Quasi-homomorphisms on mapping class groups

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Contents

1 Introduction

We survey some results on quasi-homomorphism on mapping class groups from the viewpoint of hyperbolic geometry in the sense of Gromov. Most of the results in this chapter are shown both for word-hyperbolic groups and mapping class groups by the same techniques. The mapping class group, $\mathrm{MCG}(S)$, of a compact orientable surface S is typically not word-hyperbolic, but it acts on its complex of curves $\mathcal{C}(S)$, which is δ-hyperbolic, [44]. The action is co-finite, but not proper (otherwise, the mapping class would be word-hyperbolic). Another aspect of the geometry of $\mathcal{C}(S)$ is that this space is not locally compact. Thanks to the study of $\mathcal{C}(S)$ by Masur-Minsky [44] regarding the geometry of $\mathcal{C}(S)$, we can apply the standard methods developed in the theory of word-hyperbolic groups to $\mathrm{MCG}(S)$.

1.1 Quasi-homomorphisms

Definition 1.1 (Quasi-homomorphism). Let G be a group. A *quasi-homomorphism* is a function $f : G \to \mathbb{R}$ such that

$$D(f) = \sup_{a,b\in G} |f(a) + f(b) - f(ab)| < \infty.$$

$D(f)$ is called the *defect* of f. If a quasi-homomorphism satisfies $f(a^n) = nf(a)$ for all $a \in G$ and n, it is said *homogeneous*. We denote the vector space of all homogeneous quasi-homomorphisms on G by $HQH(G)$.

Quasi-homomorphisms are also called *quasimorphisms* (for example in [3],[16]). If f is a quasi-homomorphism on G, then one can obtain a homogeneous quasi-homomorphism $\bar{f}$ as follows:

$$\bar{f}(a) = \lim_{n\to\infty} \frac{f(a^n)}{n}.$$

Note that the limit exists since the sequence $\{f(a^n)\}$ is subadditive with bounded error. For any $a \in G$, $|f(a) - \bar{f}(a)| \le D(f)$, [3, Prop 3.3.1]. Namely, a quasi-homomorphism f is (uniquely) written as sum of a homogeneous quasi-homomorphism $\bar{f}$ and a bounded function. The defect $D(\bar{f})$ is related to $D(f)$ by

$$D(\bar{f}) \le 4D(f).$$

If f is a homogeneous quasi-homomorphism, then it is easy to check that for all $a, b \in G$, $f(aba^{-1}) = f(b)$, and therefore $|f([a, b])| \le D(f)$. It turns out that there is an equality ([3, Lemma 3.6])

$$\sup_{a,b\in G} |f([a, b])| = D(f).$$

Therefore, a homogeneous quasi-homomorphism f is a homomorphism if and only if $f = 0$ on $[G, G]$, where $[G, G]$ is the commutator subgroup of G.

The following result follows from a result on bounded cohomology (see section 1.3).

Theorem 1.2 (Cor 1 [3]). *Suppose that G is an amenable group. Then, a homogeneous quasi-homomorphism on G is a homomorphism.*

Let $\mathcal{V}(G)$ be the vector space of all quasi-homomorphisms $G \to \mathbb{R}$. We denote by $BDD(G)$ and $HOM(G) = H^1(G; \mathbb{R})$ the subspaces of $\mathcal{V}(G)$ consisting of bounded functions and respectively homomorphisms. Note that $\mathrm{BDD}(G) \cap \mathrm{HOM}(G) = 0$. We will be concerned with the quotient spaces

$$QH(G) = \mathcal{V}(G)/\mathrm{BDD}(G)$$

and

$$\widetilde{\mathrm{QH}}(G) = \mathcal{V}(G)/(\mathrm{BDD}(G) + \mathrm{HOM}(G)) \cong \mathrm{QH}(G)/H^1(G; \mathbb{R}).$$

$f \in \mathcal{V}(G)$ defines $\bar{f} \in \mathrm{HQH}(G)$. This implies that $\mathrm{QH}(G) \cong \mathrm{HQH}(G)$, therefore $\widetilde{\mathrm{QH}}(G) \cong \mathrm{HQH}(G)/H^1(G; \mathbb{R})$. Theorem 1.2 says $\widetilde{\mathrm{QH}}(G)$ is trivial if G is amenable.

1.2 Stable commutator length

Let G be a group. Given $g \in [G, G]$, the *commutator length* of g, denoted by $\mathrm{cl}(g)$, is the least number of commutators in G whose product is equal to g. Namely, $\min l = \mathrm{cl}(g)$ such that $a_i, b_i \in G$ and

$$g = [a_1, b_1] \cdots [a_l, b_l].$$

The *stable commutator length*, denoted by $\mathrm{scl}(g)$, is defined by

$$\mathrm{scl}(g) = \liminf_{n \to \infty} \frac{\mathrm{cl}(g^n)}{n}.$$

Note that cl and scl are class functions, namely, they are constant on each conjugacy class in G. The function scl is defined whenever some power of g is contained in $[G, G]$. By convention, we may extend scl to all of G by setting $\mathrm{scl}(g) = \infty$ if no power of g is contained in $[G, G]$.

The following fact [3, Lemma 1.1] already appears in [47].

Proposition 1.3. *Let $f : G \to \mathbb{R}$ be a homogeneous quasi-homomorphism. If $f(a) = 1$ for $a \in [G, G]$ then $\frac{1}{2D(f)} \leq \mathrm{scl}(a)$.*

Proof. Since $f(a) = 1$, f is not a homomorphism, therefore $D(f) > 0$. Denote $D(f)$ by D. For $n > 0$, put $l(n) = \mathrm{cl}(a^n)$. a^n is a product of $l(n)$ commutators,

c_i, in G. Since f is a quasi-homomorphism,

$$n = f(a^n) \le |f(c_1)| + \cdots + |f(c_{l(n)})| + (l(n)-1)D.$$

Since f is homogeneous, $|f(c_i)| \le D$ for all i, therefore $n \le (2l(n)+1)D$. Thus, $\frac{1}{D} \le \frac{2l(n)+1}{n}$ for all $n > 0$. Letting $n \to \infty$, we obtain $\frac{1}{2D} \le \mathrm{scl}(a)$. □

Quasi-homomorphisms and stable commutator length are related by Bavard's Duality Theorem in a more precise way ([3]):

Theorem 1.4 (Bavard's Duality Theorem). *Let G be a group and $a \in [G,G]$. If $\mathrm{HQH}(G) = H^1(G;\mathbb{R})$ then $\mathrm{scl}(a) = 0$. Otherwise, we have an equality*

$$\mathrm{scl}(a) = \frac{1}{2} \sup_{\phi \in \mathrm{HQH}(G) \setminus H^1(G;\mathbb{R})} \frac{|\phi(a)|}{D(\phi)}.$$

The argument is based on the Hahn-Banach theorem. In particular, the quasi-homomorphisms promised by Bavard's theorem are typically non constructive.

By Theorems 1.4 and 1.2, if G is amenable, then $\mathrm{scl} = 0$ on $[G,G]$. On the other hand, if $F(a,b)$ is a free group with two free generators a, b, then for any $1 \neq g \in [F,F]$, $\mathrm{scl}(g) \ge 1/6$ ([18, Cor 3.3]).

A group G is called *perfect* if $G = [G,G]$ and *uniformly perfect* if G is perfect and cl is bounded on G, which implies that $\mathrm{scl} = 0$. It is known that $\mathrm{SL}_n(\mathbb{Z})$ is uniformly perfect if $n \ge 3$ (cf. [2]).

We discuss the stable commutator length in the section 7 in connection to hyperbolicity.

1.3 Bounded cohomology

To define the bounded cohomology group ([29]) of a discrete group G, let

$$C_b^k(G;\mathbb{R}) = \{f : G^k \to \mathbb{R} \mid f \text{ has bounded range}\}$$

The boundary $\delta : C_b^k(G;\mathbb{R}) \to C_b^{k+1}(G;\mathbb{R})$ is given by

$$\begin{aligned}\delta f(g_0, \ldots, g_k) &= f(g_1, \ldots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k) \\ &\quad + (-1)^{k+1} f(g_0, \ldots, g_{k-1}).\end{aligned}$$

The cohomology of the complex $\{C_b^k(G;\mathbb{R}), \delta\}$ is the *bounded cohomology group* of G, denoted by $H_b^*(G;\mathbb{R})$. See [29], [38], [49] as general references for the theory of bounded cohomology. $H_b^1(G;\mathbb{R})$ is trivial for any group G, and $H_b^n(G;\mathbb{R})$ is trivial for all $n \ge 1$ if G is amenable.

