Universal covering space of the Hilbert scheme of n points of Enriques surface

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Abstract

We work over $\mathbb{C}$. The Hilbert scheme of n points of Enriques surface has a Calabi-Yau manifold as the universal covering. We prove that every small deformations of the Calabi-Yau manifold is induced by that of the Hilbert scheme of n points of Enriques surface, and count the number of isomorphism classes of the Hilbert schemes of n points of Enriques surfaces which has X as the universal covering space when we fix one for $n \geq 3$.

Theorem 0.1. *Let E and E' be two Enriques surfaces, $E^{[n]}$ and $E'^{[n]}$ the Hilbert scheme of n points of E and E', X and X' the universal covering space of $E^{[n]}$ and $E'^{[n]}$, and $n \geq 3$. If $X \cong X'$, then $E^{[n]} \cong E'^{[n]}$, i.e. when we fix X, then there is just one isomorphism class of the Hilbert schemes of n points of Enriques surfaces such that they have it as the universal covering space. Furthermore if $X = X'$, then $E^{[n]} = E'^{[n]}$.*

Preliminaries

A $K3$ surface K is a compact complex surface with $K_K \sim 0$ and $H^1(K, \mathcal{O}_K) = 0$. An Enriques surface E is a compact complex surface with $H^1(E, \mathcal{O}_E) = 0$, $H^2(E, \mathcal{O}_E) = 0$, $K_E \not\sim 0$, and $2K_E \sim 0$. The universal covering of an Enriques surface is a $K3$ surface. A Calabi-Yau manifold X is an n-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic k-form on X for $0 < k < n$ and there is a nowhere vanishing holomorphic n-form on X.

Let S be a nonsingular surface, $S^{[n]}$ the Hilbert scheme of n points of S, $\pi_S : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism, and $p_S : S^n \to S^{(n)}$ the natural projection. We denote by D_S the exceptional divisor of π_S. Note that $S^{[n]}$ is smooth of $\dim_{\mathbb{C}} S^{[n]} = 2n$. Let Δ_S^n be the set of n-uples $(x_1, \dots, x_n) \in S^n$ with

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at least two x_i's equal, S^n_* the set of n-uples $(x_1, \dots, x_n) \in S^n$ with at most two x_i's equal. We put

$$\begin{aligned} S^{(n)}_* &:= p_S(S^n_*), \\ \Delta^{(n)}_S &:= p_S(\Delta^n_S), \\ S^{[n]}_* &:= \pi_S^{-1}(S^{(n)}_*), \\ \Delta^n_{S*} &:= \Delta^n_S \cap S^n_*, \\ \Delta^{(n)}_{S*} &:= p_S(\Delta^n_{S*}), \text{ and} \\ F_S &:= S^{[n]} \setminus S^{[n]}_*. \end{aligned}$$

Then we have $\mathrm{Blow}_{\Delta^n_{S*}} S^n_*/\mathcal{S}_n \simeq S^{[n]}_*$, F_S is an analytic closed subset, and its codimension is 2 in $S^{[n]}$ by Beauville [1, page 767-768]. Here $\mathcal{S}_n$ is the symmetric group of degree n which acts naturally on S^n by permuting of the factors.

Let E be an Enriques surface, and $E^{[n]}$ the Hilbert scheme of n points of E. By Oguiso and Schröer [5, Theorem 3.1], $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space $\pi : X \to E^{[n]}$ of degree 2. Let $\mu : K \to E$ be the universal covering space of E where K is a $K3$ surface, S_K the pullback of $\Delta^{(n)}_E$ by the morphism

$$\mu^{(n)} : K^{(n)} \ni [(x_1, \dots, x_n)] \mapsto [(\mu(x_1), \dots, \mu(x_n))] \in E^{(n)}.$$

Then we get a 2^n-sheeted unramified covering space

$$\mu^{(n)}|_{K^{(n)} \backslash S_K} : K^{(n)} \backslash S_K \to E^{(n)} \backslash \Delta^{(n)}_E.$$

