# ON THE PERIODS OF AUTOMORPHIC FORMS ON SPECIAL ORTHOGONAL GROUPS AND THE GROSS-PRASAD CONJECTURE

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*Dedicated to Professor Hiroyuki Yoshida on the occasion of his sixtieth birthday*

## Introduction

In early 90's, Gross and Prasad [12], [13] gave a series of fascinating conjectures on the restriction of automorphic representation of a special orthogonal group to a smaller special orthogonal subgroup. We now recall their global conjecture. Let $k$ be a global field with $\mathrm{char}(k) \neq 2$. Let $(V_0, Q_0) \subset (V_1, Q_1)$ be quadratic forms over $k$ with rank $n$ and $n+1$, respectively. We assume that $n \geq 2$ and that $(V_0, Q_0)$ is not isomorphic to the hyperbolic plane. We regard $G_0 = \mathrm{SO}_{Q_0}$ as a subgroup of $G_1 = \mathrm{SO}_{Q_1}$. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible tempered cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. Assume that $\mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any place $v$ of $k$. Then the global Gross-Prasad conjecture [12] asserts that

$$\langle \varphi_1|_{G_0}, \varphi_0 \rangle := \int_{G_0(k)\backslash G_0(\mathbb{A})} \varphi_1(g_0)\overline{\varphi_0(g_0)}\, dg_0 \neq 0$$

for some $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$ if and only if $L(1/2, \pi_1 \boxtimes \pi_0) \neq 0$. Here, $L(s, \pi_1 \boxtimes \pi_0)$ is the "product" $L$-function of $\pi_1$ and $\pi_0$.

In this paper, we would like to formulate a conjecture, which expresses the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ in terms of $L$-values. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_1 = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l, \chi_{Q_1}) & \text{if } \dim V_1 = 2l, \end{cases}$$

where $\chi_{Q_1}$ is the quadratic Hecke character associated with the discriminant of $Q_1$. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume, for simplicity, $\pi_1$ and $\pi_0$ are tempered. We put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s+(1/2), \pi_1, \mathrm{Ad})L(s+(1/2), \pi_0, \mathrm{Ad})},$$

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where $L(s, \pi_1, \mathrm{Ad})$ and $L(s, \pi_0, \mathrm{Ad})$ are the adjoint $L$-function of $\pi_1$ and that of $\pi_0$, respectively. We assume that the $L$-functions $L(s, \pi_1 \boxtimes \pi_0)$, $L(s, \pi_1, \mathrm{Ad})$, and $L(s, \pi_0, \mathrm{Ad})$ have meromorphic continuation. For a sufficiently large finite set of bad places $S$, we denote the partial Euler products for $\mathcal{P}_{\pi_1,\pi_0}(s)$ and $\Delta_{G_1}$ by $\mathcal{P}^S_{\pi_1,\pi_0}(s)$ and $\Delta^S_{G_1}$, respectively.

Let $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$ be cusp forms. We consider the matrix coefficients

$$\Phi_{\varphi_{1,v},\varphi_{1,v}}(g_1) = \langle \pi_{1,v}(g_1)\varphi_{1,v}, \varphi_{1,v}\rangle_v, \quad g_1 \in G_1(k_v),$$
$$\Phi_{\varphi_{0,v},\varphi_{0,v}}(g_0) = \langle \pi_{0,v}(g_0)\varphi_{0,v}, \varphi_{0,v}\rangle_v, \quad g_0 \in G_0(k_v).$$

Put

$$I(\varphi_{1,v}, \varphi_{0,v}) = \int_{G_0(k_v)} \Phi_{\varphi_{1,v},\varphi_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v},\varphi_{0,v}}(g_{0,v})}\, dg_{0,v}.$$

It will be proved that this integral is convergent (Proposition 1.1).

Then we conjecture that there exists an integer $\beta$ such that

$$(\bigstar) \qquad \frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle \langle \varphi_0, \varphi_0\rangle} = 2^\beta C_0 \Delta^S_{G_1} \mathcal{P}^S_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2},$$

where $C_0$ is a constant determined by the choice of the local and global Haar measures of $G_0(\mathbb{A})$ (Conjecture 1.5). For more precise definitions, see §1. When $n = 2$, our conjecture reduces to the theorem of Waldspurger [46].

One can give a possible interpretation of the factor $2^\beta$ in ($\bigstar$) in terms of the Arthur conjecture [2]. Let $\mathcal{L}_k$ be the hypothetical Langlands group for $k$. Then, if we admit the Arthur conjecture, for an irreducible cuspidal tempered automorphic representation $\pi_i$ of $G_i(\mathbb{A})$ ($i = 0, 1$), one can attach an $L$-homomorphism $\psi_i : \mathcal{L}_k \to {}^L G_i = \hat{G}_i \rtimes W_k$, where $W_k$ is the Weil group [45] of $k$. It is generally believed that the structure of the $L$-packet for $\pi_i$ is closely related to the finite group $\mathcal{S}_{\psi_i} = \mathrm{Cent}_{\hat{G}_i}(\mathrm{Im}(\psi_i))$. Then, we conjecture that

$$2^\beta = \frac{1}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|}.$$

(cf. Conjecture 2.1.)

This paper consists of three parts. In Part I (§§1-3), we formulate our conjecture in detail. We first formulate our conjecture in the tempered case. Then we discuss the relation with the Arthur conjecture. In particular, a possible interpretation of the factor $2^\beta$ in terms of Arthur parameter will be given. In §3, we discuss the non-tempered case. In the non-tempered case, several difficulties will arise. One is that the factor $\mathcal{P}_{\pi_1,\pi_0}(s)$ may not be holomorphic at $s = 1/2$. Another difficulty is that the integral $I(\varphi_{1,v}, \varphi_{0,v})$ may not be convergent. Nevertheless, several examples suggest that an analogue of (★) holds in non-tempered case. We give a somewhat optimistic conjecture in §3 for non-tempered case.

In Part II (§§4-5), we develop some local theory to show that our conjecture (★) makes sense. In §4, we prove that the local integral $I(\varphi_{1,v}, \varphi_{0,v})$ is convergent if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. In §5, we show that

$$I(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_1,v} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)$$

for unramified case (Theorem 1.2). In particular, the right hand side of (★) is independent of the choice of the set $S$ of bad primes. In course of the proof, we make use of the results of Ginzburg, Piatetski-Shapiro, Rallis [9] and those of Kato, Murase, Sugano [29]. We emphasise the fact that the factor $\mathcal{P}_{\pi_1,\pi_0}(s)$ already appeared in [9].

In Part III (§§6-12), we give several examples over number fields. One can also give several examples over function fields, but we do not discuss such cases in this paper. In §6, we show that our conjecture is compatible with the theorem of Waldspurger [46]. In §7, we prove our conjecture for $n = 3$ by using the first named author's result [25]. Then we show that our conjecture is compatible with the result of Watson [47] in some cases. We also discuss the relation with the conjecture of Deligne [7] and the conjecture of Shimura [39], [40]. In §8, we consider the restriction of the Yoshida lift to the diagonal subgroup. We recall the result of Gan and the first named author [8], which is compatible with our conjecture. In §9, we consider the restriction of the Saito-Kurokawa lift to the diagonal subset. We show that the first named author's result [24] is compatible with our conjecture. Note that this example is non-tempered. In §10, we consider our result on the restriction of the hermitian Maass lift to the space of Saito-Kurokawa lifts [26]. This example is also non-tempered, and is compatible with our conjecture. In §11, we consider the trivial representation. This example reduces to the mass formula for the quadratic forms. In §12, we collect the calculation over the real place, which is necessary to get the result of §7, §9, and §10.

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# Part I. Global theory

## 1. Formulation of the conjecture

In this paper, we would like to formulate a conjecture on a relation between a certain period of automorphic forms on special orthogonal groups and some $L$-value. Our conjecture can be considered as a refinement of the global Gross-Prasad conjecture [12].

Let $k$ be a global field with $\mathrm{char}(k) \neq 2$. Let $(V_1, Q_1)$ and $(V_0, Q_0)$ be quadratic forms over $k$ with rank $n+1$ and $n$, respectively. We assume $n \geq 2$. When $n = 2$, we also assume $(V_0, Q_0)$ is not isomorphic to the hyperbolic plane over $k$. We denote the special orthogonal group of $(V_i, Q_i)$ by $G_i$ $(i = 0, 1)$. From now on, the subscript $i$ will indicate either 0 or 1, except for some obvious situation. We assume there is an embedding $\iota : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal groups $\iota : G_0 \hookrightarrow G_1$. We regard $G_0$ as a subgroup of $G_1$ by this embedding. The group $G_i(k_v)$ of $k_v$-valued points of $G_i$ is denoted by $G_{i,v}$.

For even-dimensional quadratic form $(V, Q)$, the discriminant field $K_Q$ is defined by $K_Q = k(\sqrt{(-1)^{\dim V/2} \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if $\dim V_0$ is even (resp. if $\dim V_1$ is even). We call $K$ the discriminant field for the pair $(V_1, V_0)$. Let $\chi = \chi_{K/k}$ be the Hecke character associated to $K/k$ by the class field theory.

Put

$$\Delta_{G_{i,v}} = \begin{cases} \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l-2)\cdot L_v(l,\chi) & \text{if } \dim V_i = 2l, \end{cases}$$

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l,\chi) & \text{if } \dim V_i = 2l. \end{cases}$$

Note that $\Delta_{G_i} = L(M_i^\vee(1))$, where $M_i^\vee$ is the dual motive of the motive $M_i$ associated to $G_i$ by Gross [11].

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(\mathbb{A})$. There is a canonical inner product $\langle\ ,\ \rangle$ on forms on $G_i(k)\backslash G_i(\mathbb{A})$ defined by

$$\langle \varphi_i, \varphi_i' \rangle = \int_{G_i(k)\backslash G_i(\mathbb{A})} \varphi_i(g_i)\overline{\varphi_i'(g_i)}\, dg_i,$$

where $dg_i$ is the Tamagawa measure on $G_i(\mathbb{A})$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each $v$. There exists a positive number $C_i$ such that $dg_i = C_i \prod_v dg_{i,v}$, when the right hand side is well-defined. In this paper, we call $C_i$ the Haar measure constant. Since $\pi_{i,v}$ is an unitary representation, there is an inner product $\langle\ ,\ \rangle_v$ on $\pi_{i,v}$ for any place $v$ of $k$. We put $\|\varphi_{i,v}\| = \langle \varphi_{i,v}, \varphi_{i,v}\rangle_v^{1/2}$, as usual. There exists a positive constant $C_{\pi_i}$ such that $\langle \varphi_i, \varphi_i' \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi_{i,v}' \rangle_v$ for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi_i' = \otimes_v \varphi_{i,v}' \in \otimes_v \pi_{i,v}$.

We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a $\mathcal{K}_i$-finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. We are interested in the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of $\varphi_1$ to $G_0(\mathbb{A})$.

Let $S$ be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

(U1) $G_i$ is unramified over $k_v$.
(U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
(U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
(U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
(U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
(U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When $G_i$ is unramified over $k_v$, we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The $L$-group ${}^L G_i$ of $G_i$ is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, $W_k$ is the Weil group of $k$ and

$$\hat{G}_i = \begin{cases} \mathrm{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l+1, \\ \mathrm{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by st the standard representation of ${}^L G_i$. The completed standard $L$-function for $\pi_i$ is denoted by $L(s, \pi_i, \mathrm{st})$ for an irreducible automorphic representation $\pi_i$ of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, \mathrm{st})$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - \mathrm{st}(A_{\pi_{i,v}}) \cdot q_v^{-s})^{-1}$, where $A_{\pi_{i,v}}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product $L$-function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - \mathrm{st}(A_{\pi_{1,v}}) \otimes \mathrm{st}(A_{\pi_{0,v}}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\mathrm{Ad} : {}^L G_i \to \mathrm{GL}(\mathrm{Lie}(\hat{G}_i))$. The associated $L$-function $L(s, \pi_i, \mathrm{Ad})$ is called the adjoint $L$-function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \mathrm{Ad})$ can be analytically continued to the whole $s$-plane.

We put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \mathrm{Ad}) L(s + (1/2), \pi_0, \mathrm{Ad})}.$$

Let $\pi_{i,v}$ be an irreducible admissible representation of $G_{i,v}$. We denote the complex conjugate of $\pi_{i,v}$ by $\bar{\pi}_{i,v}$. It is believed that

$$\text{(MF)} \qquad \dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1$$

for any place $v$ of $k$. We do not assume (MF) in this paper. Note that an analogue of (MF) for orthogonal groups has been proved by Aizenbud, Gourevitch, Rallis, Schiffmann [1] for non-archimedean place and by Sun and Zhu [44] for irreducible Harish-Chandra smooth representations for archimedean place.

We consider the matrix coefficient

$$\Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_i) = \langle \pi_{i,v}(g_i)\varphi_{i,v}, \varphi'_{i,v}\rangle_v, \quad g_i \in G_{i,v}$$

for $\mathcal{K}_{1,v}$-finite vectors $\varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v}$ and $\mathcal{K}_{0,v}$-finite vectors $\varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v}$. Put

$$I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \int_{G_{0,v}} \Phi_{\varphi_{1,v},\varphi'_{1,v}}(g_{0,v})\overline{\Phi_{\varphi_{0,v},\varphi'_{0,v}}(g_{0,v})}\, dg_{0,v},$$

$$\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \Delta_{G_1,v}^{-1}\mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}).$$

When $\varphi_{1,v} = \varphi'_{1,v}$ and $\varphi_{0,v} = \varphi'_{0,v}$, we simply denote these objects by $I(\varphi_{1,v}, \varphi_{0,v})$ and $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$, respectively.

**Proposition 1.1.** *If both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.*

**Theorem 1.2.** *Let $v$ be a non-archimedean place. Assume that the conditions* (U1)*,* (U2)*,* (U3)*,* (U4)*,* (U5)*, and* (U6) *hold. If the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent, then we have $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$.*

The proofs of Proposition 1.1 and Theorem 1.2 will be given in Part II.

**Conjecture 1.3.** Assume that both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. Then $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We shall say that $\pi_i$ is almost locally generic if $\pi_i$ satisfies the following condition (ALG).

(ALG) For almost all $v$, the constituent $\pi_{i,v}$ is generic.

It is believed that $\pi_i$ is almost locally generic if and only if $\pi_{i,v}$ is generic for some $v$. It is also believed that $\pi_i$ is almost locally generic if and only if $\pi_i$ is tempered (the generalized Ramanujan conjecture).

**Conjecture 1.4.** Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We assume both $\pi_1$ and $\pi_0$ are almost locally generic. Then

(1) The integral $I(\varphi_{1,v}, \varphi_{0,v})$ should be absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

(2) $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now we state our global conjecture.

**Conjecture 1.5.** Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume $\pi_1$ and $\pi_0$ are almost locally generic. Then there should be an integer $\beta$ such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1} \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

We will discuss the nature of the integer $\beta$ in the next section.

*Remark* 1.6. When $\pi_1$ and $\pi_0$ are tempered, it is believed that the local $L$-factors $L(s, \pi_{1,v}, \mathrm{Ad})$, $L(s, \pi_{0,v}, \mathrm{Ad})$, and $L(s, \pi_{1,v} \boxtimes \pi_{0,v})$ are holomorphic for $\mathrm{Re}(s) > 0$. Therefore in this case our conjecture is equivalent to

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1}^S \mathcal{P}_{\pi_1,\pi_0}^S(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2},$$

where $\Delta_{G_1}^S$ and $\mathcal{P}_{\pi_1,\pi_0}^S(s)$ are the partial Euler products. In particular, the definition of the $L$-factors for bad primes plays no role in this case. Note also that it is believed that $L(1, \pi_i, \mathrm{Ad}) \neq 0$ if $\pi_i$ is tempered.

*Remark* 1.7. One can formulate Conjecture 1.5 in a different way as follows. Assume the local measure $dg_{i,v}$ and the local inner product $\langle\ ,\ \rangle_v$ are normalised so that $C_i = C_{\pi_i} = 1$. Put

$$H_{\pi_1,\pi_0} = \mathrm{Hom}_{G_0(\mathbb{A}) \times G_0(\mathbb{A})}((\pi_1 \boxtimes \tilde{\pi}_1) \otimes (\bar{\pi}_0 \boxtimes \tilde{\bar{\pi}}_0), \mathbb{C}).$$

We define two elements $L_{\pi_1,\pi_0}^{\mathrm{global}}, L_{\pi_1,\pi_0}^{\mathrm{local}} \in H_{\pi_1,\pi_0}$ by

$$L_{\pi_1,\pi_0}^{\mathrm{global}}(\varphi_1, \varphi_1'; \varphi_0, \varphi_0') = \langle \varphi_1|_{G_0}, \varphi_0 \rangle \overline{\langle \varphi_1'|_{G_0}, \varphi_0' \rangle},$$
$$L_{\pi_1,\pi_0}^{\mathrm{local}}(\varphi_1, \varphi_1'; \varphi_0, \varphi_0') = \prod_v \alpha_v(\varphi_{1,v}, \varphi_{1,v}'; \varphi_{0,v}, \varphi_{0,v}').$$

Then Conjecture 1.5 can be reformulated as

$$L_{\pi_1,\pi_0}^{\mathrm{global}} = 2^\beta \Delta_{G_1} \mathcal{P}_{\pi_1,\pi_0}(1/2) L_{\pi_1,\pi_0}^{\mathrm{local}}.$$

## 2. Relation to the Arthur conjecture

This section is devoted to a somewhat speculative argument based on the Arthur conjecture [2]. We recall the Arthur conjecture for automorphic representation of reductive algebraic groups. We assume, for simplicity, $G$ is a reductive algebraic group defined over $k$ with anisotropic center. The local Langlands group $\mathcal{L}_v$ is defined by

$$\mathcal{L}_v = \begin{cases} W_{k_v} \times \mathrm{SU}(2) & \text{if } v \text{ is non-archimedean,} \\ W_{k_v} & \text{if } v \text{ is archimedean,} \end{cases}$$

where $W_{k_v}$ is the Weil group of $k_v$. A Langlands parameter is a homomorphism $\phi_v : \mathcal{L}_v \to {}^L G$ which satisfies certain additional conditions. Two Langlands parameters are equivalent if they are conjugate by an element of $\hat{G}$. Langlands conjectured that for each equivalence class of Langlands parameter, one can associate a finite set $\Pi_{\phi_v}(G)$ of irreducible admissible representations of $G_v$. The finite set $\Pi_{\phi_v}(G)$ is called the $L$-packet for $\phi_v$. The set $\Pi(G_v)$ of all equivalence classes of irreducible admissible representations of $G_v$ should be decomposed into a disjoint union

$$\Pi(G_v) = \coprod_{\phi_v} \Pi_{\phi_v}(G),$$

where $\phi_v$ extends over the equivalence classes of Langlands parameters. The $L$-packet $\Pi_{\phi_v}(G)$ should contain a tempered representation if and only if the Langlands parameter $\phi_v$ has a bounded image, in which case $\phi_v$ is called tempered. If $\phi_v$ is tempered, then all members of $\Pi_{\phi_v}(G)$ should be tempered.

A homomorphism $\psi_v : \mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ whose restriction to $\mathrm{SL}_2(\mathbb{C})$ is holomorphic is called a (local) Arthur parameter if $\psi_v|_{\mathcal{L}_v}$ is a tempered Langlands parameter. One can consider the equivalence of Arthur parameters as in the case of Langlands parameters. Arthur conjectured that for each equivalence class of Arthur parameters $\psi_v$, one can associate a finite set of unitary representations $\Pi_{\psi_v}(G)$. The set $\Pi_{\psi_v}(G)$ is called the $A$-packet of $\psi_v$. $A$-packets are not necessarily disjoint.

For each representation $\rho_v$ of $\mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C})$, we associate an $L$-factor as follows. We may assume $\rho_v$ is irreducible. Then there exists an irreducible representation $\phi_v$ of $\mathcal{L}_v$ and an integer $t \geq 0$ such that

$$\rho_v \simeq \phi_v \boxtimes \mathrm{Sym}^t,$$

where $\mathrm{Sym}^t$ is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of degree $t+1$. We put

$$L(s, \rho_v) = \prod_{j=0}^{t} L(s - j + (t/2), \phi_v).$$

For each element $\pi_v \in \Pi_{\psi_v}(G)$ and a finite-dimensional representation $r$ of ${}^L G$, we put $L(s, \pi_v, r) = L(s, r \circ \psi_v)$. Note that $L(s, \pi_v, r)$ depends not only on $\pi_v$, but also on $\psi_v$, since $A$-packets are not necessarily disjoint, although the symbol suggests it does not.

Langlands conjectured that there exists a locally compact group $\mathcal{L}_k$ such that the equivalence classes of irreducible $n$-dimensional representation of $\mathcal{L}_k$ is in one-to-one correspondence with the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. There should be a homomorphism $\iota_v : \mathcal{L}_v \to \mathcal{L}_k$ for each $v$. A (global) Arthur parameter is a certain equivalence class of homomorphisms

$$\psi : \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

such that the image of $\mathcal{L}_k$ is bounded. Let $\Pi_\psi(G)$ be the set of square-integrable automorphic representations $\pi \simeq \otimes_v \pi_v$ of $G(\mathbb{A})$ such that $\pi_v \in \Pi_{\psi \circ \iota_v}(G)$ for each $v$. The set $\Pi_\psi(G)$ is called the $A$-packet of $\psi$. Arthur conjectured that the set of square-integrable automorphic representations of $G(\mathbb{A})$ is a union

$$\bigcup_{\psi} \Pi_\psi(G).$$

If $\pi \in \Pi_\psi(G)$, then $\psi$ is called the Arthur parameter of $\pi$. In general, $\psi$ is not uniquely determined by the equivalence class of $\pi$, but for special orthogonal groups or unitary groups, $\psi$ should be determined by $\pi$.