By definition, for each n, there is a natural homomorphism, sometimes called *comparison map*, $H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})$. An element $f \in \mathrm{QH}(G)$

defines a bounded class $[\delta f] \in H_b^2(G;\mathbb{R})$. There is an exact sequence ([3])

$$0 \to H^1(G;\mathbb{R}) \to \mathrm{QH}(G) \to H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R}).$$

Since $\widetilde{\mathrm{QH}}(G)$ is the quotient $\mathrm{QH}(G)/H^1(G;\mathbb{R})$, we see that $\widetilde{\mathrm{QH}}(G)$ can also be identified with the kernel of $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$. It follows from Theorem 1.4 that the kernel is trivial if G is uniformly perfect, [45].

If $G \to G'$ is an epimorphism then the induced maps $\mathrm{QH}(G') \to \mathrm{QH}(G)$ and $\widetilde{\mathrm{QH}}(G') \to \widetilde{\mathrm{QH}}(G)$ are injective.

Calculations of $\widetilde{\mathrm{QH}}(G)$ have been made for many groups G. In many cases $\widetilde{\mathrm{QH}}(G)$ is either 0 or infinite dimensional. We remark that there exists a group G such that $H_b^2(G;\mathbb{R})$ is nontrivial and finite dimensional ([14, Remark 25]).

If G is finitely generated by k elements, then $H^1(G;\mathbb{R})$ is at most k-dimensional, therefore $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional if $\mathrm{QH}(G)$ is infinite dimensional (cf. Theorem 5.1).

As we said, if G is amenable then $H_b^2(G;\mathbb{R}) = 0$ ([29]), therefore the kernel of $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ is trivial. In other words, $\mathrm{HQH}(G) = \widetilde{\mathrm{QH}}(G) = 0$. This is indeed how Theorem 1.2 is shown in [3]. $\widetilde{\mathrm{QH}}(G)$ also vanishes when G is an irreducible lattice in a semisimple Lie group of real rank > 1 [13] (Theorem 4.1).

2 Brooks' counting quasi-homomorphism on free groups

Our first example of a group G such that $\widetilde{\mathrm{QH}}(G)$ is non trivial is a free group.

Theorem 2.1 ([12]). *Suppose F is a free group of rank at least two. Then, $\widetilde{\mathrm{QH}}(F)$ is an infinite dimensional vector space over $\mathbb{R}$.*

We explain Brooks' construction of a quasi-homomorphism f on F which is non-trivial in $\widetilde{\mathrm{QH}}(F)$. For simplicity suppose the rank of F is two and let x, y be free generators of F. Fix a reduced word w on x, y. Any element $1 \neq a \in F$ is uniquely written as a (non-empty) reduced word on x, y, which we also denote by a. Define $|a|_w$ to be the maximal number of times that w can be seen as an (oriented) subword of a without overlapping.

Example 2.2. $|xyxyx|_{xy} = 2$. $|xyxyx|_{xyx} = 1$. $|xxyxy|_{yx} = 1$.

Let w^{-1} be the reduced word which is the inverse of w as a group element. Define $h_w(a) = |a|_w - |a|_{w^{-1}}$. h_w is a function on F. The following says that h_w is a quasi-homomorphism.

Lemma 2.3. $D(h_w) \leq 3$

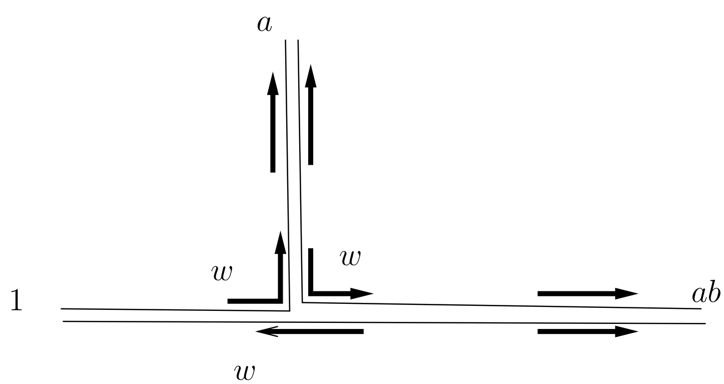

Figure 1. At most three subwords w count for $\delta h_w(a,b)$. The other pairs of subwords w cancel for δh_w (not necessarily for δc_w and $\delta c_{w^{-1}}$).

To see this, let $a, b \in F$. We think of them as reduced words too. Let $a \cdot b$ be the word which we obtain by placing the word b after a. This word represents the group element ab, but may be not reduced. If the word is reduced, we see

$$||a \cdot b|_w - |a|_w - |b|_w| \leq 1, ||a \cdot b|_{w^{-1}} - |a|_{w^{-1}} - |b|_{w^{-1}}| \leq 1.$$

Therefore $|h_w(ab) - h_w(a) - h_w(b)| \leq 2$. In general, $a \cdot b$ is not reduced, and each function $| \cdot |_w, | \cdot |_{w-1}$ on F is not a quasi-homomorphism. One verifies that h_w is a quasi-homomorphism by writing $a = a' \cdot c, b = c^{-1} \cdot b'$ such that a', b', c are reduced, therefore the above inequalities apply to $a = a' \cdot c$ and $b = c^{-1} \cdot b'$ (Use $|c|_w = |c^{-1}|_{w^{-1}}$). From this, one easily gets $D(h_w) \leq 6$ and indeed $D(h_w) \leq 3$ as Figure 1 shows.

Suppose w is cyclically reduced, namely, $w^n (n > 0)$ is reduced. Then, $|w^n|_w = n$ for all $n > 0$. On the other hand, $|w^n|_{w^{-1}} = 0$, therefore $h_w(w^n) = n$ for all $n > 0$. We find that h_w is non-trivial in $\mathrm{QH}(F)$. If we take w to represent an element in $[F, F]$, then we obtain from h_w a non-trivial element in $\widetilde{\mathrm{QH}}(F)$. For example, one can take $w = xyx^{-1}y^{-1}$. Moreover we can find a sequence of reduced and cyclically reduced words w_i such that h_{w_i} are linearly independent in $\widetilde{\mathrm{QH}}(F)$. This shows Theorem 2.1.

We remark that Lemma 2.3 and Proposition 1.3 give a uniform positive lower bound of $\mathrm{scl}(a)$ for any $1 \neq a \in [F, F]$. To see this, since scl is invariant by taking conjugates, one may assume that a, as a reduced word, is shortest among its conjugates. Then the word a is cyclically reduced, therefore $D(h_a) \leq 3$. Then $D(\overline{h_a}) \leq 12$ and $\overline{h_a}(a) = 1$. By Proposition 1.3, $\mathrm{scl}(a) \geq \frac{1}{24}$. As we said, Culler [18] shows that $\mathrm{scl}(a) \geq \frac{1}{6}$.

3 Delta-hyperbolicity and quasi-homomorphism

The construction of quasi-homomorphisms by Brooks has been generalized to the δ-hyperbolic setting. δ-hyperbolic geometry, or the hyperbolic geometry in the sense of Gromov, was invented by Gromov [28]. We only give a few basic definitions and facts. See for example [10].

Definition 3.1 (δ-hyperbolic space, δ-thin, word-hyperbolic group). Let X be a geodesic metric space and $\delta \geq 0$. We say that X is δ-*hyperbolic* if for any points a, b, c of X, and any geodesic segments $[a, b], [b, c]$ and $[c, a]$, the segment $[a, b]$ is contained in the δ-neighborhood of the union of $[b, c]$ and $[c, a]$ (then the geodesic triangle $[a, b] \cup [b, c] \cup [c, a]$ is said δ-*thin*). Note that a geodesic between two points a, b is not unique, but we denote it by $[a, b]$. If X is δ-hyperbolic, then the Hausdorff distance of any two geodesics between a, b is at most δ.

Let G be a finitely generated group with a fixed set of generators, and let Γ be its Cayley graph. We say G is word-hyperbolic if Γ is δ-hyperbolic for some δ.

If a geodesic space X is quasi-isometric (cf.[10]) to a geodesic space which is δ-hyperbolic, then there exists $\delta' \geq 0$ such that X is δ'-hyperbolic. As a consequence, the word-hyperbolicity of a finite generated group, G, does not depend on the choice of a set of generators since the Cayley graphs of G for two sets of generators are quasi-isometric to each other.

Clearly, finite groups and $\mathbb{Z}$ are word-hyperbolic. If G contains an infinite cyclic subgroup of finite index, then G is quasi-isometric to $\mathbb{Z}$ (to be precise, the Cayley graphs of those two groups are quasi-isometric to each other), therefore, G is word-hyperbolic. A group which contains a cyclic subgroup of finite index is called an *elementary* word-hyperbolic group.

