ON THE PERIODS OF AUTOMORPHIC FORMS ON SPECIAL ORTHOGONAL GROUPS AND THE GROSS-PRASAD CONJECTURE

ATSUSHI ICHINO AND TAMOTSU IKEDA

Dedicated to Professor Hiroyuki Yoshida on the occasion of his sixtieth birthday

Introduction

In early 90's, Gross and Prasad [17], [18] gave a series of fascinating conjectures on the restriction of automorphic representation of a special orthogonal group to a smaller special orthogonal subgroup. We now recall their global conjecture. Let k be a global field with $\mathrm{char}(k) \neq 2$. Let $(V_0, Q_0) \subset (V_1, Q_1)$ be quadratic forms over k with rank n and $n+1$, respectively. We assume that $n \geq 2$ and that (V_0, Q_0) is not isomorphic to the hyperbolic plane. We regard $G_0 = \mathrm{SO}_{Q_0}$ as a subgroup of $G_1 = \mathrm{SO}_{Q_1}$. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible tempered cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. Assume that $\mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any place v of k. Then the global Gross-Prasad conjecture [17] asserts that

$$\langle \varphi_1|_{G_0}, \varphi_0 \rangle := \int_{G_0(k)\backslash G_0(\mathbb{A})} \varphi_1(g_0)\overline{\varphi_0(g_0)}dg_0 \neq 0$$

for some $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$, if and only if $L(1/2, \pi_1 \boxtimes \pi_0) \neq 0$. Here, $L(s, \pi_1 \boxtimes \pi_0)$ is the "product" L-function of π_1 and π_0.

In this paper, we would like to formulate a conjecture, which expresses the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ in terms of L-values. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_1 = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l, \chi_{Q_1}) & \text{if } \dim V_1 = 2l, \end{cases}$$

where χ_{Q_1} is the quadratic Hecke character associated with the discriminant of Q_1. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume, for simplicity, π_1 and π_0 are tempered. We put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s+(1/2), \pi_1, \mathrm{Ad})L(s+(1/2), \pi_0, \mathrm{Ad})},$$

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where $L(s,\pi_1,\mathrm{Ad})$ and $L(s,\pi_0,\mathrm{Ad})$ are the adjoint L-function of π_1 and that of π_0, respectively. We assume that the L-functions $L(s,\pi_1\boxtimes\pi_0)$, $L(s,\pi_1,\mathrm{Ad})$, and $L(s,\pi_0,\mathrm{Ad})$ have meromorphic continuation. For a sufficiently large finite set of bad places S, we denote the partial Euler products for $\mathcal{P}_{\pi_1,\pi_0}(s)$ and Δ_{G_1} by $\mathcal{P}^S_{\pi_1,\pi_0}(s)$ and $\Delta^S_{G_1}$, respectively.

Let $\varphi_1=\otimes_v\varphi_{1,v}\in\pi_1$ and $\varphi_0=\otimes_v\varphi_{0,v}\in\pi_0$ be cusp forms. We consider the matrix coefficients

$$\begin{aligned}\Phi_{\varphi_{1,v},\varphi_{1,v}}(g_1)=&\langle\pi_{1,v}(g_1)\varphi_{1,v},\varphi_{1,v}\rangle_v,\quad g_1\in G_1(k_v),\\ \Phi_{\varphi_{0,v},\varphi_{0,v}}(g_0)=&\langle\pi_{0,v}(g_0)\varphi_{0,v},\varphi_{0,v}\rangle_v,\quad g_0\in G_0(k_v).\end{aligned}$$

Put

$$I(\varphi_{1,v},\varphi_{0,v})=\int_{G_0(k_v)}\Phi_{\varphi_{1,v},\varphi_{1,v}}(g_{0,v})\overline{\Phi_{\varphi_{0,v},\varphi_{0,v}}(g_{0,v})}\,dg_{0,v}.$$

It will be proved that this integral is convergent (Proposition 1.1).

Then we conjecture that there exists an integer β such that

$$(\bigstar)\qquad \frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle}=2^\beta C_0\Delta^S_{G_1}\mathcal{P}^S_{\pi_1,\pi_0}(1/2)\prod_{v\in S}\frac{I(\varphi_{1,v},\varphi_{0,v})}{\|\varphi_{1,v}\|^2\cdot\|\varphi_{0,v}\|^2},$$

where C_0 is a constant determined by the choice of local and global Haar measure of $G_0(\mathbb{A})$ (Conjecture 1.5). For more precise definitions, see § 1. When $n=2$, our conjecture reduces to the theorem of Waldspurger [65].

One can give a possible interpretation of the factor 2^β in ($\bigstar$) in terms of the Arthur conjecture [3]. Let $\mathcal{L}_k$ be the hypothetical Langlands group for k. Then, if we admit the Arthur conjecture, for an irreducible cuspidal tempered automorphic representation π_i of $G_i(\mathbb{A})$ ($i=0,1$), one can attach an L-homomorphism $\psi_i:\mathcal{L}_k\to{}^LG_i=\hat{G}_i\rtimes W_k$, where W_k is the Weil group [64] of k. It is generally believed that the structure of the L-packet for π_i is closely related to the finite group $\mathcal{S}_{\psi_i}=\mathrm{Cent}_{\hat{G}_i}(\mathrm{Im}(\psi_i))$. Then, we conjecture that

$$2^\beta=\frac{1}{|\mathcal{S}_{\psi_1}|\cdot|\mathcal{S}_{\psi_0}|}.$$

(cf. Conjecture 2.1.)

This paper consists of four parts. In Part I (§§1-3), we formulate our conjecture in detail. We first formulate our conjecture in the tempered case. Then we discuss the relation with Arthur conjecture. In particular, a possible interpretation of the factor 2^β in terms of Arthur parameter will be given. In §3, we discuss the non-tempered case. In the non-tempered case, several difficulties will arise. One is that the factor $\mathcal{P}_{\pi_1,\pi_0}(s)$ may not be holomorphic at $s=1/2$. Another difficulty is that the integral $I(\varphi_{1,v}, \varphi_{0,v})$ may not be convergent. Nevertheless, several examples suggest that an analogue of (★) holds in non-tempered case. We give a somewhat optimistic conjecture in §3 for non-tempered case.

In Part II (§§4-5), we develop some local theory to show that our conjecture (★) makes sense. In §4, we prove that the local integral $I(\varphi_{1,v}, \varphi_{0,v})$ is convergent if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. In §5, we show that

$$I(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_1,v}\mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)$$

for unramified case (Theorem 1.2). In particular, the right hand side of (★) is independent of the choice of the set S of bad primes. In course of the proof, we make use of the results of Ginzburg, Piatetski-Shapiro, Rallis [14] and those of Kato, Murase, Sugano [36]. We emphasise the fact that the factor $\mathcal{P}_{\pi_1,\pi_0}(s)$ already appeared in [14].

In Part III (§§6-11), we give several examples over number fields. In §6, we show that our conjecture is compatible with the theorem of Waldspurger [65]. In §7, we will discuss the Jacquet conjecture. The first named author proved a closely related result [31]. We show that the recent result of Watson [66] is compatible with our conjecture. We also discuss the relation with the conjecture of Deligne [9] and the conjecture of Shimura [59], [60]. In §8, we consider the restriction of the Saito-Kurokawa lifts to the diagonal subset. We show that the first named author's result [30] is compatible with our conjecture. Note that this example is non-tempered. In §9, we consider our result on the restriction of the hermitian Maass lift to the space of Saito-Kurokawa lifts [32]. This example is also non-tempered, and is compatible with our conjecture. In §10, we consider the trivial representation. This example reduces to the mass formula for the quadratic forms. In §11, we collect the calculation over the real place, which is necessary to get the result of §§7-9.

In the final part (§§12-14), we give examples over function fields. We do not pursue generality here. In fact, we consider only locally unramified examples. Nevertheless, we believe that these examples are strong evidence for our conjectures. In particular, Proposition 13.7, Proposition 13.10, and Proposition 13.13 are good examples for Conjecture 2.1.

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Part I. Global theory

1. Formulation of the conjecture

In this paper, we would like to formulate a conjecture on a relation between a certain period of automorphic forms on special orthogonal group and some L-value. Our conjecture can be considered as a refinement of global Gross-Prasad conjecture [17].

Let k be a global field with $\mathrm{char}(k) \neq 2$. Let (V_1, Q_1) and (V_0, Q_0) be quadratic forms over k with rank $n+1$ and n, respectively. We assume $n \geq 2$. When $n = 2$, we also assume (V_0, Q_0) is not isomorphic to the hyperbolic plane over k. We denote the special orthogonal group of (V_i, Q_i) by G_i $(i = 0, 1)$. From now on, the subscript i will indicate either 0 or 1, except for some obvious situation. We assume there is an embedding $\iota : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal group $\iota : G_0 \hookrightarrow G_1$. We regard G_0 as a subgroup of G_1 by this embedding. The group $G_i(k_v)$ of k_v-valued points of G_i is denoted by $G_{i,v}$.

For even-dimensional quadratic form (V, Q), the discriminant field K_Q is defined by $K_Q = k(\sqrt{(-1)^{\dim V/2} \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if $\dim V_0$ is even (resp. if $\dim V_1$ is even). We call K the discriminant field for the pair (V_1, V_0). Let $\chi = \chi_{K/k}$ be the Hecke character associated to K/k by the class field theory.

Put

$$\Delta_{G_i,v} = \begin{cases} \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l-2)\cdot L_v(l,\chi) & \text{if } \dim V_i = 2l, \end{cases}$$

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l,\chi) & \text{if } \dim V_i = 2l. \end{cases}$$

Note that $\Delta_{G_i} = L(M_i^\vee(1))$, where $M_i^\vee$ is the dual motive of the motive M_i associated to G_i by Gross [16].

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(\mathbb{A})$. There is a canonical inner product $\langle *, * \rangle$ on forms on $G_i(k)\backslash G_i(\mathbb{A})$ defined by

$$\langle \varphi_i, \varphi_i' \rangle = \int_{G_i(k)\backslash G_i(\mathbb{A})} \varphi_i(g_i)\overline{\varphi_i'(g_i)}dg_i,$$

where dg_i is the Tamagawa measure on $G_i(\mathbb{A})$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each v. There exist a positive numbers C_i such that $dg_i = C_i \prod_v dg_{i,v}$, when the right hand side is well-defined. Since $\pi_{i,v}$ is an unitary representation, there is an inner product $\langle *, * \rangle_v$ on $\pi_{i,v}$ for any place v of k. We put $\|\varphi_{i,v}\| = \langle \varphi_{i,v}, \varphi_{i,v} \rangle_v^{1/2}$, as usual. There exists a positive constant C_{π_i} such that $\langle \varphi_i, \varphi_i' \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi_{i,v}' \rangle_v$

for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi'_i = \otimes_v \varphi'_{i,v} \in \otimes_v \pi_{i,v}$

We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a $\mathcal{K}_i$-finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. We are interested in the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of φ_1 to $G_0(\mathbb{A})$.

Let S be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

- (U1) G_i is unramified over k_v.
- (U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
- (U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
- (U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
- (U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
- (U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When G_i is unramified over k_v, we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The L-group ${}^LG_i$ of G_i is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, W_k is the Weil group of k and

$$\hat{G}_i = \begin{cases} \mathrm{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l+1, \\ \mathrm{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by st the standard representation of ${}^LG_i$. The completed standard L-function for π_i is denoted by $L(s, \pi_i, \mathrm{st})$ for an irreducible automorphic representation π_i of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, \mathrm{st})$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - \mathrm{st}(A_{\pi_{i,v}}) \cdot q_v^{-s})^{-1}$, where, $A_{\pi_{i,v}}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product L-function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - \mathrm{st}(A_{\pi_{1,v}}) \otimes \mathrm{st}(A_{\pi_{0,v}}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\mathrm{Ad} : {}^LG_i \to \mathrm{GL}(\mathrm{Lie}(\hat{G}_i))$. The associated L-function $L(s, \pi_i, \mathrm{Ad})$ is called the adjoint L-function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \mathrm{Ad})$ can be analytically continued to the whole s-plane.

We put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \mathrm{Ad}) L(s + (1/2), \pi_0, \mathrm{Ad})}.$$

Let $\pi_{i,v}$ be an irreducible admissible representation of $G_{i,v}$. We denote the complex conjugate of $\pi_{i,v}$ by $\bar{\pi}_{i,v}$. It is believed that

$$\text{(MF)} \qquad \dim_{\mathbb{C}} \operatorname{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1$$

for non-archimedean place v of k. We do not assume (MF) in this paper. Recently, Aizenbud, Gourevitch, Rallis, and Schiffmann wrote a preprint [1], in which they obtained closely related results. For archimedean place, (MF) is verified in many cases, but not in general. (See e.g., [19].)

We consider the matrix coefficient

$$\Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_i) = \langle \pi_{i,v}(g_i)\varphi_{i,v}, \varphi'_{i,v}\rangle_v, \quad g_i \in G_{i,v}$$

for a $\mathcal{K}_{1,v}$-finite vector $\varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v}$ and a $\mathcal{K}_{0,v}$-finite vector $\varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v}$. Put

$$\begin{aligned} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) &= \int_{G_{0,v}} \Phi_{\varphi_{1,v},\varphi'_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v},\varphi'_{0,v}}(g_{0,v})}\, dg_{0,v}, \\ \alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) &= \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1}\, I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}). \end{aligned}$$

When $\varphi_{1,v} = \varphi'_{1,v}$ and $\varphi_{0,v} = \varphi'_{0,v}$, we simply denote these objects by $I(\varphi_{1,v}, \varphi_{0,v})$ and $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$, respectively.