Furthermore, let Γ_K be the pullback of S_K by natural projection $p_K : K^n \to K^{(n)}$. Since Γ_K is an algebraic closed set with codimension 2, then

$$\mu^{(n)} \circ p_K : K^n \backslash \Gamma_K \to E^{(n)} \backslash \Delta^{(n)}_E$$

is the $2^n n!$-sheeted universal covering space. Since $E^{[n]} \backslash D_E = E^{(n)} \backslash \Delta^{(n)}_E$ where $D_E = \pi_E^{-1}(\Delta^{(n)}_E)$, we regard the universal covering space $\mu^{(n)} \circ p_K : K^n \backslash \Gamma_K \to E^{(n)} \backslash \Delta^{(n)}_E$ as the universal covering space of $E^{[n]} \setminus D_E$:

$$\mu^{(n)} \circ p_K : K^n \backslash \Gamma_K \to E^{[n]} \backslash D_E.$$

Since $\pi : X \backslash \pi^{-1}(D_E) \to E^{[n]} \backslash D_E$ is a covering space and $\mu^{(n)} \circ p_K : K^n \backslash \Gamma_K \to E^{[n]} \setminus D_E$ is the universal covering space, there is a morphism

$$\omega : K^n \setminus \Gamma_K \to X \setminus \pi^{-1}(D_E)$$

such that $\omega : K^n \setminus \Gamma_K \to X \setminus \pi^{-1}(D_E)$ is the universal covering space and We denote the covering transformation group of $\pi \circ \omega$ by:

$$G := \{ g \in \mathrm{Aut}(K^n \setminus \Gamma_K) : \pi \circ \omega \circ g = \pi \circ \omega \}.$$

Then G is of order $2^n.n!$, since $\deg(\mu^{(n)} \circ p_K) = 2^n.n!$. Let σ be the covering involution of $\mu : K \to E$, and for

$$1 \leq k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n$$

we define automorphisms $\sigma_{i_1 \ldots i_k}$ of K^n by following. For $x = (x_i)_{i=1}^n \in K^n$,

$$\textit{the } j\textit{-th component of } \sigma_{i_1\ldots i_k}(x) = \begin{cases} \sigma(x_j) & j \in \{i_1, \cdots, i_k\} \\ x_j & j \notin \{i_1, \cdots, i_k\}. \end{cases}$$

Then $\mathcal{S}_n \subset G$, and $\{\sigma_{i_1\ldots i_k}\}_{1\leq k\leq n,\ 1\leq i_1<\ldots<i_k\leq n} \subset G$. Let H be the subgroup of G generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\leq i<j\leq n}$.

We put

$$K^n_{*\mu} := (\mu^n)^{-1}(E^n_*),$$

where $\mu^n : K^n \ni (x_i)_{i=1}^n \mapsto (\mu(x_i))_{i=1}^n \in E^n$. Recall that $\mu : K \to E$ the universal covering with σ the covering involution. We further put

$$T_{*\mu\, ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : \sigma(x_i) = x_j\},$$

$$\Delta_{K*\mu\, ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : x_i = x_j\},$$

$$T_{*\mu} := \bigcup_{1\leq i<j\leq n} T_{*\mu\, i,j}, \text{ and}$$

$$\Delta_{K*\mu} := \bigcup_{1\leq i<j\leq n} \Delta_{K*\mu\, i,j}.$$

By the definition of $K^n_{*\mu}$, H acts on $K^n_{*\mu}$, and by the definition of $\Delta_{K*\mu}$ and $T_{*\mu}$, we have $\Delta_{K*\mu} \cap T_{*\mu} = \emptyset$.

The universal covering map μ induces a local isomorphism

$$\mu^{[n]}_* : \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_{E*}} E^n_*/\mathcal{S}_n = E^{[n]}_*.$$

Here $\mathrm{Blow}_A B$ is the blow up of B along $A \subset B$.

Proposition 0.2. $\mu^{[n]}_*$ *:* $\mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_{E*}} E^n_*/\mathcal{S}_n$ *is the universal covering space, and* $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H$*.*

Theorem 0.3. *Let* E *be an Enriques surface,* $E^{[n]}$ *the Hilbert scheme of* n *points of* E*,* $\pi : X \to E^{[n]}$ *the universal covering space of* $E^{[n]}$*, and* $n \geq 2$*. Then there is a crepant resolution* $\varphi_X : X \to K^n/H$ *such that* $\varphi_X^{-1}(\Gamma_K/H) = \pi^{-1}(D_E)$*.*

Proposition 0.4. *For* $n \geq 3$*, the induced map* $\rho^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ *is identity.*

Recall that $\mu : K \to E$ is the universal covering of E where K is a $K3$ surface, and σ the covering involution of μ.