Definition 3.2 (Quasi-geodesic). Let X be a geodesic space. Let I be an interval of $\mathbb{R}$ (bounded or unbounded). A (K, ϵ)-*quasi-geodesic* in X is a map $\alpha : I \to X$ such that for all $t, s \in I$

$$\frac{|t-s|}{K} - \epsilon \leq d(\alpha(t), \alpha(s)) \leq K|t-s| + \epsilon.$$

We may denote the image of α by α.

The following fact, sometimes called Morse Lemma, is important.

Proposition 3.3 (Stability of quasi-geodesics). *(cf. [10, III.H. Theorem 1.7]) For all $\delta \geq 0, \epsilon \geq 0, K \geq 1$ there exists $L(\delta, K, \epsilon)$ with the following property: If X is a δ-hyperbolic space, α is a (K, ϵ)- quasi-geodesic in X and $[a, b]$ is a geodesic segment joining the end points of α, then the Hausdorff distance between $[a, b]$ and the image of α is at most L.*

Definition 3.4 (Hyperbolic isometry). Let X be a δ-hyperbolic space. An isometry a of X is called *hyperbolic* if there exist $x \in X$ and a constant $C > 1$ such that

$$d(x, a^n(x)) \geq Cn$$

for all $n > 1$.

Definition 3.5 (Translation length). If a is an isometry of a metric space X, the translation length of a, $\tau(a)$, is defined as follows. Let $x \in X$ be a point in X. Then,

$$\tau(a) = \liminf_{n \to \infty} \frac{d(x, a^n(x))}{n}.$$

$\tau(a)$ does not depend on the choice of x.

A finitely generated group G acts on a Cayley graph of G by isometries. It is an important fact that if G is word-hyperbolic, then each element $a \in G$ of infinite order acts as a hyperbolic isometry, [28]. Therefore, a has infinite order if and only if $\tau(a) > 0$ on the Cayley graph.

If a is a hyperbolic isometry, then there exists a quasi-geodesic α in X with $\alpha = a(\alpha)$. α is called a quasi-geodesic *axis* of a. It is not always true that α can be taken to be a geodesic. It is known that if G is word-hyperbolic and Γ is a Cayley graph, then there exists a constant P such that for any element $a \in G$ of infinite order, there exists a geodesic α such that $a^P(\alpha) = \alpha$. (For an argument, see for example [19]).

3.1 Word-hyperbolic groups

The following classification of subgroups in a word-hyperbolic group is a standard fact. We may regard it as a Tits alternative.

Theorem 3.6 (cf [10]). *Let H be a subgroup of a word-hyperbolic group G. Then one of the following holds.*

(1) *H contains a free group of rank two.*

(2) *H contains a cyclic group as a subgroup of finite index.*

A subgroup H of the second type in Theorem 3.6 is called *elementary*. In other words, H is elementary if it is finite, or if it contains $\mathbb{Z}$ as a subgroup of finite index. Note that a subgroup of a word-hyperbolic group is not necessarily word-hyperbolic. N.Brady constructed an example of a word-hyperbolic group which contains a finitely presented non-word-hyperbolic subgroup.

The following theorem is a generalization of Theorem 2.1 since a free group of rank at least two is a non-elementary word-hyperbolic group.

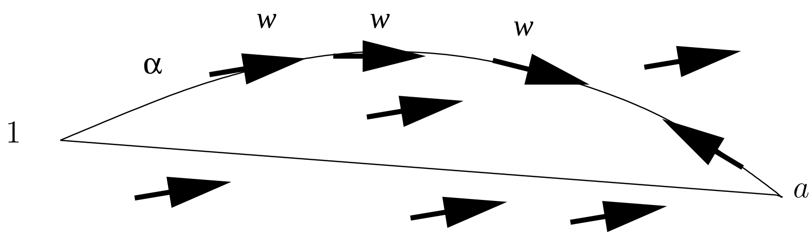

Figure 2. $|\alpha|_w = 3$

Theorem 3.7 ([19]). *Let G be a non-elementary word-hyperbolic group. Then, $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional.*

Remark 3.8. The argument in [19] shows that if H is a non-elementary subgroup of a word-hyperbolic group, then $\widetilde{\mathrm{QH}}(H)$ is infinite dimensional. H may not to be word-hyperbolic.

The argument for Theorem 3.7 is based on a generalization of the construction of quasi-homomorphisms, *counting functions*, by Brooks that we explain in section 2. We outline the argument. See [19], [24] or [6] for more details.

Suppose G is a group with a fixed symmetric generating set S, and $\Gamma = \Gamma_S(G)$ is its Cayley graph. Let w be a (reduced) word in the generating set. Let α be a (directed) path in Γ, and $|\alpha|$ its length. Define $|\alpha|_w$ to be the maximal number of times that w can be seen as an (oriented) subword of α without overlapping (see Example 2.2 and Figure 2). An (oriented) path labeled by w is called a copy of w.

The path α represents an element in G, which we denote by $\bar{\alpha}$. We can uniquely identify α and the path in Γ from 1 to $\bar{\alpha}$ with the label by α. In general, for an element $a \in G$, there is more than one geodesic, therefore reduced, path α in Γ from 1 to a. It is natural to define $|a|_w = \max |\alpha|_w$ such that α runs through all geodesics with $\bar{\alpha} = a$, but indeed we need to modify the definition to have something similar to Lemma 2.3.

Let $0 < W < |w|$ be a constant. For $x, y \in \Gamma$, define

$$c_{w,W}(x, y) = d(x, y) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where α ranges over all the paths from x to y. If the infimum is attained by α, we say α is a *realizing path* for $c_{w,W}$ from x to y. If γ is a geodesic from x to y, then define $c(\gamma) = c(x, y)$.

Fix a point $x \in \Gamma$. (We may take $x = 1$.) Define for $a \in G$

$$c_{w,W}(a) = c_{w,W}(x, a(x)).$$

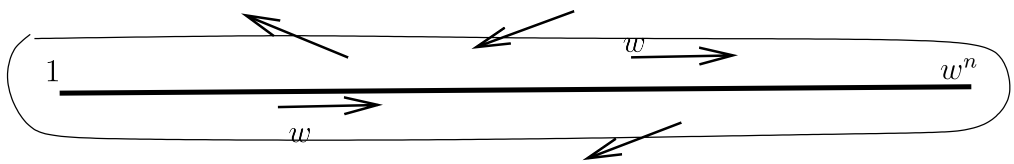

Figure 3. copies of w with the opposite direction do not fit in the L-neighborhood of a geodesic from 1 to w^n.

$c_{w,W}$ is called the *counting function* for the pair (w, W). Let w^{-1} denote the inverse word of w. We define

$$h_{w,W} = c_{w,W} - c_{w^{-1},W}.$$

In [19], the normalization $W = 1$ is used. This is an appropriate choice of constant when $w^* := \cdots wwww \cdots$ is a bi-infinite geodesic. Then w^* is a geodesic axis for w. In spirit, $h_{w,1}$ is same as h_w which is defined in Section 2 for free groups.

The following fact is not so difficult to prove. This does not require that Γ is δ-hyperbolic.

Proposition 3.9 (cf. Lemma 3.3 [24], Prop 3.9 [19]). *If α is a realizing path for $c_{w,W}$, then it is a (K, ϵ)-quasigeodesic, where*

$$K = \frac{|w|}{|w| - W}\ ,\ \epsilon = \frac{2W|w|}{|w| - W}.$$

Since Γ is δ-hyperbolic, Proposition 3.3 applies to realizing paths. Let $L = L\left(\delta, \frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W}\right)$. Let α be a geodesic from x to y. From Proposition 3.9 we deduce that a realizing path for α must be contained in the L-neighborhood of α. Consequently, if the L-neighborhood of α does not contain a copy of w, then $c_{w,W}(\alpha) = 0$.

Remark 3.10. We will use this fact later in our argument to avoid "reverse counting". Roughly speaking, let w be a word such that w^n is a geodesic. Then, for $n > 0$,

$$c_{w,W}(w^n) \geq Wn$$

because $|w^n|_w = n$.

Suppose the L-neighborhood of w^n does not contain a copy of w^{-1} (see Figure 3). Here we are thinking of the L-neighborhood of w^n, for large n, like a long narrow tube whose core has a definite orientation, agreeing with the orientation on w. By "a copy of w^{-1}", we mean a copy of w whose orientation *disagrees* with that of the core of the tube. We will find a necessary and sufficient algebraic condition for $\bar{w}$ to satisfy regarding this combinatorial/geometric property (see Condition 6.2, cf. Example 3.12).

It follows that $c_{w^{-1},W}(w^n) = 0$ because for a realizing path α for $c_{w^{-1},W}$ at w^n we must have $|\alpha|_{w^{-1}} = 0$. We thus obtain for all $n > 0$ an inequality $h_{w,W}(w^n) \geq nW$.