Proposition 1.1. *If both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.*

Theorem 1.2. *Let v be a non-archimedean place. Assume that the conditions* (U1)*,* (U2)*,* (U3)*,* (U4)*,* (U5)*, and* (U6) *hold. If the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent, then we have $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$.*

The proofs of Proposition 1.1 and Theorem 1.2 will be given in Part II.

Conjecture 1.3. Assume that both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. Then $\dim_{\mathbb{C}} \operatorname{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now let $\pi_i \simeq \otimes_v \pi_{i,v}$ be irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We shall say that π_i is almost locally generic if π_i satisfies the following condition (ALG).

(ALG) For almost all v, the constituent $\pi_{i,v}$ is generic.

It is believed that π_i is almost locally generic if and only if π_i is tempered (generalized Ramanujan conjecture).

Conjecture 1.4. Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We assume both π_1 and π_0 are almost locally generic. Then

(1) The integral $I(\varphi_{1,v}, \varphi_{0,v})$ should be absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.
(2) $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now we state our global conjecture.

Conjecture 1.5. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ are irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume π_1 and π_0 are almost locally generic. Then there should be an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = 2^{\beta} C_0 \Delta_{G_1} \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

We will discuss the nature of the integer β in the next section.

Remark 1.6. When π_1 and π_0 are tempered, it is believed that the local L-factors $L(s, \pi_{1,v}, \mathrm{Ad})$, $L(s, \pi_{0,v}, \mathrm{Ad})$, and $L(s, \pi_{1,v} \boxtimes \pi_{0,v})$ are holomorphic for $\mathrm{Re}(s) > 0$. Therefore in this case our conjecture is equivalent to

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = 2^{\beta} C_0 \Delta_{G_1}^S \mathcal{P}_{\pi_1,\pi_0}^S(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2},$$

where $\Delta_{G_1}^S$ and $\mathcal{P}_{\pi_1,\pi_0}^S(s)$ are the partial Euler products. In particular, the definition of the L-factors for bad primes plays no role in this case. Note also that it is believed that $L(1, \pi_i, \mathrm{Ad}) \neq 0$ if π_i is tempered.

Remark 1.7. One can formulate Conjecture 1.5 in a different way as follows. Assume the local measure $dg_{i,v}$ and the local inner product $\langle *, * \rangle_v$ are normalised so that $C_i = C_{\pi_i} = 1$. Put

$$H_{\pi_1,\pi_0} = \mathrm{Hom}_{G_0(\mathbb{A}) \times G_0(\mathbb{A})}((\pi_1 \boxtimes \tilde{\pi}_1) \otimes (\bar{\pi}_0 \boxtimes \tilde{\bar{\pi}}_0), \mathbb{C}).$$

We define two elements $L_{\pi_1,\pi_0}^{\mathrm{global}}, L_{\pi_1,\pi_0}^{\mathrm{local}} \in H_{\pi_1,\pi_0}$ by

$$\begin{aligned} L_{\pi_1,\pi_0}^{\mathrm{global}}(\varphi_1, \varphi_1'; \varphi_0, \varphi_0') &= \langle \varphi_1|_{G_0}, \varphi_0\rangle \overline{\langle \varphi_1'|_{G_0}, \varphi_0'\rangle}, \\ L_{\pi_1,\pi_0}^{\mathrm{local}}(\varphi_1, \varphi_1'; \varphi_0, \varphi_0') &= \prod_v \alpha_v(\varphi_{1,v}, \varphi_{1,v}'; \varphi_{0,v}, \varphi_{0,v}'). \end{aligned}$$

Then Conjecture 1.5 can be reformulated as

$$L_{\pi_1,\pi_0}^{\mathrm{global}} = 2^{\beta} \Delta_{G_1} \mathcal{P}_{\pi_1,\pi_0}(1/2) L_{\pi_1,\pi_0}^{\mathrm{local}}.$$

2. Relation to the Arthur conjecture

This section is devoted to a somewhat speculative argument based on the Arthur conjecture [3]. We recall the Arthur conjecture for automorphic representation of reductive algebraic groups. We assume, for simplicity, G is a reductive algebraic group defined over k with anisotropic center. The local Langlands group $\mathcal{L}_v$ is defined by

$$\mathcal{L}_v = \begin{cases} W_{k_v} \times \mathrm{SU}(2) & \text{if } v \text{ is non-archimedean,} \\ W_{k_v} & \text{if } v \text{ is archimedean,} \end{cases}$$

where W_{k_v} is the Weil group of k_v. A Langlands parameter is a homomorphism $\phi_v : \mathcal{L}_v \to {}^L G$ which satisfies certain additional conditions. Two Langlands parameters are equivalent if they are conjugate by an element of $\hat{G}$. Langlands conjectured that for each equivalence class of Langlands parameter, one can associate a finite set $\Pi_{\phi_v}(G)$ of irreducible admissible representations of G_v. The finite set $\Pi_{\phi_v}(G)$ is called the L-packet for ϕ_v. The set $\Pi(G_v)$ of all equivalence classes of irreducible admissible representations of G_v should be decomposed into a disjoint union

$$\Pi(G_v) = \coprod_{\phi_v} \Pi_{\phi_v}(G),$$

where ϕ_v extends over the equivalence classes of Langlands parameters. The L-packet $\Pi_{\phi_v}(G)$ should contain a tempered representation if and only if the Langlands parameter ϕ_v has a bounded image, in which case ϕ_v is called tempered. If ϕ_v is tempered, then all members of $\Pi_{\phi_v}(G)$ should be tempered.

A homomorphism $\psi_v : \mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ whose restriction to $\mathrm{SL}_2(\mathbb{C})$ is holomorphic is called a (local) Arthur parameter if $\psi_v|_{\mathcal{L}_v}$ is a tempered Langlands parameter. One can consider the equivalence of Arthur parameters as in the case of Langlands parameters. Arthur conjectured that for each equivalence class of Arthur parameters ψ_v, one can associate a finite set of unitary representations $\Pi_{\psi_v}(G)$. The set $\Pi_{\psi_v}(G)$ is called the A-packet of ψ_v. A-packets are not necessarily disjoint.

For each representation ρ_v of $\mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C})$, we associate an L-factor as follows. We may assume ρ_v is irreducible. Then there exists an irreducible representation ϕ_v of $\mathcal{L}_v$ and an integer $t \geq 0$ such that

$$\rho_v \simeq \phi_v \boxtimes \mathrm{Sym}^t,$$

where Sym^t is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of degree $t+1$. We put

$$L(s,\rho_v) = \prod_{j=0}^{t} L(s-j+(t/2),\phi_v).$$

For each element $\pi_v \in \Pi_{\psi_v}(G)$ and a finite-dimensional representation r of ${}^LG$, we put $L(s,\pi_v,r) = L(s, r\circ\psi_v)$. Note that $L(s,\pi_v,r)$ may depend not only on π_v, but also on ψ_v, although the symbol suggests it does not.

Langlands conjectured that there exists a locally compact group $\mathcal{L}_k$ such that the equivalence classes of irreducible n-dimensional representation of $\mathcal{L}_k$ is in one-to-one correspondence with the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. There should be a homomorphism $\iota_v : \mathcal{L}_v \to \mathcal{L}_k$ for each v. A (global) Arthur parameter is a certain equivalence class of homomorphisms

$$\psi : \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$$

such that the image of $\mathcal{L}_k$ is bounded. Let $\Pi_\psi(G)$ be the set of square-integrable automorphic representations $\pi \simeq \otimes_v \pi_v$ of $G(\mathbb{A})$ such that $\pi_v \in \Pi_{\psi\circ\iota_v}(G)$ for each v. The set $\Pi_\psi(G)$ is called the A-packet of ψ. Arthur conjectured that the set of square-integrable automorphic representations of $G(\mathbb{A})$ is a union

$$\bigcup_{\psi} \Pi_\psi(G).$$

If $\pi \in \Pi_\psi(G)$, then ψ is called the Arthur parameter of π. In general, ψ is not uniquely determined by the equivalence class of π, but for special orthogonal groups or unitary groups, ψ should be determined by π.

It is believed that the Arthur parameter $\psi : \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ associated with a square-integrable automorphic representation should be elliptic in the sense that $\mathrm{Im}(\psi)$ is not contained in any proper Levi subgroup of ${}^LG$. This is the case if and only if $\mathrm{Cent}_{\hat{G}}(\mathrm{Im}(\psi))$ is finite. If ψ is an elliptic Arthur parameter such that $\Pi_\psi(G)$ is non-empty, the A-packet $\Pi_\psi(G)$ consists of only irreducible tempered cuspidal automorphic representations if and only if the restriction $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial. In this case, the Arthur parameter ψ said to be tempered. For an elliptic Arthur parameter ψ, we put

$$\mathcal{S}_\psi = \mathrm{Cent}_{\hat{G}}(\mathrm{Im}(\psi)).$$

Now we go back to the situation that $G_1 = \mathrm{SO}(n+1)$ and $G_0 = \mathrm{SO}(n)$. Let ψ_i be an elliptic Arthur parameter for the group G_i. In this case, the group $\mathcal{S}_{\psi_i}$ can be calculated as follows. Let st be the standard representation of ${}^L G_i$. Then $\mathrm{st} \circ \psi_i$ can be decomposed into a direct sum of irreducible representations of $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C})$:

$$\mathrm{st} \circ \psi_i = \bigoplus_{j=1}^{r} \psi_i^{(j)}.$$

Here, the representations $\psi_i^{(1)}, \ldots, \psi_i^{(r)}$ are mutually distinct orthogonal (resp. symplectic) representations of $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C})$ if $\dim V_i$ is even (resp. odd). Then

$$\mathcal{S}_{\psi_i} \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } \dim V_i \text{ is even and } \operatorname{rank} \psi_i^{(j)} \text{ is odd for some } j. \\ (\mathbb{Z}/2\mathbb{Z})^{r} & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{S}_{\psi_i}$ is an elementary 2-abelian group.

Now we admit the Arthur conjecture. Let π_i be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$, which satisfies the condition (ALG). Then corresponding Arthur parameter ψ_i must be tempered, since otherwise $\pi_{i,v}$ cannot be generic for any v.

Conjecture 2.1. Assume that π_i is an irreducible tempered cuspidal automorphic representation of $G_i(\mathbb{A})$ with Arthur parameter ψ_i. Then the constant 2^β in Conjecture 1.5 should be equal to $1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$. Equivalently, the equation

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{C_0 \Delta_{G_1}}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|} \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

holds.

3. The non-tempered case

Let $\pi_{i,v}$ be an irreducible representation of $G_{i,v}$, which we do not assume to be unitary for a moment. Note that if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ gives an element of

$$\mathrm{Hom}_{G_{0,v} \times G_{0,v}}((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C}),$$

where $\tilde{\pi}_{i,v}$ is the contragredient of $\pi_{i,v}$.

Conjecture 3.1. The quantity $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ should be somehow "analytically continued" for any $\pi_{1,v}$ and $\pi_{0,v}$. If $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$, then the continuation $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ is unique and gives an element of

$$\mathrm{Hom}_{G_{0,v} \times G_{0,v}}((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C}).$$

Now we consider the global situation. Let π_i be an square-integrable automorphic representation of $G_i(\mathbb{A})$, which may not be almost locally generic. We assume that $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any v. For $v \notin S$, we may assume $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$ by Theorem 1.2, as long as it is meaningful.

Conjecture 3.2. Let π_i be as above. Then

(1) The integral $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ should be convergent for any $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$.

(2) There should be an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero decomposable vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

Remark 3.3. Contrary to the almost locally generic case, the factor 2^β depends not only on global data, but also on local data. See the examples in §8, §9, §10, and §14.

Part II. Local theory

Until §5, we consider only local objects and drop subscript v.

4. Convergence of the integral: Proof of Proposition 1.1

In this section, we assume that k is a local field with $\mathrm{char}(k) \neq 2$. Let (V, Q) be a non-degenerate quadratic space over k. We denote the anisotropic kernel of (V, Q) by $(V^{\mathrm{an}}, Q^{\mathrm{an}})$. Then there is a decomposition $V = X \oplus V^{\mathrm{an}} \oplus Y$, where X and Y are totally isotropic subspaces. The Witt rank r of (V, Q) is, by definition, equal to the dimension of X or Y. We put $d = \dim V^{\mathrm{an}}$. Choosing a basis of X, we get a minimal parabolic subgroup $P_{\min} = M_{\min} N_{\min}$ of G. The Levi factor $M_{\min}$ is isomorphic to $(k^\times)^r \times \mathrm{SO}_{Q^{\mathrm{an}}}$. The split component $A_{\min}$ of $M_{\min}$ is isomorphic to $(k^\times)^r$, and the Weyl group $W(G, A_{\min})$ is of type B or D according as $d \neq 0$ or $d = 0$. We will denote an element of $A_{\min} \simeq (k^\times)^r$ by $x = (x_1, \ldots, x_r)$. The simple roots of $(P_{\min}, A_{\min})$ are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \ldots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1},$$

$$\alpha_r(x) = \begin{cases} x_r & \text{if } d \neq 0 \\ x_{r-1} x_r & \text{if } d = 0. \end{cases}$$

These roots are also regarded as a character of $M_{\min}$. Let $\delta_{P_{\min}}(x)$ be modulus character of $P_{\min}$. Then

$$\delta_{P_{\min}}(x) = \prod_{i=1}^{r} |x_i|^{d+2r-2i}.$$

Fix a special maximal compact subgroup $\mathcal{K}$ of G. Then we have a Cartan decomposition $G = \mathcal{K}M_{\min}^{+}\mathcal{K}$, where

$$M_{\min}^{+} = \{m \in M_{\min} | \, |\alpha_i(m)| \leq 1 \, (i = 1, \dots, r)\}.$$

Fix a suitable embedding $\eta : G \to \mathrm{GL}_m$. Then the height function $\sigma(g)$ (with respect to the embedding η) is given by

$$\sigma(g) = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} (\log |\eta(g)_{ij}|, \log |\eta(g^{-1})_{ij}|).$$

When k is non-archimedean, the following integral formula holds

$$\int_G f(g)dg = \int_{M_{\min}^{+}} \mu(m) \int_{\mathcal{K}\times\mathcal{K}} f(k_1 m k_2) dk_1 dk_2 dm, \quad f \in L^1(G)$$

where $\mu(m) = \mathrm{Vol}(\mathcal{K}m\mathcal{K})/\mathrm{Vol}(\mathcal{K})$. Moreover, there are positive constants A such that $A^{-1}\delta_{P_{\min}}^{-1}(m) \leq \mu(m) \leq A\delta_{P_{\min}}^{-1}(m)$ for $m \in M_{\min}^{+}$. (See Silberger [62] p. 149.)