It is believed that the Arthur parameter $\psi : \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ associated with a square-integrable automorphic representation should be elliptic in the sense that $\mathrm{Im}(\psi)$ is not contained in any proper Levi subgroup of ${}^L G$. This is the case if and only if $\mathrm{Cent}_{\hat{G}}(\mathrm{Im}(\psi))$ is finite. If $\psi$ is an elliptic Arthur parameter such that $\Pi_\psi(G)$ is non-empty, the $A$-packet $\Pi_\psi(G)$ consists of only irreducible tempered cuspidal automorphic representations if and only if the restriction $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial. In this case, the Arthur parameter $\psi$ said to be tempered. For an elliptic Arthur parameter $\psi$, we put

$$\mathcal{S}_\psi = \mathrm{Cent}_{\hat{G}}(\mathrm{Im}(\psi)).$$

Now we go back to the situation that $G_1 = \mathrm{SO}(n+1)$ and $G_0 = \mathrm{SO}(n)$. Let $\psi_i$ be an elliptic Arthur parameter for the group $G_i$. In this case, the group $\mathcal{S}_{\psi_i}$ can be calculated as follows. Let st be the standard representation of ${}^L G_i$. Then $\mathrm{st} \circ \psi_i$ can be decomposed into a direct sum of irreducible representations of $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C})$:

$$\mathrm{st} \circ \psi_i = \bigoplus_{j=1}^{r} \psi_i^{(j)}.$$

Here, the representations $\psi_i^{(1)}, \dots, \psi_i^{(r)}$ are mutually distinct orthogonal (resp. symplectic) representations of $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C})$ if $\dim V_i$ is even (resp. odd). Then

$$\mathcal{S}_{\psi_i} \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } \dim V_i \text{ is even and } \mathrm{rank}\, \psi_i^{(j)} \text{ is odd for some } j, \\ (\mathbb{Z}/2\mathbb{Z})^{r} & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{S}_{\psi_i}$ is an elementary 2-abelian group.

Now we admit the Arthur conjecture. Let $\pi_i$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$, which satisfies the condition (ALG). Then corresponding Arthur parameter $\psi_i$ must be tempered, since otherwise $\pi_{i,v}$ cannot be generic for any $v$.

**Conjecture 2.1.** Assume that $\pi_i$ is an irreducible tempered cuspidal automorphic representation of $G_i(\mathbb{A})$ with Arthur parameter $\psi_i$. Then the constant $2^\beta$ in Conjecture 1.5 should be equal to $1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$. Equivalently, the equation

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{C_0 \Delta_{G_1}}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|} \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

holds.

## 3. The non-tempered case

Let $\pi_{i,v}$ be an irreducible representation of $G_{i,v}$, which we do not assume to be unitary for a moment. Note that if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ gives an element of

$$\mathrm{Hom}_{G_{0,v} \times G_{0,v}}((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C}),$$

where $\tilde{\pi}_{i,v}$ is the contragredient of $\pi_{i,v}$.

**Conjecture 3.1.** The quantity $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ should be somehow "analytically continued" for any $\pi_{1,v}$ and $\pi_{0,v}$. If $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$, then the continuation $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ is unique and gives an element of

$$\mathrm{Hom}_{G_{0,v} \times G_{0,v}}((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C}).$$

Now we consider the global situation. Let $\pi_i$ be an square-integrable automorphic representation of $G_i(\mathbb{A})$, which may not be almost locally generic. We assume that $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any $v$. For $v \notin S$, we may assume $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$ by Theorem 1.2, as long as it is meaningful.

**Conjecture 3.2.** Let $\pi_i$ be as above. Then

(1) The integral $\langle \varphi_1|_{G_0}, \varphi_0\rangle$ should be convergent for any $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$.

(2) There should be an integer $\beta$ such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = 2^\beta \Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v\in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero decomposable vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

*Remark* 3.3. Contrary to the almost locally generic case, the factor $2^\beta$ is not necessarily equal to $1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$, and depends not only on global data, but also on local data. See the examples in §9, §10, and §11.

## Part II. Local theory

Until §5, we consider only local objects and drop subscript $v$.

## 4. Convergence of the integral: Proof of Proposition 1.1

In this section, we assume that $k$ is a local field with $\mathrm{char}(k) \neq 2$. Let $(V, Q)$ be a non-degenerate quadratic space over $k$. We denote the anisotropic kernel of $(V, Q)$ by $(V^{\mathrm{an}}, Q^{\mathrm{an}})$. Then there is a decomposition $V = X \oplus V^{\mathrm{an}} \oplus Y$, where $X$ and $Y$ are totally isotropic subspaces. The Witt rank $r$ of $(V, Q)$ is, by definition, equal to the dimension of $X$ or $Y$. We put $d = \dim V^{\mathrm{an}}$. Choosing a basis of $X$, we get a minimal parabolic subgroup $P_{\min} = M_{\min} N_{\min}$ of $G$. The Levi factor $M_{\min}$ is isomorphic to $(k^\times)^r \times \mathrm{SO}_{Q^{\mathrm{an}}}$. The split component $A_{\min}$ of $M_{\min}$ is isomorphic to $(k^\times)^r$, and the Weyl group $W(G, A_{\min})$ is of type B or D according as $d \neq 0$ or $d = 0$. We will denote an element of $A_{\min} \simeq (k^\times)^r$ by $x = (x_1, \ldots, x_r)$. The simple roots of $(P_{\min}, A_{\min})$ are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \ldots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1},$$

$$\alpha_r(x) = \begin{cases} x_r & \text{if } d \neq 0 \\ x_{r-1} x_r & \text{if } d = 0. \end{cases}$$

These roots are also regarded as a character of $M_{\min}$. Let $\delta_{P_{\min}}(x)$ be the modulus character of $P_{\min}$. Then

$$\delta_{P_{\min}}(x) = \prod_{i=1}^{r} |x_i|^{d+2r-2i}.$$

Fix a special maximal compact subgroup $\mathcal{K}$ of $G$. Then we have a Cartan decomposition $G = \mathcal{K} M_{\min}^{+} \mathcal{K}$, where

$$M_{\min}^{+} = \{m \in M_{\min} \mid |\alpha_i(m)| \le 1 \, (i = 1, \dots, r)\}.$$

Fix a suitable embedding $\eta : G \to \mathrm{GL}_m$. Then the height function $\sigma(g)$ (with respect to the embedding $\eta$) is given by

$$\sigma(g) = \max_{\substack{1 \le i \le m \\ 1 \le j \le m}} (\log |\eta(g)_{ij}|, \log |\eta(g^{-1})_{ij}|).$$

When $k$ is non-archimedean, the following integral formula holds

$$\int_G f(g)\, dg = \int_{M_{\min}^{+}} \mu(m) \int_{\mathcal{K} \times \mathcal{K}} f(k_1 m k_2)\, dk_1\, dk_2\, dm, \qquad f \in L^1(G)$$

where $\mu(m) = \mathrm{Vol}(\mathcal{K} m \mathcal{K}) / \mathrm{Vol}(\mathcal{K})$. Moreover, there exists a positive constant $A$ such that $A^{-1} \delta_{P_{\min}}^{-1}(m) \le \mu(m) \le A \delta_{P_{\min}}^{-1}(m)$ for any $m \in M_{\min}^{+}$. (See Silberger [42] p. 149.)

When $k$ is archimedean, similar integral formula holds. (See e.g., Helgason, [21], Theorem 5.8.) In particular, there exists a non-negative function $\mu(m)$ on $M_{\min}^{+}$ such that

$$\int_G f(g)\, dg = \int_{M_{\min}^{+}} \mu(m) \int_{\mathcal{K} \times \mathcal{K}} f(k_1 m k_2)\, dk_1\, dk_2\, dm, \qquad f \in L^1(G).$$

Moreover, there exists a constant $A > 0$ such that $\mu(m) \le A \delta_{P_{\min}}^{-1}(m)$ for $m \in M_{\min}^{+}$.

Harish-Chandra's spherical function $\Xi(g)$ of $G$ is given by

$$\Xi(g) = \int_{\mathcal{K}} h_0(kg)\, dk$$

where $h_0 \in \mathrm{Ind}_{P_{\min}}^{G} 1$ is a function whose restriction to $\mathcal{K}$ is identically equal to 1. Note that $\Xi$ is a matrix coefficient of a tempered representation $\mathrm{Ind}_{P_{\min}}^{G} 1$. It is known that there exists positive constants $A$, $B$ such that

$$A^{-1} \delta_{P_{\min}}^{1/2}(m) \le \Xi(m) \le A \delta_{P_{\min}}^{1/2}(m)(1 + \sigma(m))^B$$

for any $m \in M_{\min}^{+}$. (See Silberger [42], p. 154, Theorem 4.2.1 and Harish-Chandra [14], p. 129, Lemma 1 in Section 10.)

Recall that a function $f(g)$ on $G$ satisfies the weak inequality if

$$|f(g)| \leq A\Xi(g)(1+\sigma(g))^B$$

for some positive constant $A$, $B$. A matrix coefficient of a tempered representation satisfies the weak inequality.

Applying these results for $G_1 = \mathrm{SO}(n+1)$ and $G_0 = \mathrm{SO}(n)$, we can now prove Proposition 1.1. As before, we define $P_{i,\min}$, $A_{i,\min}$, $r_i$, etc., for the group $G_i$.

*Proof of Proposition 1.1.* Let $\pi_1$ and $\pi_0$ be irreducible tempered representations of $G_1$ and $G_0$, respectively. We may assume $A_{0,\min} \subset A_{1,\min}$. Then we have estimates

$$|\Phi_{\varphi_1,\varphi_1'}(m)| \leq A\delta_{P_{1,\min}}^{1/2}(m)(1+\sigma(m))^B, \qquad (m \in M_{1,\min}^+),$$
$$|\Phi_{\varphi_0,\varphi_0'}(m)| \leq A\delta_{P_{0,\min}}^{1/2}(m)(1+\sigma(m))^B, \qquad (m \in M_{0,\min}^+)$$

for some positive constants $A$, $B$. When $W(G_0, A_{0,\min})$ is of type B, it is enough to show the following integral

$$\int_{A_{0,\min}^+} \delta_{P_{0,\min}}^{-1/2}(m)\delta_{P_{1,\min}}^{1/2}(m)(1+\sigma(m))^{2B}\, dm$$

is convergent. This is reduced to the convergence of

$$\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0}|\leq 1} |x_1x_2\cdots x_{r_0}|^{1/2}(1-\sum_{j=1}^{r_0}\log|x_j|)^{2B}\, d^\times x_1\, d^\times x_2\cdots d^\times x_{r_0}.$$

One can easily prove the convergence of this integral. Note that when $W(G_0, A_{0,\min})$ is of type D, $A_{0,\min}^+$ is not contained in $A_{1,\min}^+$. In this case, one need to consider the integral

$$\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0}|\leq 1} |x_1x_2\cdots x_{r_0}|^{1/2}(1-\sum_{j=1}^{r_0}\log|x_j|)^{2B}\, d^\times x_1\, d^\times x_2\cdots d^\times x_{r_0}$$
$$+\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0-1}|\leq|x_{r_0}|^{-1}\leq 1} |x_1x_2\cdots x_{r_0-1}x_{r_0}^{-1}|^{1/2}$$
$$\times (1-\sum_{j=1}^{r_0-1}\log|x_j|+\log|x_{r_0}|)^{2B}\, d^\times x_1\, d^\times x_2\cdots d^\times x_{r_0}.$$

One can show the convergence of this integral similarly.

To prove the latter part of the proposition, we make use of the result of He [20]. Let $\Xi_1$ and $\Xi_0$ be Harish-Chandra's spherical function for $G_1$ and $G_0$, respectively. Then the function $g_0 \mapsto \Xi_1(g_0)\Xi_0(g_0)$ belongs to $L^1(G_0)$ by the first part of the proposition. Note that Harish-Chandra's spherical function is a matrix coefficient of a tempered representation.

Then the latter part of the proposition follows from Theorem 2.1 of He's paper [20]. Note that He [20] used the estimates of almost $L^2$ matrix coefficients [6], which is valid for $p$-adic groups as well. □

## 5. Calculation of the unramified integral: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We assume the conditions (U1)–(U6) in §1 holds. In particular, both $G_1$ and $G_0$ are quasi-split. We should consider the following two cases:

$$\text{(Case A)} \qquad G_1 = \mathrm{SO}(2l+1) \quad \text{and} \quad G_0 = \mathrm{SO}(2l),$$

$$\text{(Case B)} \qquad G_1 = \mathrm{SO}(2l+2) \quad \text{and} \quad G_0 = \mathrm{SO}(2l+1).$$

Let $K$ be the discriminant field. Note that $K$ is equal to either $k$ or the unramified quadratic extension of $k$. Let $q$ be the number of elements of the residue field of $k$. The local zeta function $\zeta(s)$ is defined by $(1-q^{-s})^{-1}$.

Let $B_i = T_i N_i$ be a Borel subgroup of $G_i$, where $T_i$ and $N_i$ are a maximal torus of $G_i$ and the unipotent radical of $B_i$, respectively. Let $A_i \subset T_i$ be the maximal split subtorus. Without loss of generality, we may assume $N_0 \subset N_1$ and $A_0 \subset A_1$.

Let $\pi_1 = I(\Xi) = \mathrm{Ind}_{B_1}^{G_1}(\Xi)$ and $\pi_0 = I(\xi) = \mathrm{Ind}_{B_0}^{G_0}(\xi)$ be unramified principal series of $G_1$ and $G_0$, respectively. Here, $\Xi$ and $\xi$ are unramified quasi-characters of $T_1$ and $T_0$, respectively. Let $\Phi_\Xi$ and $\Phi_\xi$ be the class-one matrix coefficients of $I(\Xi)$ and $I(\xi)$ such that $\Phi_\Xi(1) = \Phi_\xi(1) = 1$, respectively. We consider the integral

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \int_{G_0} \Phi_\Xi(g_1^{-1} g_0) \Phi_\xi(g_0)\, dg_0.$$

We assume that both $\Xi$ and $\xi$ are sufficiently close to the unitary axis. As shown in §4, this condition implies that the integral $I(g_1; \Phi_\Xi, \Phi_\xi)$ is absolutely convergent. In this section, we calculate the value of $I(g_1; \Phi_\Xi, \Phi_\xi)$ at $g_1 = 1$.

Let $f_\Xi \in I(\Xi)$ and $f_\xi \in I(\xi)$ be the class-one vectors such that $f_\Xi(1) = f_\xi(1) = 1$. Then we have

$$\Phi_\Xi(g_1) = \int_{\mathcal{K}_1} f_\Xi(k_1 g_1)\, dk_1, \quad g_1 \in G_1,$$

$$\Phi_\xi(g_0) = \int_{\mathcal{K}_0} f_\xi(k_0 g_0)\, dk_0, \quad g_0 \in G_0.$$

We recall the theory of Shintani functions [29]. We denote the Hecke algebra $\mathcal{H}(\mathcal{K}_i \backslash G_i / \mathcal{K}_i)$ by $\mathcal{H}_i$. By the Satake isomorphism, there are algebra homomorphisms

$$\omega_1 : \mathcal{H}_1 \longrightarrow \mathbb{C} \quad \text{and} \quad \omega_0 : \mathcal{H}_0 \longrightarrow \mathbb{C}$$

corresponding to the unramified principal series $\pi_1$ and $\pi_0$, respectively. Recall that a smooth function $S$ on $G_1$ is called a Shintani function for $\pi_1$ and $\pi_0$, if the following conditions are satisfied:

- $\mathcal{L}(k_0)\mathcal{R}(k_1)S = S$ for any $k_1 \in \mathcal{K}_1$ and $k_0 \in \mathcal{K}_0$.
- $\mathcal{L}(\varphi_0)\mathcal{R}(\varphi_1)S = \omega_0(\varphi_0)\omega_1(\varphi_1)S$ for any $\varphi_0 \in \mathcal{H}_0$ and $\varphi_1 \in \mathcal{H}_1$.

Here, $\mathcal{L}$ and $\mathcal{R}$ are the left regular representation and the right regular representation, respectively. Note that $I(g_1; \Phi_\Xi, \Phi_\xi)$ is a Shintani function for $\tilde{\pi}_1$ and $\tilde{\pi}_0$. Kato, Murase, and Sugano [29] have proved that if both $G_1$ and $G_0$ are split, then a Shintani function exists and is unique up to scalar. In this paper, we do not use the uniqueness of Shintani functions.

Recall that the double coset $B_1\backslash G_1/B_0$ has a unique open orbit and the open orbit has a representative $\eta \in \mathcal{K}_1$ (cf. [9], §7). Note that $\eta^{-1}B_1\eta \cap B_0 = \{1\}$. Let $Y_{\Xi,\xi}$ be the function on $G_1$ determined by the following conditions.

(1) $Y_{\Xi,\xi}(b_1 g_1 b_0) = (\Xi^{-1}\delta_1^{1/2})(b_1)(\xi\delta_0^{-1/2})(b_0)Y_{\Xi,\xi}(g_1)$ for any $b_1 \in B_1$ and $b_0 \in B_0$.
(2) $Y_{\Xi,\xi}(\eta) = 1$.
(3) $Y_{\Xi,\xi}(g_1) = 0$ if $g_1 \notin B_1\eta B_0$.

Here, $\delta_i$ is the modulus character of $B_i$. Note that a function satisfying (1) and (3) is unique up to scalar. We define $l_{\Xi,\xi} \in \mathrm{Hom}_{G_0}(\pi_1, \tilde{\pi}_0) = \mathrm{Hom}_{G_0}(I(\Xi), I(\xi^{-1}))$ by

$$l_{\Xi,\xi}(\mathrm{pr}_1(f))(g_0) = \int_{G_1} f(g_1 g_0) Y_{\Xi,\xi}(g_1)\, dg_1, \quad g_0 \in G_0.$$

Here, $\mathrm{pr}_1 : C_c^\infty(G_1) \to \pi_1 = I(\Xi)$ is given by

$$\mathrm{pr}_1(f)(g_1) = \int_{B_1} (\Xi^{-1}\delta_1^{1/2})(b_1) f(b_1 g_1)\, db_1.$$

Let $\langle\ ,\ \rangle$ be the natural pairing on $\pi_0 \times \tilde{\pi}_0$ defined by

$$\langle \varphi_0, \varphi_0' \rangle = \int_{\mathcal{K}_0} \varphi_0(k_0)\varphi_0'(k_0)\, dk_0$$

for $\varphi_0 \in \pi_0$ and $\varphi_0' \in \tilde{\pi}_0$. Put

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1) f_\Xi) \rangle.$$

Then $S_{\Xi,\xi}$ is a Shintani function, and we have

$$\begin{aligned} S_{\Xi,\xi}(g_1) &= \int_{\mathcal{K}_0} f_\xi(k_0) \int_{G_1} \mathbf{1}_{\mathcal{K}_1}(g_1' k_0 g_1) Y_{\Xi,\xi}(g_1')\, dg_1'\, dk_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi,\xi}(k_1 g_1^{-1} k_0)\, dk_1\, dk_0. \end{aligned}$$

Here, $\mathbf{1}_{\mathcal{K}_1}$ is the characteristic function of $\mathcal{K}_1$. Put

$$T_{\Xi,\xi}(g_1) = \begin{cases} \displaystyle\int_{G_0} f_\Xi(g_1 g_0) f_\xi(g_0)\, dg_0 & \text{if } g_1 \in B_1 \eta B_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $T_{\Xi,\xi}(g_1) = T_{\Xi,\xi}(\eta) \cdot Y_{\Xi^{-1},\xi^{-1}}(g_1)$, since $T_{\Xi,\xi}$ satisfies the conditions (1) and (3) for $\Xi^{-1}$ and $\xi^{-1}$. Therefore we have

$$\begin{aligned} I(g_1; \Phi_\Xi, \Phi_\xi) &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1 g_1^{-1} g_0) f_\xi(k_0 g_0)\, dk_0\, dk_1\, dg_0 \\ &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1 g_1^{-1} k_0 g_0) f_\xi(g_0)\, dk_0\, dk_1\, dg_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} T_{\Xi,\xi}(k_1 g_1^{-1} k_0)\, dk_1\, dk_0 \\ &= T_{\Xi,\xi}(\eta) \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi^{-1},\xi^{-1}}(k_1 g_1^{-1} k_0)\, dk_1\, dk_0 \\ &= T_{\Xi,\xi}(\eta) S_{\Xi^{-1},\xi^{-1}}(g_1). \end{aligned}$$

In particular, $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(g_1)$ are convergent if $\Xi$ and $\xi$ are sufficiently close to the unitary axis. Indeed, since the first part of Proposition 1.1 holds for $I(|\Xi|)$ and $I(|\xi|)$ if $\Xi$ and $\xi$ are sufficiently close to the unitary axis, $I(g_1; \Phi_{|\Xi|}, \Phi_{|\xi|})$ is convergent, and hence the above integral is absolutely convergent. It follows that, for each $g_1 \in G_1$, $T_{\Xi,\xi}(k_1 g_1^{-1} k_0)$ is convergent for almost all $k_1 \in \mathcal{K}_1$ and $k_0 \in \mathcal{K}_0$ such that $k_1 g_1^{-1} k_0 \in B_1 \eta B_0$. By definition, $T_{\Xi,\xi}(g_1)$ is convergent for some $g_1 \in B_1 \eta B_0$ if and only if $T_{\Xi,\xi}(g_1)$ is convergent for all $g_1 \in B_1 \eta B_0$. Therefore $T_{\Xi,\xi}(\eta)$ is convergent, and the convergence of the above integral also implies that $S_{\Xi^{-1},\xi^{-1}}(g_1)$ is convergent.