Consider a triangle of realizing paths. We have observed that it is L-close to a geodesic triangle, which is δ-thin. Therefore the triangle of realizing paths is $(\delta + 2L)$-thin. The following inequality on the defect then follows. This is an analogue of Lemma 2.3. The argument is same in spirit.

Proposition 3.11 (cf. Prop 3.10 [24], Prop 2.13 [19]).

$$D(h_{w,W}) \leq 12L + 6W + 48\delta.$$

Note that the defect only depends on $|w|, W$ and δ. If we take $W = 1$, then L depends only on δ if $|w| \geq 2$. In particular, the upper bound in Proposition 3.11 depends only on δ.

Although h_w is unbounded if w is cyclically reduced in Section 2, $h_{w,W}$ may be bounded.

Example 3.12. Let $G = \langle a, b | a^2 = b^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$. The group G is an elementary word-hyperbolic group. Since G is generated by torsion elements a, b, there is no non-trivial homomorphism. It follows that any quasi-homomorphism is bounded (use Theorem 1.2. G is amenable).

Indeed, this conclusion can be thought of as a consequence of an algebraic property. Let h be a homogeneous quasi-homomorphism. To see $h(w) = 0$ for all w, we may assume that w is either a, b or $(ab)^n$ since w is conjugate to one of those. $h(a) = h(b) = 0$ since $a = a^{-1}, b = b^{-1}$. Since ab is conjugate to $ba = (ab)^{-1}$ by a, $h(ab) = 0$. What is essential in this argument is the algebraic property that $(ab)^n$ is conjugate to $(ab)^{-n}$. We will state this as an axiom in Condition 6.2. This property can be thought of as a dynamical property concerning the action of G on its Cayley graph. Namely, the points $(ab)^n$ are on a geodesic axis α for the action of ab, which is flipped by a to α with the opposite direction.

The following result (cf. [19], and [6] for WPD-actions) guarantees that there are many choices w such that $h_{w,1}$ are unbounded quasi-homomorphisms. We already know that G contains a (quasi-convex) free group F of rank two by Theorem 3.6. Proposition 3.13 says that one can take F to satisfy an additional dynamical property (no flip of an axis), which is explained in Example 3.12. This property is critical to show (2). See Remark 3.10. For the counting functions $c_{w,1}, c_{w^{-1},1}$ for $1 \neq w \in F$ to make sense, we take a geodesic path/word from 1 to w, which we also denote w. For the definition of quasi-convexity, see [28], [10].

Proposition 3.13. *Let G be a non-elementary word-hyperbolic group. Then there exist a quasi-convex subgroup $F < [G,G]$ which is isomorphic to a rank-two free group and a constant D such that for each non-trivial element $w \in F$ we have the following:*

(1) $c_{w,1}(w^n) \geq n/2$ *for all* $n > 0$.

(2) $c_{w^{-1},1}(w^n) = 0$ *for all* $n > 0$.

(3) $D(h_{w,1}) \leq D$, *where* $h_{w,1} = c_{w,1} - c_{w^{-1},1}$.

In particular, $h_{w,1}$ is an unbounded quasi-homomorphism. Moreover, one can show (see [19]) that there is a sequence of elements $w_i \in F$ such that the corresponding quasi-homomorphisms h_i are linearly independent. This proves Theorem 3.7. Since $w \in [G,G]$, it follows that $\bar{h} \in \mathrm{HQH}(G)$ is not a homomorphism.

3.2 Mapping class groups and curve complexes

We apply the construction of quasi-homomorphisms in Section 3.1 to mapping class groups.

Let S be a compact orientable surface of genus g and p punctures. The *mapping class group* of S, $\mathrm{MCG}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$. This group acts on the *curve complex* $\mathcal{C}(S)$ of S defined by Harvey [35] and successfully used in the study of mapping class groups by Harer [34], [33]. For our purposes, we will restrict to the 1-skeleton of Harvey's complex, so that $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential, nonparallel, nonperipheral, simple closed curves in S and two distinct vertices are joined by an edge if they can be realized simultaneously by pairwise disjoint curves. If a non-empty (finite) collection of vertices are realized simultaneously by pairwise disjoint curves, we call it a *curve system* (or *multi curve*). (The actual curve complex of S is the flag complex made from $\mathcal{C}(S)$, and it is quasi-isometric to $\mathcal{C}(S)$. A curve system defines a simplex in the curve complex.)

In certain *sporadic* cases $\mathcal{C}(S)$ as defined above is 0-dimensional or empty. This happens when there are no curve systems consisting of two curves, i.e. when $g = 0$, $p \leq 4$ and when $g = 1$, $p \leq 1$. One could rectify the situation by declaring that in those cases two vertices are joined by an edge if the corresponding curves can be realized with only one intersection point.

The mapping class group $\mathrm{MCG}(S)$ acts on $\mathcal{C}(S)$ by $a \cdot [c] = [a(c)]$, where $a \in \mathrm{MCG}(S)$ and $[c]$ is the isotopy class of a simple closed curve c on S. A classification of each element a in $\mathrm{MCG}(S)$ is known (cf [36, Section 7.1]):

(1) a has finite order.

(2) There exists a curve system M on S such that the simplex that M defines is invariant by a (maybe its vertices are permuted). Then a is called *reducible*.

(3) a is not reducible and has infinite order. a is called *pseudo-Anosov*.

Two pseudo-Anosov elements a, b are called *independent* if the subgroup generated by a, b does not contain $\mathbb{Z}$ as a subgroup of finite index.

H. Masur and Y. Minsky proved the following remarkable result.

Theorem 3.14 ([44]). *Let S be a nonsporadic surface. The curve complex $\mathcal{C}(S)$ is δ-hyperbolic. An element of* $\mathrm{MCG}(S)$ *acts hyperbolically on $\mathcal{C}(S)$ if and only if it is pseudo-Anosov .*

It follows that $a \in \mathrm{MCG}(S)$ has positive (indeed, uniformly positive by [9]) translation length on $\mathcal{C}(S)$ (Definition 3.5) if and only if a is pseudo-Anosov .

Remark 3.15. Theorem 3.14 is generalized to a non-orientable surface [7]. When a surface S is non-orientable, we consider the group of isotopy classes of all homeomorphisms $S \to S$. This group is called the *extended mapping class group* of S. When S is orientable, the extended mapping class group contains $\mathrm{MCG}(S)$ as a subgroup of index two.

The action of $\mathrm{MCG}(S)$ on $\mathcal{C}(S)$ is not proper. We introduce the following notion.

Definition 3.16 (WPD). We say that the action of G on a δ-hyperbolic space X satisfies WPD (weak proper discontinuity) if

- G contains at least one element that acts on X as a hyperbolic isometry, and
- for every hyperbolic element $g \in G$, for every $x \in X$, and for every $C > 0$, there exists $N > 0$ such that the set
$$\{\gamma \in G | d(x, \gamma(x)) \leq C, d(g^N(x), \gamma g^N(x)) \leq C\}$$
is finite.

Proposition 3.17 ([6]). *Let S be a nonsporadic surface. The action of* $\mathrm{MCG}(S)$ *on the curve complex $\mathcal{C}(S)$ satisfies WPD.*

The following is a generalization of Theorem 3.7, which is the case when the action of G on X is proper (and co-compact). As we point out in Remark 3.8, that the action is co-compact is not important.

Theorem 3.18 ([6]). *Let X be a δ-hyperbolic space and suppose G acts on X by isometry and WPD. If G contains a hyperbolic isometry and is not virtually $\mathbb{Z}$, then $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional.*

The argument for Theorem 3.18 is similar to Theorem 3.7. To construct counting functions on G using its action on X, we modify the definition of counting functions (section 3.1) as follows. Let w be a path in X and call $a(w)$ for $a \in G$ a *copy of* w. For a path α in Γ, define $|\alpha|_w$ to be the maximal number of disjoint oriented copies of w which can be obtained as subpaths of α. All other definitions are the same as before. To find many elements w which give unbounded quasi-homomorphisms, we prove something similar to Proposition 3.13. This is where WPD is essentially used.

By Theorem 3.14 and Proposition 3.17, we can apply Theorem 3.18 to the action of $\mathrm{MCG}(S)$ on $\mathcal{C}(S)$. We obtain the following. This settles Morita's conjectures 6.19 and 6.21 [52] in the affirmative.

Theorem 3.19 ([6]). *Let S be a compact orientable surface. Suppose $G < \mathrm{MCG}(S)$ is a subgroup. If G is not virtually abelian, then $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional.*

In the argument for Theorem 3.19, we use the following classification of subgroups of a mapping class group (see [46]).