When k is archimedean, similar integral formula holds. (See e.g., Helgason, [27], Theorem 5.8.) In particular, there exists a non-negative function $\mu(m)$ on $M_{\min}^{+}$ such that

$$\int_G f(g)dg = \int_{M_{\min}^{+}} \mu(m) \int_{\mathcal{K}\times\mathcal{K}} f(k_1 m k_2) dk_1 dk_2 dm, \quad f \in L^1(G).$$

Moreover, there exists a constant $A > 0$ such that $\mu(m) \leq A\delta_{P_{\min}}^{-1}(m)$ for $m \in M_{\min}^{+}$.

Harish-Chandra's spherical function $\Xi(g)$ of G is given by

$$\Xi(g) = \int_{\mathcal{K}} h_0(kg)dk$$

where $h_0 \in \mathrm{Ind}_{P_{\min}}^{G} 1$ is a function whose restriction to $\mathcal{K}$ is identically equal to 1. Note that Ξ is a matrix coefficient of a tempered representation $\mathrm{Ind}_{P_{\min}}^{G} 1$. It is known that there exists positive constants A, B such that

$$A^{-1}\delta_{P_{\min}}^{1/2}(m) \leq \Xi(m) \leq A\delta_{P_{\min}}^{1/2}(m)(1 + \sigma(m))^B$$

for any $m \in M_{\min}^{+}$. (See Silberger [62], p. 154, Theorem 4.2.1 and Harish-Chandra [20] p.129, Lemma 1 in section 10.)

Recall that a function $f(g)$ on G satisfies the weak inequality if

$$|f(g)| \leq A\Xi(g)(1+\sigma(g))^B$$

for some positive constant A, B. A matrix coefficient of a tempered representation satisfies the weak inequality.

Applying these results for $G_1 = \mathrm{SO}(n+1)$ and $G_0 = \mathrm{SO}(n)$, we can now prove Proposition 1.1. As before, we define $P_{i,\min}$, $A_{i,\min}$, r_i, etc. for the group G_i.

Proof of Proposition 1.1. Let π_1 and π_0 be irreducible tempered representations of G_1 and G_0, respectively. We may assume $A_{0,\min} \subset A_{1,\min}$. Then we have an estimate

$$|\Phi_{\varphi_1,\varphi_1'}(m)| \leq A\delta_{P_{1,\min}}^{1/2}(m)(1+\sigma(m))^B, \qquad (m \in M_{1,\min}^+),$$
$$|\Phi_{\varphi_0,\varphi_0'}(m)| \leq A\delta_{P_{0,\min}}^{1/2}(m)(1+\sigma(m))^B, \qquad (m \in M_{0,\min}^+)$$

for some positive constants A, B. When $W(G_0, A_{0,\min})$ is of type B, it is enough to show the following integral

$$\int_{A_{0,\min}^+} \delta_{P_{0,\min}}^{-1/2}(m)\delta_{P_{1,\min}}^{1/2}(m)(1+\sigma(m))^{2B} dm$$

is convergent. This is reduced to the convergence of

$$\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0}|\leq 1} |x_1x_2\cdots x_{r_0}|^{1/2}(1-\sum_{j=1}^{r_0}\log|x_j|)^{2B} d^\times x_1 d^\times x_2 \cdots d^\times x_{r_0}.$$

One can easily prove the convergence of this integral. Note that when $W(G_0, A_{0,\min})$ is of type D, $A_{0,\min}^+$ is not contained in $A_{1,\min}^+$. In this case, one need to consider the integral

$$\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0}|\leq 1} |x_1x_2\cdots x_{r_0}|^{1/2}(1-\sum_{j=1}^{r_0}\log|x_j|)^{2B} d^\times x_1\, d^\times x_2 \cdots d^\times x_{r_0}$$
$$+ \int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_0-1}|\leq|x_{r_0}|^{-1}\leq 1} |x_1x_2\cdots x_{r_0-1}x_{r_0}^{-1}|^{1/2}$$
$$\times (1-\sum_{j=1}^{r_0-1}\log|x_j| + \log|x_{r_0}|)^{2B} d^\times x_1\, d^\times x_2 \cdots d^\times x_{r_0}.$$

One can show the convergence of this integral similarly.

To prove the latter part of the proposition, we make use of the result of He [26]. Let Ξ_1 and Ξ_0 by Harish-Chandra's spherical function for G_1 and G_0, respectively. Then the function $g_0 \mapsto \Xi_1(g_0)\Xi_0(g_0)$ belongs to $L^1(G_0)$ by the first part of the proposition. Note that Harish-Chandra's spherical function is a matrix coefficient of a tempered representation.

Then the latter part of the proposition follows from Theorem 2.1 of He's paper [26]. Note that He [26] used the estimates of almost L^2 matrix coefficients [8], which is valid for p-adic groups as well.

□

5. Calculation of the unramified integral: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We assume the conditions (U1) – (U6) in §1 holds. In particular, both G_1 and G_0 are quasi-split. We should consider the following two cases:

(Case A) $\qquad G_1 = \mathrm{SO}(2l+1)$ and $G_0 = \mathrm{SO}(2l)$,

(Case B) $\qquad G_1 = \mathrm{SO}(2l+2)$ and $G_0 = \mathrm{SO}(2l+1)$.

Let K be the discriminant field. Note that K is equal to either k or the unramified quadratic extension of k. Let q be the number of elements of the residue field of k. The local zeta function $\zeta(s)$ is defined by $(1-q^{-s})^{-1}$.

Let $B_i = T_i N_i$ be a Borel subgroup of G_i, where T_i and N_i are a maximal torus of G_i and the unipotent radical of B_i, respectively. Let $A_i \subset T_i$ be the maximal split subtorus. Without loss of generality, we may assume $N_0 \subset N_1$ and $A_0 \subset A_1$.

Let $\pi_1 = I(\Xi) = \mathrm{Ind}_{B_1}^{G_1}(\Xi)$ and $\pi_0 = I(\xi) = \mathrm{Ind}_{B_0}^{G_0}(\xi)$ be unramified principal series of G_1 and G_0, respectively. Here, Ξ and ξ are unramified quasi-characters of T_1 and T_0, respectively. Let Φ_Ξ and Φ_ξ be the class-one matrix coefficients of $I(\Xi)$ and $I(\xi)$ such that $\Phi_\Xi(1) = \Phi_\xi(1) = 1$, respectively. We consider the integral

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \int_{G_0} \Phi_\Xi(g_1^{-1} g_0) \Phi_\xi(g_0) dg_0.$$

We assume that both Ξ and ξ are sufficiently close to the unitary axis. As shown in §4, this condition implies that the integral $I(g_1; \Phi_\Xi, \Phi_\xi)$ is absolutely convergent. In this section, we calculate the value of $I(g_1; \Phi_\Xi, \Phi_\xi)$ at $g_1 = 1$.

Let $f_\Xi \in I(\Xi)$ and $f_\xi \in I(\xi)$ be the class-one vectors such that $f_\Xi(1) = f_\xi(1) = 1$. Then we have

$$\Phi_\Xi(g_1) = \int_{\mathcal{K}_1} f_\Xi(k_1 g_1) dk_1, \quad g_1 \in G_1,$$
$$\Phi_\xi(g_0) = \int_{\mathcal{K}_0} f_\Xi(k_0 g_0) dk_0, \quad g_0 \in G_0.$$

We recall the theory of Shintani functions [36]. We denote the Hecke algebra $\mathcal{H}(\mathcal{K}_i\backslash G_i/\mathcal{K}_i)$ by $\mathcal{H}_i$. By the Satake isomorphism, there are algebra homomorphisms

$$\omega_1 : \mathcal{H}_1 \longrightarrow \mathbb{C}$$

and

$$\omega_0 : \mathcal{H}_0 \longrightarrow \mathbb{C}$$

corresponding to the unramified principal series π_1 and π_0, respectively. Recall that a smooth function S on G_1 is called a Shintani function for π_1 and π_0, if the following conditions are satisfied:

- $\mathcal{L}(k_0)\mathcal{R}(k_1)S = S$ for any $k_1 \in \mathcal{K}_1$ and $k_0 \in \mathcal{K}_0$.
- $\mathcal{L}(\varphi_0)\mathcal{R}(\varphi_1)S = \omega_0(\varphi_0)\omega_1(\varphi_1)S$ for any $\varphi_0 \in \mathcal{H}_0$ and $\varphi_1 \in \mathcal{H}_1$.

Here, $\mathcal{L}$ and $\mathcal{R}$ are the left regular representation and the right regular representation, respectively. S.-I. Kato, Murase, and Sugano [36] has proved that if both G_1 and G_0 are split, then a Shintani function exists and is unique up to scalar. In this paper, we do not use the uniqueness of Shintani functions.

Recall that the double coset $B_1\backslash G_1/B_0$ has a unique open orbit and the open orbit has a representative $\eta \in \mathcal{K}_1$ (cf. [14], §7). Note that $\eta^{-1}B_1\eta \cap B_0 = \{1\}$. Let $Y_{\Xi,\xi}$ be the function on G_1 determined by the following conditions.

(1) $Y_{\Xi,\xi}(b_1g_1b_0) = (\Xi^{-1}\delta_1^{1/2})(b_1)(\xi\delta_0^{-1/2})(b_0)Y_{\Xi,\xi}(g_1)$ for any $b_1 \in B_1$ and $b_0 \in B_0$.
(2) $Y_{\Xi,\xi}(\eta) = 1$.
(3) $Y_{\Xi,\xi}(g_1) = 0$ if $g_1 \notin B_1\eta B_0$.

Here, δ_i is the modulus character of B_i. Note that a function satisfying (1) and (3) is unique up to scalar. We define $l_{\Xi,\xi} \in \mathrm{Hom}_{G_0}(\pi_1, \tilde{\pi}_0) = \mathrm{Hom}_{G_0}(I(\Xi), I(\xi^{-1}))$ by

$$l_{\Xi,\xi}(\mathrm{pr}_1(f))(g_0) = \int_{G_1} f(g_1g_0)Y_{\Xi,\xi}(g_1)dg_1, \quad g_0 \in G_0.$$

Here, $\mathrm{pr}_1 : C_c^\infty(G_1) \to \pi_1 = I(\Xi)$ is given by

$$\mathrm{pr}_1(f)(g_1) = \int_{B_1} (\Xi^{-1}\delta_1^{1/2})(b_1)f(b_1g_1)db_1.$$

Let $\langle\ ,\ \rangle$ be the natural pairing on $\pi_0 \times \tilde{\pi}_0$ defined by

$$\langle\varphi_0, \varphi_0'\rangle = \int_{\mathcal{K}_0} \varphi_0(k_0)\varphi_0'(k_0)\, dk_0$$

for $\varphi_0 \in \pi_0$ and $\varphi_0' \in \tilde{\pi}_0$. Note that the defining integral of $l_{\Xi,\xi}(\mathrm{pr}_1(f))$ is convergent by [36] Proposition 4.8. Put

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1) f_\Xi)\rangle.$$

Then $S_{\Xi,\xi}$ is a Shintani function, and we have

$$\begin{aligned} S_{\Xi,\xi}(g_1) &= \int_{\mathcal{K}_0} f_\xi(k_0) \int_{G_1} \mathbf{1}_{\mathcal{K}_1}(g_1' k_0 g_1) Y_{\Xi,\xi}(g_1') dg_1' dk_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi,\xi}(k_1 g_1^{-1} k_0) dk_1 dk_0. \end{aligned}$$

Here, $\mathbf{1}_{\mathcal{K}_1}$ is the characteristic function of $\mathcal{K}_1$. Put

$$T_{\Xi,\xi}(g_1) = \begin{cases} \displaystyle\int_{G_0} f_\Xi(g_1 g_0) f_\xi(g_0) dg_0 & \text{if } g_1 \in B_1 \eta B_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $T_{\Xi,\xi}(g_1) = T_{\Xi,\xi}(\eta) \cdot Y_{\Xi^{-1},\xi^{-1}}(g_1)$, since $T_{\Xi,\xi}$ satisfies the conditions (1) and (3) for Ξ^{-1} and ξ^{-1}. Therefore we have

$$\begin{aligned} I(g_1; \Phi_\Xi, \Phi_\xi) &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1 g_1^{-1} g_0) f_\xi(k_0 g_0) dk_0 dk_1 dg_0 \\ &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1 g_1^{-1} k_0 g_0) f_\xi(g_0) dk_0 dk_1 dg_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} T_{\Xi,\xi}(k_1 g_1^{-1} k_0) dk_1 dk_0 \\ &= T_{\Xi,\xi}(\eta) \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi^{-1},\xi^{-1}}(k_1 g_1^{-1} k_0) dk_1 dk_0 \\ &= T_{\Xi,\xi}(\eta) S_{\Xi^{-1},\xi^{-1}}(g_1). \end{aligned}$$

In particular, $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(g_1)$ are convergent if Ξ and ξ are sufficiently close to the unitary axis.