We first assume that the residual characteristic of $k$ is not 2. We consider the case when $K = k$. In this case, both $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(1)$ are already calculated. Note that

$$\begin{aligned} T_1 = A_1 &\simeq \begin{cases} (k^\times)^l & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (k^\times)^{l+1} & \text{if } G_1 = \mathrm{SO}(2l+2), \end{cases} \\ T_0 = A_0 &\simeq (k^\times)^l \qquad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1). \end{aligned}$$

We write

$$\Xi = \begin{cases} (\Xi_1, \dots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_1, \dots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2), \end{cases}$$
$$\xi = (\xi_1, \dots, \xi_l) \qquad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1).$$

There exists a quadratic space $(\tilde{V}_1, \tilde{Q}_1) \subset (V_0, Q_0)$ such that $V_1$ is isomorphic to the direct sum of $\tilde{V}_1$ and the hyperbolic plane. Without loss of generality, we may assume that $(V_0, \tilde{V}_1)$ satisfies the conditions (U1)–(U6). Put

$$\tilde{\Xi} = \begin{cases} (\Xi_2, \dots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_2, \dots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2). \end{cases}$$

Since $T_{\Xi,\xi}(\eta)$ is independent of the choice of $\eta$, we set $\zeta(\Xi, \xi) = T_{\Xi,\xi}(\eta)$. By Ginzburg, Piatetski-Shapiro, and Rallis, [9], p. 22, Corollary to Lemma 1.1 and p. 179, Corollary 1 to Lemma 7.2, we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A)} \\ 1 & \text{(Case B).} \end{cases}$$

Here, $L(s, I(\xi), \Xi_1)$ is the standard $L$-factor of $I(\xi)$ twisted by the character $\Xi_1$. By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) = & \prod_{i=1}^{l} L(1, \Xi_i^2)^{-1} \prod_{1 \le i < j \le l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i < j \le l} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i \le j \le l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \\ & \times \prod_{1 \le j < i \le l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case A, and

$$\begin{aligned}\zeta(\Xi,\xi) = & \prod_{1\le i<j\le l+1} L(1,\Xi_i\Xi_j)^{-1} L(1,\Xi_i\Xi_j^{-1})^{-1} \\ & \times \prod_{i=1}^{l} L(1,\xi_i^2)^{-1} \prod_{1\le i<j\le l} L(1,\xi_i\xi_j)^{-1} L(1,\xi_i\xi_j^{-1})^{-1} \\ & \times \prod_{1\le i\le j\le l} L(1/2,\Xi_i\xi_j) L(1/2,\Xi_i\xi_j^{-1}) \\ & \times \prod_{1\le j<i\le l+1} L(1/2,\Xi_i\xi_j) L(1/2,\Xi_i^{-1}\xi_j)\end{aligned}$$

in Case B. On the other hand, Theorem 10.8 of [29] implies

$$\begin{aligned}& S_{\Xi^{-1},\xi^{-1}}(1) \\ & = \Delta_{G_1}\zeta(1)^{-2l} \prod_{i=1}^{l} L(1,\Xi_i^{-2})^{-1} \prod_{1\le i<j\le l} L(1,\Xi_i^{-1}\Xi_j^{-1})^{-1} L(1,\Xi_i^{-1}\Xi_j)^{-1} \\ & \quad \times \prod_{1\le i<j\le l} L(1,\xi_i^{-1}\xi_j^{-1})^{-1} L(1,\xi_i^{-1}\xi_j)^{-1} \\ & \quad \times \prod_{1\le i\le j\le l} L(1/2,\Xi_i^{-1}\xi_j^{-1}) L(1/2,\Xi_i^{-1}\xi_j) \\ & \quad \times \prod_{1\le j<i\le l} L(1/2,\Xi_i^{-1}\xi_j^{-1}) L(1/2,\Xi_i\xi_j^{-1})\end{aligned}$$

in Case A, and

$$\begin{aligned}& S_{\Xi^{-1},\xi^{-1}}(1) \\ & = \Delta_{G_1}\zeta(1)^{-2l-1} \prod_{1\le i<j\le l+1} L(1,\Xi_i^{-1}\Xi_j^{-1})^{-1} L(1,\Xi_i^{-1}\Xi_j)^{-1} \\ & \quad \times \prod_{i=1}^{l} L(1,\xi_i^{-2})^{-1} \prod_{1\le i<j\le l} L(1,\xi_i^{-1}\xi_j^{-1})^{-1} L(1,\xi_i^{-1}\xi_j)^{-1} \\ & \quad \times \prod_{1\le i\le j\le l} L(1/2,\Xi_i^{-1}\xi_j^{-1}) L(1/2,\Xi_i^{-1}\xi_j) \\ & \quad \times \prod_{1\le j<i\le l+1} L(1/2,\Xi_i^{-1}\xi_j^{-1}) L(1/2,\Xi_i\xi_j^{-1})\end{aligned}$$

in Case B. Combining these results, we have

$$I(1;\Phi_\Xi,\Phi_\xi) = \Delta_{G_1}\mathcal{P}_{\pi_1,\pi_0}(1/2),$$

when both $G_1$ and $G_0$ are split. Thus we have proved Theorem 1.2 in the case $2 \nmid q$ and both $G_1$ and $G_0$ are split.

Now we consider the case when the discriminant field $K$ is equal to the unramified quadratic extension of $k$. Note that the character $\chi$ of $k^\times$ associated to $K/k$ by the class field theory is equal to the unique unramified quasi-character of order 2. As in the split case, we should consider the following two cases:

$$\text{(Case A)} \qquad G_1 = \mathrm{SO}(2l+1) \quad \text{and} \quad G_0 = \mathrm{SO}(2l),$$
$$\text{(Case B)} \qquad G_1 = \mathrm{SO}(2l+2) \quad \text{and} \quad G_0 = \mathrm{SO}(2l+1).$$

Note that

$$\begin{cases} A_1 \simeq (k^\times)^l,\ A_0 \simeq (k^\times)^{l-1} & \text{(Case A)}, \\ A_1 \simeq A_0 \simeq (k^\times)^l & \text{(Case B)}. \end{cases}$$

The unramified characters $\Xi$ and $\xi$ are determined by their restriction to $A_1$ and $A_0$, respectively. We write

$$\Xi = (\Xi_1, \ldots, \Xi_l)$$
$$\xi = \begin{cases} (\xi_1, \ldots, \xi_{l-1}) & \text{(Case A)}, \\ (\xi_1, \ldots, \xi_l) & \text{(Case B)}. \end{cases}$$

Put $\tilde{\Xi} = (\Xi_2, \ldots, \Xi_l)$. We set $\zeta(\Xi, \xi) = T_{\Xi,\xi}(\eta)$. As before, we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A)} \\ 1 & \text{(Case B)} \end{cases}$$

by [9], p. 22, Corollary to Lemma 1.1 and p. 179, Corollary 1 to Lemma 7.2. By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) = & \prod_{i=1}^{l} L(1, \Xi_i^2)^{-1} \prod_{1 \le i < j \le l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ & \times \prod_{i=1}^{l-1} L(1, \xi_i)^{-1} L(1, \chi\xi_i)^{-1} \prod_{1 \le i < j \le l-1} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i \le j \le l-1} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^{l-1} L(1/2, \Xi_i) L(1/2, \chi\Xi_i) \\ & \times \prod_{1 \le j < i \le l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case A, and

$$\begin{aligned}\zeta(\Xi,\xi) = &\prod_{i=1}^{l} L(1,\Xi_i)^{-1}L(1,\chi\Xi_i)^{-1} \prod_{1\le i<j\le l} L(1,\Xi_i\Xi_j)^{-1}L(1,\Xi_i\Xi_j^{-1})^{-1} \\ &\times \prod_{i=1}^{l} L(1,\xi_i^2)^{-1} \prod_{1\le i<j\le l} L(1,\xi_i\xi_j)^{-1}L(1,\xi_i\xi_j^{-1})^{-1} \\ &\times \prod_{1\le i\le j\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i\xi_j^{-1}) \prod_{i=1}^{l} L(1/2,\xi_i)L(1/2,\chi\xi_i) \\ &\times \prod_{1\le j<i\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i^{-1}\xi_j)\end{aligned}$$

in Case B. As for $S_{\Xi,\xi}(1)$, we can prove the following lemma.

**Lemma 5.1.** *We have*

$$S_{\Xi,\xi}(1) = \Delta_{G_1}\zeta(1)^{-\dim A_1-\dim A_0}L(1,\chi)^{-1}\zeta(\Xi,\xi).$$

The proof of this lemma will be given in the appendix to this section. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(1/2) = \zeta(1)^{-\dim A_1-\dim A_0}L(1,\chi)^{-1}\zeta(\Xi,\xi)\zeta(\Xi^{-1},\xi^{-1}).$$

We would like to emphasise that this relation has been already noted by Ginzburg, Piatetski-Shapiro, and Rallis [9]. Combining these results, we have $I(1;\Phi_\Xi,\Phi_\xi) = \Delta_{G_1}\mathcal{P}_{\pi_1,\pi_0}(1/2)$. Thus we have proved Theorem 1.2 in the case $2\nmid q$.

Now we consider the case $2\mid q$. It is enough to prove that $I(1;\Phi_\Xi,\Phi_\xi)$ is an element of $\mathbb{Q}(q^{1/2},\Xi,\xi)$. More precisely, we will show that there exists a rational function $\mathcal{I}(t,X_1,\ldots,x_1,\ldots) \in \mathbb{Q}(t,X_1,\ldots,x_1,\ldots)$, where $t,X_1,\ldots,x_1,\ldots$ are indeterminants, such that if the order of residue field of $k$ is $q$, then

$$I(1;\Phi_\Xi,\Phi_\xi) = \mathcal{I}(q^{1/2},\Xi_1,\ldots,\xi_1,\ldots).$$

To prove this, we make use of Macdonald's formula for the spherical function. Recall that Macdonald's formula ([5], p. 403, Theorem 4.2) says that the spherical functions $\Phi_\Xi$ and $\Phi_\xi$ are of the form

$$\Phi_\Xi(m_1) = Q_1^{-1}\sum_{w_1\in W_1}\gamma_1(w_1\Xi)\cdot((w_1\Xi)\delta_1^{-1/2})(m_1), \quad m_1\in A_1^+,$$

$$\Phi_\xi(m_0) = Q_0^{-1}\sum_{w_0\in W_0}\gamma_0(w_0\xi)\cdot((w_0\xi)\delta_0^{-1/2})(m_0), \quad m_0\in A_0^+.$$

Here, $Q_1$, $Q_0$, $\gamma_1(\Xi)$, $\gamma_0(\xi) \in \mathbb{Q}(q^{1/2}, \Xi, \xi)$ and $\delta_i$ is the modulus function of the Borel subgroup $B_i$. The integral $I(1; \Phi_\Xi, \Phi_\xi)$ is equal to

$$\int_{A_0^+} \Phi_\Xi(m_0)\Phi_\xi(m_0)\mathrm{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0)\, dm_0.$$

Note that $\mathrm{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0) = [\mathcal{K}_0 : \mathcal{K}_0 \cap m_0 \mathcal{K}_0 m_0^{-1}]$. One can show easily this integral gives an element of $\mathbb{Q}(q^{1/2}, \Xi, \xi)$. Therefore the proof of Theorem 1.2 is complete.

**Appendix to §5: Proof of Lemma 5.1.**

In this appendix, we prove Lemma 5.1. The proof of Lemma 5.1 consists of three steps.

**Step 1. The Weyl invariance.**

The Weyl group $W_1 \times W_0$ acts on the character group of $A_1 \times A_0$ by $(\Xi, \xi) \mapsto (w_1\Xi, w_0\xi)$.

**Lemma 5.2.** *The quantity $S_{\Xi,\xi}(g_1)\zeta(\Xi,\xi)^{-1}$ is $W_1 \times W_0$-invariant as a function of $\Xi$ and $\xi$. (cf.* [29] *Theorem 10.8.)*

*Proof.* Note that both $\zeta(\Xi,\xi)\zeta(\Xi^{-1},\xi^{-1})$ and

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \zeta(\Xi,\xi) S_{\Xi^{-1},\xi^{-1}}(g_1)$$

are $W_1 \times W_0$-invariant. It follows that

$$\frac{I(g_1; \Phi_\Xi, \Phi_\xi)}{\zeta(\Xi,\xi)\zeta(\Xi^{-1},\xi^{-1})} = \frac{S_{\Xi^{-1},\xi^{-1}}(g_1)}{\zeta(\Xi^{-1},\xi^{-1})}$$

is also $W_1 \times W_0$-invariant. Hence the lemma. □

**Step 2. An explicit formula for $S_{\Xi,\xi}(g_1)$.**

Now we closely follow the argument of [29]. Fix a hyperspecial maximal compact subgroup $\mathcal{K}_i \subset G_i$ and a maximal split torus $A_i \subset G_i$. Then the centralizer $T_i$ of $A_i$ is a maximally split maximal torus of $G_i$. We assume $\mathcal{K}_0 \subset \mathcal{K}_1$ and $A_0 \subset A_1$. Note that $T_0$ need not be a subgroup of $T_1$. Choose a Borel subgroup $B_i = T_i N_i \subset G_i$. We also assume $N_0 \subset N_1$. The opposite Borel subgroup of $B_i = T_i N_i$ is denoted by $\bar{B}_i = T_i \bar{N}_i$. We put $T_i^{(0)} = T_i \cap \mathcal{K}_i$, $N_i^{(0)} = N_i \cap \mathcal{K}_i$, and $\bar{N}_i^{(0)} = \bar{N}_i \cap \mathcal{K}_i$. Choose a longest element $w_{i,\mathrm{long}}$ of the Weyl group $W_i = W(G_i, A_i)$. We assume $w_{i,\mathrm{long}} \in \mathcal{K}_i$. There exists an Iwahori subgroup $\mathcal{B}_i \subset \mathcal{K}_i$ such that $N_i^{(0)} \subset \mathcal{B}_i$. We put $\bar{N}_i^{(1)} = \bar{N}_i \cap \mathcal{B}_i$ and $N_i^{(1)} = w_{i,\mathrm{long}}^{-1} \bar{N}_i^{(1)} w_{i,\mathrm{long}}$. Then we have an Iwahori decomposition $\mathcal{B}_i = \bar{N}_i^{(1)} T_i^{(0)} N_i^{(0)}$.

Recall that the element $\eta \in G_1$ is a representative of the unique open orbit of $B_1 \backslash G_1 / B_0$ such that $\eta \in \mathcal{K}_1$. Let $\mathfrak{o}$ and $\mathfrak{o}_K$ be the ring of integers of $k$ and $K$, respectively. The maximal ideal of $\mathfrak{o}$ and $\mathfrak{o}_K$ are denoted by $\mathfrak{p}$ and $\mathfrak{p}_K$, respectively.

**Lemma 5.3.** *One can choose the representative $\eta$ of the open orbit of $B_1\backslash G_1/B_0$ such that the following conditions hold.*

(1) $\eta\bar{N}_0^{(1)} \subset \mathcal{B}_1\eta$,
(2) $\bar{N}_1^{(1)}\eta \subset T_1^{(0)}N_1^{(0)}\eta T_0^{(0)}N_0^{(0)}$.

*Proof.* We first consider Case B. Note that in this case $N_0$ is a normal subgroup of $N_1$. By [9], p. 171, Lemma 7.1, $N_1/N_0$ is isomorphic to $k^{l-1} \times (K/k)$ as a left module of $A_0 = A_1 \simeq (k^\times)^l$. We fix an isomorphism $N_1/N_0 \simeq k^{l-1} \times (K/k)$, which induces an isomorphism $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^{l-1} \times (\mathfrak{o}_K/\mathfrak{o})$. Since $K/k$ is unramified, $\mathfrak{o}_K/\mathfrak{o}$ is isomorphic to $\mathfrak{o}$, and so $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. There exists a cross section (i.e., "épinglage") $\iota$ of the map $N_1^{(0)} \to N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. Let $\eta'$ be the image of the cross section of $(1, 1, \ldots, 1) \in \mathfrak{o}^l$. We put $\eta = w_{1,\mathrm{long}}\eta'$. Then $\eta$ is a representative of the open orbit of $B_1\backslash G_1/B_0$. Let $\mathcal{U}_1$ be the group generated by $N_1^{(1)}$ and $\bar{N}_1^{(1)}$. Then $\mathcal{U}_1$ is a normal subgroup of $\mathcal{K}_1$. It follows that $\eta\bar{N}_0^{(1)} \subset \eta\mathcal{U}_1 = \mathcal{U}_1\eta \subset \mathcal{B}_1\eta$. As for (2), $\bar{N}_1^{(1)}\eta = w_{1,\mathrm{long}}N_1^{(1)}\eta' \subset w_{1,\mathrm{long}}\iota(\mathfrak{p}^l)\eta' N_0^{(1)}$. It suffices to prove that $\iota(\mathfrak{p}^l)\eta' \subset T_1^{(0)}\eta' T_0^{(0)}$. This is easily seen by the facts $1 + \mathfrak{p} \subset \mathfrak{o}^\times$.

Now we consider Case A. Let $P_1$ be the standard parabolic subgroup of $G_1$ with Levi factor $(k^\times)^{l-1} \times \mathrm{SO}(3) \simeq (k^\times)^{l-1} \times \mathrm{PGL}_2$. Let $N_{P_1}$ be the unipotent radical of $P_1$. Then as in Case B, $N_{P_1}/N_0$ is isomorphic to $k^{l-1}$ as a left module of $A_0 \simeq (k^\times)^{l-1}$. We fix an isomorphism $N_{P_1}/N_0 \simeq k^{l-1}$, which induces an isomorphism $(N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Take a cross section $\iota$ of the map $(N_{P_1} \cap N_1^{(0)}) \to (N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Put $\eta = w_{1,\mathrm{long}}\iota((1, 1, \ldots, 1))$. Then $\eta$ is a representative of the open orbit of $B_1\backslash G_1/B_0$, since $\mathrm{PGL}_2 = (\mathrm{PGL}_2 \cap N_1) \cdot (\mathrm{PGL}_2 \cap T_0)$ (cf. [9], Appendix 1 to §7). One can prove (1) in the same way as in Case B. As for (2), observe that $\bar{N}_1^{(1)} = (\bar{N}_1^{(1)} \cap \bar{N}_{P_1}) \cdot (\bar{N}_1^{(1)} \cap \mathrm{PGL}_2)$, where $\bar{N}_{P_1}$ is the unipotent radical of the opposite parabolic subgroup of $P_1$ with respect to the Levi subgroup $(k^\times)^{l-1} \times \mathrm{PGL}_2$. One can prove that $(\bar{N}_1^{(1)} \cap \bar{N}_{P_1})\eta \subset T_1^{(0)}\eta T_0^{(0)}N_0^{(0)}$ in the same way as in Case B. Now (2) follows from the fact $(T_1^{(0)}N_1^{(0)} \cap \mathrm{PGL}_2) \cdot (T_0^{(0)} \cap \mathrm{PGL}_2) = \mathcal{K}_1 \cap \mathrm{PGL}_2$. □

**Lemma 5.4.** *We have*

$$\mathcal{B}_0\eta^{-1}\mathcal{B}_1 \subset T_0^{(0)}N_0^{(0)}\eta^{-1}T_1^{(0)}N_1^{(0)}.$$

*Proof.* By Lemma 5.3, we have

$$\begin{aligned}\mathcal{B}_0\eta^{-1}\mathcal{B}_1 &= T_0^{(0)}N_0^{(0)}\bar{N}_0^{(1)}\eta^{-1}\mathcal{B}_1\\ &\subset T_0^{(0)}N_0^{(0)}\eta^{-1}\mathcal{B}_1\\ &= T_0^{(0)}N_0^{(0)}\eta^{-1}\bar{N}_1^{(1)}T_1^{(0)}N_1^{(0)}\\ &\subset T_0^{(0)}N_0^{(0)}\eta^{-1}T_1^{(0)}N_1^{(0)}.\end{aligned}$$

□

Put

$$\begin{aligned}A_1^+ &= \{t \in A_1 \,|\, |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_1, A_1)\},\\ A_0^+ &= \{t \in A_0 \,|\, |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_0, A_0)\}.\end{aligned}$$

Then we have Cartan decompositions $G_1 = \mathcal{K}_1 A_1^+ \mathcal{K}_1$, $G_0 = \mathcal{K}_0 A_0^+ \mathcal{K}_0$.

For each positive root $\alpha$ of $G_1$ (resp. $G_0$), we denote Harish-Chandra's $c$-function (cf. e.g., Casselman [5]) by $c_\alpha(\Xi)$ (resp. $c_\alpha(\xi)$). We put

$$\bar{c}_{w_1}(\Xi) = \prod_{\substack{\alpha>0\\ w_1\alpha>0}} c_\alpha(\Xi), \qquad \left(\text{resp. } \bar{c}_{w_0}(\xi) = \prod_{\substack{\alpha>0\\ w_0\alpha>0}} c_\alpha(\xi)\right).$$

When $w_1$ (resp. $w_0$) is the identity element, we set

$$\mathbf{c}_1(\Xi) = \prod_{\alpha>0} c_\alpha(\Xi), \qquad \left(\text{resp. } \mathbf{c}_0(\xi) = \prod_{\alpha>0} c_\alpha(\xi)\right).$$

**Lemma 5.5.** *There exists a basis $\{g_{1,w_1}\}_{w_1\in W_1}$ of $I(\Xi)^{\mathcal{B}_1}$ with the following properties.*

- ($1_1$) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t^{-1}\mathcal{B}_1})g_{1,w_1} = \mathrm{Vol}(\mathcal{B}_1 t\mathcal{B}_1)\cdot(w_1\Xi)^{-1}\delta_1^{1/2}(t)\cdot g_{1,w_1}$ *for any* $t \in A_1^+$.
- ($2_1$) *The restriction of $g_{1,1}$ to $\mathcal{K}_1$ is the characteristic function of $\mathcal{B}_1$.*
- ($3_1$) $f_\Xi = [N_1^{(0)} : N_1^{(1)}]\sum_{w_1\in W_1}\bar{c}_{w_1}(\Xi)\cdot g_{1,w_1}$.