Theorem 3.20. *Let G be a subgroup of the mapping class group of an orientable surface S. Then one of the following holds:*

(1) *G contains two pseudo-Anosov elements which are independent. (Called sufficiently large.) Then, G contains a free group of rank two.*
(2) *G contains $\mathbb{Z}$ as a subgroup of finite index.*
(3) *G fixes a multi curve on S. (called reducible).*

From this classification, a Tits alternative follows (cf. Theorem 3.6), namely, either G contains a free group of rank two, or else G contains a free abelian group of finite rank as a subgroup of finite index ([11]).

3.3 Rank-1 manifolds

Let M be a complete Riemannian manifold of non-positive sectional curvature of finite volume, and $G = \pi_1(M)$. We briefly discuss $\widetilde{\mathrm{QH}}(G)$ in this section. Suppose $\dim M \geq 2$. Assume that G is *irreducible*, namely, it does not contain a subgroup H of finite index such that H is product of two infinite groups. If M is a locally symmetric space, namely the universal cover $\tilde{M}$ is a symmetric space (cf [10]), then $\widetilde{\mathrm{QH}}(G)$ is trivial if the rank of M is at least two (Theorem 4.1), or $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional if the rank is one (see the proof of Theorem 5.4, cf Theorem 3.18).

Indeed the converse of Theorem 4.1 is true. In other words, $\widetilde{\mathrm{QH}}(\Gamma) = 0$ characterizes locally symmetric spaces of rank at least two.

Theorem 3.21 ([8]). *Let M be a complete Riemannian manifold of nonpositive curvature and finite volume. Assume that $\Gamma = \pi_1(M)$ is finitely generated and does not contain a subgroup of finite index which is cyclic or a Cartesian product of two infinite groups. Then the universal cover $\tilde{M}$ is a symmetric space of rank at least two if and only if $\widetilde{\mathrm{QH}}(\Gamma) = 0$. Otherwise, $\widetilde{\mathrm{QH}}(\Gamma)$ is infinite-dimensional.*

The proof uses the celebrated Rank Rigidity Theorem ([1]), as well as a new construction of quasi-homomorphisms on groups that act on CAT(0) spaces and contain rank-1 elements, which can be thought of as a generalization of Theorem 3.18. (See [10],[1] for the definitions of CAT(0) spaces and rank-1 elements.) In connection to Theorem 4.1, we remark that a symmetric space of non-compact type is CAT(0), and if it has rank at least two then any hyperbolic isometry of the space is not rank-1.

4 Rigidity

We discuss a version of superrigidity for mapping class groups. Theorem 4.2 was conjectured by N.V. Ivanov and proved by Kaimanovich and Masur [39] using random walks in the case when the image group contains independent pseudo-Anosov elements and it was extended to the general case by Farb and Masur [22] using the classification of subgroups of $\mathrm{MCG}(S)$ (see section 3.2). We give an argument based on the work of M. Burger and N. Monod [13] on bounded cohomology of lattices.

Theorem 4.1 ([13],[14]). *Let Γ be an irreducible lattice in a connected semi-simple Lie group G with no compact factors, with finite center, and of rank > 1. Then the kernel of $H^2_b(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ is trivial.*

They indeed show that $\widetilde{\mathrm{QH}}(\Gamma)$ is trivial. Their approach is out of the range of this chapter. It was known that $H^1(\Gamma;\mathbb{R})$ is trivial by Matsushima and others.

Theorem 4.2. *Let Γ be an irreducible lattice in a connected semi-simple Lie group G with no compact factors, with finite center, and of rank > 1. Then every homomorphism $\Gamma \to \mathrm{MCG}(S)$ has finite image.*

Proof. Let $\phi : \Gamma \to \mathrm{MCG}(S)$ be a homomorphism. By the Margulis-Kazhdan theorem [55, Theorem 8.1.2] either the image of ϕ is finite or the kernel of ϕ is contained in the center. When Γ is a nonuniform lattice, the proof is easier and was known to Ivanov before the work of Kaimanovich-Masur (see Ivanov's comments to Problem 2.15 on Kirby's list). Since the rank is ≥ 2 the lattice Γ then contains a solvable subgroup N which does not become abelian after quotienting out a finite normal subgroup. If the kernel is finite, then $\phi(N)$ is a solvable subgroup of $\mathrm{MCG}(S)$ which is not virtually abelian, contradicting [11] (see the classification of subgroups in mapping class groups in section 3.2).

Now assume that Γ is a uniform lattice. If the kernel $Ker(\phi)$ is finite then there is an unbounded quasi-homomorphism $q : Im(\phi) \to \mathbb{R}$ by Theorem 3.19. But then $q\phi : \Gamma \to \mathbb{R}$ is an unbounded quasi-homomorphism contradicting Theorem 4.1 that every quasi-homomorphism $\Gamma \to \mathbb{R}$ is bounded. $\square$

In connection to Theorem 4.1, we ask a question.

Question 4.3. Let Γ be as in Theorem 4.1. Is there a constant C such that for all $a \in [\Gamma, \Gamma]$, $\mathrm{cl}(a) \leq C$?

Note that $[\Gamma, \Gamma]$ has finite index in Γ since $H^1(\Gamma; \mathbb{R})$ is trivial. The answer is yes if Γ is $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ (see section 1.2).

5 Bounded generation

A group G is said *boundedly generated* if there exist finitely many elements $g_1, \cdots, g_k \in G$ such that for any $g \in G$ there exist $n_i \in \mathbb{Z}$ with

$$g = g_1^{n_1} \cdots g_k^{n_k}.$$

One may say G is boundedly generated by $g_1, \cdots, g_k$.

Kotschick related bounded generation of a group G and $\mathrm{HQH}(G)$ as follows.

Theorem 5.1 (Prop 5 [41]). *If G is boundedly generated by $g_1, \cdots, g_k$ then the dimension of* $\mathrm{HQH}(G)$ *as a vector space is at most k.*

If G is generated by k elements, then the vector space of all homomorphisms from G to $\mathbb{R}$ is at most k-dimensional. One may see this theorem as a generalization. He combined this result and Theorem 3.19, and gave a new proof to the following theorem by Farb-Lubotzky-Minsky.

Theorem 5.2 ([23]). *The mapping class group* MCG *of a closed orientable surface S of genus at least one is not boundedly generated.*

In fact, since Theorem 3.19 applies to all subgroups in $\mathrm{MCG}(S)$, a subgroup G in $\mathrm{MCG}(S)$ is not boundedly generated if G is not virtually abelian (cf. [26]).

It is observed in [23] that non-elementary word-hyperbolic group G is not boundedly generated. Their argument uses the deep result by Gromov [28] such that such G has an infinite quotient which is a torsion group. Clearly, a boundedly generated group cannot have an infinite torsion quotient. By Theorem 5.1 and Theorem 3.7 (and Remark 3.8), we have the following([26]).

Theorem 5.3. *A non-elementary subgroup in a word-hyperbolic group is not boundedly generated.*

It follows that a uniform lattice G in a simple Lie group of rank one is not boundedly generated since G is non-elementary word-hyperbolic. Margulis and Vinberg [43] showed that many discrete subgroups in a rank-1 simple Lie group are virtually mapped by homomorphisms to non-abelian free groups, so that they are not boundedly generated. A group is said to *virtually* have some property if some subgroup of finite index in the group has this property. In fact we have the following.

Theorem 5.4 ([26]). *Let G be a discrete subgroup in a rank-1 simple Lie group. If G does not contain a nilpotent subgroup of finite index then it is not boundedly generated.*

Proof. G acts on the rank-1 symmetric space, which is δ-hyperbolic. The action is proper. If G is not virtually nilpotent, then G contains a hyperbolic isometry (we use a classification of discrete subgroups in a rank-1 simple Lie group). Then a theorem from [24] (the theorem applies to proper G-actions on δ-hyperbolic spaces. Or one can use Theorem 3.19) says that $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional since G is not virtually cyclic. $\square$

Note that Theorem 5.4 gives a classification of virtually nilpotent subgroups among discrete subgroups in terms of bounded generation since the converse is true. It is not hard to check that a finitely generated nilpotent group is boundedly generated. It then follows that a finitely generated virtually nilpotent group is boundedly generated. If G as in the theorem is virtually nilpotent, then it is finitely generated, therefore, boundedly generated.

Non-uniform lattices in a Lie group of rank at least two are known to be boundedly generated (cf. [54]). For example, $\mathrm{SL}(n,\mathbb{Z}), n > 2$ and $\mathrm{SL}(2,\mathbb{Z}[1/p])$ such that p is a prime number are boundedly generated.

There is a more direct way to show Theorem 5.2, 5.3, 5.4 using quasi-homomorphisms. We discuss it in the next section (for example see Remark 6.7).

6 Separation by quasi-homomorphisms

Definition 6.1 (Separation, [53]). Let G be a group and $a \neq b \in G$. If there exists a homogeneous quasi-homomorphism f on G such that $f(a) = 1$ and $f(b) = 0$, then we say that a is *separated* from b (by f).

