f^* \circ \rho'^* \circ f^{-1^*}$ as an automorphism of $\mathrm{H}^2(X, \mathbb{C})$. Since the eigenspace for the eigenvalue $-1$ of $\rho^*$ is $\mathbb{C}\langle h_1 - h_2\rangle$, we have*

$$\begin{aligned} -(h_1 - h_2) &= \rho^*(h_1 - h_2) \\ &= f^* \circ \rho'^* \circ f^{-1^*}(h_1 - h_2) \text{ in } \mathrm{H}^2(X, \mathbb{C}). \end{aligned}$$

*Thus for a linear isomorphism $f^*$ from $\mathrm{H}^2(X', \mathbb{C})$ to $\mathrm{H}^2(X, \mathbb{C})$, we obtain*

$$\rho'^*(f^{-1^*}(h_1 - h_2)) = -f^{-1^*}(h_1 - h_2) \text{ in } \mathrm{H}^2(X', \mathbb{C}).$$

*Since the eigenspace for the eigenvalue $-1$ of the linear mapping $\rho'^*$ is $\mathbb{C}\langle h'_1 - h'_2\rangle$, there is some $a \in \mathbb{C}$ such that*

$$f^*(h'_1 - h'_2) = a(h_1 - h_2) \text{ in } \mathrm{H}^2(X, \mathbb{C}).$$

*Since $X$ and $X'$ are Calabi-Yau manifolds, $Pic(X)$ and $Pic(X')$ are torsion free and the natural maps $Pic(X) \to H^2(X, \mathbb{Z})$ and $Pic(X') \to H^2(X', \mathbb{Z})$ are isomorphic. Thus there are some non zero integer $t \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}\backslash\{0\}$ such that $a = \frac{s}{t}$, i.e.*

$$f^*(\mathcal{O}_{X'}(t(D'_1 - D'_2))) \cong \mathcal{O}_X(s(D_1 - D_2)) \text{ as a line bundle.}$$

*Since $D_1$ and $D_2$ are the exceptional divisors of $X \to \mathrm{Blow}_{T \cup \Delta_K} K^2/H$, we get that $f(D_1 \cup D_2) = D'_1 \cup D'_2$.*

## References


[1] A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom. 18 (1983), no. 4, 755-782.

[2] D. Arapura and S. Archava: Kodaira dimension of symmetric powers. Proc. Amer. Math. Soc. 131 (2003), no. 5, 1369-1372 (electronic).

[3] S. Boissiere: Automorphismes naturels de l'espace de Douady de points sur une surface. Canad. J. Math. 64 (2012), no. 1, 3-23.

[4] B. Fantechi: Deformation of Hilbert schemes of points on a surface. Compositio Math. 98 (1995), 205-217.

[5] J. Fogarty: Families on an Algebraic Surface. American Journal of Mathematics Vol. 90, No. 2 (Apr., 1968), pp. 511-521.

[6] L. Göttsche: The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. 286, 193-207 (1990).

[7] L. Göttsche, W. Soergel: Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. Math. Ann. 296 (1993), 235-245.

[8] T. Hayashi: Universal covering calabi-yau manifolds of the Hilbert schemes of n points of Enriques surfaces. arXiv:1502.02231.

[9] K. Oguiso: On automorphisms of the punctual Hilbert schemes of K3 surfaces. Eur. J. Math. 2 (2016), no. 1, 246-261.

[10] H. Ohashi: On the number of Enriques quotients of a K3 surface. Publ. Res. Inst. Math. Sci. 43 (2007), no. 1, 181-200. 14J28.

[11] K. Yoshioka: Moduli spaces of stable sheaves on abelian surfaces. Math. Ann. 321 (2001), no. 4, 817-884.