*Similarly, there exists a basis $\{g_{0,w_0}\}_{w_0\in W_0}$ of $I(\xi)^{\mathcal{B}_0}$ with the following properties.*

- ($1_0$) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_0 t^{-1}\mathcal{B}_0})g_{0,w_0} = \mathrm{Vol}(\mathcal{B}_0 t\mathcal{B}_0)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t)\cdot g_{0,w_0}$ *for any* $t \in A_0^+$.
- ($2_0$) *The restriction of $g_{0,1}$ to $\mathcal{K}_0$ is the characteristic function of $\mathcal{B}_0$.*
- ($3_0$) $f_\xi = [N_0^{(0)} : N_0^{(1)}]\sum_{w_0\in W_0}\bar{c}_{w_0}(\xi)\cdot g_{0,w_0}$.

*Proof.* See [29] p. 8, Proposition 1.10. □

**Lemma 5.6.** *We have*

$$S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1}) = \mathrm{Vol}(\mathcal{B}_0 t_0^{-1}\mathcal{B}_0)^{-1}\mathrm{Vol}(\mathcal{B}_1 t_1^{-1}\mathcal{B}_1)^{-1} \times (\mathcal{L}(\mathbf{1}_{\mathcal{B}_0 t_0^{-1}\mathcal{B}_0})\mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t_1^{-1}\mathcal{B}_1})S_{\Xi,\xi})(\eta^{-1})$$

*for* $t_0 \in A_0^+$, $t_1 \in A_1^+$.

*Proof.* It suffices to show that

$$(\mathcal{B}_0 t_0 \mathcal{B}_0)\eta^{-1}(\mathcal{B}_1 t_1^{-1}\mathcal{B}_1) \subset \mathcal{K}_0 t_0 \eta^{-1} t_1^{-1}\mathcal{K}_1$$

for $t_0 \in A_0^+$, $t_1 \in A_1^+$. By Lemma 5.4, we have

$$\mathcal{B}_0 t_0 \mathcal{B}_0 \eta^{-1}\mathcal{B}_1 t_1^{-1}\mathcal{B}_1 \subset \mathcal{B}_0 t_0 T_0^{(0)} N_0^{(0)} \eta^{-1} T_1^{(0)} N_1^{(0)} t_1^{-1}\mathcal{B}_1.$$

Since $t_i T_i^{(0)} N_i^{(0)} t_i^{-1} \subset T_i^{(0)} N_i^{(0)}$, the lemma follows. □

Recall that

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1)f_\Xi)\rangle.$$

By $(1_1)$, $(3_1)$, $(1_0)$, and $(3_0)$ of Lemma 5.5, we have

$$\begin{aligned} S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1}) = {} & [N_1^{(0)} : N_1^{(1)}][N_0^{(0)} : N_0^{(1)}] \\ & \times \sum_{\substack{w_1\in W_1 \\ w_0 \in W_0}} \bar{c}_{w_1}(\Xi)\bar{c}_{w_0}(\xi)(w_1\Xi)^{-1}\delta_1^{1/2}(t_1)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t_0) \\ & \times \int_{\mathcal{K}_0\times\mathcal{K}_1} g_{0,w_0}(k_0)g_{1,w_1}(k_1)Y_{\Xi,\xi}(k_0\eta k_1)\,dk_0\,dk_1. \end{aligned}$$

By $(2_1)$ and $(2_0)$ of Lemma 5.5, we have

$$\begin{aligned} & \int_{\mathcal{K}_0\times\mathcal{K}_1} g_{0,1}(k_0)g_{1,1}(k_1)Y_{\Xi,\xi}(k_0\eta k_1)\,dk_0\,dk_1 \\ & = \mathrm{Vol}(\mathcal{B}_1)\mathrm{Vol}(\mathcal{B}_0) \\ & = \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1 - \dim A_0}L(1,\chi)^{-1}/([N_1^{(0)} : N_1^{(1)}][N_0^{(0)} : N_0^{(1)}]). \end{aligned}$$

Put $\mathbf{c}_{\mathrm{WS}}(\Xi,\xi)=\mathbf{c}_1(\Xi)\mathbf{c}_0(\xi)\zeta(\Xi,\xi)^{-1}=\mathbf{b}(\Xi,\xi)\mathbf{d}_1(\Xi)^{-1}\mathbf{d}_0(\xi)^{-1}$, where

$$\begin{aligned}
\mathbf{b}(\Xi,\xi)^{-1} &= \prod_{1\le i\le j\le l-1} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i\xi_j^{-1})\prod_{i=1}^{l-1} L(1/2,\Xi_i)L(1/2,\chi\Xi_i)\\
&\quad\times\prod_{1\le j<i\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i^{-1}\xi_j)\\
\mathbf{d}_1(\Xi)^{-1} &= \prod_{i=1}^{l} L(0,\Xi_i^2)\prod_{1\le i<j\le l} L(0,\Xi_i\Xi_j)L(0,\Xi_i\Xi_j^{-1})\\
\mathbf{d}_0(\xi)^{-1} &= \prod_{i=1}^{l-1} L(0,\xi_i)L(0,\chi\xi_i)\prod_{1\le i<j\le l-1} L(0,\xi_i\xi_j)L(0,\xi_i\xi_j^{-1})
\end{aligned}$$

in Case A, and

$$\begin{aligned}
\mathbf{b}(\Xi,\xi)^{-1} &= \prod_{1\le i\le j\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i\xi_j^{-1})\prod_{i=1}^{l} L(1/2,\xi_i)L(1/2,\chi\xi_i)\\
&\quad\times\prod_{1\le j<i\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i^{-1}\xi_j)\\
\mathbf{d}_1(\Xi)^{-1} &= \prod_{i=1}^{l} L(0,\Xi_i)L(0,\chi\Xi_i)\prod_{1\le i<j\le l} L(0,\Xi_i\Xi_j)L(0,\Xi_i\Xi_j^{-1})\\
\mathbf{d}_0(\xi)^{-1} &= \prod_{i=1}^{l} L(0,\xi_i^2)\prod_{1\le i<j\le l} L(0,\xi_i\xi_j)L(0,\xi_i\xi_j^{-1})
\end{aligned}$$

in Case B. By the Weyl-invariance, we have

$$\begin{aligned}
\frac{S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1})}{\zeta(\Xi,\xi)} &= \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1-\dim A_0}L(1,\chi)^{-1}\\
&\quad\times\sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi,w_0\xi)\cdot(w_1\Xi)^{-1}\delta_1^{1/2}(t_1)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t_0).
\end{aligned}$$

(cf. [29], Theorem 10.7.) Note that

$$\mathbf{b}(\Xi,\xi),\ \mathbf{d}_1(\Xi),\ \mathbf{d}_0(\xi)\in\mathbb{Z}[q^{\pm1/2},\Xi_1,\Xi_2,\ldots,\xi_1,\xi_2,\ldots].$$

Here and from now on, we identify an unramified quasi-character of $k^\times$ with its value at a prime element.

**Step 3. Calculation of $S_{\Xi,\xi}(1)/\zeta(\Xi,\xi)$.**

Our next task is to prove the following lemma.

**Lemma 5.7.** *The sum*

$$\frac{S_{\Xi,\xi}(1)}{\zeta(\Xi,\xi)} = \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1 - \dim A_0}L(1,\chi)^{-1} \times \sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi, w_0\xi)$$

*is independent of* $\Xi$ *and* $\xi$.

*Proof.* We shall prove the lemma only in Case B. One can handle Case A in a similar way. Put

$$A_{\Xi,\xi} = \sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi, w_0\xi).$$

We are going to prove that $A_{\Xi,\xi}$ is independent of $\Xi$ and $\xi$. Put

$$\mathcal{D}(\Xi) = \Xi^{-\rho_1}\mathbf{d}_1(\Xi) = \sum_{w_1\in W_1} \mathrm{sgn}(w_1)\cdot(w_1\Xi)^{-\rho_1}$$

$$\mathcal{D}(\xi) = \xi^{-\rho_0}\mathbf{d}_0(\xi) = \sum_{w_0\in W_0} \mathrm{sgn}(w_0)\cdot(w_0\xi)^{-\rho_0},$$

where

$$\rho_1 = \rho_0 = (l, l-1, \dots, 1).$$

Then we have $\mathcal{D}(w_1\Xi) = \mathrm{sgn}(w_1)\mathcal{D}(\Xi)$ and $\mathcal{D}(w_0\xi) = \mathrm{sgn}(w_0)\mathcal{D}(\xi)$ for $w_1 \in W_1$ and $w_0 \in W_0$. Note that $\rho_1$ and $\rho_0$ are half the sum of the positive roots of type C. It follows that $A_{\Xi,\xi}$ is equal to

$$(\mathcal{D}(\Xi)\mathcal{D}(\xi))^{-1} \sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathrm{sgn}(w_1)\mathrm{sgn}(w_0)\cdot(w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0}\mathbf{b}(w_1\Xi, w_0\xi).$$

Put $B_{\Xi,\xi} = \Xi^{-\rho_1}\xi^{-\rho_0}\mathbf{b}(\Xi,\xi)$. Observe that $B_{\Xi,\xi}$ is equal to

$$\prod_{1\le j\le l}(\xi_j^{-1} - q^{-1}\xi_j) \prod_{1\le i\le j\le l}(\Xi_i^{-1} - q^{-1/2}\xi_j^{-1}) \times \prod_{1\le j<i\le l}(\xi_j^{-1} - q^{-1/2}\Xi_i^{-1}) \prod_{\substack{1\le i\le l\\ 1\le j\le l}}(1 - q^{-1/2}\Xi_i\xi_j).$$

We express $B_{\Xi,\xi}$ as a sum of monomials

$$B_{\Xi,\xi} = \sum_{\lambda,\mu} c_{\lambda,\mu}\Xi^\lambda\xi^\mu, \quad \lambda,\ \mu \in \mathbb{Z}^l,\ c_{\lambda,\mu} \in \mathbb{Z}[q^{\pm 1/2}].$$

We say that a monomial $\Xi^\lambda\xi^\mu$ is regular if $\Xi^{w_1\lambda}\xi^{w_0\mu} = \Xi^\lambda\xi^\mu$ implies $w_1 = w_0 = 1$. We also say that a monomial is singular if it is not regular. Here the action of the Weyl group on $\mathbb{Z}^l$ is given by $(w_1\Xi)^{w_1\lambda} = \Xi^\lambda$, $(w_0\xi)^{w_0\mu} = \xi^\mu$, as usual.

We would like to show that if a regular monomial $\Xi^\lambda\xi^\mu$ appears in $B_{\Xi,\xi}$, then it is of the form $\Xi^{w_1\rho_1}\xi^{w_0\rho_0}$ with $w_1 \in W_1$, $w_0 \in W_0$. It is enough to show $|\lambda_i|$, $|\mu_j| \le l$, since such a monomial is either singular or Weyl-equivalent to $\Xi^{\rho_1}\xi^{\rho_0}$. Choose $i_0$, $j_0 \in \{1, 2, \dots, l\}$. The positive contribution of $\Xi_{i_0}$ comes from

$$\prod_{1\le j\le l}(1 - q^{-1/2}\Xi_{i_0}\xi_j),$$

and the negative contribution of $\Xi_{i_0}$ comes from

$$\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1})\prod_{1\le j<i_0}(\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}).$$

Therefore $|\lambda_{i_0}| \le l$. Similarly, the positive contribution of $\xi_{j_0}$ comes from

$$(\xi_{j_0}^{-1} - q^{-1}\xi_{j_0})\prod_{1\le i\le l}(1 - q^{-1/2}\Xi_i\xi_{j_0})$$

and the negative contribution of $\xi_{j_0}$ comes from

$$(\xi_{j_0}^{-1} - q^{-1}\xi_{j_0})\prod_{1\le i\le j_0}(\Xi_i^{-1} - q^{-1/2}\xi_{j_0}^{-1})\prod_{j_0<i\le l}(\xi_{j_0}^{-1} - q^{-1/2}\Xi_i^{-1}).$$

Therefore $|\mu_{j_0}| \le l + 1$. It follows that if a regular monomial $\Xi^\lambda\xi^\mu$ occurs in $B_{\Xi,\xi}$, then $l \le |\mu_{j_0}| \le l + 1$ for some $j_0$. We will show that no regular monomial $\Xi^\lambda\xi^\mu$ such that $|\mu_{j_0}| > l$ occurs in $B_{\Xi,\xi}$. Assume that the monomial $\Xi^\lambda\xi^\mu$ occurs in $B_{\Xi,\xi}$ and $|\mu_{j_0}| > l$. We must show that such a monomial $\Xi^\lambda\xi^\mu$ is singular. Note that the monomial $\Xi^\lambda\xi^\mu$ occurs in

$$\begin{aligned}&q^{-1}\xi_{j_0}\cdot\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1})\\&\times\prod_{1\le j<i_0}(\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1})\cdot q^{-1/2}\Xi_{i_0}\xi_{j_0}\prod_{\substack{1\le j\le l\\ j\ne j_0}}(1 - q^{-1/2}\Xi_{i_0}\xi_j)\\&\times(\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0}).\end{aligned}$$

In particular, we have $\lambda_{i_0} \ne -l$. If $\lambda_{i_0} = l$, then the factor $\xi_{j_0}^{-1}$ must occur in the factor

$$\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1})\prod_{1\le j<i_0}(\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}),$$

which would contradicts to the condition $\mu_{j_0} > l$. It follows that the condition $\mu_{j_0} > l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial such that $\mu_{j_0} > l$ occurs in $B_{\Xi,\xi}$. Assume now $\mu_{j_0} < -l$. Then the monomial $\Xi^\lambda \xi^\mu$ occurs in

$$\begin{aligned}
&\xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{\substack{i_0 \le j \le l \\ j \ne j_0}} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\
&\times \xi_{j_0}^{-l+j_0} \prod_{1 \le j < i_0} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \le j \le l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\
&\times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0})
\end{aligned}$$

if $i_0 \le j_0$, and

$$\begin{aligned}
&\xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{i_0 \le j \le l} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\
&\times \xi_{j_0}^{-l+j_0} \prod_{\substack{1 \le j < i_0 \\ j \ne j_0}} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \le j \le l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\
&\times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0})
\end{aligned}$$

if $i_0 > j_0$. In particular, $\lambda_{i_0} \ne -l$. If $\lambda_{i_0} = l$, then the factor $\xi_{j_0}$ occurs, and so the condition $\mu_{j_0} < -l$ fails. It follows that the condition $\mu_{j_0} < -l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial $\Xi^\lambda \xi^\mu$ such that $\mu_{j_0} < -l$ occurs in $B_{\Xi,\xi}$.

We have proved that the regular monomials $\Xi^\lambda \xi^\mu$ which occur in $B_{\Xi,\xi}$ are of the form $(w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0}$, for some $w_1 \in W_1$ and $w_0 \in W_0$. Therefore, up to a constant, $A_{\Xi,\xi}$ is equal to

$$(\mathcal{D}(\Xi)\mathcal{D}(\xi))^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathrm{sgn}(w_1)\mathrm{sgn}(w_0) \cdot (w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0} = 1.$$

Hence the lemma. □

Recall that

$$A_{\Xi,\xi} = \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi, w_0\xi).$$

**Lemma 5.8.** *The constant $A_{\Xi,\xi}$ is equal to $\Delta_{G_0}^{-1}$.*

*Proof.* We shall prove the lemma only in Case B. One can handle Case A in a similar way. We put

$$\begin{aligned}
\tilde{\Xi} &= (q^{-l}, q^{-l+1}, \dots, q^{-1}), \\
\tilde{\xi} &= (q^{-l+(1/2)}, q^{-l+(3/2)}, \dots, q^{-1/2}).
\end{aligned}$$

As in the proof of [29], Lemma 11.9, we shall prove that $\mathbf{b}(w_1\tilde{\Xi}, w_0\tilde{\xi}) \neq 0$ implies $w_1 = w_0 = 1$. Note that $\mathbf{b}(\Xi, \xi)$ is equal to

$$\prod_{1\le i\le j\le l}(1-q^{-1/2}\Xi_i\xi_j^{-1})\prod_{1\le j<i\le l}(1-q^{-1/2}\Xi_i^{-1}\xi_j)\prod_{\substack{1\le i\le l\\1\le j\le l}}(1-q^{-1/2}\Xi_i\xi_j)$$
$$\times\prod_{1\le j\le l}(1-q^{-1}\xi_j^2).$$

Note that $W_1 \simeq W_0 \simeq \{\pm 1\}^l \rtimes \mathfrak{S}_l$, where $\mathfrak{S}_l$ is the symmetric group. Therefore, for every $w_1 \in W_1$, $w_0 \in W_0$, one can find $\sigma, \tau \in \mathfrak{S}_l$ and $\varepsilon_i, \varepsilon'_j \in \{\pm 1\}$ such that

$$w_1\Xi = (\Xi_{\sigma(1)}^{\varepsilon_1}, \dots, \Xi_{\sigma(l)}^{\varepsilon_l}),$$
$$w_0\xi = (\xi_{\tau(1)}^{\varepsilon'_1}, \dots, \xi_{\tau(l)}^{\varepsilon'_l}).$$

Put $i_s = \sigma^{-1}(l+1-s)$, $j_t = \tau^{-1}(l+1-t)$. Then we have

$$(w_1\tilde{\Xi})_{i_s} = \tilde{\Xi}_{l+1-s}^{\varepsilon_{i_s}} = q^{-\varepsilon_{i_s}\cdot s},$$
$$(w_0\tilde{\xi})_{j_t} = \tilde{\xi}_{l+1-t}^{\varepsilon'_{j_t}} = q^{-\varepsilon'_{j_t}(t-(1/2))}.$$

Assume $\mathbf{b}(w_1\tilde{\Xi}, w_0\tilde{\xi}) \neq 0$. Firstly, $1 - q^{-1}(w_0\tilde{\xi})_{j_1}^2 \neq 0$ implies $\varepsilon'_{j_1} = 1$. Secondly, $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}(w_0\tilde{\xi})_{j_s} \neq 0$ and $1-q^{-1/2}(w_1\tilde{\Xi})_{i_{t+1}}(w_0\tilde{\xi})_{j_t} \neq 0$ imply

$$\varepsilon'_{j_1} = \varepsilon_{i_1} = \varepsilon'_{j_2} = \varepsilon_{i_2} = \cdots = \varepsilon'_{j_l} = \varepsilon_{i_l} = 1.$$

Now, if $j_s < i_s$, then the second factor would contain the factor $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}^{-1}(w_0\tilde{\xi})_{j_s} = 0$, therefore we have $j_s \geq i_s$. Similarly, if $i_s \leq j_{s+1}$, the first factor would contain the factor $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}(w_0\tilde{\xi})_{j_{s+1}}^{-1} = 0$, therefore we have $i_s > j_{s+1}$. It follows that

$$j_1 \geq i_1 > j_2 \geq i_2 > \cdots > j_l \geq i_l,$$

and so $w_1 = w_0 = 1$. It follows that $A_{\Xi,\xi} = \mathbf{b}(\tilde{\Xi}, \tilde{\xi})\mathbf{d}_1(\tilde{\Xi})^{-1}\mathbf{d}_0(\tilde{\xi})^{-1}$. By direct calculation, one can easily show that it is $\Delta_{G_0}^{-1}$. □

Now Lemma 5.1 follows from Lemma 5.7 and Lemma 5.8.

## Part III. Examples

In §§6-12, $k$ is an algebraic number field. The Dedekind zeta function of $k$ is denoted by $\zeta_k(s)$. The $\Gamma$-factors of $L$-functions are normalized as in Tate [45]. In particular, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The completed Dedekind zeta function of $k$ is denoted by $\xi_k(s)$. When $k = \mathbb{Q}$, the subscript $k$ is dropped. The symbol $L(s, \pi, r)$

is the Euler product $\prod_{v<\infty} L(s, \pi_v, r)$ and the completed $L$-function for $L(s, \pi, r)$ is denoted by $\Lambda(s, \pi, r)$.

## 6. Waldspurger's theorem

The following example is due to Waldspurger [46]. Let $D$ be a quaternion algebra over an algebraic number field $k$. Then $G_1 = D^\times/k^\times$ can be considered as a special orthogonal group associated to a 3-dimensional quadratic space over $k$. Note that $\Delta_{G_1} = \xi_k(2)$. Let $G_0 = T$ be an anisotropic torus of $G_1$. Then $T$ can be considered as a special orthogonal group associated to a 2-dimensional quadratic space over $k$. Let $K$ be a splitting field of $T$ over $k$. Then there exists an exact sequence

$$1 \longrightarrow k^\times \longrightarrow K^\times \longrightarrow T \longrightarrow 1.$$

By means of this exact sequence, a character $\omega$ of $T(\mathbb{A})/T(k)$ can be regarded as a character of $\mathbb{A}_K^\times/K^\times$ whose restriction to $\mathbb{A}_k^\times/k^\times$ is trivial. As in [46], we choose a Haar measure of $T(k_v)$ as follows. Fix a non-trivial additive character $\psi$ of $\mathbb{A}/k$. Then we give the Haar measure $\zeta_v(1)^{-1}|t|_v^{-1}\,dt_v$ on $k_v^\times$, where $dt_v$ is the self-dual Haar measure of $k_v$ with respect to $\psi_v$. We give a Haar measure on $K_v^\times$ in a similar way. Then the Haar measure on $T(k_v)$ is defined by the exact sequence

$$1 \longrightarrow k_v^\times \longrightarrow K_v^\times \longrightarrow T(k_v) \longrightarrow 1.$$

Let $C_0$ be the Haar measure constant. It is easily seen that $C_0 = \Lambda(1, \chi_{K/k})^{-1}$ for this choice of measure. Note that in [46], Waldspurger considered the measure on $T(\mathbb{A})$ such that $\mathrm{Vol}(T(\mathbb{A})/T(k)) = 2\Lambda(1, \chi_{K/k})$.