We first assume that the residual characteristic of k is not 2. We consider the case when $K = k$. In this case, both $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(1)$ are already calculated. Note that

$$\begin{aligned} T_1 = A_1 &\simeq \begin{cases} (k^\times)^l & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (k^\times)^{l+1} & \text{if } G_1 = \mathrm{SO}(2l+2), \end{cases} \\ T_0 = A_0 &\simeq (k^\times)^l \qquad\qquad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1). \end{aligned}$$

We write

$$\Xi = \begin{cases} (\Xi_1, \ldots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_1, \ldots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2), \end{cases}$$
$$\xi = (\xi_1, \ldots, \xi_l) \qquad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1).$$

There exists a quadratic space $(\tilde{V}_1, \tilde{Q}_1) \subset (V_0, Q_0)$ such that V_1 is isomorphic to the direct sum of $\tilde{V}_1$ and the hyperbolic plane. Without loss of generality, we may assume that $(V_0, \tilde{V}_1)$ satisfies the conditions (U1) – (U6). Put

$$\tilde{\Xi} = \begin{cases} (\Xi_2, \ldots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_2, \ldots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2). \end{cases}$$

Since $T_{\Xi,\xi}(\eta)$ is independent of the choice of η, we set $\zeta(\Xi, \xi) = T_{\Xi,\xi}(\eta)$. By Ginzburg, Piatetski-Shapiro, and Rallis, [14], p. 22, Corollary to Lemma 1.1, we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A)} \\ 1 & \text{(Case B).} \end{cases}$$

Here, $L(s, I(\xi), \Xi_1)$ is the standard L-factor of $I(\xi)$ twisted by the character Ξ_1. By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) = & \prod_{i=1}^{l} L(1, \Xi_i^2)^{-1} \prod_{1 \le i < j \le l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i < j \le l} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i \le j \le l} L(1/2, , \Xi_i \xi_j) L(1/2, , \Xi_i \xi_j^{-1}) \\ & \times \prod_{1 \le j < i \le l} L(1/2, , \Xi_i \xi_j) L(1/2, , \Xi_i^{-1} \xi_j), \end{aligned}$$

in Case A, and

$$\zeta(\Xi,\xi) = \prod_{1\le i<j\le l+1} L(1,\Xi_i\Xi_j)^{-1}L(1,\Xi_i\Xi_j^{-1})^{-1}$$
$$\times \prod_{i=1}^{l} L(1,\xi_i^2)^{-1} \prod_{1\le i<j\le l} L(1,\xi_i\xi_j)^{-1}L(1,\xi_i\xi_j^{-1})^{-1}$$
$$\times \prod_{1\le i\le j\le l} L(1/2,,\Xi_i\xi_j)L(1/2,,\Xi_i\xi_j^{-1})$$
$$\times \prod_{1\le j<i\le l+1} L(1/2,,\Xi_i\xi_j)L(1/2,,\Xi_i^{-1}\xi_j),$$

in Case B. On the other hand, Theorem 10.8 of [36] implies

$$S_{\Xi^{-1},\xi^{-1}}(1)$$
$$=\Delta_{G_1}\zeta(1)^{-2l}\prod_{i=1}^{l} L(1,\Xi_i^{-2})^{-1} \prod_{1\le i<j\le l} L(1,\Xi_i^{-1}\Xi_j^{-1})^{-1}L(1,\Xi_i^{-1}\Xi_j)^{-1}$$
$$\times \prod_{1\le i<j\le l} L(1,\xi_i^{-1}\xi_j^{-1})^{-1}L(1,\xi_i^{-1}\xi_j)^{-1}$$
$$\times \prod_{1\le i\le j\le l} L(1/2,,\Xi_i^{-1}\xi_j^{-1})L(1/2,,\Xi_i^{-1}\xi_j)$$
$$\times \prod_{1\le j<i\le l} L(1/2,,\Xi_i^{-1}\xi_j^{-1})L(1/2,,\Xi_i\xi_j^{-1}),$$

in Case A, and

$$S_{\Xi^{-1},\xi^{-1}}(1)$$
$$=\Delta_{G_1}\zeta(1)^{-2l-1} \prod_{1\le i<j\le l+1} L(1,\Xi_i^{-1}\Xi_j^{-1})^{-1}L(1,\Xi_i^{-1}\Xi_j)^{-1}$$
$$\times \prod_{i=1}^{l} L(1,\xi_i^{-2})^{-1} \prod_{1\le i<j\le l} L(1,\xi_i^{-1}\xi_j^{-1})^{-1}L(1,\xi_i^{-1}\xi_j)^{-1}$$
$$\times \prod_{1\le i\le j\le l} L(1/2,\Xi_i^{-1}\xi_j^{-1})L(1/2,,\Xi_i^{-1}\xi_j)$$
$$\times \prod_{1\le j<i\le l+1} L(1/2,,\Xi_i^{-1}\xi_j^{-1})L(1/2,,\Xi_i\xi_j^{-1})$$

in Case B. Combining these results, we have

$$I(1;\Phi_\Xi,\Phi_\xi) = \Delta_{G_1}\mathcal{P}_{\pi_1,\pi_0}(1/2),$$

when both G_1 and G_0 are split. Thus we have proved Theorem 1.2 in the case $2 \nmid q$ and both G_1 and G_0 are split.

Now we consider the case when the discriminant field K is equal to the unramified quadratic extension of k. Note that the character χ associated to $k^\times$ is equal to the unique unramified quasi-character of order 2. As in the split case, we should consider the following two cases:

$$\text{(Case A)} \qquad G_1 = \mathrm{SO}(2l+1) \text{ and } G_0 = \mathrm{SO}(2l),$$
$$\text{(Case B)} \qquad G_1 = \mathrm{SO}(2l+2) \text{ and } G_0 = \mathrm{SO}(2l+1).$$

Note that

$$\begin{cases} A_1 \simeq (k^\times)^l, A_0 \simeq (k^\times)^{l-1} & \text{(Case A)}, \\ A_1 \simeq A_0 \simeq (k^\times)^l & \text{(Case B)}. \end{cases}$$

The unramified characters Ξ and ξ are determined by their restriction to A_1 and A_0, respectively. We write

$$\Xi = (\Xi_1, \ldots, \Xi_l)$$
$$\xi = \begin{cases} (\xi_1, \ldots, \xi_{l-1}) & \text{(Case A)}, \\ (\xi_1, \ldots, \xi_l) & \text{(Case B)}. \end{cases}$$

Put $\tilde{\Xi} = (\Xi_2, \ldots, \Xi_l)$. We set $\zeta(\Xi, \xi) = T_{\Xi,\xi}(\eta)$. As before, we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A)} \\ 1 & \text{(Case B)} \end{cases}$$

by [14], p. 22, Corollary to Lemma 1.1. By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) = & \prod_{i=1}^{l} L(1, \Xi_i^2)^{-1} \prod_{1 \le i < j \le l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ & \times \prod_{i=1}^{l-1} L(1, \xi_i)^{-1} L(1, \chi \xi_i)^{-1} \prod_{1 \le i < j \le l-1} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ & \times \prod_{1 \le i \le j \le l-1} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^{l-1} L(1/2, \Xi_i) L(1/2, \chi \Xi_i) \\ & \times \prod_{1 \le j < i \le l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case A, and

$$\zeta(\Xi,\xi) = \prod_{i=1}^{l} L(1,\Xi_i)^{-1} L(1,\chi\Xi_i)^{-1} \prod_{1\le i<j\le l} L(1,\Xi_i\Xi_j)^{-1} L(1,\Xi_i\Xi_j^{-1})^{-1}$$
$$\times \prod_{i=1}^{l} L(1,\xi_i^2)^{-1} \prod_{1\le i<j\le l} L(1,\xi_i\xi_j)^{-1} L(1,\xi_i\xi_j^{-1})^{-1}$$
$$\times \prod_{1\le i\le j\le l} L(1/2,\Xi_i\xi_j) L(1/2,\Xi_i\xi_j^{-1}) \prod_{i=1}^{l} L(1/2,\xi_i) L(1/2,\chi\xi_i)$$
$$\times \prod_{1\le j<i\le l} L(1/2,\Xi_i\xi_j) L(1/2,\Xi_i^{-1}\xi_j)$$

in Case B. As for $S_{\Xi,\xi}(1)$, we can prove the following lemma.

Lemma 5.1. *We have*

$$S_{\Xi,\xi}(1) = \Delta_{G_1} \zeta(1)^{-\dim A_1 - \dim A_0} L(1,\chi)^{-1} \zeta(\Xi,\xi).$$

The proof of this lemma will be given in the appendix to this section. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(1/2) = \zeta(1)^{-\dim A_1 - \dim A_0} L(1,\chi)^{-1} \zeta(\Xi,\xi) \zeta(\Xi^{-1},\xi^{-1}).$$

We would like to emphasise that this relation has been already noted by Ginzburg, Piatetski-Shapiro, and Rallis [14]. Combining these results, we have $I(1;\Phi_\Xi,\Phi_\xi) = \Delta_{G_1}\mathcal{P}_{\pi_1,\pi_0}(1/2)$. Thus we have proved Theorem 1.2 in the case $2 \nmid q$.

Now we consider the case $2|q$. It is enough to prove that $I(1;\Phi_\Xi,\Phi_\xi)$ is an element of $\mathbb{Q}(q^{1/2},\Xi,\xi)$. More precisely, we will show that there exists a rational function $\mathcal{I}(t,X_1,\dots,x_1,\dots) \in \mathbb{Q}(t,X_1,\dots,x_1,\dots)$, where $t,X_1,\dots,x_1,\dots$ are indeterminants, such that if the order of residue field of k is q, then

$$I(1;\Phi_\Xi,\Phi_\xi) = \mathcal{I}(q^{1/2},\Xi_1,\dots,\xi_1,\dots).$$

To prove this, we make use of Macdonald's formula for the spherical function. Recall that Macdonald's formula ([6], p. 403, Theorem 4.2) says that the spherical function Φ_Ξ and Φ_ξ is of the form

$$\Phi_\Xi(m_1) = Q_1^{-1} \sum_{w_1\in W_1} \gamma_1(w_1\Xi)\cdot((w_1\Xi)\delta_1^{-1/2})(m_1), \qquad m_1 \in A_1^+,$$
$$\Phi_\xi(m_0) = Q_0^{-1} \sum_{w_0\in W_0} \gamma_0(w_0\xi)\cdot((w_0\xi)\delta_0^{-1/2})(m_0), \qquad m_0 \in A_0^+.$$

Here, $Q_1, Q_0, \gamma_1(\Xi), \gamma_0(\xi) \in \mathbb{Q}(q^{1/2}, \Xi, \xi)$ and δ_i is the modulus function of the Borel subgroup B_i. The integral $I(1; \Phi_\Xi, \Phi_\xi)$ is equal to

$$\int_{A_0^+} \Phi_\Xi(m_0)\Phi_\xi(m_0)\mathrm{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0) dm_0.$$

Note that $\mathrm{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0) = [\mathcal{K}_0 : \mathcal{K}_0 \cap m_0 \mathcal{K}_0 m_0^{-1}]$. One can show easily this integral gives an element of $\mathbb{Q}(q^{1/2}, \Xi, \xi)$. Therefore the proof of Theorem 1.2 is complete.

Appendix to §5: Proof of Lemma 5.1

In this appendix, we prove Lemma 5.1. The proof of Lemma 5.1 consists of three steps.

Step 1. The Weyl invariance.

The Weyl group $W_1 \times W_0$ acts on the character group of $A_1 \times A_0$ by $(\Xi, \xi) \mapsto (w_1\Xi, w_0\xi)$.

Lemma 5.2. *The quantity $S_{\Xi,\xi}(g_1)\zeta(\Xi, \xi)^{-1}$ is $W_1 \times W_0$-invariant as a function of Ξ and ξ. (cf.* [36] *Theorem 10.8.)*

Proof. Note that both $\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})$ and

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \zeta(\Xi, \xi) S_{\Xi^{-1},\xi^{-1}}(g_1)$$

are $W_1 \times W_0$-invariant. It follows that

$$\frac{I(g_1; \Phi_\Xi, \Phi_\xi)}{\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})} = \frac{S_{\Xi^{-1},\xi^{-1}}(g_1)}{\zeta(\Xi^{-1}, \xi^{-1})}$$

is also $W_1 \times W_0$-invariant. Hence the lemma. □

Step 2. An explicit formula for $S_{\Xi,\xi}(g_1)$.