Let $B \subset G$ be a set of elements such that $a \notin B$. If there exists a homogeneous quasi-homomorphism f on G such that $f(a) = 1$ and $f(b) = 0$ for all $b \in B$, then we say that a is *separated* from B (by f).

The condition that a quasi-homomorphism f is homogeneous is necessary, otherwise, one can always separate a from b (by letting $f(a) = 1$ and $f(c) = 0$ for all $c \neq a$). On the other hand, as long as $f(a) \neq 0$, one can always normalize f such that $f(a) = 1$. The normalization $f(a) = 1$ becomes important when one tries to bound the defect $D(f)$ from above. See (the second part of) Theorem 7.3 and 7.4.

Our separation property has a similar flavor to the residual finiteness of a group. A group G is said to be *residually finite* if for any non-trivial element $a \in G$, there exists a finite group F and a homomorphism $f : G \to F$ such that $f(a)$ is non-trivial. Similarly, we may try to separated two elements by a homomorphism to $\mathbb{Z}$. But, for example, if $G \simeq \mathrm{SL}(2,\mathbb{Z})$, then any homomorphism $G \to \mathbb{R}$ is trivial since G is generated by two torsion elements. Therefore, it is impossible to separate two elements by a homomorphism to $\mathbb{Z}$. On the other hand, we know that $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional, and moreover we can separate two elements by a map in $\widetilde{\mathrm{QH}}(G)$ (G is non-elementary word-hyperbolic. Apply Theorem 3.7).

Suppose that one can separate a from b by a homogeneous quasi-homomorphism f such that $f(a) = 1, f(b) = 0$. Then the elements a and b must satisfy the following condition since f is a class function.

Condition 6.2. (1) For all $n \neq m$ and $c \in G$, $a^n \neq ca^mc^{-1}$.

(2) For all $n \neq 0, m$ and $c \in G$, $a^n \neq cb^mc^{-1}$.

Note that by Condition (1), a has infinite order. It is interesting to know if Condition 6.2 is sufficient to separate a from b by a homogeneous quasi-homomorphism. An affirmative answer is found by Polterovich and Rudnick [53] for $\mathrm{SL}(2,\mathbb{Z})$.

Theorem 6.3. *Suppose $a, b \in \mathrm{SL}(2,\mathbb{Z})$ satisfy Condition 6.2. Then, there is a homogeneous quasi-homomorphism f such that $f(a) = 1, f(b) = 0$.*

Polterovich and Rudnick asked if one can generalize the theorem to word-hyperbolic groups.

Theorem 6.4 ([16] cf.[19]). *Let G be a word-hyperbolic group. Suppose $a, b \in G$ satisfy Condition 6.2. Then, there is a homogeneous quasi-homomorphism f on G such that $f(a) = 1, f(b) = 0$.*

Moreover, let $B \subset G$ be a finite collection of elements such that for a and each $b \in B$ Condition 6.2 holds. Then, there is a homogeneous quasi-homomorphism f on G such that $f(a) = 1$ and for all $b \in B, f(b) = 0$.

We also show a separation theorem for mapping class groups.

Theorem 6.5 ([16] cf.[6]). *Let S be a compact orientable surface and let $\mathrm{MCG}(S)$ be its mapping class group. Suppose $a, b \in \mathrm{MCG}(S)$ satisfy Condition 6.2 and a is a pseudo-Anosov element. Then, there is a homogeneous quasi-homomorphism f on G such that $f(a) = 1, f(b) = 0$.*

Moreover, let $B \subset \mathrm{MCG}(S)$ be a collection of elements such that Condition 6.2 holds for a and each $b \in B$. Suppose there exists T such that the translation length of each $b \in B$ on $\mathcal{C}(S)$ is at most T. Then, there is a homogeneous quasi-homomorphism f on G such that $f(a) = 1$ and for all $b \in B, f(b) = 0$.

In fact, Theorem 6.4, 6.5 are part of Theorem 7.3, 7.4, in which we obtain upper bounds on the defect of f.

Note that it is free to assume that the set B contains all non-pseudo-Anosov elements. This is because if $c \in \mathrm{MCG}(S)$ is not pseudo-Anosov, then the translation length of c on $\mathcal{C}(S)$ is zero as c has a bounded orbit. It follows that a homogeneous quasi-homomorphism f obtained in Theorem 6.5 satisfies $f(c) = 0$.

To explain the connection of separation and bounded generation, we need one definition.

Definition 6.6 (Product of subgroups). Let G be a group and $H_1, \cdots, H_n < G$ subgroups. Then, *product*, $H_1 \cdots H_n$, is a subset of G defined as follows:

$$H_1 \cdots H_n = \{h_1 \cdots h_n | h_i \in H_i\}.$$

Remark 6.7. One can show Theorem 5.3 using Theorem 6.4 as follows. Let G be a non-elementary word-hyperbolic group. Suppose that elements $b_1, \cdots, b_n \in G$ are given. Then, one can find an element $a \in G$ such that a and each b_i satisfy Condition 6.2 (this is not trivial). By Theorem 6.4, there exists a homogeneous quasi-homomorphism f with $f(a) = 1$ and $f(b_i) = 0$ for all i. Then, $|f|$ is bounded, by $(n-1)D(f)$, on the following subset in G.

$$\langle b_1 \rangle \cdots \langle b_n \rangle$$

Since f is unbounded on $\langle a \rangle$, we have $G \neq \langle b_1 \rangle \cdots \langle b_n \rangle$. Therefore G is not boundedly generated by $b_1, \cdots, b_n$.

Similarly, one can show that $\mathrm{MCG}(S)$ is not boundedly generated using Theorem 6.5.

7 Gaps in stable commutator length

We discuss the image, or the spectrum, of the function scl on $[G, G]$.

7.1 word-hyperbolic groups

D. Calegari [15] shows the following theorem.

Theorem 7.1. *For every dimension n and any $\epsilon > 0$, there is a constant $\delta(\epsilon, n) > 0$ such that if M is a complete hyperbolic n-manifold and $a \in \pi_1(M)$ has stable commutator length $\leq \delta(\epsilon, n)$, then a is represented by a closed geodesic in M with length $\leq \epsilon$.*

Since there are only finitely many closed geodesics of length at most ϵ in M, this theorem says that there is a gap (at zero) in the spectrum of stable commutator length. Calegari uses pleated surfaces in M to estimate stable commutator length from below. A similar argument appears in [28], where Gromov asserts that the hyperbolicity implies the positivity of scl. The existence of a gap at zero was found by Calegari.

Via Theorem 1.4, Theorem 7.1 is related to quasi-homomorphisms on $\pi_1(M)$. In some way, the following result [16] is a generalization to word-hyperbolic groups.

Theorem 7.2 (Gap Theorem in hyperbolic groups, weak version [16]). *Let G be a word-hyperbolic group which is δ-hyperbolic with respect to a symmetric generating set S with $|S|$ generators. Then there is a constant $C(\delta, |S|) > 0$ such that for every $a \in G$, either $\mathrm{scl}(a) \geq C$ or else there is some positive integer n and some $b \in G$ such that $ba^{-n}b^{-1} = a^n$.*

Note that $\mathrm{scl}(a) = 0$ if the condition $ba^{-n}b^{-1} = a^n$ holds for $n > 0$ (cf. Condition 6.2 (1). This condition is called *mirror condition* in [16]). It follows from this condition that b has finite order if a has infinite order. Therefore the condition never holds in the fundamental group of a hyperbolic manifold since there is no nontrivial torsion element (cf. Theorem 7.1).

Theorem 7.2 is a consequence of the first part of the following theorem by Proposition 1.3 (cf. Theorem 1.4) with $C = \frac{1}{2D}$. The second part of the theorem can be thought of a separation theorem (see section 6).

Theorem 7.3 (Gap Theorem in hyperbolic groups, strong version [16]). *Let G be a word-hyperbolic group which is δ-hyperbolic with respect to a symmetric generating set S with $|S|$ generators. There exists a constant $D(\delta, |S|)$ with the following property. Let $a \in G$ be a (non-torsion) element. Assume there is no $n > 0$ and no $b \in G$ with $ba^{-n}b^{-1} = a^n$. Then there is a homogeneous quasi-homomorphism h on G such that*

(1) *$h(a) = 1$.*

(2) *The defect of h is $\leq D(\delta, |S|)$.*

Moreover, let $a_i \in G$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n \neq 0, m$ and all elements $b \in G$ and indices i, that there is an inequality

$$ba^n b^{-1} \neq a_i^m.$$

Then there is a homogeneous quasi-homomorphism h on G such that

(1) *$h(a) = 1$, and $h(a_i) = 0$ for all i;*

(2) *the defect of h is $\leq D'(\delta, |S|, T, \tau(a))$.*

Note that the translation length τ concerns the Cayley graph of G with respect to S. The argument for Theorems 7.2, 7.3 is a refinement of the one for Theorem 3.7. We construct a quasi-homomorphisms f by counting functions, and the issue is to bound the defect of f.