An irreducible cuspidal automorphic representation $\pi$ of $G_1(\mathbb{A})$ can be considered as a representation of $D^\times(\mathbb{A})$ with trivial central character. We assume $\pi$ is almost locally generic. The base change of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_K)$ is denoted by $\Pi$. Choose a non-zero cusp form $\varphi = \otimes_v \varphi_v \in \pi \simeq \otimes_v \pi_v$.

Then among other things, Waldspurger ([46], Proposition 7) proved that the integral $I(\varphi_v, \omega_v)$ is convergent and that

$$\begin{aligned} \frac{|\langle \varphi|_{G_0}, \omega\rangle|^2}{\langle \varphi, \varphi\rangle\langle \omega, \omega\rangle} &= \frac{1}{4}\Delta_{G_1} C_0 \frac{\Lambda(1/2, \Pi\otimes\omega^{-1})}{\Lambda(1, \pi, \mathrm{Ad})\Lambda(1, \chi_{K/k})} \prod_{v\in S} \frac{\alpha_v(\varphi_v, \omega_v)}{\|\varphi_v\|^2} \\ &= \frac{1}{4}\Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v\in S} \frac{\alpha_v(\varphi_v, \omega_v)}{\|\varphi_v\|^2\cdot\|\omega_v\|^2}, \end{aligned}$$

where $\pi_1 = \pi$, $\pi_0 = \omega$. Thus Conjecture 1.5 is true for $n = 2$. Note that we have $|\mathcal{S}_{\psi_1}| = |\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture. Thus Waldspurger's result is compatible with Conjecture 2.1.

## 7. The case $n = 3$

In this section, we prove Conjecture 1.5 for $n = 3$. Let $D$ be an quaternion algebra over an algebraic number field $k$. Let $k'$ be either $k \times k$ or a quadratic extension of $k$. We put

$$\begin{aligned}\tilde{G}_1 &= (D \otimes_k k')^\times / k^\times,\\ G_1 &= \{g \in (D \otimes_k k')^\times \mid \nu(g) \in k^\times\}/k^\times,\\ G_0 &= D^\times / k^\times.\end{aligned}$$

Here $\nu$ is the reduced norm of $D$. Then $G_1$ (resp. $G_0$) can be considered as a special orthogonal group associated to a 4-dimensional (resp. 3-dimensional) quadratic space over $k$. We regard $G_0$ as a subgroup of $G_1$. Note that

$$\Delta_{G_1} = \begin{cases} \xi_k(2)^2 & \text{if } k' = k \times k, \\ \xi_{k'}(2) & \text{otherwise.} \end{cases}$$

Let $Z_{\tilde{G}_1}$ be the identity component of the center of $\tilde{G}_1$.

Let $\pi_i$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$ on the space $\mathcal{V}_{\pi_i}$. We assume $\pi_i$ is almost locally generic. By the result of Hiraga and Saito [22], Theorem 4.13, there exists an irreducible unitary cuspidal automorphic representation $\tau$ of $\tilde{G}_1(\mathbb{A})$ on the space $\mathcal{V}_\tau$ such that $\mathcal{V}_{\pi_1} \subset \mathcal{V}_\tau^1|_{G_1(\mathbb{A})}$. Here, $\mathcal{V}_\tau^1$ is the subspace of $\mathcal{V}_\tau$ on which the group

$$\mathfrak{X}_\tau = \{\omega \in \mathrm{Hom}_{\mathrm{cont}}(Z_{\tilde{G}_1}(\mathbb{A})G_1(\mathbb{A})\tilde{G}_1(k)\backslash \tilde{G}_1(\mathbb{A}), \mathbb{C}^\times) \mid \tau \otimes \omega \simeq \tau\}$$

acts trivially, and $\mathcal{V}_\tau^1|_{G_1(\mathbb{A})}$ is the restriction of $\mathcal{V}_\tau^1$ to $G_1(\mathbb{A})$ as functions. Note that $\mathfrak{X}_\tau$ is an elementary 2-abelian group.

Let $\langle\ ,\ \rangle$ be the canonical inner product on $\mathcal{V}_\tau$ and $\langle\ ,\ \rangle_v$ an inner product on $\tau_v$ for any place $v$ of $k$. Then Ichino's result ([25] Theorem 1.1) says

$$\frac{|\langle \tilde{\varphi}|_{G_0(\mathbb{A})}, \varphi_0\rangle|^2}{\langle \tilde{\varphi}, \tilde{\varphi}\rangle\langle \varphi_0, \varphi_0\rangle} = 2^{\tilde{\beta}} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{\tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v})}{\langle \tilde{\varphi}_v, \tilde{\varphi}_v\rangle_v \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \tau$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$. Here,

$$\begin{aligned}\tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v}) &= \Delta_{G_1,v}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} \\ &\quad \times \int_{G_{0,v}} \langle \tau_v(g_{0,v})\tilde{\varphi}_v, \tilde{\varphi}_v\rangle_v \overline{\langle \pi_{0,v}(g_{0,v})\varphi_{0,v}, \varphi_{0,v}\rangle_v}\, dg_{0,v}\end{aligned}$$

and

$$\tilde{\beta} = \begin{cases} -3 & \text{if } k' = k \times k, \\ -2 & \text{otherwise.} \end{cases}$$

Choose a non-zero cusp form $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathcal{V}_{\pi_1}$. We choose $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \mathcal{V}_\tau^1$ such that $\tilde{\varphi}|_{G_1(\mathbb{A})} = \varphi_1$. We may assume $\tilde{\varphi}$ belongs to the $\otimes_v \pi_{1,v}$-isotypic subspace of $\mathcal{V}_\tau^1$. Then we have

$$\frac{\tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v})}{\langle \tilde{\varphi}_v, \tilde{\varphi}_v \rangle_v} = \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2}.$$

By Remark 4.20 of [22], we have

$$\langle \tilde{\varphi}, \tilde{\varphi} \rangle = \frac{1}{|\mathfrak{X}_\tau|} \langle \varphi_1, \varphi_1 \rangle \times \begin{cases} 2 & \text{if } k' = k \times k, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we obtain the following theorem.

**Theorem 7.1.** *We have*

$$\frac{|\langle \varphi_1|_{G_0(\mathbb{A})}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{1}{4|\mathfrak{X}_\tau|} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

*for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.*

Thus Conjecture 1.5 is true for $n = 3$. Note that we have $|\mathcal{S}_{\psi_1}| = 2|\mathfrak{X}_\tau|$ and $|\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture.

We show that Theorem 7.1 is compatible with the result of Watson [47] in some cases. Put $G_1 = \mathrm{SO}(2,2)$ and $G_0 = \mathrm{SO}(2,1) = \mathrm{PGL}_2$, defined over $k = \mathbb{Q}$. By definition, we have $\Delta_{G_1} = \xi(2)^2$. When $v$ is non-archimedean, the local measure $dg_{0,v}$ of $G_{0,v}$ is the standard measure. In particular, the volume of the hyperspecial maximal compact subgroup $\mathcal{K}_v = \mathcal{K}_{0,v} = \mathrm{PGL}_2(\mathbb{Z}_v)$ is 1. For the real place, we choose a Haar measure as follows. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty = \mathcal{K}_{0,\infty} = \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(1))$ be a maximal compact subgroup of $G_0(\mathbb{R})$. We put $\mathcal{K}_\infty^0 = G_0(\mathbb{R})^0 \cap \mathcal{K}_\infty$. Then $G_0(\mathbb{R})^0 / \mathcal{K}_\infty^0$ can be identified with the upper-half plane $\mathfrak{H}_1$. Let $dk$ be the Haar measure on $\mathcal{K}_\infty^0$ with total volume 1. Then the Haar measure $dg_{0,\infty}$ on $G_0(\mathbb{R})^0$ is such that $dg_{0,\infty}/dk$ induces the measure $y^{-2}\, dx\, dy$ on $G_0(\mathbb{R})^0 / \mathcal{K}_\infty^0 \simeq \mathfrak{H}_1$. The Haar measure $dg_{0,\infty}$ can be naturally extended to $G_0(\mathbb{R})$. Let $G_0(\mathbb{R})^0 = AN\mathcal{K}_\infty^0$ be an Iwasawa decomposition, which induces a bijection $\mathfrak{H}_1 \simeq AN$. Let $X \subset AN$ be an image of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1$. Then there is a bijection

$$X \times \mathcal{K}_\infty^0 \times \prod_{v < \infty} \mathcal{K}_v \simeq G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}).$$

It follows that

$$\int_{G_0(\mathbb{Q}) \backslash G_0(\mathbb{A})} \prod_{v \leq \infty} dg_{0,v} = \mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1) = 2\xi(2).$$

Therefore we have $C_0 = \xi(2)^{-1} = 6\pi^{-1}$, where $C_0$ is the Haar measure constant.

Let $f_j \in S_{\kappa_j}(\mathrm{SL}_2(\mathbb{Z}))$ $(j = 1, 2, 3)$ be normalized Hecke eigenforms. We assume $\kappa_1 + \kappa_2 = \kappa_3$. We denote the automorphic form on $\mathrm{GL}_2(\mathbb{A})$ corresponding to $f_j$ by $\mathbf{f}_j$. Let $\tau_j$ be the irreducible automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ generated by $\mathbf{f}_j$. Note that $\varphi_1 = \mathbf{f}_1 \times \mathbf{f}_2$ induces a cusp form on $\mathrm{SO}(2, 2)(\mathbb{A})$ and its restriction to $\mathrm{SO}(2, 1)$ is $\mathbf{f}_1\mathbf{f}_2$. Put $\pi_1 = \tau_1 \boxtimes \tau_2$, $\pi_0 = \tau_3$ and $\varphi_0 = \mathbf{f}_3$. By the result of Watson [47], (see also Harris-Kudla [18]), we have

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) = 2^{2\kappa_3+2}\langle f_1 f_2, f_3\rangle^2.$$

It is well-known that $\Lambda(1, \tau_j, \mathrm{Ad}) = 2^{\kappa_j}\langle f_j, f_j\rangle$. Here $\langle\ ,\ \rangle$ is the usual Petersson inner product.

As both the Tamagawa numbers of $\mathrm{SO}(2, 2)$ and $\mathrm{SO}(2, 1)$ are equal to 2, we have

$$\begin{aligned}\frac{|\langle\varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle\varphi_1, \varphi_1\rangle\langle\varphi_0, \varphi_0\rangle} &= 2\xi(2)\frac{|\langle f_1 f_2, f_3\rangle|^2}{\prod_{j=1}^3\langle f_j, f_j\rangle} \\ &= \frac{1}{2}\xi(2)\frac{\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3)}{\prod_{j=1}^3 \Lambda(1, \tau_j, \mathrm{Ad})}.\end{aligned}$$

By easy calculation,

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{\Lambda(s, \tau_1 \times \tau_2 \times \tau_3)}{\prod_{j=1}^3 \Lambda(s + (1/2), \tau_j, \mathrm{Ad})},$$

$$\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2) = \frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3 - 1)}{\Gamma_{\mathbb{R}}(2)^3\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3)} = \frac{2\pi^3}{\kappa_3 - 1}.$$

**Proposition 7.2.** *Let $\tau_{j,\infty}$ $(j = 1, 2, 3)$ be the holomorphic discrete series of $\mathrm{SO}(2, 1) \simeq \mathrm{PGL}_2(\mathbb{R})$ with lowest weight $\pm\kappa_j$. Put $\pi_{1,\infty} = \tau_{1,\infty} \boxtimes \tau_{2,\infty}$ and $\pi_{0,\infty} = \tau_{3,\infty}$. Let $\varphi_{1,\infty} \in \pi_{1,\infty}$ be the vector with weight $(\kappa_1, \kappa_2)$. Let $\varphi_{0,\infty} \in \pi_{0,\infty}$ be the vector with weight $\kappa_3$. We assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$. Then we have*

$$\begin{aligned}I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 4\pi(\kappa_3 - 1),\\ \alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 2.\end{aligned}$$

The proof of this proposition will be given in §12. Putting together, we recover Theorem 7.1 in this case. Note that we have $|\mathfrak{X}_\tau| = 1$.

In fact, Watson [47] obtained a more general result. Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$. The reduced discriminant $d_B$ of $B$ is, by definition, the product of primes which ramify in $B$. Let $N$ be a square-free integer such that $(N, d_B) = 1$. Put $S_{\mathrm{f}}$ be the set of primes which divide $d_B N$. Let $\tau_j = \otimes_v \tau_{j,v}$ $(j = 1, 2, 3)$ be an irreducible cuspidal automorphic representation of $\mathbb{A}^\times\backslash B^\times(\mathbb{A})$ with new vector $f_j = \otimes_v f_{j,v}$ which satisfies the following conditions:

(1) When $v < \infty$ and $v \notin S_{\mathrm{f}}$, the local components $\tau_{j,v}$ $(j = 1, 2, 3)$ are unramified representations and $f_{j,v}$ are unramified vectors.
(2) When $v \mid d_B$, the local component $\tau_{j,v}$ $(j = 1, 2, 3)$ are one-dimensional representations of the form $\chi_j \circ \nu_{B_v}$, where $\chi_j$ are unramified quadratic characters and $\nu_{B_v}$ is the reduced norm. We also assume $\chi_1\chi_2\chi_3 = 1$.
(3) When $v \mid N$, the local component $\tau_{j,v}$ $(j = 1, 2, 3)$ are representations of the form $\chi_j \otimes$ (Steinberg), where $\chi_j$ are unramified quadratic characters. We assume that $\chi_1\chi_2\chi_3$ is the unique unramified character of order 2 and that $f_{j,v}$ are Iwahori fixed vectors.
(4) When $v = \infty$, we assume that $\tau_{j,v}$ $(j = 1, 2, 3)$ are discrete series representations with minimal weight $\pm\kappa_j$. We assume $\kappa_3 = \kappa_1 + \kappa_2$ and $f_{j,v}$ have weight $\kappa_j > 0$.

Then Watson's result ([47] Theorem 3) says

$$\frac{|\int_X f_1(z)f_2(z)\overline{f_3(z)}\mathrm{Im}(z)^{\kappa_3-2}\,dz|^2}{\prod_{j=1}^3 \int_X |f_j(z)|^2\mathrm{Im}(z)^{\kappa_j-2}\,dz} = \frac{2^{|S_{\mathrm{f}}|-2}}{(d_B N)^2}\frac{\Lambda(1/2, \tau_1\times\tau_2\times\tau_3)}{\prod_{j=1}^3 \Lambda(1, \tau_j, \mathrm{Ad})}.$$

Here, $X = \mathcal{O}^{(1)}(d_B, N)\backslash\mathfrak{H}_1$, where $\mathcal{O}^{(1)}(d_B, N)$ is the arithmetic subgroup defined in Watson [47], Ch. 1. Watson proved that

$$\mathrm{Vol}(X) = 2\xi(2)\prod_{p\mid d_B}(p-1)\prod_{p\mid N}(p+1).$$

Watson also considered the cases when $\tau_{j,\infty}$ are not discrete series, but we do not discuss such cases.

Let $V_1$ be the vector space $B$ equipped with the reduced norm form $\nu_B$. The subspace $V_0 \subset V_1$ is defined by the space of elements of reduced trace 0. Then we have

$$G_1 = \{(g_1, g_2) \in B^\times \times B^\times \mid \nu_B(g_1) = \nu_B(g_2)\}/\mathbb{Q}^\times,$$
$$G_0 = B^\times/\mathbb{Q}^\times.$$

As in the case of $\mathrm{SO}(2,2)$, we regard $\pi_1 = \tau_1 \boxtimes \tau_2$ as a representation of $G_1(\mathbb{A})$, and $\pi_0 = \tau_3$ as a representation of $G_0(\mathbb{A})$. We put $\varphi_1 = f_1 \times f_2$, and $\varphi_0 = f_3$. We may assume $\|\varphi_{1,v}\| = \|\varphi_{0,v}\| = 1$ for any $v$. Note that

Watson's result implies

$$\begin{aligned}\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} &= \mathrm{Vol}(X)\frac{|\int_X f_1(z)f_2(z)\overline{f_3(z)}\mathrm{Im}(z)^{\kappa_3-2}\,dz|^2}{\prod_{j=1}^3\int_X |f_j(z)|^2\mathrm{Im}(z)^{\kappa_j-2}\,dz}\\ &= 2^{-1}\xi(2)\mathcal{P}_{\pi_1,\pi_0}(1/2)\\ &\quad\times\prod_{p|d_B}((2p^{-1}(1-p^{-1}))\prod_{p|N}(2p^{-1}(1+p^{-1})).\end{aligned}$$

We describe local calculations below. Since $G_0$ is an inner form of $\mathrm{PGL}_2$, we can transfer the local measure of $\mathrm{PGL}_2(\mathbb{Q}_v)$ to $G_{0,v} = B^\times(\mathbb{Q}_v)/\mathbb{Q}_v^\times$. Note that $\Delta_{G_{1,v}} = \zeta_v(2)^2$ and $C_0 = 6\pi^{-1}$ are unchanged. When $p \mid d_B$, we have

$$\begin{aligned}\mathrm{Vol}(G_{0,p}) = I(\varphi_{1,p},\varphi_{0,p}) &= 2p^{-1}(1-p^{-1})^{-1},\\ \mathcal{P}_{\pi_{1,p},\pi_{0,p}}(1/2) &= \zeta_p(1)^2\zeta_p(2)^{-2}.\end{aligned}$$

It follows that $\alpha_p(\varphi_{1,p},\varphi_{0,p}) = 2p^{-1}(1-p^{-1})$ for $p \mid d_B$. When $p \mid N$, let $\varepsilon_p$ be the unique unramified character of $\mathbb{Q}_p^\times$ of order 2. Then we have

$$\begin{aligned}\mathcal{P}_{\pi_{1,p},\pi_{0,p}}(1/2) &= L(1,\varepsilon_p)^2L(2,\varepsilon_p)\zeta_p(2)^{-3}\\ &= (1+p^{-1})^{-2}(1+p^{-2})^{-1}(1-p^{-2})^3.\end{aligned}$$

The integral $I(\varphi_{1,p},\varphi_{0,p})$ can be calculated as follows (cf. Godement and Jacquet [10] §7). The image of $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is denoted by $\begin{bmatrix} a & b\\ c & d\end{bmatrix}$. Let

$$I = \left\{ \begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{PGL}_2(\mathbb{Q}_p) \,\middle|\, a,\, b,\, d\in\mathbb{Z}_p,\, c\in p\mathbb{Z}_p\right\}$$

be an Iwahori subgroup of $G_{0,p} = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let $W_a$ be the affine Weyl group generated by $w_1 = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & p^{-1}\\ p & 0\end{bmatrix}$. The extended affine Weyl group $\tilde{W}$ is defined by $\tilde{W} = W_a \rtimes \Omega$, where $\Omega$ is the group of order 2 generated by $\omega = \begin{bmatrix} 0 & 1\\ p & 0\end{bmatrix}$. Then we have a Bruhat decomposition $G_{0,p} = \coprod_{w\in\tilde{W}} IwI$. The extended Weyl group $\tilde{W}$ has a length function $l(w)$ such that $l(w_1) = l(w_2) = 1$, $l(\omega) = 0$. The Poincaré series $\sum_{w\in W_a} t^{l(w)}$ is equal to $(1+t)(1-t)^{-1}$. Then the function

$$\Phi(b_1\omega^j wb_2) = (-1)^j(-p^{-1})^{l(w)},\quad b_1,\, b_2\in I,\; j\in\{0,1\},\; w\in W_a$$

is a bi-$I$-invariant matrix coefficient of the Steinberg representation of $G_0$. From this, we have

$$\begin{aligned} I(\varphi_{1,p},\varphi_{0,p}) &= \sum_{j=0}^{1}(-1)^j \sum_{w\in W_a} \mathrm{Vol}(I\omega^j wI)\Phi(\omega^j w)^3 \\ &= 2(p+1)^{-1}\sum_{w\in W_a}(-p^{-2})^{l(w)} \\ &= 2p^{-1}(1-p^{-1})(1+p^{-2})^{-1}. \end{aligned}$$

Note that

$$\mathrm{Vol}(IwI) = (1+p)^{-1}p^{l(w)}, \quad w\in\tilde{W}.$$

It follows that $\alpha_p(\varphi_{1,p},\varphi_{0,p}) = 2p^{-1}(1+p^{-1})$ for $p \mid N$.

Putting together, we recover Theorem 7.1 in this case. Note that we have $|\mathfrak{X}_\tau| = 1$, since the Steinberg representation does not come from a quadratic field.

We remark that Theorem 7.1 is compatible with algebraicity results for the triple product $L$-functions. For $j = 1,2,3$, let $f_j$ be a primitive cusp form with weight $\kappa_j$, level $N_j$, and character $\varepsilon_j$. We assume that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and $\kappa_1 \le \kappa_2 \le \kappa_3$. We denote by $\tau_j$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $f_j$.

We use the symbol $a \sim b$ for $a$, $b\in\mathbb{C}$, which means that $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$. It is well-known that $\Lambda(1,\tau_j,\mathrm{Ad}) \sim \langle f_j, f_j\rangle$. Then Harris-Kudla [18] proved that

$$\Lambda(1/2,\tau_1\times\tau_2\times\tau_3) \sim p(f_1,f_2,f_3),$$

where

$$p(f_1,f_2,f_3) = \begin{cases} \langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle & \text{if } \kappa_3 < \kappa_1+\kappa_2 \\ \langle f_3,f_3\rangle^2 & \text{if } \kappa_3 \ge \kappa_1+\kappa_2. \end{cases}$$

We assume $\Lambda(1/2,\tau_1\times\tau_2\times\tau_3) \neq 0$. They also proved the Jacquet conjecture which states that there exist a unique quaternion algebra $D$ and some automorphic forms $F_j^D \in \tau_i^D$ such that

$$\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})} F_1^D(g)F_2^D(g)F_3^D(g)\,dg \neq 0.$$

Here $\tau_j^D$ is the Jacquet-Langlands-Shimizu correspondence of $\tau_j$. Assume that $\varepsilon_1\varepsilon_2 = \varepsilon_3 = 1$ and $F_j^D \in \tau_j^D$. Then $\varphi_0 = F_3^D$ can be regarded as an automorphic form on $G_0 = D^\times/\mathbb{Q}^\times$ and $\varphi_1 = F_1^D \times F_2^D$ can be regarded as an automorphic form on

$$G_1 = \{(d_1,d_2)\in D^\times\times D^\times \mid \nu(d_1)=\nu(d_2)\}/\mathbb{Q}^\times.$$

Here $\nu$ is the reduced norm of $D$. As before, we transfer the Haar measure $dg_v$ on $\mathrm{GL}_2(\mathbb{Q}_v)$ to $G_0(\mathbb{Q}_v)$. In particular, $C_0 = 6/\pi$.