Proposition 0.5. *Let E be an Enriques surface which does not have numerically trivial involutions, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, ρ the covering involution of π, and $n \geq 3$. Let ι be an involution of X which acts on $H^2(X, \mathbb{C})$ as* id*, then $\iota = \rho$.*

We suppose that E has numerically trivial involutions. By [4, Proposition 1.1], there is just one automorphism of E, denoted υ, such that its order is 2, and υ^* acts on $H^2(E, \mathbb{C})$ as id. For υ, there are just two involutions of K which are liftings of υ, one acts on $H^0(K, \Omega_K^2)$ as id, and another acts on $H^0(K, \Omega_K^2)$ as $-$id, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$. Let $\upsilon^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς', respectively.

Lemma 0.6. *For ς and ς', one acts on $H^0(X, \Omega_X^{2n})$ as* id*, and another act on $H^0(X, \Omega_X^{2n})$ as $-$*id*.*

We put $\varsigma_+ \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as id and $\varsigma_- \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as $-$id.

Proposition 0.7. *Suppose E has numerically trivial involutions. Let $E^{[n]}$ be the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, ρ the covering involution of π, and $n \geq 3$. Let ι be an involution of X which ι^* acts on $H^2(X, \mathbb{C})$ as* id *and on $H^0(X, \Omega_X^{2n})$ as $-$*id*, and $\iota \neq \rho$. Then we have $\iota = \varsigma_-$.*

Theorem 0.8. *Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and $n \geq 3$. If X has a involution ι which ι^* acts on $H^2(X, \mathbb{C})$ as* id*, and $\iota \neq \rho$. Then E has a numerically trivial involution.*

Since X and K^n/H are projective, K^n/H is a V-manifold, and π is a surjective, $\pi^* : H^{p,q}(K^n/H, \mathbb{C}) \to H^{p,q}(X, \mathbb{C})$ is injective

By [4, Proposition 1.1], there is just one automorphism of E, denoted υ, such that its order is 2, and υ^* acts on $H^2(E, \mathbb{C})$ as id. For υ, there are just two involutions of K which are liftings of υ, one acts on $H^0(K, \Omega_K^2)$ as id, and another acts on $H^0(K, \Omega_K^2)$ as $-$id, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$. Let $\upsilon^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς' respectively. Then they satisfies $\varsigma = \varsigma' \circ \sigma$, and each order of ς and ς' is 2. From 0.6, one acts on $H^0(X, \Omega_X^{2n})$ as id, and another act on $H^0(X, \Omega_X^{2n})$ as $-$id. We put $\varsigma_+ \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as id and $\varsigma_- \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as $-$id.

Theorem 0.9. *Let E and E' be two Enriques surfaces, $E^{[n]}$ and $E'^{[n]}$ the Hilbert scheme of n points of E and E', X and X' the universal covering space of $E^{[n]}$ and $E'^{[n]}$, and $n \geq 3$. If $X \cong X'$, then $E^{[n]} \cong E'^{[n]}$, i.e. when we fix X, then there is just one isomorphism class of the Hilbert schemes of n points of Enriques surfaces such that they have it as the universal covering space. Furthermore if $X = X'$, then $E^{[n]} = E'^{[n]}$.*

Proof. For an involution of X which is the covering involution of some the Hilbert scheme of n points of Enriques surfaces acts on $H^2(X, \mathbb{C})$ as id, $H^0(X, \Omega_X^{2n})$ as $-$id, and $H^{2n}(X, \mathbb{C})^{2n-1,1}$ as $-$id. From Proposition 0.7, the automorphisms which acts on $H^2(X, \mathbb{C})$ as id, $H^0(X, \Omega_X^{2n})$ as $-$id, are only ρ and ς_-. From the definition of ς_- and Lemma 0.6, ς_- does not act on $H^{2n}(X, \mathbb{C})^{2n-1,1}$ as $-$id. Therefore an automorphism g of X which acts on $H^2(X, \mathbb{C})$ as id, $H^0(X, \Omega_X^{2n})$ as $-$id, and $H^{2n}(X, \mathbb{C})^{2n-1,1}$ as $-$id is the covering involution ρ of $X \to E^{[n]}$. Thus we have an argument. □

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