7.2 Mapping class groups

We show a theorem similar to Theorem 7.3 for mapping class groups.

Theorem 7.4 ([16]). *Let S be a compact orientable surface of hyperbolic type and $\mathrm{MCG}(S)$ its mapping class group. Then there is a positive integer P depending on S such that for any pseudo-Anosov element a, either there is an $0 < n \leq P$ and an element $b \in \mathrm{MCG}(S)$ with $ba^{-n}b^{-1} = a^n$, or else there exists a homogeneous quasi-homomorphism h on $\mathrm{MCG}(S)$ such that $h(a) = 1$ and the defect of h is $\leq D(S)$, where $D(S)$ depends only on S.*

Moreover, let $a_i \in \mathrm{MCG}(S)$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n \neq 0, m$ and all elements $b \in \mathrm{MCG}(S)$ and indices i, that there is an inequality

$$ba^n b^{-1} \neq a_i^m$$

Then there is a homogeneous quasi-homomorphism h on $\mathrm{MCG}(S)$ such that

(1) *$h(a) = 1$, and $h(a_i) = 0$ for all i.*

(2) *The defect of h is $\leq D'(S, T, \tau(a))$.*

The construction of a quasi-homomorphism is the same as in Theorem 3.19, but to have the desired bound on the defect, we need extra ingredients. This extra part is more difficult than for word-hyperbolic groups since the action of $\mathrm{MCG}(S)$ on $\mathcal{C}(S)$ is not proper, and $\mathcal{C}(S)$ is not locally finite. The standard argument which has been developed in the theory of word-hyperbolic groups does not apply immediately. To compensate this difficulty, we use the notion of *tight geodesics*, which is introduced by Masur-Minsky [44]. They show a certain local finiteness property in terms of tight geodesics. Bowditch [9] obtains more refined information than [44], which we use.

Theorem 7.5 ([9]). *Let S be a compact orientable surface and* $\mathrm{MCG}(S)$ *its mapping class group. For $R > 0$, there exist $D(R), K(R)$, which depends on S, such that for any two vertices $x, y \in \mathcal{C}(S)$ with $d(x, y) \geq D$, the following set contains at most K elements:*

$$\{a \in \mathrm{MCG}(S) | d(x, a(x)) \leq R, d(y, a(y)) \leq R\}.$$

Proposition 3.17 also follows from Theorem 7.5.

Theorem 7.6 ([9]). *Let S be a compact orientable surface and* $\mathrm{MCG}(S)$ *its mapping class group. There exists a constant $M = M(S) > 0$ such that for any pseudo-Anosov element $a \in \mathrm{MCG}(S)$, there exists a geodesic $\alpha \subset \mathcal{C}(S)$ with $a^M(\alpha) = \alpha$.*

A similar result is known for word-hyperbolic groups in terms their action on their Cayley graphs (for example, see [19, Theorem 5.1]).

Combining the first part of Theorem 7.4 and Proposition 1.3, we obtain the following with $C(S) = \frac{1}{2D(S)}$.

Theorem 7.7 (Gap theorem [16]). *Let S be a compact orientable surface of hyperbolic type and* $\mathrm{MCG}(S)$ *its mapping class group. Then there exists $C(S) > 0$ such that for any pseudo-Anosov element $a \in \mathrm{MCG}(S)$, either there is an $0 < n \leq P(S)$ and an element $b \in \mathrm{MCG}(S)$ with $ba^{-n}b^{-1} = a^n$ (then $\mathrm{scl}(a) = 0$), or else $\mathrm{scl}(a) \geq C$.*

This theorem is complementary to the following results.

Theorem 7.8. *Let S be a closed orientable surface of genus $g \geq 2$.*

(1) *[20] (cf. [40]) If $a \in \mathrm{MCG}(S)$ is a Dehn-twist along a separating simple closed curve, then $\mathrm{scl}(a) \geq \frac{1}{6(3g-1)}$.*

(2) *[21] There exists $a \in \mathrm{MCG}(S)$ such that for all $n > 0$ and $c \in \mathrm{MCG}(S)$, $a^n \neq ca^{-n}c^{-1}$ and that $\mathrm{scl}(a) = 0$.*

Note that the element a in (2) is not pseudo-Anosov by Theorem 7.7. It follows from (1) that $\mathrm{MCG}(S)$ is not uniformly perfect, and that $H_b^2(\mathrm{MCG}(S); \mathbb{R})$ is not trivial (and indeed infinite dimensional by Theorem 3.19).

8 Appendix. Bounded cohomology

The theory of bounded cohomology was developed in Gromov's seminal work [29]. We already mentioned in Section 1.3 that the space of quasi-homomorphisms on a group is closely related to the second bounded cohomology of the group. We review a part of the theory in this chapter. We recommend survey articles [5] and [50] for interested readers. All spaces and manifolds in this section are connected.

8.1 Riemannian geometry

In [29], Gromov defined the minimal volume, $\mathrm{MinVol}(M)$, of a compact manifold M to be the infimum of the volume of all Riemannian metric g on M such that the sectional curvature K_g satisfies $-1 \leq K_g \leq 1$. If $\dim M = 2$, then Gauss-Bonnet formula gives

$$\int_M K_g dv_g = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. It immediately follows that $\mathrm{MinVol}(M) = 2\pi|\chi(M)|$, and if $\chi(M) < 0$, then the minimal volume is attained (only) by a metric of constant curvature -1.

It is difficult to compute $\mathrm{MinVol}(M)$ in general. To give a lower bound for $\mathrm{MinVol}(M)$, Gromov defined the simplicial volume, $||M||$, of M, which can be used in general as a replacement of the Euler characteristic of a surface. Let $c = \sum r_i c_i (r_i \in \mathbb{R})$ be a real singular chain of M. Consider the l^1-norm defined by $||c||_1 = \sum |r_i|$. For a homology class $\alpha \in H_*(M;\mathbb{R})$, define a semi-norm by

$$||\alpha|| = \inf\{||z|| : z \text{ is closed and } [z] = \alpha\}.$$

If M is orientable, define $||M|| = ||[M]||$, where $[M]$ is the fundamental n-class. If M is not orientable, then pass to the double cover M' and define $||M|| = \frac{1}{2}||M'||$.

Theorem 8.1 ([29]). *If M is a compact n-dimensional manifold, then*

$$C_n||M|| \leq \mathrm{MinVol}(M),$$

where $C_n > 0$ is a constant which depends only on n.

Of course, if $||M|| = 0$, then this estimate is useless. Suppose $f : M \to N$ is a continuous map such that M and N are compact orientable manifolds of the same dimension. Then it is easy to see from the definition that

$$||M|| \geq |\deg f| \cdot ||N||.$$

It follows that if there exists a continuous map $g : M \to M$ such that $\deg g \neq 0, \pm 1$, then $||M|| = 0$ (if M is compact). For example, if M is a sphere or a torus, then $||M|| = 0$.

There are examples of M with $||M|| > 0$.

Theorem 8.2 (Gromov-Thurston [29]). *Let (M, g) be an n-dimensional complete Riemannian manifold with finite volume. Suppose there exists a constant k such that $-k \leq K_g \leq -1$. Then,*

$$vol(M, g) \leq c_n ||M||,$$

where c_n is a constant which depends only on n.

Moreover, if $K_g = -1$, then

$$vol(M, g) = T_n ||M||,$$

where T_n is the supremum of the volume of all geodesic n-simplices in the n-dimensional real hyperbolic space, $\mathbb{H}^n$.

It is shown in [29] that one can take $c_n = (n-1)^n n!$. A simplex is called geodesic if all of its faces are totally geodesic. The proof is by "straightening" (into a geodesic one in the case $K_g = -1$) the lift of an n-simplex contained in $[M]$ in the universal cover of M. That's how T_n comes into the estimate. It is known by now ([31]) that T_n is equal to the volume of ideal regular n-simplices in $\mathbb{H}^n$. Thus one needs to consider only regular (namely, all edges have same length) geodesic n-simplices in the definition of T_n.

We explain the connection of the simplicial volume and the bounded cohomology. The definition of bounded cohomology of a topological space X differs from the one for the ordinary real singular cohomology in that one considers only the set of singular cochains each of which is bounded as a function.

Let $S_n(X)$ be the set of n-dimensional singular simplexes in X. Real n-dimensional singular cochains are functions $S_n(X) \to \mathbb{R}$. They form a vector space over $\mathbb{R}$, which we denote $C^n(X)$. Let δ be the standard coboundary map $C^n(X) \to C^{n+1}(X)$ for each n. The real singular cohomology of X, $H^*(X; \mathbb{R})$ (sometimes we omit $\mathbb{R}$ in this chapter), is the cohomology of this cochain complex.