For each finite prime $p$, the component $\pi_p$ has a $\bar{\mathbb{Q}}$-structure. Note that for $\bar{\mathbb{Q}}$-rational vectors $\varphi_{1,p}$ and $\varphi_{0,p}$, the quantity $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) \in \bar{\mathbb{Q}}$.

In the balanced case $\kappa_3 < \kappa_1 + \kappa_2$, the quaternion algebra $D$ is definite. We choose arithmetic automorphic forms $F_j^D \in \tau_j^D$. Then we have

$$\langle \varphi_1, \varphi_1 \rangle,\ \langle \varphi_0, \varphi_0 \rangle \in \bar{\mathbb{Q}}^\times, \quad \langle \varphi_1|_{G_0}, \varphi_0 \rangle \in \bar{\mathbb{Q}}.$$

Note that in this case we have

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)^{-1} \cdot \mathrm{Vol}(G_0(\mathbb{R})) \sim \pi^{-1}.$$

Note that $\mathrm{Vol}(G_0(\mathbb{R})) = \mathrm{Vol}(\mathrm{U}(2)/(\mathrm{U}(1) \times \mathrm{U}(1))) \sim \pi$. Therefore in this case Theorem 7.1 is compatible with the known result

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle.$$

Now we consider the unbalanced case $\kappa_3 \geq \kappa_1 + \kappa_2$. We choose arithmetic holomorphic automorphic form $F_3^D \in \tau_3^D$ of weight $\kappa_3$ and arithmetic nearly anti-holomorphic forms $F_1^D \in \tau_1^D$ and $F_2^D \in \tau_2^D$ with some weight. Then we have (see Shimura [38])

$$\begin{aligned} \langle \varphi_0, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle, \\ \langle \varphi_1, \varphi_1 \rangle &\sim \xi(2)^{-2} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle, \\ \langle \varphi_1|_{G_0}, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle. \end{aligned}$$

Note that in this case, we have $\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim 1$. Therefore in this case Theorem 7.1 is compatible with the known result

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_3, f_3 \rangle^2.$$

*Remark* 7.3. More generally, Theorem 7.1 is compatible with Shimura's conjecture [39], [40] for Hilbert modular forms, which was proved by Harris [15], [16], [17], and Yoshida [49] in most cases.

## 8. Restriction of the Yoshida lift to the diagonal subgroup

In this section, we recall the result of Gan and Ichino [8], in which a formula for the restriction of the Yoshida lift [48] to the diagonal subgroup by Böcherer, Furusawa, Schulze-Pillot [3] has been generalized. They have proved Conjecture 1.5 for $n = 4$ in some cases and given strong evidence for Conjecture 2.1.

Let $k$ be a totally real algebraic number field. Let $k'$ be either $k \times k$ or a totally real quadratic extension of $k$. We put

$$G_1 = \mathrm{PGSp}_2,$$
$$\tilde{G}_0 = \mathrm{GL}_2(k')/k^\times,$$
$$G_0 = \{g \in \mathrm{GL}_2(k') \mid \det g \in k^\times\}/k^\times.$$

Then $G_1$ (resp. $G_0$) can be considered as a special orthogonal group associated to a 5-dimensional (resp. 4-dimensional) quadratic space over $k$. We regard $G_0$ as a subgroup of $G_1$. Note that $\Delta_{G_1} = \xi_k(2)\xi_k(4)$.

Let $(V, Q)$ be another 4-dimensional quadratic space over $k$ with discriminant field $K_Q$. We put $H = \mathrm{GO}_Q$ and

$$k'' = \begin{cases} k \times k & \text{if } K_Q = k, \\ K_Q & \text{if } [K_Q : k] = 2. \end{cases}$$

Then there exists a quaternion algebra $D$ over $k$ such that

$$1 \longrightarrow k''^\times \longrightarrow (D \otimes_k k'')^\times \times k^\times \longrightarrow H^0 \longrightarrow 1$$

(cf. e.g., Roberts [37] §2). Here, $H^0$ is the identity component of $H$.

Let $\sigma$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ with trivial central character. We assume the following conditions:

- The Jacquet-Langlands lift of $\sigma|_{D^\times(\mathbb{A}_{k''})}$ to $\mathrm{GL}_2(\mathbb{A}_{k''})$ is cuspidal.
- $\sigma_v \otimes \mathrm{sgn} \simeq \sigma_v$ for some $v$.
- If $\sigma_v \otimes \mathrm{sgn} \not\simeq \sigma_v$, then $\sigma_v \not\simeq \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of $H_v^0$ (cf. [8], Definition 5.4).

Let $\pi_1$ be the theta lift of $\sigma$ to $G_1(\mathbb{A})$. Note that $\pi_1$ is a non-zero irreducible cuspidal automorphic representation of $G_1(\mathbb{A})$. This theta lift was first considered by Yoshida [48] in a certain case. Later, it was considered by Howe and Piatetski-Shapiro [23], Böcherer and Schulze-Pillot [4], Harris, Soudry, and Taylor [19], Roberts [37] more generally. For this reason, we call $\pi_1$ the Yoshida lift of $\sigma$.

Let $\pi_0$ be an irreducible cuspidal automorphic representation of $G_0(\mathbb{A})$. As in §7, we choose an irreducible unitary cuspidal automorphic representation $\tau$ of $\tilde{G}_0(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_{k'})/\mathbb{A}^\times$ such that $\mathcal{V}_{\pi_0} \subset \mathcal{V}_\tau^1|_{G_0(\mathbb{A})}$. We assume the following conditions:

- The base change $\mathcal{BC}(\tau)$ of $\tau$ to $\tilde{G}_0(\mathbb{A}_{k''}) = \mathrm{GL}_2(\mathbb{A}_{k' \otimes_k k''})/\mathbb{A}_{k''}^\times$ is cuspidal.
- The Jacquet-Langlands lift of $\mathcal{BC}(\tau)$ to $D^\times(\mathbb{A}_{k' \otimes_k k''})/\mathbb{A}_{k''}^\times$ exists.

Then Theorem 1.1 of [8] says

$$\frac{|\langle \varphi_1|_{G_0(\mathbb{A})}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = \frac{1}{2^{\beta'}|\mathfrak{X}_\tau|}\Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2)\prod_{v\in S}\frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2\cdot\|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$. Here,

$$\beta' = \begin{cases} 3 & \text{if } K_Q = k, \\ 2 & \text{if } [K_Q : k] = 2, \end{cases}$$

and $\mathfrak{X}_\tau$ is the elementary 2-group as in §7. Thus Conjecture 1.5 is true in this case. Note that we have

$$|\mathcal{S}_{\psi_1}| = \begin{cases} 4 & \text{if } K_Q = k, \\ 2 & \text{if } [K_Q : k] = 2, \end{cases}$$

and $|\mathcal{S}_{\psi_0}| = 2|\mathfrak{X}_\tau|$, if we admit the Arthur conjecture.

## 9. Restriction of the Saito-Kurokawa lift to the diagonal subset $\mathfrak{H}_1 \times \mathfrak{H}_1$

Let $\kappa > 0$ be an odd integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. We denote the Kohnen plus subspace by $S^+_{\kappa+(1/2)}(\Gamma_0(4)) \subset S_{\kappa+(1/2)}(\Gamma_0(4))$ (cf. Kohnen [30]). Let $h \in S^+_{\kappa+(1/2)}(\Gamma_0(4))$ be a Hecke eigenform associated to $f$ by Shimura correspondence. Let $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of $h$. Let $\tau$ and $\sigma$ be the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $f$ and $g$, respectively. Then it is shown in Ichino [24] that

$$\Lambda(1/2, \mathrm{Ad}(\sigma)\boxtimes\tau) = 2^{\kappa+1}\frac{\langle f, f\rangle}{\langle h, h\rangle}\frac{|\langle \mathcal{F}|_{\mathfrak{H}_1\times\mathfrak{H}_1}, g\times g\rangle|^2}{\langle g, g\rangle^2}.$$

Here, $\langle\ ,\ \rangle$ is the usual Petersson inner product on $\mathfrak{H}_1$. We interpret this result in terms of automorphic representations. Let $\varphi_1$ be the automorphic form on $G_1(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(3,2)(\mathbb{A}_\mathbb{Q})$ corresponding to $\mathcal{F}$. Similarly, let $\varphi_0$ be the automorphic form on $G_0(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(2,2)(\mathbb{A}_\mathbb{Q})$ corresponding to $g \times g$. As in §7, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$ for $v < \infty$. Let $G_0(\mathbb{R})^0$ be the topological identity component of $G_0(\mathbb{R})$. The maximal compact subgroup $\mathcal{K}^0_\infty$ of $G_0(\mathbb{R})^0$ is defined by $\mathcal{K}^0_\infty = \mathrm{SO}(2)\times\mathrm{SO}(2)$. Let $dg_{0,\infty}$ be the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(y_1y_2)^{-2}\,dx_1\,dx_2\,dy_1\,dy_2$ on $G_0(\mathbb{R})^0/\mathcal{K}^0_\infty \simeq \mathfrak{H}_1\times\mathfrak{H}_1$. Here, $dk$ is the Haar measure on $\mathcal{K}^0_\infty$ with total measure 1. The Haar measure $dg_{0,\infty}$ can be naturally extended to $G_0(\mathbb{R})$. We calculate the Haar measure constant $C_0$. Let $G_0(\mathbb{R})^0 = AN\mathcal{K}^0_\infty$ be an Iwasawa decomposition, and $X \subset AN$ be a set bijective to a fundamental domain for $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1)^2$. Then each element of

$G_0(\mathbb{Q})\backslash G_0(\mathbb{A})$ has exactly two representatives in $X \times \mathcal{K}_\infty^0 \times \prod_{v<\infty} \mathcal{K}_{0,v}$. It follows that

$$\int_{G_0(\mathbb{Q})\backslash G_0(\mathbb{A})} \prod_{v\le\infty} dg_{0,v} = \frac{1}{2}\mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1)^2 = 2\xi(2)^2.$$

Therefore we have $C_0 = \xi(2)^{-2} = 36\pi^{-2}$. Note that $\Delta_{G_1} = \xi(2)\xi(4)$. Note also that the volume of $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathfrak{H}_2$ is $2\xi(2)\xi(4)$, where $\mathfrak{H}_2$ is the Siegel upper-half space of genus 2. It follows that

$$\begin{aligned} \langle\varphi_1,\varphi_1\rangle &= \frac{\langle\mathcal{F},\mathcal{F}\rangle}{\xi(2)\xi(4)}, \\ \langle\varphi_0,\varphi_0\rangle &= \frac{\langle g,g\rangle^2}{2\xi(2)^2}, \\ \frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} &= \frac{\xi(4)}{2\xi(2)}\frac{|\langle\mathcal{F}|_{\mathfrak{h}_1\times\mathfrak{h}_1}, g\times g\rangle|^2}{\langle\mathcal{F},\mathcal{F}\rangle\langle g,g\rangle^2}. \end{aligned}$$

As noticed in §7, it is well-known that $\langle f,f\rangle = 2^{-2\kappa}\Lambda(1,\mathrm{Ad}(\tau))$. By Kohnen-Skoruppa [31], we have

$$\frac{\langle\mathcal{F},\mathcal{F}\rangle}{\langle h,h\rangle} = 2^{\kappa-2}\pi^{-1}\xi(2)\Lambda(3/2,\tau).$$

(Note that there is a minor error in the unfolding argument of [31], p. 547. Since the action of the center of $\mathrm{Sp}_2(\mathbb{Z})$ on $\mathfrak{H}_2$ is trivial, the right hand side of the equation of [31] p. 547, line 23 must be multiplied by 2.) It follows that

$$\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} = \pi\cdot\frac{\xi(4)}{\xi(2)}\cdot\frac{\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\tau))}.$$

It is easy to check that

$$\begin{aligned} \Lambda(s,\pi_0) &= \Lambda(s,\mathrm{Ad}(\sigma))\xi(s), \\ \Lambda(s,\pi_1) &= \Lambda(s,\tau)\xi(s+(1/2))\xi(s-(1/2)), \\ \Lambda(s,\pi_0,\mathrm{Ad}) &= \Lambda(s,\mathrm{Ad}(\sigma))^2, \\ \Lambda(s,\pi_1,\mathrm{Ad}) &= \Lambda(s,\mathrm{Ad}(\tau))\Lambda(s+(1/2),\tau)\Lambda(s-(1/2),\tau) \\ &\qquad \times\xi(s+1)\xi(s)\xi(s-1). \end{aligned}$$

From this, one can show that $\mathcal{P}_{\pi_1,\pi_0}(s)$ is equal to

$$\frac{\Lambda(s-(1/2),\mathrm{Ad}(\sigma))\Lambda(s,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(s+(3/2))\Lambda(s+1,\tau)\Lambda(s+(1/2),\mathrm{Ad}(\sigma))\Lambda(s+(1/2),\mathrm{Ad}(\tau))}.$$

It follows that

$$\mathcal{P}_{\pi_1,\pi_0}(1/2) = \frac{\Lambda(0,\mathrm{Ad}(\sigma))\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\sigma))\Lambda(1,\mathrm{Ad}(\tau))}$$
$$= \frac{\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\tau))}.$$

Observe that

$$\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2) = \frac{\Gamma_{\mathbb{R}}(1)\Gamma_{\mathbb{C}}(\kappa)\cdot\Gamma_{\mathbb{C}}(\kappa)\Gamma_{\mathbb{C}}(2\kappa)\Gamma_{\mathbb{C}}(1)}{\Gamma_{\mathbb{R}}(2)\cdot\Gamma_{\mathbb{C}}(\kappa+1)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa+1)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2\kappa)}$$
$$= 4\kappa^{-2}\pi^4.$$

**Proposition 9.1.** *Let $\pi_{1,\infty}$ be the irreducible holomorphic discrete series representation of* $\mathrm{SO}(3,2)$ *with lowest $K$-type* $(\det)^{\pm(\kappa+1)}$. *Let $\pi_{0,\infty}$ be the irreducible discrete series representation of* $\mathrm{SO}(2,2)$ *with lowest $K$-type $\pm(\kappa+1,\kappa+1)$. Choose lowest weight vectors $\varphi_{1,\infty}\in\pi_{1,\infty}$ and $\varphi_{0,\infty}\in\pi_{0,\infty}$ such that $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$. Then we have*

$$I(\varphi_{1,\infty},\varphi_{0,\infty}) = 16\kappa^{-2}\pi^2,$$
$$\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty}) = 4\pi.$$

The proof of Proposition 9.1 will be given in §12. Using Proposition 9.1, we have

$$\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} = \frac{1}{4}\Delta_{G_1}C_0\mathcal{P}_{\pi_1,\pi_0}(1/2)\cdot\frac{\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2\cdot\|\varphi_{0,\infty}\|^2}.$$

Therefore in this case, it seems Conjecture 3.2 holds with $2^\beta = 1/4$. Note that we have $|\mathcal{S}_{\psi_1}| = 4$ and $|\mathcal{S}_{\psi_0}| = 2$, and hence $2^\beta \neq 1/(|\mathcal{S}_{\psi_1}|\cdot|\mathcal{S}_{\psi_0}|)$, if we admit the Arthur conjecture.

*Remark* 9.2. Now choose another normalized Hecke eigenform $g'\in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ such that $g\neq g'$. Let $\sigma'$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $g'$. Let $\varphi_1$ be as before and $\varphi_0$ the lifting of $g\times g'$ to $G_0(\mathbb{A})$. Then we have $\langle\varphi_1|_{G_0},\varphi_0\rangle = 0$. Note that $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v}\otimes\bar\pi_{0,v},\mathbb{C}) = \{0\}$ for some $v$ (See e.g., [27] Proposition 3.1). After a little calculation, one can show the numerator of $\mathcal{P}_{\pi_1,\pi_0}(s)$ is equal to

$$\Lambda(s,\tau\times\sigma\times\sigma')\Lambda(s+(1/2),\sigma\times\sigma')\Lambda(s-(1/2),\sigma\times\sigma')$$

and the denominator is

$$\Lambda(s+(1/2),\mathrm{Ad}(\tau))\Lambda(s+1,\tau)\Lambda(s,\tau)$$
$$\times\,\xi(s+(3/2))\xi(s+(1/2))\xi(s-(1/2))$$
$$\times\,\Lambda(s+(1/2),\mathrm{Ad}(\sigma))\Lambda(s+(1/2),\mathrm{Ad}(\sigma')).$$

Note that as far as we know, any relation between $\mathrm{ord}_{s=1/2}\Lambda(s, \tau\times\sigma\times\sigma')$ and $\mathrm{ord}_{s=1/2}\Lambda(s,\tau)$ are not known, and so $\mathcal{P}_{\pi_1,\pi_0}(s)$ might have a pole at $s=1/2$. It seems this example suggests that there is no relation between the period $\langle\varphi_1|_{G_0},\varphi_0\rangle$ and the $L$-value $\mathcal{P}_{\pi_1,\pi_0}(1/2)$, when $\pi_1$ or $\pi_0$ is non-tempered and the condition $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v}\otimes\bar{\pi}_{0,v},\mathbb{C})\neq\{0\}$ fails. Note that when both $\pi_1$ and $\pi_0$ are tempered, Conjecture 1.5 still makes sense even if the condition $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v}\otimes\bar{\pi}_{0,v},\mathbb{C})\neq\{0\}$ fails, since it is believed that $\mathcal{P}_{\pi_1,\pi_0}(s)$ is holomorphic at $s=1/2$.

## 10. Restriction of the hermitian Maass lift to $\mathfrak{H}_2$

Now we discuss the case $n=5$ and $k=\mathbb{Q}$. We put $G_0=\mathrm{SO}(3,2)\simeq\mathrm{PGSp}_2$. Let $\kappa>0$ be an odd integer and $f\in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$, $h\in S^+_{\kappa+(1/2)}(\Gamma_0(4))$, $\mathcal{F}\in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$, and $\tau$ be as in §9. Let

$$h(\tau)=\sum_{\substack{n>0\\ -n\equiv 0,1\,(4)}} c(n)q^n$$

be the Fourier expansion of $h(\tau)$.

Let $K$ be an imaginary quadratic field with discriminant $-D$. We assume that $c(D)\neq 0$. We denote by $\chi$ and $w_K$ the associated Dirichlet character for $K/\mathbb{Q}$ and the number of units in $K$, respectively. We put $G_1=\mathrm{SO}(4,2)_{K/\mathbb{Q}}\simeq\mathrm{SU}(2,2)_{K/\mathbb{Q}}/\{\pm1\}$.

Now let $\Gamma_K=\mathrm{SU}(2,2)(\mathbb{Q})\cap\mathrm{GL}_4(\mathcal{O}_K)$ be the special hermitian modular group, where $\mathcal{O}_K$ is the integer ring of $K$.

By using the fact that the Tamagawa number of $\mathrm{SU}(2,2)$ is 1, one can show that the volume of the fundamental domain for $\Gamma_K$ is equal to

$$\mathrm{Vol}(\Gamma_K\backslash\mathcal{H}_2)=2^{-3}D^{5/2}(4,w_K)\xi(2)\Lambda(3,\chi)\xi(4),$$

where $\mathcal{H}_2$ is the hermitian upper-half space of degree 2. Here, we have given an invariant measure on $\mathcal{H}_2$ as follows. Put $X=(Z+{}^t\bar{Z})/2$, $Y=(Z-{}^t\bar{Z})/(2\sqrt{-1})$ for $Z\in\mathcal{H}_2$. The measure $dX$ on the space of hermitian matrices is defined by $dX=\prod_{i\leq j}dX^{(r)}_{ij}\prod_{i<j}dX^{(i)}_{ij}$, where $X=X^{(r)}+\sqrt{-1}X^{(i)}$, $X^{(r)}_{ij},X^{(i)}_{ij}\in\mathbb{R}$. Then the invariant measure is given by $(\det Z)^{-4}\,dX\,dY$. This calculation will be carried out in the appendix to this section.