Now we closely follow the argument of [36]. Fix a hyperspecial maximal compact subgroups $\mathcal{K}_i \subset G_i$ and a maximal split torus $A_i \subset G_i$. Then the centralizer T_i of A_i is a maximally split maximal torus of G_i. We assume $\mathcal{K}_0 \subset \mathcal{K}_1$ and $A_0 \subset A_1$. Note that T_0 need not be a subgroup of T_1. Choose a Borel subgroup $B_i = T_i N_i \subset G_i$. We also assume $N_0 \subset N_1$. The opposite Borel subgroup of $B_i = T_i N_i$ is denoted by $\bar{B}_i = T_i \bar{N}_i$. We put $T_i^{(0)} = T_i \cap \mathcal{K}_i$, $N_i^{(0)} = N_i \cap \mathcal{K}_i$, and $\bar{N}_i^{(0)} = \bar{N}_i \cap \mathcal{K}_i$. Choose a longest element $w_{i,\mathrm{long}}$ of the Weyl group $W_i = W(G_i, A_i)$. We assume $w_{i,\mathrm{long}} \in \mathcal{K}_i$. There exists a Iwahori subgroup $\mathcal{B}_i \subset \mathcal{K}_i$ such that $N_i^{(0)} \subset \mathcal{B}_i$. We put $\bar{N}_i^{(1)} = \bar{N}_i \cap \mathcal{B}_i$ and $N_i^{(1)} = w_{i,\mathrm{long}}^{-1} \bar{N}_i^{(1)} w_{i,\mathrm{long}}$. Then we have an Iwahori decomposition $\mathcal{B}_i = \bar{N}_i^{(1)} T_i^{(0)} N_i^{(0)}$.

Recall that the element $\eta \in G_1$ is a representative of the unique open orbit of $B_1 \backslash G_1 / B_0$ such that $\eta \in \mathcal{K}_1$. Let $\mathfrak{o}$ and $\mathfrak{o}_K$ be the ring of integers of k and K, respectively. The maximal ideal of $\mathfrak{o}$ and $\mathfrak{o}_K$ are denoted by $\mathfrak{p}$ and $\mathfrak{p}_K$, respectively.

Lemma 5.3. *One can choose the representative η of the open orbit of $B_1\backslash G_1/B_0$ such that the following conditions hold.*

(1) $\eta\bar{N}_0^{(1)} \subset \mathcal{B}_1\eta$,
(2) $\bar{N}_1^{(1)}\eta \subset T_1^{(0)}N_1^{(0)}\eta T_0^{(0)}N_0^{(0)}$.

Proof. We first consider Case B. Note that in this case N_0 is a normal subgroup of N_1. By [14], p.171, Lemma 7.1, N_1/N_0 is isomorphic to $k^{l-1} \times (K/k)$ as a left module of $A_0 = A_1 \simeq (k^\times)^l$. We fix an isomorphism $N_1/N_0 \simeq k^{l-1} \times (K/k)$, which induces an isomorphism $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^{l-1} \times (\mathfrak{o}_K/\mathfrak{o})$. Since K/k is unramified, $\mathfrak{o}_K/\mathfrak{o}$ is isomorphic to $\mathfrak{o}$, and so $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. There exists a cross section (i.e., "épinglage") ι of the map $N_1^{(0)} \to N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. Let η' be the image of the cross section of $(1,1,\dots,1) \in \mathfrak{o}^l$. We put $\eta = w_{1,\mathrm{long}}\,\eta'$. Then η is a representative of the open orbit of $B_1\backslash G_1/B_0$. Let $\mathcal{U}_1$ be the group generated by $N_1^{(1)}$ and $\bar{N}_1^{(1)}$. Then $\mathcal{U}_1$ is a normal subgroup of $\mathcal{K}_1$. It follows that $\eta\bar{N}_0^{(1)} \subset \eta\mathcal{U}_1 = \mathcal{U}_1\eta \subset \mathcal{B}_1\eta$. As for (2), $\bar{N}_1^{(1)}\eta = w_{1,\mathrm{long}}N_1^{(1)}\eta' \subset w_{1,\mathrm{long}}\iota(\mathfrak{p}^l)\eta' N_0^{(1)}$. It suffices to prove that $\iota(\mathfrak{p}^l)\eta' \subset T_1^{(0)}\eta' T_0^{(0)}$. This is easily seen by the facts $1+\mathfrak{p} \subset \mathfrak{o}^\times$.

Now we consider Case A. Let P_1 be the standard parabolic subgroup of G_1 with Levi factor $(k^\times)^{l-1} \times \mathrm{SO}(3) \simeq (k^\times)^{l-1} \times \mathrm{PGL}_2$. Let N_{P_1} be the unipotent radical of P_1. Then as in Case B, N_{P_1}/N_0 is isomorphic to k^{l-1} as a left module of $A_0 \simeq (k^\times)^{l-1}$. We fix an isomorphism $N_{P_1}/N_0 \simeq k^{l-1}$, which induces an isomorphism $(N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Take a cross section ι of the map $(N_{P_1} \cap N_1^{(0)}) \to (N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Put $\eta = w_{1,\mathrm{long}}\iota((1,1,\dots,1))$. Then η is a representative of the open orbit of $B_1\backslash G_1/B_0$, since $\mathrm{PGL}_2 = (\mathrm{PGL}_2 \cap N_1)\cdot(\mathrm{PGL}_2 \cap T_0)$ (cf. [14], Appendix 1 to §7). One can prove (1) in the same way as in Case B. As for (2), observe that $\bar{N}_1^{(1)} = (\bar{N}_1^{(1)} \cap \bar{N}_{P_1})\cdot(\bar{N}_1^{(1)} \cap \mathrm{PGL}_2)$, where $\bar{N}_{P_1}$ is the unipotent radical of the opposite parabolic subgroup of P_1 with respect to the Levi subgroup $(k^\times)^{l-1} \times \mathrm{PGL}_2$. One can prove that $(\bar{N}_1^{(1)} \cap \bar{N}_{P_1})\eta \subset T_1^{(0)}\eta T_0^{(0)}N_0^{(0)}$ in the same way as in Case B. Now (2) follows from the fact $(T_1^{(0)}N_1^{(0)} \cap \mathrm{PGL}_2)\cdot(T_0^{(0)} \cap \mathrm{PGL}_2) = \mathcal{K}_1 \cap \mathrm{PGL}_2$. □

Lemma 5.4. *We have*

$$\mathcal{B}_0\eta^{-1}\mathcal{B}_1 \subset T_0^{(0)}N_0^{(0)}\eta^{-1}T_1^{(0)}N_1^{(0)}.$$

Proof. By Lemma 5.3, we have

$$\begin{aligned}\mathcal{B}_0\eta^{-1}\mathcal{B}_1 &= T_0^{(0)}N_0^{(0)}\bar{N}_0^{(1)}\eta^{-1}\mathcal{B}_1\\ &\subset T_0^{(0)}N_0^{(0)}\eta^{-1}\mathcal{B}_1\\ &= T_0^{(0)}N_0^{(0)}\eta^{-1}\bar{N}_1^{(1)}T_1^{(0)}N_1^{(0)}\\ &\subset T_0^{(0)}N_0^{(0)}\eta^{-1}T_1^{(0)}N_1^{(0)}.\end{aligned}$$

□

Put

$$\begin{aligned}A_1^+ &= \{t \in A_1 \,|\, |\alpha(t)| \le 1 \text{ for any positive root } \alpha \text{ of } (G_1, A_1)\},\\ A_0^+ &= \{t \in A_0 \,|\, |\alpha(t)| \le 1 \text{ for any positive root } \alpha \text{ of } (G_0, A_0)\}.\end{aligned}$$

Then we have Cartan decompositions $G_1 = \mathcal{K}_1 A_1^+ \mathcal{K}_1$, $G_0 = \mathcal{K}_0 A_0^+ \mathcal{K}_0$.

For each positive root α of G_1 (resp. G_0), we denote Harish-Chandra's c-function (cf. e.g., Casselman [6]) by $c_\alpha(\Xi)$ (resp. $c_\alpha(\xi)$). We put

$$\bar{c}_{w_1}(\Xi) = \prod_{\substack{\alpha>0\\ w_1\alpha>0}} c_\alpha(\Xi), \qquad \bar{c}_{w_0}(\xi) = \prod_{\substack{\alpha>0\\ w_0\alpha>0}} c_\alpha(\xi).$$

Lemma 5.5. *There exists a basis $\{g_{1,w_1}\}_{w_1\in W_1}$ of $I(\Xi)^{\mathcal{B}_1}$ with the following properties.*

(1_1) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t^{-1}\mathcal{B}_1})g_{1,w_1} = \mathrm{Vol}(\mathcal{B}_1 t \mathcal{B}_1)\cdot(w_1\Xi)^{-1}\delta_1^{1/2}(t)\cdot g_{1,w_1}$ *for any* $t \in A_1^+$.

(2_1) *The restriction of $g_{1,1}$ to $\mathcal{K}_1$ is the characteristic function of $\mathcal{B}_1$.*

(3_1) $f_\Xi = [N_1^{(0)} : N_1^{(1)}]\sum_{w_1\in W_1}\bar{c}_{w_1}(\Xi)\cdot g_{1,w_1}$.

Similarly, there exists a basis $\{g_{0,w_0}\}_{w_0\in W_0}$ of $I(\xi)^{\mathcal{B}_0}$ with the following properties.

(1_0) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_0 t^{-1}\mathcal{B}_0})g_{0,w_0} = \mathrm{Vol}(\mathcal{B}_0 t \mathcal{B}_0)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t)\cdot g_{0,w_0}$ *for any* $t \in A_0^+$.

(2_0) *The restriction of $g_{0,1}$ to $\mathcal{K}_0$ is the characteristic function of $\mathcal{B}_0$.*

(3_0) $f_\xi = [N_0^{(0)} : N_0^{(1)}]\sum_{w_0\in W_0}\bar{c}_{w_0}(\xi)\cdot g_{0,w_0}$.

Proof. See [36] p. 8, Proposition 1.10. □

Lemma 5.6. *We have*

$$\begin{aligned}S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1}) =& \mathrm{Vol}(\mathcal{B}_0 t_0^{-1}\mathcal{B}_0)^{-1}\mathrm{Vol}(\mathcal{B}_1 t_1^{-1}\mathcal{B}_1)^{-1}\\ &\times(\mathcal{L}(\mathbf{1}_{\mathcal{B}_0 t_0^{-1}\mathcal{B}_0})\mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t_1^{-1}\mathcal{B}_1})S_{\Xi,\xi})(\eta^{-1})\end{aligned}$$

for $t_0 \in A_0^+$, $t_1 \in A_1^+$.

Proof. It suffices to show that

$$(\mathcal{B}_0 t_0 \mathcal{B}_0)\eta^{-1}(\mathcal{B}_1 t_1^{-1} \mathcal{B}_1) \subset \mathcal{K}_0 t_0 \eta^{-1} t_1^{-1} \mathcal{K}_1$$

for $t_0 \in A_0^+$, $t_1 \in A_1^+$. By Lemma 5.4, we have

$$\mathcal{B}_0 t_0 \mathcal{B}_0 \eta^{-1} \mathcal{B}_1 t_1^{-1} \mathcal{B}_1 \subset \mathcal{B}_0 t_0 T_0^{(0)} N_0^{(0)} \eta^{-1} T_1^{(0)} N_1^{(0)} t_1^{-1} \mathcal{B}_1.$$

Since $t_i T_i^{(0)} N_i^{(0)} t_i^{-1} \subset T_i^{(0)} N_i^{(0)}$, the lemma follows. □

Recall that

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1) f_\Xi)\rangle.$$

By (1_1), (3_1), (1_0), and (3_0) of Lemma 5.5, we have

$$\begin{aligned} S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1}) = {}& [N_1^{(0)} : N_1^{(1)}][N_0^{(0)} : N_0^{(1)}] \\ & \times \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \bar{c}_{w_1}(\Xi)\bar{c}_{w_0}(\xi)(w_1\Xi)^{-1}\delta_1^{1/2}(t_1) \cdot (w_0\xi)^{-1}\delta_0^{1/2}(t_0) \\ & \times \int_{\mathcal{K}_0 \times \mathcal{K}_1} g_{0,w_0}(k_0) g_{1,w_1}(k_1) Y_{\Xi,\xi}(k_0\eta k_1) dk_0\, dk_1. \end{aligned}$$

By (2_1) and (2_0) of Lemma 5.5, we have

$$\begin{aligned} & \int_{\mathcal{K}_0 \times \mathcal{K}_1} g_{0,1}(k_0) g_{1,1}(k_1) Y_{\Xi,\xi}(k_0\eta k_1) dk_0\, dk_1 \\ = {}& \mathrm{Vol}(\mathcal{B}_1)\mathrm{Vol}(\mathcal{B}_0) \\ = {}& \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1 - \dim A_0} L(1,\chi)^{-1}/([N_1^{(0)} : N_1^{(1)}][N_0^{(0)} : N_0^{(1)}]) \end{aligned}$$

Put $\mathbf{c}_{\mathrm{WS}}(\Xi,\xi) = \mathbf{b}(\Xi,\xi)\mathbf{d}_1(\Xi)^{-1}\mathbf{d}_0(\xi)^{-1}$, where

$$\begin{aligned} \mathbf{b}(\Xi,\xi)^{-1} = {}& \prod_{1 \le i \le j \le l-1} L(1/2, \Xi_i\xi_j) L(1/2, \Xi_i\xi_j^{-1}) \prod_{i=1}^{l-1} L(1/2,\Xi_i) L(1/2, \chi\Xi_i) \\ & \times \prod_{1 \le j < i \le l} L(1/2, \Xi_i\xi_j) L(1/2, \Xi_i^{-1}\xi_j) \end{aligned}$$

$$\mathbf{d}_1(\Xi)^{-1} = \prod_{i=1}^{l} L(0, \Xi_i^2) \prod_{1 \le i < j \le l} L(0, \Xi_i\Xi_j) L(0, \Xi_i\Xi_j^{-1})$$

$$\mathbf{d}_0(\xi)^{-1} = \prod_{i=1}^{l-1} L(0,\xi_i) L(0,\chi\xi_i) \prod_{1 \le i < j \le l-1} L(0, \xi_i\xi_j) L(0, \xi_i\xi_j^{-1})$$

in Case A, and

$$\mathbf{b}(\Xi,\xi)^{-1} = \prod_{1\le i\le j\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i\xi_j^{-1}) \prod_{i=1}^{l} L(1/2,\xi_i)L(1/2,\chi\xi_i)$$
$$\times \prod_{1\le j<i\le l} L(1/2,\Xi_i\xi_j)L(1/2,\Xi_i^{-1}\xi_j)$$
$$\mathbf{d}_1(\Xi)^{-1} = \prod_{i=1}^{l} L(0,\Xi_i)L(0,\chi\Xi_i) \prod_{1\le i<j\le l} L(0,\Xi_i\Xi_j)L(0,\Xi_i\Xi_j^{-1})$$
$$\mathbf{d}_0(\xi)^{-1} = \prod_{i=1}^{l} L(0,\xi_i^2) \prod_{1\le i<j\le l} L(0,\xi_i\xi_j)L(0,\xi_i\xi_j^{-1})$$

in Case B. Then it is easy to see

$$\mathbf{c}_{\mathrm{WS}}(\Xi,\xi) = \frac{\bar{c}_1(\Xi)\bar{c}_0(\xi)}{\zeta(\Xi,\xi)}.$$

By the Weyl-invariance, we have

$$\frac{S_{\Xi,\xi}(t_0\eta^{-1}t_1^{-1})}{\zeta(\Xi,\xi)} = \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1-\dim A_0}L(1,\chi)^{-1}$$
$$\times \sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi,w_0\xi)\cdot(w_1\Xi)^{-1}\delta_1^{1/2}(t_1)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t_0).$$

(cf. [36], Theorem 10.7.) Note that

$$\mathbf{b}(\Xi,\xi),\mathbf{d}_1(\Xi),\mathbf{d}_0(\xi) \in \mathbb{Z}[q^{\pm1/2},\Xi_1,\Xi_2,\ldots,\xi_1,\xi_2,\ldots].$$

Here and from now on, we identify an unramified quasi-character of $k^\times$ with its value at a primes element.