Now let $B^n(X) \subset C^n(X)$ be the set of all bounded functions on $S^n(X)$. Each element in $B^n(X)$ is called a bounded n-cochain. It is easy to see that $\delta(c) \in B^{n+1}(X)$ if $c \in B^n(X)$. The cohomology of the complex $B^*(X)$ is the *bounded cohomology* of X, denoted by $H_b^*(X)$. Each element $c \in C^n(X)$ has a natural l^∞-norm.

$$||c||_\infty = \sup_{\sigma \in S^n(X)} c(\sigma) \leq \infty.$$

For an element $\beta \in H^*(X)$, define

$$||\beta|| = ||\beta||_\infty = \inf_y ||y||_\infty \leq \infty,$$

where y are all cochains such that $\delta y = 0$ and $[y] = \beta$.

The inclusion $B^n(X) \to C^n(X)$ induces a canonical map $H_b^n(X) \to H^n(X)$, the comparison map. We say $\beta \in H^n(X)$ is *bounded* if it is contained in the image of this map, in other words, $||\beta||_\infty < \infty$.

The following two results in [29] are fundamental. There is a detailed account of the argument in [38], where he discusses a countable CW-complex X.

Theorem 8.3. *Let X be a topological space. Then,*

$$H_b^n(K(\pi_1(X), 1); \mathbb{R}) \simeq H_b^n(X; \mathbb{R})$$

for all n.

$H_b^n(K(\pi_1(X), 1); \mathbb{R})$ can be computed as $H_b^n(\pi_1(X); \mathbb{R})$ using the definition of the bounded cohomology of a group in Section 1.3. We obtain the following theorem, which says that the bounded cohomology depends only on the fundamental group.

Theorem 8.4. *Let X be a topological space. Then,*

$$H_b^n(X; \mathbb{R}) \simeq H_b^n(\pi_1(X); \mathbb{R}).$$

By this theorem, if M is a closed Riemannian manifold of negative sectional curvature, then $H_b^2(M; \mathbb{R})$ is infinite dimensional, in particular, nontrivial. This is because $G = \pi_1(M)$ is non-elementary word-hyperbolic, therefore $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional by Theorem 3.7, so that $H_b^2(G; \mathbb{R})$ is also infinite dimensional since $\widetilde{\mathrm{QH}}(G)$ is a subspace as a vector space over $\mathbb{R}$ in $H_b^2(G; \mathbb{R})$ (see Section 1.3).

The simplicial volume of a manifold M is related to the bounded cohomology of M as follows.

Theorem 8.5. *Let M be an n-dimensional closed orientable manifold and $\alpha \in H^n(M; \mathbb{R})$ the fundamental class such that $\langle \alpha, [M] \rangle = 1$. Then,*

$$||M||^{-1} = ||\alpha||_\infty.$$

In particular, if α is bounded, namely $||\alpha||_\infty < \infty$, then $||M|| \neq 0$.

It follows that if M is simply connected, then $||M|| = 0$. This is because $H_b^n(M; \mathbb{R})$ is trivial since $\pi_1(M)$ is trivial. Therefore, $||\alpha||_\infty = \infty$.

The following is also proved using straightening.

Theorem 8.6 ([29]). *Let M be a closed Riemannian manifold such that the sectional curvature is negative. Then the map $H_b^n(M;\mathbb{R}) \to H^n(M;\mathbb{R})$ is surjective for all $n > 1$.*

If M is an n-dimensional closed hyperbolic manifold ($K_g = -1$), then by Theorems 8.1 and 8.2, $\frac{C_n}{T_n} vol(M) \le \mathrm{MinVol}(M)$. The following result was conjectured in [29].

Theorem 8.7 ([4]). *Let (M,g) be a closed Riemannian manifold such that $K_g = -1$. Then,* $\mathrm{MinVol}(M) = vol(M,g)$ *and a metric which attains* $\mathrm{MinVol}(M)$ *is isometric to g.*

We record one more recent progress. This is an answer in affirmative to a question in [29].

Theorem 8.8 ([42]). *Let M be a closed locally symmetric space of non-compact type. Then $||M|| > 0$.*

In particular it follows that $\mathrm{MinVol}(M) > 0$ for such manifolds by Theorem 8.1, which was known for most cases([30], [17]).

8.2 Group theory

Theorem 8.6 is generalized to word-hyperbolic groups. In general, $H_b^1(G;\mathbb{R}) = 0$ since a bounded homomorphism from G to $\mathbb{R}$ is trivial.

Theorem 8.9 ([48]). *Let G be a non-elementary word-hyperbolic group. Then the map $H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})$ is surjective for all $n > 1$.*

In this chapter, we have seen several examples of a group G such that $\widetilde{\mathrm{QH}}(G,\mathbb{R})$ is infinite dimensional. Those groups have infinite dimensional $H_b^2(G,\mathbb{R})$. Here is a list of such G.

(1) Free groups of rank at least two (Theorem 2.1).

(2) Non-elementary subgroups of a word-hyperbolic group (Theorem 3.7, Remark 3.8).

(3) Subgroups in $\mathrm{MCG}(S)$ which are not virtually abelian (Theorem 3.19).

(4) Discrete subgroups in a rank-1 simple Lie group which are not virtually nilpotent (see the proof of Theorem 5.4).

(5) The fundamental group G of a complete Riemannian manifold of M of dimension at least two such that $vol(M) < \infty$, the sectional curvature is non-positive, M is not locally symmetric of rank at least two and G is irreducible (Theorem 3.21).

(6) $G = A *_C B$ such that $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$; or $G = A*_{C,\phi}$ such that $|A/C| \geq 2$ and $|A/\phi(C)| \geq 2$ (see [25]).

If there is a surjective homomorphism $h : G \to F$, where F is a rank two free group (sometimes then G is called *large*), then $\widetilde{\mathrm{QH}}(G)$, therefore, $H_b^2(G, \mathbb{R})$ is infinite dimensional. This is because if $f : F \to \mathbb{R}$ is a homogeneous quasi-homomorphism, then $f \circ h : G \to \mathbb{R}$ is a homogeneous quasi-homomorphism. (We do not need that G is finitely generated. $\widetilde{\mathrm{QH}}(F)$ is indeed infinite dimensional if we restrict it to $[F, F]$ as well.) For example, this argument applies to the fundamental group of a closed orientable surface of genus at least two, which is non-elementary word-hyperbolic. By the same reason, if a group G has a surjective homomorphism to one of the groups in the list, then $\widetilde{\mathrm{QH}}(G)$ is infinite dimensional.

Not much is known about $H_b^n(G; \mathbb{R})$ for $n > 2$. If M is an n-dimensional closed locally symmetric space, then $H_b^n(\pi_1(M); \mathbb{R})$ is non-trivial by Theorems 8.4, 8.6, 8.8. It is not known in general if the dimension of $H_b^n(\pi_1(M); \mathbb{R})$ is finite.

There is a new direction of study of the second bounded cohomology with non-trivial coefficient. It is revealed that it has a connection to rigidity in terms of orbit equivalence of actions.

Let Γ and Λ be countable groups and $(X, \mu), (Y, \nu)$ probability $\Gamma-$ and $\Lambda-$spaces respectively. A measurable isomorphism $F : X \to Y$ is said to be *orbit equivalence* (OE) of the actions if for a.e. $x \in X$, $F(\Gamma x) = \Lambda F(x)$. (See [50], [51].)

Let C_{reg} be the class of countable groups G such that $H_b^2(G, \ell^2(G)) \neq 0$.

Theorem 8.10 ([51]). *A countable group G belongs to C_{reg} if it admits one of the following actions.*

(1) *A non-elementary simplicial action on a simplicial tree, proper on the set of edges,*

(2) *a non-elementary, proper isometric action on a proper CAT(-1) space,*

(3) *a non-elementary, proper isometric action on a δ-hyperbolic graph with bounded valency.*

In particular, a countable group which is free of rank at least two, a non-trivial free product of two countable groups except for $\mathbb{Z}_2 * \mathbb{Z}_2$, and a non-elementary subgroup of a word-hyperbolic group are in C_{reg}.

Among many rigidity theorems, they showed the following.

Theorem 8.11 ([51]). *Let Γ_1, Γ_2 be torsion-free groups in C_{reg}, $\Gamma = \Gamma_1 \times \Gamma_2$, and let (X, μ) be an irreducible probability Γ-space. Let (Y, ν) be any other probability Γ-space. If the Γ-actions on X and Y are orbit equivalent, then they are isomorphic with respect to an automorphism of Γ.*

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