Let $g\in S_\kappa(\Gamma_0(D),\chi)$ be a primitive form and $\mathcal{G}\in S_{\kappa+1}(\Gamma_K)$ the hermitian Maass lift of $g$ (cf. Kojima [33], Krieg [34], Ikeda [28]). We assume that $\mathcal{G}\neq 0$. Let $\rho$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $g$. By using Sugano [43], Corollary 8.3 and Ikeda [28] §15, we have

$$\langle\mathcal{G},\mathcal{G}\rangle=2^{-2\kappa-7}D^{\kappa+2}\pi^{-2}(4,w_K)\xi(2)\Lambda(2,\mathrm{Sym}^2(\rho))\Lambda(1,\mathrm{Ad}(\rho)).$$

One can prove this formula using Raghavan-Sengupta [36]. The main theorem of Ichino and Ikeda [26] says

$$|c(D)|^2 \frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F}\rangle|^2}{\langle \mathcal{F}, \mathcal{F}\rangle^2} = 2^{-4\kappa-2} D^{2\kappa-1} \frac{\Lambda(1/2, \rho \times \rho \times \tau)}{\langle f, f\rangle^2}.$$

Combining these result and the Kohnen-Zagier formula [32]

$$|c(D)|^2 \frac{\langle f, f\rangle}{\langle h, h\rangle} = 2^{\kappa-1} D^{\kappa-(1/2)} \Lambda(1/2, \tau \otimes \chi),$$

we have

$$\begin{aligned}\frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F}\rangle|^2}{\langle \mathcal{G}, \mathcal{G}\rangle \langle \mathcal{F}, \mathcal{F}\rangle} &= 2\pi \cdot \mathrm{Vol}(\Gamma_K \backslash \mathcal{H}_2)^{-1} \xi(2) \Lambda(3, \chi) \xi(4) \\ &\quad \times \frac{\Lambda(1/2, \mathrm{Sym}^2(\rho) \boxtimes \tau) \Lambda(3/2, \tau)}{\Lambda(2, \mathrm{Sym}^2(\rho)) \Lambda(1, \mathrm{Ad}(\rho)) \Lambda(1, \mathrm{Ad}(\tau))}.\end{aligned}$$

We translate these results to adelic language. Let $\varphi_1$ (resp. $\varphi_0$) be the automorphic form on $G_1(\mathbb{A})$ (resp. $G_0(\mathbb{A})$) corresponding to $\mathcal{G}$ (resp. $\mathcal{F}$). We put $S = S_{\mathrm{f}} \cup \{\infty\}$, where $S_{\mathrm{f}}$ is the set of primes which divide $D$. When $v < \infty$, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty^0 = \mathrm{SO}(3) \times \mathrm{SO}(2)$ be a maximal compact subgroup of $G_0(\mathbb{R})^0$. Let $dk$ be the Haar measure of $\mathcal{K}_\infty^0$ with the total measure 1, and $dg_{0,\infty}$ the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(\det Y)^{-3}\, dX\, dY$ on $\mathfrak{H}_2 \simeq G_0(\mathbb{R})^0/\mathcal{K}_\infty^0$. Then we have $\mathrm{Vol}(\mathrm{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) = \mathrm{Vol}(\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2) = 2\xi(2)\xi(4)$. Let $C_0$ be the Haar measure constant. It follows that $C_0 = \xi(2)^{-1}\xi(4)^{-1} = 540\pi^{-3}$, since there is a bijection $G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}) \simeq (\mathrm{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) \times \prod_{p<\infty} \mathcal{K}_{0,p}$. Note also that $\Delta_{G_1} = \xi(2)\Lambda(3, \chi)\xi(4)$.

Let $\pi_1$ (resp. $\pi_0$) be the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_{\mathbb{Q}})$ (resp. $G_0(\mathbb{A}_{\mathbb{Q}})$) generated by $\varphi_1$ (resp. $\varphi_0$). Note that both $\pi_1$ and $\pi_0$ are non-tempered. It is easy to check that

$$\begin{aligned}\Lambda(s, \pi_1) &= \Lambda(s, \mathrm{Sym}^2(\rho)) \xi(s+1) \xi(s) \xi(s-1), \\ \Lambda(s, \pi_0) &= \Lambda(s, \tau) \xi(s + (1/2)) \xi(s - (1/2)), \\ \Lambda(s, \pi_1, \mathrm{Ad}) &= \Lambda(s+1, \mathrm{Sym}^2(\rho)) \Lambda(s, \mathrm{Sym}^2(\rho)) \Lambda(s-1, \mathrm{Sym}^2(\rho)) \\ &\quad \times \Lambda(s, \mathrm{Ad}(\rho)) \xi(s+1) \xi(s) \xi(s-1), \\ \Lambda(s, \pi_0, \mathrm{Ad}) &= \Lambda(s, \mathrm{Ad}(\tau)) \Lambda(s + (1/2), \tau) \Lambda(s - (1/2), \tau) \\ &\quad \times \xi(s+1) \xi(s) \xi(s-1).\end{aligned}$$

It follows that $\mathcal{P}_{\pi_1,\pi_0}(s) = R(s)/Q(s)$, where

$$\begin{aligned} R(s) &= \Lambda(s, \mathrm{Sym}^2(\rho) \boxtimes \tau)\Lambda(s-1, \tau)\xi(s-(3/2)), \\ Q(s) &= \Lambda(s+(3/2), \mathrm{Sym}^2(\rho))\Lambda(s+(1/2), \mathrm{Ad}(\rho)) \\ &\qquad \times \Lambda(s+(1/2), \mathrm{Ad}(\tau))\xi(s+(3/2)). \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{P}_{\pi_1,\pi_0}(1/2) &= \frac{\Lambda(1/2, \mathrm{Sym}^2(\rho) \boxtimes \tau)\Lambda(-1/2, \tau)\xi(-1)}{\Lambda(2, \mathrm{Sym}^2(\rho))\Lambda(1, \mathrm{Ad}(\rho))\Lambda(1, \mathrm{Ad}(\tau))\xi(2)} \\ &= -\frac{\Lambda(1/2, \mathrm{Sym}^2(\rho) \boxtimes \tau)\Lambda(3/2, \tau)}{\Lambda(2, \mathrm{Sym}^2(\rho))\Lambda(1, \mathrm{Ad}(\rho))\Lambda(1, \mathrm{Ad}(\tau))} \end{aligned}$$

by the functional equations $\Lambda(1-s, \tau) = -\Lambda(s, \tau)$, $\xi(1-s) = \xi(s)$.

We consider the local factor $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$. For $v \notin S$, we may consider $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$. For $v \in S_{\mathrm{f}}$, the conditions (U1) and (U2) in §1 fail. Instead of (U1) and (U2), we consider the following conditions:

(U1′) $G_{i,v}$ is quasi-split.
(U2′) $\mathcal{K}_{i,v}$ is a special maximal compact subgroup of $G_{i,v}$.

**Lemma 10.1.** *Assume $n = 5$. Let $v$ be a non-archimedean place such that the conditions (U1′), (U2′), (U3), (U4), (U5), and (U6) hold. Then we have $I(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_{1,v}} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)$, if it is convergent.*

The authors have verified this lemma by using computer calculation. By this lemma we may consider $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$ by "analytic continuation".

For $v = \infty$, one can easily see that $\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)$ is equal to

$$\begin{aligned} &\frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa)\Gamma_{\mathbb{C}}(2\kappa-1) \cdot \Gamma_{\mathbb{C}}(\kappa-1) \cdot \Gamma_{\mathbb{R}}(-1)}{\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa+1) \cdot \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa) \cdot \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2\kappa) \cdot \Gamma_{\mathbb{R}}(2)} \\ &= -\frac{16\pi^7}{\kappa(\kappa-1)(2\kappa-1)}. \end{aligned}$$

Note that $\pi_{1,\infty}$ is a discrete series representation of $\mathrm{SO}(4,2)$, and the $K$-type of $\varphi_{1,\infty}$ is the lowest $K$-type. Similarly, $\pi_{0,\infty}$ is a discrete series representation of $\mathrm{SO}(3,2)$, and $\varphi_{0,\infty}$ is a lowest $K$-type vector. We may assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$.

**Proposition 10.2.** *We have*

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= \frac{64\pi^3}{\kappa(\kappa-1)(2\kappa-1)}, \\ \alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) &= -4\pi. \end{aligned}$$

A proof of Proposition 10.2 will be given in §12.

By Proposition 10.2, we have

$$\begin{aligned}\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} &= \frac{1}{4}\Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2)\cdot \frac{\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2\cdot\|\varphi_{0,\infty}\|^2} \\ &= \frac{1}{4}\Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2)\prod_{v\in S}\frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2\cdot\|\varphi_{0,v}\|^2}\end{aligned}$$

under the assumption $c(D)\neq 0$. Therefore in this case, Conjecture 3.2 seems to hold with $2^\beta = 1/4$. Note that we have $|\mathcal{S}_{\psi_1}| = 2$ and $|\mathcal{S}_{\psi_0}| = 4$, and hence $2^\beta \neq 1/(|\mathcal{S}_{\psi_1}|\cdot|\mathcal{S}_{\psi_0}|)$, if we admit the Arthur conjecture.

**Appendix to §10: Calculation of the volume of the fundamental domain for $\Gamma_K\backslash\mathcal{H}_2$.**

In this appendix, we calculate the volume of the fundamental domain for the hermitian modular group. Let $K=\mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$. We put $K_p = K\otimes\mathbb{Q}_p$ and $\mathcal{O}_p = \mathcal{O}_K\otimes\mathbb{Z}_p$, where $\mathcal{O}_K$ is the integer ring of $K$.

Let $\Gamma_K^{(n)} = \mathrm{SU}(n,n)(\mathbb{Q})\cap\mathrm{GL}_{2n}(\mathcal{O}_K)$ be the special hermitian modular group. By using the fact that the Tamagawa number of $\mathrm{SU}(n,n)$ is 1, we shall show that

$$\mathrm{Vol}(\Gamma_K^{(n)}\backslash\mathcal{H}_n) = 2^{-n^2+1}D^{(2n^2-n-1)/2}(2n, w_K)\prod_{i=2}^{2n}\Lambda(i,\chi^i),$$

where $\mathcal{H}_n$ is the hermitian upper half space of degree $n$.

Put $\mathfrak{G} = \mathrm{SU}(n,n)$. Then

$$\mathrm{Lie}(\mathfrak{G}) = \{X\in \mathrm{M}_{2n}(K)\,|\,XJ + J{}^t\bar{X} = 0,\ \mathrm{tr}(X)=0\},$$

where $J=\begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & 0\end{pmatrix}$. We choose a basis of the $\mathrm{Lie}(\mathfrak{G})$ as follows. Let $E[i,j]\in\mathrm{M}_n(\mathbb{Z})$ be the $(i,j)$-elementary matrix of size $n$. Set

$$S[i,j] = \begin{cases} E[i,i] & (i=j),\\ E[i,j]+E[j,i] & (i\neq j),\end{cases}$$
$$A[i,j] = E[i,j]-E[j,i].$$

Put

$$X_{ij} = \begin{pmatrix} E[i,j] & 0 \\ 0 & -E[j,i] \end{pmatrix},$$
$$Y_{ij} = \begin{pmatrix} 0 & S[i,j] \\ 0 & 0 \end{pmatrix},$$
$$Y'_{ij} = \begin{pmatrix} 0 & 0 \\ S[i,j] & 0 \end{pmatrix},$$
$$V_{ij} = \sqrt{-D} \begin{pmatrix} 0 & A[i,j] \\ 0 & 0 \end{pmatrix},$$
$$V'_{ij} = -\sqrt{-D} \begin{pmatrix} 0 & 0 \\ A[i,j] & 0 \end{pmatrix},$$
$$W_{ij} = \sqrt{-D} \begin{pmatrix} E[i,j] & 0 \\ 0 & E[j,i] \end{pmatrix},$$
$$W'_{i} = \sqrt{-D} \begin{pmatrix} E[i,i]-E[i+1,i+1] & 0 \\ 0 & E[i,i]-E[i+1,i+1] \end{pmatrix}.$$

The following vectors make up a basis of $\mathrm{Lie}(\mathfrak{G})$.

$$\begin{aligned} X_{ij} &\quad (1 \le i,j \le n), \\ Y_{ij} &\quad (1 \le i \le j \le n), \\ Y'_{ij} &\quad (1 \le i \le j \le n), \\ V_{ij} &\quad (1 \le i < j \le n), \\ V'_{ij} &\quad (1 \le i < j \le n), \\ W_{ij} &\quad (1 \le i < j \le n), \\ W'_{i} &\quad (1 \le i < n). \end{aligned}$$

Let $\mathfrak{L} \subset \mathrm{Lie}(\mathfrak{G})$ be the lattice generated by this basis. This basis determines a Haar measure $dg_v$ on $\mathfrak{G}(\mathbb{Q}_v)$ for each place $v$, and the product measure $\prod_v dg_v$ is the Tamagawa measure on $\mathfrak{G}(\mathbb{A})$. For each prime $p$, we define a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_p}$ of $\mathfrak{G}(\mathbb{Q}_p)$ by $\mathcal{K}_{\mathfrak{G}_p} = \mathfrak{G}(\mathbb{Q}_p) \cap \mathrm{GL}_{2n}(\mathcal{O}_p)$. Since $[\mathcal{O}_p : \mathbb{Z}_p + \sqrt{-D}\mathbb{Z}_p] = (2,p)$, we have

$$[\mathrm{Lie}(\mathfrak{G})(\mathbb{Q}_p) \cap \mathrm{M}_{2n}(\mathcal{O}_p) : \mathfrak{L} \otimes \mathbb{Z}_p] = (2,p)^{2n^2-n-1}.$$

It follows that the volume of $\mathcal{K}_{\mathfrak{G}_p}$ is equal to $(2,p)^{2n^2-n-1} \prod_{i=2}^{2n} L(i, \chi_p^i)^{-1}$.

For the real place, the vectors

$$\begin{aligned} X_{ij} - X_{ji} &\quad (1 \le i < j \le n), \\ Y_{ij} - Y'_{ij} &\quad (1 \le i \le j \le n), \\ V_{ij} + V'_{ij} &\quad (1 \le i < j \le n), \\ W_{ij} + W_{ji} &\quad (1 \le i < j \le n), \\ W'_i &\quad (1 \le i < n) \end{aligned}$$

generate the Lie algebra of a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ of $\mathfrak{G}(\mathbb{R})$. The maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ is isomorphic to

$$\{(u_1, u_2) \in \mathrm{U}(n) \times \mathrm{U}(n) \mid \det u_1 \cdot \det u_2 = 1\}.$$

This isomorphism is explicitly given by $\mathrm{Ad}(A) : \kappa \mapsto A\kappa A^{-1}$, where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_n & -\sqrt{-1} \cdot \mathbf{1}_n \\ \mathbf{1}_n & \sqrt{-1} \cdot \mathbf{1}_n \end{pmatrix}.$$

Note that

$$\begin{aligned} \mathrm{Ad}(A)(X_{ij} - X_{ji}) &= \begin{pmatrix} A[i,j] & 0 \\ 0 & A[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(Y_{ij} - Y'_{ij}) &= \sqrt{-1} \begin{pmatrix} S[i,j] & 0 \\ 0 & -S[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(V_{ij} + V'_{ij}) &= \sqrt{D} \begin{pmatrix} -A[i,j] & 0 \\ 0 & A[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(W_{ij} + W_{ji}) &= \sqrt{-D} \begin{pmatrix} S[i,j] & 0 \\ 0 & S[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(W'_i) &= W'_i. \end{aligned}$$

Let $dk_\infty$ be the Haar measure on $\mathcal{K}_{\mathfrak{G}_\infty}$ determined by these vectors. By Macdonald [35], the volume of $\mathrm{U}(n)$ is equal to $(2\pi)^{n(n+1)/2} \prod_{i=1}^n \Gamma(i)^{-1}$, if the Haar measure is normalized by a Chevalley basis of $\mathrm{Lie}(\mathrm{U}(n)) \otimes \mathbb{C}$. Using this, we have

$$\mathrm{Vol}(\mathcal{K}_{\mathfrak{G}_\infty}; dk_\infty) = D^{(-n^2+1)/2} 2^{-n^2+2n} \pi^{n^2+n-1} \prod_{i=1}^n \Gamma(i)^{-2}.$$

We now consider the invariant measure on the hermitian upper half space $\mathcal{H}_n$. We define an invariant measure on $\mathcal{H}_n$ as follows. Let $\mathrm{Her}_n(\mathbb{C}/\mathbb{R})$ be the space of hermitian matrices of size $n$. Then the Haar measures $dX$ and $dY$ on $\mathrm{Her}_n(\mathbb{C}/\mathbb{R})$ are such that the covolume of the lattice $\mathrm{Her}_n(\mathbb{C}/\mathbb{R}) \cap \mathrm{M}_n(\mathbb{Z}[\sqrt{-1}])$ is 1. Then the measure $(\det Y)^{-2n} \, dX \, dY$ is invariant under the action of $\mathfrak{G}(\mathbb{R}) = \mathrm{SU}(n,n)(\mathbb{R})$.

Note that $\mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. We claim that $dg_\infty/dk_\infty$ is equal to $2^{-n}D^{-(n^2-n)/2}(\det Y)^{-2n}\,dX\,dY$. To prove this, we consider the Iwasawa decomposition $\mathfrak{G}(\mathbb{R}) = A_{\mathfrak{G}_\infty}N_{\mathfrak{G}_\infty}\mathcal{K}_{\mathfrak{G}_\infty}$, where $A_{\mathfrak{G}_\infty}$ and $N_{\mathfrak{G}_\infty}$ are Lie subgroup of $\mathfrak{G}(\mathbb{R})$ corresponding to the Lie algebras generated by

$$\{X_{ii}\,|\,1 \le i \le n\}$$

and

$$\{X_{ij}, V_{ij}, W_{ij}\,|\,1 \le i < j \le n\} \cup \{Y_{ij}\,|\,1 \le i \le j \le n\},$$

respectively. Then it is easy to check the left invariant Haar measure determined by these basis induces $2^{-n}D^{-(n^2-n)/2}(\det Y)^{-2n}\,dX\,dY$ on $\mathcal{H}_n$, which implies the claim.

Now we consider the adele space $\mathfrak{G}(\mathbb{A})$. Let $\mathfrak{X}$ be a fundamental domain for $\Gamma_K^{(n)}\backslash\mathcal{H}_n$. We regard $\mathfrak{X}$ as a subset of $A_{\mathfrak{G}_\infty}N_{\mathfrak{G}_\infty}$ by the bijection $A_{\mathfrak{G}_\infty}N_{\mathfrak{G}_\infty} \simeq \mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. Then each fibre of the map

$$(\prod_p \mathcal{K}_{\mathfrak{G}_p}) \times \mathfrak{X} \times \mathcal{K}_{\mathfrak{G}_\infty} \to \mathfrak{G}(\mathbb{Q})\backslash\mathfrak{G}(\mathbb{A})$$

has exactly $|Z(\Gamma_K^{(n)})|$ elements, where $Z(\Gamma_K^{(n)})$ is the center of $\Gamma_K^{(n)}$. Note that $|Z(\Gamma_K^{(n)})| = (2n, w_K)$. It follows that

$$(2n, w_K)^{-1} \cdot 2^{2n^2-n-1}\prod_{i=2}^{2n} L(i,\chi^i)^{-1} \cdot D^{(-n^2+1)/2}2^{-n^2+2n}\pi^{n^2+n-1}\prod_{i=1}^{n}\Gamma(i)^{-2}$$
$$\times\, 2^{-n}D^{-(n^2-n)/2}\mathrm{Vol}(\mathfrak{X}) = 1.$$

It follows that

$$\mathrm{Vol}(\Gamma_K^{(n)}\backslash\mathcal{H}_n) = 2^{-n^2+1}D^{(2n^2-n-1)/2}(2n, w_K)\prod_{i=2}^{2n}\Lambda(i,\chi^i),$$

as desired.

## 11. The trivial representation

Let $k$ be a totally real algebraic number field and $S$ the set of archimedean places of $k$. The discriminant of $k$ is denoted by $D_k$. Recall that the completed Dedekind zeta function $\xi_k(s)$ satisfies the functional equation $\xi_k(1-s) = D_k^{s-(1/2)}\xi_k(s)$. Put $d = [k : \mathbb{Q}]$. We assume the following conditions:

(a) Both $G_1$ and $G_0$ are unramified over $k_v$ for each $v \notin S$.
(b) $G_{0,v}$ is compact for each $v \in S$.

Note that such a pair $G_0 \subset G_1$ exists if and only if the following (i), (ii), and (iii) hold:

(i) The discriminant field $K$ is unramified over $k$.

(ii) $K$ is totally real if $n \equiv 0 \bmod 4$, and totally imaginary if $n \equiv 2 \bmod 4$.

(iii) $d$ is even if $n \equiv 3, 4, 5, 6 \bmod 8$.

Let $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v}$ be a maximal compact subgroup of $G_0(\mathbb{A})$. We assume $\mathcal{K}_{0,v}$ is a hyperspecial maximal compact subgroup for $v \notin S$. For $v \notin S$, we give the standard Haar measure $dg_{0,v}$ on $G_{0,v}$. For $v \in S$, we give the Haar measure $dg_{0,v}$ with total volume 1 on $\mathcal{K}_{0,v} = G_{0,v}$. The Haar measure constant $C_0$ can be calculated directly, but here we make use of the mass formula. There exists a finite subset $\mathfrak{B} \subset G_0(\mathbb{A})$ such that $G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} G_0(k)x\mathcal{K}_0$. For each $x \in \mathfrak{B}$, the group $\Gamma^x = x^{-1}G_0(k)x \cap \mathcal{K}_0$ is a finite group. The left coset $G_0(k)\backslash G_0(\mathbb{A})$ is decomposed into a disjoint union

$$G_0(k)\backslash G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} x \cdot (\Gamma^x \backslash \mathcal{K}_0).$$

Let $e_x$ be the order of the group $\Gamma^x$. The mass $M$ is defined by $M = \sum_{x \in \mathfrak{B}} e_x^{-1}$. Then Shimura's exact mass formula (Shimura [41], p. 27, Theorem 5.8) says that

$$M = 2D_k^{m^2-(m/2)}[(2\pi)^{-m}\Gamma(m)]^d L(m,\chi) \prod_{j=1}^{m-1} \left\{[(2\pi)^{-2j}\Gamma(2j)]^d \zeta_k(2j)\right\}$$

if $n = 2m$ is even, and that

$$M = 2^{1-md} D_k^{m^2+(m/2)} \prod_{j=1}^{m} \left\{[(2\pi)^{-2j}\Gamma(2j)]^d \zeta_k(2j)\right\}$$

if $n = 2m+1$ is odd.

Then we have

$$\int_{G_0(k)\backslash G_0(\mathbb{A})} \prod_v dg_{0,v} = M.$$

Since the Tamagawa number of $G_0$ is 2, we have $C_0 = 2M^{-1}$.