Step 3. Calculation of $S_{\Xi,\xi}(1)/\zeta(\Xi,\xi)$.

Our next task is to prove the following lemma.

Lemma 5.7. *The sum*

$$\frac{S_{\Xi,\xi}(1)}{\zeta(\Xi,\xi)} = \Delta_{G_1}\Delta_{G_0}\zeta(1)^{-\dim A_1-\dim A_0}L(1,\chi)^{-1}$$
$$\times \sum_{\substack{w_1\in W_1\\ w_0\in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi,w_0\xi)$$

is independent of Ξ *and* ξ.

Proof. We shall prove the lemma only in Case B. One can handle Case A in a similar way. Put

$$A_{\Xi,\xi} = \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi, w_0\xi).$$

We are going to prove that $A_{\Xi,\xi}$ is independent of Ξ and ξ. Put

$$\mathcal{D}(\Xi) = \Xi^{-\rho_1}\mathbf{d}_1(\Xi) = \sum_{w_1 \in W_1} \mathrm{sgn}(w_1) \cdot (w_1\Xi)^{-\rho_1}$$
$$\mathcal{D}(\xi) = \xi^{-\rho_0}\mathbf{d}_0(\xi) = \sum_{w_0 \in W_0} \mathrm{sgn}(w_0) \cdot (w_0\xi)^{-\rho_0},$$

where

$$\rho_1 = \rho_0 = (l, l-1, \dots, 1).$$

Then we have $\mathcal{D}(w_1\Xi) = \mathrm{sgn}(w_1)\mathcal{D}(\Xi)$ and $\mathcal{D}(w_0\xi) = \mathrm{sgn}(w_0)\mathcal{D}(\xi)$ for $w_1 \in W_1$ and $w_0 \in W_0$. Note that ρ_1 and ρ_0 are half the sum of the positive roots of type C. It follows that $A_{\Xi,\xi}$ is equal to

$$(\mathcal{D}(\Xi)\mathcal{D}(\xi))^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathrm{sgn}(w_1)\mathrm{sgn}(w_0) \cdot (w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0}\mathbf{b}(w_1\Xi, w_0\xi).$$

Put $B_{\Xi,\xi} = \Xi^{-\rho_1}\xi^{-\rho_0}\mathbf{b}(\Xi, \xi)$. Observe that $B_{\Xi,\xi}$ is equal to

$$\prod_{1 \le j \le l} (\xi_j^{-1} - q^{-1}\xi_j) \prod_{1 \le i \le j \le l} (\Xi_i^{-1} - q^{-1/2}\xi_j^{-1})$$
$$\times \prod_{1 \le j < i \le l} (\xi_j^{-1} - q^{-1/2}\Xi_i^{-1}) \prod_{\substack{1 \le i \le l \\ 1 \le j \le l}} (1 - q^{-1/2}\Xi_i\xi_j).$$

We express $B_{\Xi,\xi}$ as a sum of monomials

$$B_{\Xi,\xi} = \sum_{\lambda,\mu} c_{\lambda,\mu}\Xi^\lambda\xi^\mu, \quad \lambda, \mu \in \mathbb{Z}^l, c_{\lambda,\mu} \in \mathbb{Z}[q^{\pm 1/2}].$$

We say that a monomial $\Xi^\lambda\xi^\mu$ is regular if $\Xi^{w_1\lambda}\xi^{w_0\mu} = \Xi^\lambda\xi^\mu$ implies $w_1 = w_0 = 1$. We also say that a monomial is singular if it is not regular. Here the action of the Weyl group on $\mathbb{Z}^l$ is given by $(w_1\Xi)^{w_1\lambda} = \Xi^\lambda$, $(w_0\xi)^{w_0\mu} = \xi^\mu$, as usual.

We would like to show that if a regular monomial $\Xi^\lambda\xi^\mu$ appears in $B_{\Xi,\xi}$, then it is of the form $\Xi^{w_1\rho_1}\xi^{w_0\rho_0}$ with $w_1 \in W_1$, $w_0 \in W_0$. It is enough to show $|\lambda_i|, |\mu_j| \le l$, since such a monomial is either singular or Weyl-equivalent to $\Xi^{\rho_1}\xi^{\rho_0}$. Choose $i_0, j_0 \in \{1, 2, \ldots, l\}$. The positive contribution of Ξ_{i_0} comes from

$$\prod_{1\le j\le l}(1-q^{-1/2}\Xi_{i_0}\xi_j),$$

and the negative contribution of Ξ_{i_0} comes from

$$\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1}-q^{-1/2}\xi_j^{-1})\prod_{1\le j<i_0}(\xi_j^{-1}-q^{-1/2}\Xi_{i_0}^{-1}).$$

Therefore $|\lambda_{i_0}| \le l$. Similarly, the positive contribution of ξ_{j_0} comes from

$$(\xi_{j_0}^{-1}-q^{-1}\xi_{j_0})\prod_{1\le i\le l}(1-q^{-1/2}\Xi_i\xi_{j_0})$$

and the negative contribution of ξ_{j_0} comes from

$$(\xi_{j_0}^{-1}-q^{-1}\xi_{j_0})\prod_{1\le i\le j_0}(\Xi_i^{-1}-q^{-1/2}\xi_{j_0}^{-1})\prod_{j_0<i\le l}(\xi_{j_0}^{-1}-q^{-1/2}\Xi_i^{-1}).$$

Therefore $|\mu_{j_0}| \le l+1$. It follows that if a regular monomial $\Xi^\lambda\xi^\mu$ occurs in $B_{\Xi,\xi}$, then $l \le |\mu_{j_0}| \le l+1$ for some j_0. We will show that no regular monomial $\Xi^\lambda\xi^\mu$ such that $|\mu_{j_0}| > l$ occurs in $B_{\Xi,\xi}$. Assume that the monomial $\Xi^\lambda\xi^\mu$ occurs in $B_{\Xi,\xi}$ and $|\mu_{j_0}| > l$. We must show that such a monomial $\Xi^\lambda\xi^\mu$ is singular. Note that the monomial $\Xi^\lambda\xi^\mu$ occurs in

$$\begin{aligned}&q^{-1}\xi_{j_0}\cdot\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1}-q^{-1/2}\xi_j^{-1})\\&\times\prod_{1\le j<i_0}(\xi_j^{-1}-q^{-1/2}\Xi_{i_0}^{-1})\cdot q^{-1/2}\Xi_{i_0}\xi_{j_0}\prod_{\substack{1\le j\le l\\ j\ne j_0}}(1-q^{-1/2}\Xi_{i_0}\xi_j)\\&\times(\text{terms not containing }\Xi_{i_0}\text{ or }\xi_{j_0}).\end{aligned}$$

In particular, we have $\lambda_{i_0} \ne -l$. If $\lambda_{i_0} = l$, then the factor $\xi_{j_0}^{-1}$ must occur in the factor

$$\prod_{i_0\le j\le l}(\Xi_{i_0}^{-1}-q^{-1/2}\xi_j^{-1})\prod_{1\le j<i_0}(\xi_j^{-1}-q^{-1/2}\Xi_{i_0}^{-1}),$$

which would contradicts to the condition $\mu_{j_0} > l$. It follows that the condition $\mu_{j_0} > l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial such that $\mu_{j_0} > l$ occurs in $B_{\Xi,\xi}$. Assume now $\mu_{j_0} < -l$. Then the monomial

$\Xi^\lambda \xi^\mu$ occurs in

$$\begin{aligned}
&\xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{\substack{i_0 \le j \le l \\ j \neq j_0}} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\
&\times \xi_{j_0}^{-l+j_0} \prod_{1 \le j < i_0} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \le j \le l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\
&\times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0})
\end{aligned}$$

if $i_0 \le j_0$, and

$$\begin{aligned}
&\xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{i_0 \le j \le l} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\
&\times \xi_{j_0}^{-l+j_0} \prod_{\substack{1 \le j < i_0 \\ j \neq j_0}} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \le j \le l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\
&\times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0})
\end{aligned}$$

if $i_0 > j_0$. In particular, $\lambda_{i_0} \neq -l$. If $\lambda_{i_0} = l$, then the factor ξ_{j_0} occurs, and so the condition $\mu_{j_0} < -l$ fails. It follows that the condition $\mu_{j_0} < -l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial $\Xi^\lambda \xi^\mu$ such that $\mu_{j_0} < -l$ occurs in $B_{\Xi,\xi}$.

We have proved that the regular monomials $\Xi^\lambda \xi^\mu$ which occur in $B_{\Xi,\xi}$ are of the form $(w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0}$, for some $w_1 \in W_1$ and $w_0 \in W_0$. Therefore, up to a constant, $A_{\Xi,\xi}$ is equal to

$$(\mathcal{D}(\Xi)\mathcal{D}(\xi))^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathrm{sgn}(w_1)\mathrm{sgn}(w_0) \cdot (w_1\Xi)^{-\rho_1}(w_0\xi)^{-\rho_0} = 1.$$

Hence the lemma. □

Recall that

$$A_{\Xi,\xi} = \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\mathrm{WS}}(w_1\Xi, w_0\xi).$$

Lemma 5.8. *The constant $A_{\Xi,\xi}$ is equal to $\Delta_{G_0}^{-1}$.*

Proof. We shall prove the lemma only in Case B. One can handle Case A in a similar way. We put

$$\begin{aligned}
\tilde{\Xi} &= (q^{-l}, q^{-l+1}, \dots, q^{-1}), \\
\tilde{\xi} &= (q^{-l+(1/2)}, q^{-l+(3/2)}, \dots, q^{-1/2}).
\end{aligned}$$

As in the proof of [36], Lemma 11.9, we shall prove that $\mathbf{b}(w_1\tilde{\Xi}, w_0\tilde{\xi}) \neq 0$ implies $w_1 = w_0 = 1$. Note that $\mathbf{b}(\Xi, \xi)$ is equal to

$$\prod_{1\le i\le j\le l}(1-q^{-1/2}\Xi_i\xi_j^{-1})\prod_{1\le j<i\le l}(1-q^{-1/2}\Xi_i^{-1}\xi_j)\prod_{\substack{1\le i\le l\\1\le j\le l}}(1-q^{-1/2}\Xi_i\xi_j)$$
$$\times\prod_{1\le j\le l}(1-q^{-1}\xi_j^2).$$

Note that $W_1 \simeq W_0 \simeq \{\pm1\}^l \rtimes \mathfrak{S}_l$, where $\mathfrak{S}_l$ is the symmetric group. Therefore, for every $w_1 \in W_1$, $w_0 \in W_0$, one can find $\sigma, \tau \in \mathfrak{S}_l$ and $\varepsilon_i, \varepsilon'_j \in \{\pm1\}$ such that

$$\begin{aligned} w_1\Xi =&(\Xi_{\sigma(1)}^{\varepsilon_1}, \dots, \Xi_{\sigma(l)}^{\varepsilon_l}),\\ w_0\xi =&(\xi_{\tau(1)}^{\varepsilon'_1}, \dots, \xi_{\tau(l)}^{\varepsilon'_l}).\end{aligned}$$

Put $i_s = \sigma^{-1}(l+1-s)$, $j_t = \tau^{-1}(l+1-t)$. Then we have

$$\begin{aligned}(w_1\tilde{\Xi})_{i_s} =&\tilde{\Xi}_{l+1-s}^{\varepsilon_{i_s}} = q^{-\varepsilon_{i_s}\cdot s},\\ (w_0\tilde{\xi})_{j_t} =&\tilde{\xi}_{l+1-t}^{\varepsilon'_{j_t}} = q^{-\varepsilon'_{j_t}(t-(1/2))}.\end{aligned}$$

Assume $\mathbf{b}(w_1\tilde{\Xi}, w_0\tilde{\xi}) \neq 0$. Firstly, $1 - q^{-1}(w_0\tilde{\xi})_{j_1}^2 \neq 0$ implies $\varepsilon'_{j_1} = 1$. Secondly, $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}(w_0\tilde{\xi})_{j_s} \neq 0$ and $1-q^{-1/2}(w_1\tilde{\Xi})_{i_{t+1}}(w_0\tilde{\xi})_{j_t} \neq 0$ imply

$$\varepsilon'_{j_1} = \varepsilon_{i_1} = \varepsilon'_{j_2} = \varepsilon_{i_2} = \cdots = \varepsilon'_{j_l} = \varepsilon_{i_l} = 1.$$

Now, if $j_s < i_s$, then the second factor would contains the factor $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}^{-1}(w_0\tilde{\xi})_{j_s} = 0$, therefore we have $j_s \geq i_s$. Similarly, if $i_s \leq j_{s+1}$, the first factor would contains the factor $1-q^{-1/2}(w_1\tilde{\Xi})_{i_s}(w_0\tilde{\xi})_{j_{s+1}}^{-1} = 0$, therefore we have $i_s > j_{s+1}$. It follows that

$$j_1 \geq i_1 > j_2 \geq i_2 > \cdots > j_l \geq i_l,$$

and so $w_1 = w_0 = 1$. It follows that $A_{\Xi,\xi} = \mathbf{b}(\tilde{\Xi}, \tilde{\xi})\mathbf{d}_1(\tilde{\Xi})^{-1}\mathbf{d}_0(\tilde{\xi})^{-1}$. By direct calculation, one can easily show that it is $\Delta_{G_0}^{-1}$. □

Now Lemma 5.1 follows from Lemma 5.7 and Lemma 5.8.

Part III. Examples over number fields

In §§6-11, k is an algebraic number field. The Dedekind zeta function of k is denoted by $\zeta_k(s)$. The Γ-factors of L-functions are normalized as in Tate [64]. In particular, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The completed Dedekind zeta function of k is denoted by $\xi_k(s)$. When $k = \mathbb{Q}$, the subscript k is dropped. The symbol $L(s, \pi, r)$

is the Euler product $\prod_{v<\infty} L(s,\pi_v,r)$ and the completed L-function for $L(s,\pi,r)$ is denoted by $\Lambda(s,\pi,r)$.

6. Waldspurger's theorem

The following example is due to Waldspurger [65]. Let D be a quaternion algebra over an algebraic number field k. Then $G_1 = D^\times/k^\times$ can be considered as a special orthogonal group associated to a 3-dimensional quadratic space over k. Note that $\Delta_{G_1} = \xi_k(2)$. Let $G_0 = T$ be an anisotropic torus of G_1. Then T can be considered as a special orthogonal group associated to a 2-dimensional quadratic space over k. Let K be a splitting field of T over k. Then there exists an exact sequence

$$1 \to k^\times \to K^\times \to T \to 1.$$

By means of this exact sequence, a character ω of $T(\mathbb{A})/T(k)$ can be regarded as a character of $\mathbb{A}_K^\times/K^\times$ whose restriction to $\mathbb{A}_k^\times/k^\times$ is trivial. As in [65], we choose a Haar measure of $T(k_v)$ as follows. Fix a nontrivial additive character ψ of $\mathbb{A}/k$. Then we give the Haar measure $\zeta_v(1)^{-1}|t|_v^{-1}dt_v$ on $k_v^\times$, where dt_v is the self-dual Haar measure of k_v with respect to ψ_v. We give a Haar measure on $K_v^\times$ in a similar way. Then the Haar measure of $T(k_v)$ is defined by the exact sequence

$$1 \to k_v^\times \to K_v^\times \to T(k_v) \to 1.$$

It is easily seen that $C_0 = \Lambda(1,\chi_{K/k})^{-1}$ for this choice of measure. Note that in [65], Waldspurger considered the measure on $T(\mathbb{A})$ such that $\mathrm{Vol}(T(\mathbb{A})/T(k)) = 2\Lambda(1,\chi_{K/k})$.

An irreducible cuspidal automorphic representation π of $G_1(\mathbb{A})$ can be considered as a representation of $D^\times(\mathbb{A})$ with trivial central character. The base change of π to $\mathrm{GL}_2(\mathbb{A}_K)$ is denoted by Π. Choose a cusp form $\varphi = \otimes_v \varphi_v \in \pi \simeq \otimes_v \pi_v$.

Then among other things, Waldspurger ([65], Proposition 7) proved that the integral $I(\varphi_v,\omega_v)$ is convergent and that

$$\begin{aligned}\frac{|\langle \varphi|_{G_0},\omega\rangle|^2}{\langle\varphi,\varphi\rangle\langle\omega,\omega\rangle} &= \frac{1}{4}\Delta_{G_1} C_0 \frac{\Lambda(1/2,\Pi\otimes\omega^{-1})}{\Lambda(1,\pi,\mathrm{Ad})\Lambda(1,\chi_{K/k})} \prod_{v\in S}\frac{\alpha_v(\varphi_v,\omega_v)}{\|\varphi_v\|^2} \\ &= \frac{1}{4}\Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v\in S}\frac{\alpha_v(\varphi_v,\omega_v)}{\|\varphi_v\|^2\cdot\|\omega_v\|^2},\end{aligned}$$

where $\pi_1 = \pi$, $\pi_0 = \omega$. Thus Conjecture 1.5 is true for $n = 2$. Note that we have $|\mathcal{S}_{\psi_1}| = |\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture. Thus Waldspurger's result is compatible with Conjecture 2.1.

7. The Jacquet conjecture

In this section, we consider some examples such that $n = 3$. We first consider the example due to Harris and Kudla [24]. Put $G_1 = \mathrm{SO}(2,2)$ and $G_0 = \mathrm{SO}(2,1) = \mathrm{PGL}_2$, defined over $k = \mathbb{Q}$. By definition, we have $\Delta_{G_1} = \xi(2)^2$. When v is non-archimedean, the local measure $dg_{0,v}$ of $G_{0,v}$ is the standard measure. In particular, the volume of the hyperspecial maximal compact subgroup $\mathcal{K}_v = \mathcal{K}_{0,v} = \mathrm{PGL}_2(\mathbb{Z}_v)$ is 1. For the real place, we choose a Haar measure as follows. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty = \mathcal{K}_{0,\infty} = \mathrm{SO}(2,1) \cap \mathrm{SO}(3)$ be a maximal compact subgroup of $G_0(\mathbb{R})$. We put $\mathcal{K}_\infty^0 = G_0(\mathbb{R})^0 \cap \mathcal{K}_\infty$. Then $G_0(\mathbb{R})^0/\mathcal{K}_\infty^0$ can be identified with the upper-half plane $\mathfrak{H}_1$. Let dk be the Haar measure on $\mathcal{K}_\infty^0$ with total volume 1. Then the Haar measure $dg_{0,\infty}$ on $G_0(\mathbb{R})^0$ is such that $dg_{0,\infty}/dk$ induces the measure $y^{-2}dx\,dy$ on $G_0(\mathbb{R})^0/\mathcal{K}_\infty^0 \simeq \mathfrak{H}_1$. The Haar measure $dg_{0,\infty}$ can be naturally extended to $G_0(\mathbb{R})$. Let $G_0(\mathbb{R})^0 = AN\mathcal{K}_\infty^0$ be an Iwasawa decomposition, which induces a bijection $\mathfrak{H}_1 \simeq AN$. Let $X \subset AN$ be an image of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1$. Then there is a bijection

$$X \times \mathcal{K}_\infty^0 \times \prod_{v<\infty} \mathcal{K}_v \simeq G_0(\mathbb{Q})\backslash G_0(\mathbb{A}).$$

It follows that

$$\int_{G_0(\mathbb{Q})\backslash G_0(\mathbb{A})} \prod_{v\leq\infty} dg_{0,v} = \mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1) = 2\xi(2).$$

Therefore we have $C_0 = \xi(2)^{-1} = 6\pi^{-1}$.

Let $f_j \in S_{\kappa_j}(\mathrm{SL}_2(\mathbb{Z}))$ $(j = 1,2,3)$ be normalized Hecke eigenforms. We assume $\kappa_1 + \kappa_2 = \kappa_3$. We denote the automorphic form on $\mathrm{GL}_2(\mathbb{A})$ corresponding to f_j by $\mathbf{f}_j$. Let τ_j be the irreducible automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ generated by $\mathbf{f}_j$. Note that $\varphi_1 = \mathbf{f}_1 \times \mathbf{f}_2$ induces a cusp form on $\mathrm{SO}(2,2)(\mathbb{A})$ and its restriction to $\mathrm{SO}(2,1)$ is $\mathbf{f}_1\mathbf{f}_2$. Put $\pi_1 = \tau_1 \boxtimes \tau_2$, $\pi_0 = \tau_3$ and $\varphi_0 = \mathbf{f}_3$. By the result of Watson [66], (see also Harris-Kudla [24]), we have

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) = 2^{2\kappa_3+2}\langle f_1 f_2, f_3\rangle^2.$$

It is well-known that $\Lambda(1, \tau_j, \mathrm{Ad}) = 2^{\kappa_j}\langle f_j, f_j\rangle$. Here the $\langle\ ,\ \rangle$ is the usual Petersson inner product.

As both the Tamagawa numbers of $\mathrm{SO}(2,2)$ and $\mathrm{SO}(2,1)$ are equal to 2, we have

$$\begin{aligned}\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} =& 2\xi(2)\frac{|\langle f_1 f_2, f_3\rangle|^2}{\prod_{j=1}^3 \langle f_j, f_j\rangle} \\ =& \frac{1}{2}\xi(2)\frac{\Lambda(1/2, \tau_1\times\tau_2\times\tau_3)}{\prod_{j=1}^3 \Lambda(1, \tau_j, \mathrm{Ad})}.\end{aligned}$$

By easy calculation,

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(s) =& \frac{\Lambda(s, \tau_1\times\tau_2\times\tau_3)}{\prod_{j=1}^3 \Lambda(s+(1/2), \tau_j, \mathrm{Ad})}, \\ \mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2) =& \frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3-1)}{\Gamma_{\mathbb{R}}(2)^3\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3)} = \frac{2\pi^3}{\kappa_3-1}.\end{aligned}$$

Proposition 7.1. *Let $\tau_{j,\infty}$ ($j=1,2,3$) be the holomorphic discrete series of $\mathrm{SO}(2,1)\simeq \mathrm{PGL}_2(\mathbb{R})$ with lowest weight $\pm\kappa_j$. Put $\pi_{1,\infty} = \tau_{1,\infty}\boxtimes\tau_{2,\infty}$ and $\pi_{0,\infty}=\tau_{3,\infty}$. Let $\varphi_{1,\infty}\in\pi_{1,\infty}$ be the vector with weight (κ_1,κ_2). Let $\varphi_{0,\infty}\in\pi_{0,\infty}$ be the vector with weight κ_3. We assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$. Then we have*

$$\begin{aligned}I(\varphi_{1,\infty}, \varphi_{0,\infty}) =& 4\pi(\kappa_3-1), \\ \alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) =& 2.\end{aligned}$$

The proof of this proposition will be given in §11. Putting altogether, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = \frac{1}{4}\Delta_{G_1}C_0\mathcal{P}_{\pi_1,\pi_0}(1/2)\cdot\frac{\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2\cdot\|\varphi_{0,\infty}\|^2}.$$

Note that we have $|\mathcal{S}_{\psi_1}| = |\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture.

In fact, Watson [66] obtained a more general result. Let B be an indefinite quaternion algebra over $\mathbb{Q}$. The reduced discriminant d_B of B is, by definition, the product of primes which ramifies in B. Let N be a square-free integer such that $(N, d_B)=1$. Put S_{f} be the set of primes which divide $d_B N$. Let $\tau_j = \otimes_v \tau_{j,v}$ ($j=1,2,3$) be irreducible cuspidal automorphic representation of $\mathbb{A}^\times\backslash B^\times(\mathbb{A})$ with new vector $f_j = \otimes_v f_{j,v}$ which satisfies the following conditions:

(1) When $v<\infty$ and $v\notin S_{\mathrm{f}}$, the local components $\tau_{j,v}$ ($j=1,2,3$) are unramified representations and $f_{j,v}$ are unramified vectors.
(2) When $v|d_B$, the local component $\tau_{j,v}$ ($j=1,2,3$) are one-dimensional representations of the form $\chi_j\circ\nu_{B_v}$, where χ_j are unramified quadratic characters and ν_{B_v} is the reduced norm. We also assume $\chi_1\chi_2\chi_3=1$.

(3) When $v|N$, the local component $\tau_{j,v}$ ($j = 1, 2, 3$) are representations of the form $\chi_j \otimes$ (Steinberg), where χ_j are unramified quadratic characters. We assume that $\chi_1\chi_2\chi_3$ is the unique unramified character of order 2 and that $f_{j,v}$ are Iwahori fixed vectors.

(4) When $v = \infty$, we assume that $\tau_{j,v}$ ($j = 1, 2, 3$) are discrete series representations with minimal weight $\pm\kappa_j$. We assume $\kappa_3 = \kappa_1 + \kappa_2$ and $f_{j,v}$ have weight $\kappa_j > 0$.