By definition, we have

$$\Delta_{G_1} = \begin{cases} \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m \text{ is even,} \\ \Lambda(m+1,\chi) \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

We now put $\varphi_1 = 1$ and $\varphi_0 = 1$. Then $\pi_i$ is the trivial representation of $G_i(\mathbb{A})$. Obviously, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 1.$$

The $L$-function of the trivial representation of $G_0$ is given by

$$\Lambda(s,\pi_0)=\begin{cases}\Lambda(s,\chi)\prod_{j=1}^{2m-1}\xi_k(s-m+j) & \text{if } n=2m \text{ is even,}\\ \prod_{j=1}^{2m}\xi_k(s-m+j-(1/2)) & \text{if } n=2m+1 \text{ is odd.}\end{cases}$$

Similarly, we have

$$\Lambda(s,\pi_1)=\begin{cases}\prod_{j=1}^{2m}\xi_k(s-m+j-(1/2)) & \text{if } n=2m \text{ is even,}\\ \Lambda(s,\chi)\prod_{j=1}^{2m+1}\xi_k(s-m+j-1) & \text{if } n=2m+1 \text{ is odd.}\end{cases}$$

When $n=2m$ is even, we have

$$\begin{aligned}\Lambda(s,\pi_1\boxtimes\pi_0)&=\prod_{i=1}^{2m}\Lambda(s-m+i-(1/2),\chi)\\ &\quad\times\prod_{i=1}^{2m}\prod_{j=1}^{2m-1}\xi_k(s-2m+i+j-(1/2))\\ \Lambda(s,\pi_0,\mathrm{Ad})&=\prod_{i=1}^{2m-1}\Lambda(s-m+i,\chi)\\ &\quad\times\prod_{1\le i<j\le 2m-1}\xi_k(s-2m+i+j)\\ \Lambda(s,\pi_1,\mathrm{Ad})&=\prod_{1\le i\le j\le 2m}\xi_k(s-2m+i+j-1).\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(s)&=\frac{\Lambda(s-m+(1/2),\chi)}{\xi_k(s+2m-(1/2))}\prod_{j=1}^{m-1}\frac{\xi_k(s-2j+(1/2))}{\xi_k(s+2j-(1/2))},\\ \mathcal{P}_{\pi_1,\pi_0}(1/2)&=\frac{\Lambda(1-m,\chi)}{\xi_k(2m)}\prod_{j=1}^{m-1}\frac{\xi_k(-2j+1)}{\xi_k(2j)}\\ &=D_k^{m^2-(m/2)}\frac{\Lambda(m,\chi)}{\xi_k(2m)}\end{aligned}$$

if $n=2m$ is even. A similar calculation shows that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(s)&=\Lambda(s+m+(1/2),\chi)^{-1}\prod_{j=1}^{m}\frac{\xi_k(s-2j+(1/2))}{\xi_k(s+2j-(1/2))},\\ \mathcal{P}_{\pi_1,\pi_0}(1/2)&=\Lambda(m+1,\chi)^{-1}\prod_{j=1}^{m}\frac{\xi_k(-2j+1)}{\xi_k(2j)}\\ &=D_k^{m^2+(m/2)}\Lambda(m+1,\chi)^{-1},\end{aligned}$$

if $n = 2m + 1$ is odd. When $v \in S$, the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is clearly equal to 1. It follows that

$$\begin{aligned}\alpha_v(\varphi_{1,v}, \varphi_{0,v}) &= \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} \\ &= \begin{cases} \Gamma_{\mathbb{R}}(1-m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m \equiv 0 \bmod 4, \\ \Gamma_{\mathbb{R}}(2-m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m \equiv 2 \bmod 4, \\ \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m+1 \text{ is odd.} \end{cases}\end{aligned}$$

Therefore we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^{\beta} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \alpha_v(\varphi_{1,v}, \varphi_{0,v}),$$

where

$$\beta = \begin{cases} -md & \text{if } n = 2m \text{ is even,} \\ -2md & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Note that the integer $\beta$ depends on the number of bad places.

## 12. Calculation for the real place

In this section, we carry out the calculation of the archimedean local integrals which appeared in §7, §9, and §10. Every algebraic group is defined over $\mathbb{R}$ in this section.

We first consider the case $G_0 = \mathrm{SO}(2,1) \simeq \mathrm{PGL}_2(\mathbb{R})$. The (topological) identity component of $G_0$ is denoted by $G_0(\mathbb{R})^0$. Note that $G_0(\mathbb{R})^0 \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. The image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{R})$ is denoted by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The maximal compact subgroup $\mathrm{O}(2)/\{\pm 1\} \subset \mathrm{PGL}_2(\mathbb{R})$ is denoted by $\mathcal{K}$. Put $\mathcal{K}^0 = \mathrm{SO}(2)/\{\pm 1\} \subset \mathcal{K}$. The Haar measure $dk$ on $\mathcal{K}^0$ is such that the total measure is 1. By Iwasawa decomposition, an element $g \in G_0(\mathbb{R})^0$ can be uniquely written as

$$g = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} k,$$

$t$, $n \in \mathbb{R}$, $k \in \mathcal{K}^0$. We choose a Haar measure $dg$ on $G_0(\mathbb{R})^0$ such that $dg/dk$ induces the measure $y^{-2}\, dx\, dy$ on the upper half plane $\mathfrak{H}_1 \simeq G_0(\mathbb{R})/\mathcal{K}^0$. Note that $dg = 2\, dt\, dn\, dk$. The Haar measure $dg$ can be naturally extended to $G_0(\mathbb{R})$. We put

$$A^+ = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \,\middle|\, t \geq 0 \right\}.$$

We consider the map

$$\mathcal{K}^0 \times A^+ \times \mathcal{K}^0 \longrightarrow G_0(\mathbb{R})^0$$
$$\left(k, \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, k'\right) \longmapsto k \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} k'.$$

By Cartan decomposition, this map is bijective outside the boundary of $A^+$. It is well-known (e.g., [21], Theorem 5.8) that

$$dg = C \cdot \sinh(2t)\, dk\, dt\, dk'$$

for some constant $C > 0$. Let $A(T)$ be the area of the small disc with radius $T$ and center $\sqrt{-1} \in \mathfrak{H}_1$. Then we have $A(T) \sim C \int_0^{T/2} \sinh(2t)\, dt$ when $T \to 0$, and so we have $C = 4\pi$.

Let $\tau_j$ be the (limit of) discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with minimal weight $\pm\kappa_j$. Let $\Phi_j$ be the matrix coefficient of $\tau_{j,\infty}$ with respect to the lowest weight vector with norm 1. Then the support of $\Phi_j$ is contained in $G_0(\mathbb{R})^0$ and

$$\Phi_j\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\right) = \cosh(t)^{-\kappa_j}.$$

*Proof of Proposition 7.2.* Let $\varphi_{1,\infty}$ and $\varphi_{0,\infty}$ be as in Proposition 7.2. Then we have

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 4\pi \int_0^\infty \cosh(t)^{-2\kappa_3} \sinh(2t)\, dt \\ &= 4\pi(\kappa_3 - 1)^{-1}. \end{aligned}$$

For the latter part of the proposition,

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)^{-1} I(\varphi_{1,\infty}, \varphi_{0,\infty}) = 2.$$

□

Next, we consider the case $G_0 = \mathrm{SO}(2,2)$. Put

$$\mathrm{GL}_2^{(2)} = \{(h_1, h_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det h_1 = \det h_2\}.$$

Then, we have $\mathrm{SO}(2,2) \simeq \mathrm{GL}_2^{(2)}(\mathbb{R})/\mathbb{R}^\times$. We denote the image of $(h_1, h_2) \in \mathrm{GL}_2^{(2)}(\mathbb{R})$ in $\mathrm{SO}(2,2)$ by $[h_1, h_2]$. Put

$$\begin{aligned} A &= \left\{ \left[ \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right] \,\middle|\, t_1,\, t_2 \in \mathbb{R} \right\}, \\ N &= \left\{ \left[ \begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} \right] \,\middle|\, n_1,\, n_2 \in \mathbb{R} \right\}, \\ \mathcal{K} &= \{[k_1, k_2] \mid k_1,\, k_2 \in \mathrm{O}(2),\, \det k_1 = \det k_2\}. \end{aligned}$$

For each $(t_1, t_2) \in \mathbb{R}^2$, we put

$$m(t_1, t_2) = \left[ \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right].$$

The connected component $\mathrm{SO}(2,2)^0$ is equal to the image of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Put $\mathcal{K}^0 = \mathcal{K} \cap \mathrm{SO}(2,2)^0$. Then we have an Iwasawa decomposition $\mathrm{SO}(2,2)^0 = AN\mathcal{K}^0$. Then $\mathrm{SO}(2,2)^0/\mathcal{K}^0$ can be identified with $\mathfrak{H}_1 \times \mathfrak{H}_1$. The Haar measure $dk$ on $\mathcal{K}^0$ is the Haar measure such that the total volume is 1. We choose a Haar measure $dg$ on $\mathrm{SO}(2,2)^0$ such that the induced measure $dg/dk$ on $\mathfrak{H}_1 \times \mathfrak{H}_1$ is equal to $y_1^{-2} y_2^{-2}\, dx_1\, dx_2\, dy_1\, dy_2$. Then $dg = 4\, dt_1\, dt_2\, dn_1\, dn_2\, dk$. The Haar measure $dg$ can be naturally extended to $G_0(\mathbb{R}) = \mathrm{SO}(2,2)$. Put $A^+ = \{m(t_1, t_2) \,|\, t_1,\, t_2 \geq 0\}$. Consider the map

$$\begin{aligned} \lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\longrightarrow \mathrm{SO}(2,2)^0 \\ (k, m(t_1, t_2), k') &\longmapsto k \cdot m(t_1, t_2) \cdot k'. \end{aligned}$$

Let $\partial A^+$ be the boundary of $A^+$. If $g \in G_0(\mathbb{R})^0$ is not in the image of $\partial A^+$, then $\lambda^{-1}(g)$ consists of two elements. In terms of the map $\lambda$, we have

$$\begin{aligned} &\int_{G_0(\mathbb{R})^0} f(g)\, dg \\ &= 16\pi^2 \int_{\mathcal{K}^0 \times A^+ \times \mathcal{K}^0} f(\lambda(k, m(t_1, t_2), k')) \sinh(2t_1) \sinh(2t_2)\, dk\, dt_1\, dt_2\, dk' \end{aligned}$$

for any integrable function $f$ on $G_0(\mathbb{R})^0$.

*Proof of Proposition 9.1.* We need to calculate the matrix coefficient of $\varphi_{1,\infty} \in \pi_{1,\infty}$. In fact, it is enough to consider the pullback of the matrix coefficient by the map $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2,2) \subset \mathrm{SO}(3,2)$, since $A^+$ is contained in the image of this map. Note that the image of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is contained in the identity component $\mathrm{SO}(3,2)^0 = \mathrm{Sp}_2(\mathbb{R})/\{\pm 1\}$. The restriction of $\pi_{1,\infty}$ is a direct sum of a holomorphic discrete series and an anti-holomorphic discrete series. Since the holomorphic discrete series is a lowest weight representation, its pullback to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a direct sum of lowest weight representations. We denote $\tau_\lambda$ the holomorphic discrete series of $\mathrm{SL}_2(\mathbb{R})$ with lowest weight $\lambda$. Since the lowest weight $(\kappa+1, \kappa+1)$ occurs with multiplicity one, the summand contains $\tau_{\kappa+1} \boxtimes \tau_{\kappa+1}$ exactly once, and the other summands are of the form $\tau_{\lambda_1} \boxtimes \tau_{\lambda_2}$, where $\lambda_1,\, \lambda_2 \geq \kappa+1$ and $(\lambda_1, \lambda_2) \neq (\kappa+1, \kappa+1)$. (In fact, the precise decomposition of the restriction is known in this case.) Therefore the value of the matrix coefficient at $m(t_1, t_2) \in A^+$ is equal to $\cosh(t_1)^{-\kappa-1} \cosh(t_2)^{-\kappa-1}$.

It follow that

$$\begin{aligned}
I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 16\pi^2 \left( \int_0^\infty \cosh(t)^{-2\kappa-2} \sinh(2t)\, dt \right)^2 \\
&= 16\pi^2/\kappa^2, \\
\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) &= \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)^{-1} I(\varphi_{1,\infty}, \varphi_{0,\infty}) \\
&= 4\pi.
\end{aligned}$$

□

Now, we consider the case $G_0 = \mathrm{SO}(3,2) = \mathrm{GSp}_2(\mathbb{R})/\mathbb{R}^\times$. We denote the image of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{R})$ in $G_0(\mathbb{R})$ by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Put

$$\begin{aligned}
A &= \left\{ \left[ \begin{array}{cc|cc} e^{t_1} & 0 & & \\ 0 & e^{t_2} & \multicolumn{2}{c}{0} \\ \hline & & e^{-t_1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & e^{-t_2} \end{array} \right] \;\middle|\; t_1,\, t_2 \in \mathbb{R} \right\}, \\
N' &= \left\{ \left[ \begin{array}{cc|cc} 1 & n'_1 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & -n'_1 & 1 \end{array} \right] \;\middle|\; n'_1 \in \mathbb{R} \right\}, \\
N'' &= \left\{ \left[ \begin{array}{c|cc} \mathbf{1}_2 & n''_{11} & n''_{12} \\ & n''_{12} & n''_{22} \\ \hline 0 & \multicolumn{2}{c}{\mathbf{1}_2} \end{array} \right] \;\middle|\; n''_{11},\, n''_{12},\, n''_{22} \in \mathbb{R} \right\}, \\
\mathcal{K}^0 &= \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \;\middle|\; A + \sqrt{-1}B \in \mathrm{U}(2) \right\}.
\end{aligned}$$

Then the topological identity component $G_0(\mathbb{R})^0 = \mathrm{SO}(3,2)^0$ has an Iwasawa decomposition $G_0(\mathbb{R})^0 = AN\mathcal{K}^0$, where $N = N'N''$. Note that $G_0(\mathbb{R})^0/\mathcal{K}^0$ can be identified with $\mathfrak{H}_2$. We take the Haar measures $dk$ on $\mathcal{K}^0$ with the total volume 1. We choose the Haar measure $dg$ of $G_0(\mathbb{R})^0$ such that the induced measure $dg/dk$ is equal to $(\det Y)^{-3}\, dX\, dY$. Then we have

$$dg = 4\, dt_1\, dt_2\, dn'_1\, dn''_{11}\, dn''_{12}\, dn''_{22}\, dk.$$

The Haar measure $dg$ can be naturally extended to $G_0(\mathbb{R})$. Put $\mathfrak{a} = \mathrm{Lie}(A)$. Then $\mathfrak{a}$ can be identified with $\mathbb{R}^2$ and we put

$$m(t_1, t_2) = \left[\begin{array}{cc|cc} e^{t_1} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & e^{t_2} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & e^{-t_1} & 0 \\ & & 0 & e^{-t_2} \end{array}\right]$$

for each $(t_1, t_2) \in \mathbb{R}^2 \simeq \mathfrak{a}$. The positive chamber $A^+$ is defined by $A^+ = \{m(t_1, t_2) \in A \,|\, t_1 \geq t_2 \geq 0\}$. Then the map

$$\begin{aligned} \lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\longrightarrow \mathrm{SO}(3,2)^0 \\ (k, m(t_1, t_2), k') &\longmapsto k \cdot m(t_1, t_2) \cdot k' \end{aligned}$$

is a double covering outside the boundary of $A^+$. In terms of this map, we have (cf. [21], Theorem 5.8)

$$dg = C \sinh(2t_1) \sinh(2t_2) \sinh(t_1 - t_2) \sinh(t_1 + t_2)\, dk\, dt_1\, dt_2\, dk'.$$

for some positive constant $C > 0$.

The constant $C$ can be calculated as follows. We recall the argument of [21], Ch. I, Theorem 5.8. We shall calculate the Jacobian of the induced map

$$\bar{\lambda} : \mathcal{K}^0 \times A^+ \longrightarrow G_0(\mathbb{R})^0 / \mathcal{K}^0 \simeq AN$$

at $(k, m(t_1, t_2)) \in \mathcal{K}^0 \times A^+$. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(\mathrm{SO}(3,2)^0)$. Then the tangent space of $\mathcal{K}^0 \times A^+$ at $(k, m(t_1, t_2))$ can be identified with $\mathfrak{k} + \mathfrak{a}$ by left translation. Let $\Sigma^+$ be the set of positive roots for $(G_0(\mathbb{R})^0, A)$. Then for each $\alpha \in \Sigma^+$, we put

$$\mathfrak{k}_\alpha = \{T \in \mathfrak{k} \,|\, \mathrm{ad}((x_1, x_2))^2 T = \alpha((x_1, x_2))^2 T \text{ for all } (x_1, x_2) \in \mathfrak{a}\}.$$

Then $\dim \mathfrak{a}_\alpha = 1$ for any $\alpha \in \Sigma^+$. Choose a non-zero vector $T_\alpha \in \mathfrak{k}_\alpha$ for each $\alpha \in \Sigma^+$. For example, we can choose

$$T_{\varepsilon_1 - \varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right), \qquad T_{2\varepsilon_1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$T_{\varepsilon_1 + \varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \qquad T_{2\varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right).$$

For each $\alpha \in \Sigma^+$,

$$U_\alpha = \alpha((t_1, t_2))^{-1} \mathrm{ad}((t_1, t_2))(T_\alpha)$$

belongs to $\mathfrak{p}$, and does not depend on $(t_1, t_2) \in \mathfrak{a}$. Note that

$$U_{\varepsilon_1-\varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \end{array}\right), \qquad U_{2\varepsilon_1} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 0 \\ \hline 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \end{array}\right),$$

$$U_{\varepsilon_1+\varepsilon_2} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right), \qquad U_{2\varepsilon_2} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 1 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 1 & & \end{array}\right).$$

Then

$$T_\alpha \ (\alpha \in \Sigma^+), \quad (1,0),\ (0,1) \in \mathfrak{a}$$

make up a basis of $\mathfrak{k} + \mathfrak{a}$, and

$$U_\alpha \ (\alpha \in \Sigma^+), \quad (1,0),\ (0,1) \in \mathfrak{a}$$

make up a basis of $\mathfrak{p}$. By the proof of [21], Ch. I, Theorem 5.8,

$$|\det(d\bar{\lambda}_{(k,m(t_1,t_2))})| = \prod_{\alpha \in \Sigma^+} \sinh(\alpha(t_1, t_2))$$

with respect to these basis.

Let $\omega_\alpha$ $(\alpha \in \Sigma^+)$ be the basis of the space of left invariant 1-forms on $\mathcal{K}^0$ dual to $T_\alpha$ $(\alpha \in \Sigma^+)$. Then it is easy to check that

$$\int_{\mathcal{K}^0} |\bigwedge_{\alpha \in \Sigma^+} \omega_\alpha| = 2\pi^3.$$

On the other hand, the dual basis of

$$(1,0),\ (0,1) \in \mathfrak{a}, \quad U_\alpha \ (\alpha \in \Sigma^+)$$

induces

$$\frac{1}{16}\, dt_1\, dt_2\, dn'_1\, dn''_{11}\, dn''_{12}\, dn''_{22}$$

on $AN \simeq G_0(\mathbb{R})^0/\mathcal{K}^0$. It follows that $C = 64\pi^3$.

*Proof of Proposition 10.2.* As in the proof of Proposition 9.1, the value of the matrix coefficient $\langle \pi_{1,\infty}(g_0)\varphi_{1,\infty}, \varphi_{1,\infty}\rangle$ at $g_0 = m(t_1, t_2)$ is equal to $\cosh(t_1)^{-\kappa-1}\cosh(t_2)^{-\kappa-1}$. It follows that

$$\begin{aligned} I(\varphi_{1,\infty},\varphi_{0,\infty}) &= 64\pi^3 \int_{t_1\geq t_2\geq 0} \cosh(t_1)^{-2\kappa-2}\cosh(t_2)^{-2\kappa-2} \\ &\qquad \times \sinh(2t_1)\sinh(2t_2)\sinh(t_1+t_2)\sinh(t_1-t_2)\,dt_1\,dt_2 \\ &= 64\pi^3 \int_0^\infty\int_0^\infty \cosh(x+y)^{-2\kappa-2}\cosh(y)^{-2\kappa-2} \\ &\qquad \times \sinh(2x+2y)\sinh(2y)\sinh(x+2y)\sinh(x)\,dx\,dy. \end{aligned}$$

By using the formulas

$$\sinh(2a) = 2\sinh(a)\cosh(a),$$
$$\sinh(a+b)\sinh(a-b) = \cosh^2(a) - \cosh^2(b),$$

one can show that the integral $I(\varphi_{1,\infty},\varphi_{0,\infty})$ is equal to

$$\begin{aligned} & 256\pi^3 \int_0^\infty \cosh(y)^{-2\kappa-1}\sinh(y) \\ & \quad \times \int_0^\infty \cosh(x+y)^{-2\kappa-1}\sinh(x+y)[\cosh^2(x+y)-\cosh^2(y)]\,dx\,dy \\ & = 256\pi^3 \int_0^\infty \cosh(y)^{-2\kappa-1}\sinh(y) \\ & \quad \times \left\{ \left[-\frac{u^{-2\kappa+2}}{2\kappa-2}\right]_{u=\cosh(y)}^{\infty} - \cosh^2(y)\left[-\frac{u^{-2\kappa}}{2\kappa}\right]_{u=\cosh(y)}^{\infty} \right\} dy \\ & = \frac{128\pi^3}{\kappa(\kappa-1)} \int_0^\infty \cosh(y)^{-4\kappa+1}\sinh(y)\,dy \\ & = \frac{64\pi^3}{\kappa(\kappa-1)(2\kappa-1)}. \end{aligned}$$

Since $\Delta_{G_{1,\infty}} = \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{R}}(4)^2 = \pi^{-5}$, we have $\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty}) = -4\pi$. □

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