Then Watson's result ([66] Theorem 3) says

$$\frac{\left|\int_X f_1(z)f_2(z)\overline{f_3(z)}\mathrm{Im}(z)^{\kappa_3-2}dz\right|^2}{\prod_{j=1}^3 \int_X |f_j(z)|^2\mathrm{Im}(z)^{\kappa_j-2}dz} = \frac{2^{\sharp S_{\mathrm{f}}-2}}{(d_BN)^2}\frac{\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3)}{\prod_{j=1}^3 \Lambda(1, \tau_j, \mathrm{Ad})}.$$

Here, $X = \mathcal{O}^{(1)}(d_B, N)\backslash\mathfrak{H}_1$, where $\mathcal{O}^{(1)}(d_B, N)$ is the arithmetic subgroup defined in Watson [66], Ch. 1. Watson proved that

$$\mathrm{Vol}(X) = 2\xi(2)\prod_{p|d_B}(p-1)\prod_{p|N}(p+1).$$

Watson also considered the cases when $\tau_{j,\infty}$ are not discrete series, but we do not discuss such cases.

We now interpret Watson's result in terms of our conjecture. Let V_1 be the vector space B equipped with the reduced norm form ν_B. The subspace $V_0 \subset V_1$ is defined by the space of elements of reduced trace 0. Then we have

$$G_1 = \{(g_1, g_2) \in B^\times \times B^\times \mid \nu_B(g_1) = \nu_B(g_2)\}/\mathbb{Q}^\times,$$
$$G_0 = B^\times/\mathbb{Q}^\times.$$

As in the case of $\mathrm{SO}(2, 2)$, we regard $\pi_1 = \tau_1 \boxtimes \tau_2$ as a representation of $G_1(\mathbb{A})$, and $\pi_0 = \tau_3$ as a representation of $G_0(\mathbb{A})$. We put $\varphi_1 = f_1 \times f_2$, and $\varphi_0 = f_3$. We may assume $\|\varphi_{1,v}\| = \|\varphi_{0,v}\| = 1$ for any v. Note that Watson's result implies

$$\begin{aligned}\frac{|\langle\varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle\varphi_1, \varphi_1\rangle\langle\varphi_0, \varphi_0\rangle} &= \mathrm{Vol}(X)\frac{\left|\int_X f_1(z)f_2(z)\overline{f_3(z)}\mathrm{Im}(z)^{\kappa_3-2}dz\right|^2}{\prod_{j=1}^3 \int_X |f_j(z)|^2\mathrm{Im}(z)^{\kappa_j-2}dz} \\ &= 2^{-1}\xi(2)\mathcal{P}_{\pi_1,\pi_0}(1/2) \\ &\quad \times \prod_{p|d_B}\left(2p^{-1}(1-p^{-1})\right)\prod_{p|N}\left(2p^{-1}(1+p^{-1})\right).\end{aligned}$$

We describe local calculations below. Since G_0 is an inner form of PGL_2, we can transfer the local measure of $\mathrm{PGL}_2(\mathbb{Q}_v)$ to $G_{0,v} = B^\times(\mathbb{Q}_v)/\mathbb{Q}_v^\times$. Note that $\Delta_{G_{1,v}} = \zeta_v(2)^2$ and $C_0 = 6\pi^{-1}$ are unchanged.

When $p|d_B$, we have

$$\mathrm{Vol}(G_{0,p}) = I(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1-p^{-1})^{-1},$$
$$\mathcal{P}_{\pi_{1,p},\pi_{0,p}}(1/2) = \zeta_p(1)^2\zeta_p(2)^{-2}.$$

It follows that $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1-p^{-1})$ for $p|d_B$. When $p|N$, let ε_p be the unique unramified character of $\mathbb{Q}_p^\times$ of order 2. Then we have

$$\begin{aligned}\mathcal{P}_{\pi_{1,p},\pi_{0,p}}(1/2) &= L(1,\varepsilon_p)^2 L(2,\varepsilon_p)\zeta_p(2)^{-3} \\ &= (1+p^{-1})^{-2}(1+p^{-2})^{-1}(1-p^{-2})^3.\end{aligned}$$

The integral $I(\varphi_{1,p}, \varphi_{0,p})$ can be calculated as follows (cf. Godement and Jacquet [15] §7). The image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is denoted by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let

$$I = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{Q}_p) \,\middle|\, a,b,d \in \mathbb{Z}_p, c \in p\mathbb{Z}_p \right\}$$

be an Iwahori subgroup of $G_{0,p} = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let W_a be the affine Weyl group generated by $w_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & p^{-1} \\ p & 0 \end{bmatrix}$. The extended affine Weyl group $\tilde{W}$ is defined by $\tilde{W} = W_a \rtimes \Omega$, where Ω is the group of order 2 generated by $\omega = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$. Then we have a Bruhat decomposition $G_{0,p} = \coprod_{w \in \tilde{W}} IwI$. The extended Weyl group $\tilde{W}$ has a length function $l(w)$ such that $l(w_1) = l(w_2) = 1$, $l(\omega) = 0$. The Poincaré series $\sum_{w \in W_a} t^{l(w)}$ is equal to $(1+t)(1-t)^{-1}$. Then the function

$$\Phi(b_1\omega^j w b_2) = (-1)^j(-p^{-1})^{l(w)}, \quad b_1, b_2 \in I, j \in \{0,1\},\, w \in W_a$$

is a bi-I-invariant matrix coefficient of the Steinberg representation of G_0. From this, we have

$$\begin{aligned}I(\varphi_{1,p}, \varphi_{0,p}) &= \sum_{j=0}^{1}(-1)^j \sum_{w \in W_a} \mathrm{Vol}(I\omega^j wI)\Phi(\omega^j w)^3 \\ &= 2(p+1)^{-1}\sum_{w \in W_a}(-p^{-2})^{l(w)} \\ &= 2p^{-1}(1-p^{-1})(1+p^{-2})^{-1}.\end{aligned}$$

Note that

$$\mathrm{Vol}(IwI) = (1+p)^{-1}p^{l(w)}, \quad w \in \tilde{W}.$$

It follows that $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1+p^{-1})$ for $p|N$.

Putting together, Conjecture 1.5 holds in this case with $2^{\beta} = 1/4$. Note that the orders $|\mathcal{S}_{\psi_1}|$ and $|\mathcal{S}_{\psi_0}|$ must be equal to 2, since the Steinberg representation does not come from a quadratic field. Note also that Conjecture 1.5 holds for any vector $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$, since $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) = 1$ for any v (see Prasad [54]).

We remark that Conjecture 1.5 is compatible with algebraicity results for the triple product L-functions. For $j = 1, 2, 3$, let f_j be a primitive cusp form with weight κ_j, level N_j, and character ε_j. We assume that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and $\kappa_1 \leq \kappa_2 \leq \kappa_3$. We denote by τ_j the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f_j.

We use the symbol $a \sim b$ for $a, b \in \mathbb{C}$, which means that $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$. It is well-known that $\Lambda(1, \tau_j, \mathrm{Ad}) \sim \langle f_j, f_j \rangle$. Then Harris-Kudla [24] proved that

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim p(f_1, f_2, f_3),$$

where

$$p(f_1, f_2, f_3) = \begin{cases} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle & \text{if } \kappa_3 < \kappa_1 + \kappa_2 \\ \langle f_3, f_3 \rangle^2 & \text{if } \kappa_3 \geq \kappa_1 + \kappa_2. \end{cases}$$

We assume $\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \neq 0$. They also proved the Jacquet conjecture which states that there exist a unique quaternion algebra D and some automorphic forms $F_j^D \in \tau_i^D$ such that

$$\int_{\mathbb{A}^\times D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A})} F_1^D(g) F_2^D(g) F_3^D(g) dg \neq 0.$$

Here τ_j^D is the Jacquet-Langlands-Shimizu correspondence of τ_j. Assume that $\varepsilon_1\varepsilon_2 = \varepsilon_3 = 1$ and $F_j^D \in \tau_j^D$. Then $\varphi_0 = F_3^D$ can be regarded as an automorphic form on $G_0 = D^\times/\mathbb{Q}^\times$ and $\varphi_1 = F_1^D \times F_2^D$ can be regarded as an automorphic form on

$$G_1 = \{(d_1, d_2) \in D^\times \times D^\times \mid \nu(d_1) = \nu(d_2)\}/\mathbb{Q}^\times.$$

Here ν is the reduced norm of D. As before, we transfer the Haar measure dg_v on $\mathrm{GL}_2(\mathbb{Q}_v)$ to $G_0(\mathbb{Q}_v)$. In particular, $C_0 = 6/\pi$.

For finite prime p, the component π_p has a $\bar{\mathbb{Q}}$-structure. Note that for $\bar{\mathbb{Q}}$-rational vectors $\varphi_{1,p}$ and $\varphi_{0,p}$, the quantity $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) \in \bar{\mathbb{Q}}$.

In the balanced case $\kappa_3 < \kappa_1 + \kappa_2$, the quaternion algebra D is definite. We choose arithmetic automorphic forms $F_j^D \in \tau_j^D$. Then we have

$$\langle \varphi_1, \varphi_1 \rangle, \langle \varphi_0, \varphi_0 \rangle \in \bar{\mathbb{Q}}^\times, \quad \langle \varphi_1|_{G_0}, \varphi_0 \rangle \in \bar{\mathbb{Q}}.$$

Note that in this case we have

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty}, \pi_{0,\infty}}(1/2)^{-1} \cdot \mathrm{Vol}(G_0(\mathbb{R})) \sim \pi^{-1}.$$

Note that $\mathrm{Vol}(G_0(\mathbb{R})) = \mathrm{Vol}(\mathrm{U}(2)/(\mathrm{U}(1) \times \mathrm{U}(1))) \sim \pi$. Therefore in this case our conjecture is compatible with the known results

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle.$$

Now we consider the unbalanced case $\kappa_3 \geq \kappa_1 + \kappa_2$. We choose arithmetic holomorphic automorphic form $F_3^D \in \tau_3^D$ of weight κ_3 and arithmetic nearly anti-holomorphic forms $F_1^D \in \tau_1^D$ and $F_2^D \in \tau_2^D$ with some weight. Then we have (see Shimura [58])

$$\begin{aligned} \langle \varphi_0, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle, \\ \langle \varphi_1, \varphi_1 \rangle &\sim \xi(2)^{-2} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle, \\ \langle \varphi_1|_{G_0}, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle. \end{aligned}$$

Note that in this case, we have $\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim 1$. Therefore in this case our conjecture is compatible with the known results

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_3, f_3 \rangle^2.$$

Remark 7.2. More generally, Conjecture 1.5 is compatible with Shimura's conjecture [59], [60] for Hilbert modular forms, which was proved by Harris [21], [22], [23], and Yoshida [68] in most cases.

8. Restriction of the Saito-Kurokawa lift to the diagonal subset $\mathfrak{H}_1 \times \mathfrak{H}_1$

Let $\kappa > 0$ be an odd integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. We denote the Kohnen plus subspace by $S^+_{\kappa+(1/2)}(\Gamma_0(4)) \subset S_{\kappa+(1/2)}(\Gamma_0(4))$ (cf. Kohnen [38]). Let $h \in S^+_{\kappa+(1/2)}(\Gamma_0(4))$ be a Hecke eigenform associated to f by Shimura correspondence. Let $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of h. Let τ and σ be the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f and g, respectively. Then it is shown in Ichino [30] that

$$\Lambda(1/2, \mathrm{Ad}(\sigma) \boxtimes \tau) = 2^{\kappa+1} \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle \mathcal{F}|_{\mathfrak{H}_1 \times \mathfrak{H}_1}, g \times g \rangle|^2}{\langle g, g \rangle^2}.$$

Here, $\langle \ , \ \rangle$ is the usual Petersson inner product on $\mathfrak{H}_1$. We interpret this result in terms of automorphic representations. Let φ_1 be the automorphic form on $G_1(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(3,2)(\mathbb{A}_\mathbb{Q})$ corresponding to $\mathcal{F}$. Similarly, let φ_0 be the automorphic form on $G_0(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(2,2)(\mathbb{A}_\mathbb{Q})$ corresponding to $g \times g$. As in the last section, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$ for $v < \infty$. Let $G_0(\mathbb{R})^0$ be the topological identity component of $G_0(\mathbb{R})$. The maximal compact subgroup $\mathcal{K}^0_\infty$ of $G_0(\mathbb{R})^0$ is defined by $\mathcal{K}^0_\infty = G_0(\mathbb{R})^0 \cap \mathrm{SO}(4)$. Let $dg_{0,\infty}$ be the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(y_1 y_2)^{-2} dx_1 dx_2 dy_1 dy_2$ on

$G_0(\mathbb{R})^0/\mathcal{K}_\infty^0 \simeq \mathfrak{H}_1 \times \mathfrak{H}_1$. Here, dk is the Haar measure of $\mathcal{K}_\infty^0$ with total measure 1. The Haar measure $dg_{0,\infty}$ can be naturally extended to $G_0(\mathbb{R})$. We calculate the constant C_0. Let $G_0(\mathbb{R})^0 = AN\mathcal{K}_\infty^0$ be an Iwasawa decomposition, and $X \subset AN$ be a set bijective to a fundamental domain for $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1)^2$. Then each element of $G_0(\mathbb{Q})\backslash G_0(\mathbb{A})$ has exactly two representatives in $X \times \mathcal{K}_\infty^0 \times \prod_{v<\infty} \mathcal{K}_{0,v}$. It follows that

$$\int_{G_0(\mathbb{Q})\backslash G_0(\mathbb{A})} \prod_{v\leq\infty} dg_{0,v} = \frac{1}{2}\mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}_1)^2 = 2\xi(2)^2.$$

Therefore we have $C_0 = \xi(2)^{-2} = 36\pi^{-2}$. Note that $\Delta_{G_1} = \xi(2)\xi(4)$. Note also that the volume of $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathfrak{H}_2$ is $2\xi(2)\xi(4)$, where $\mathfrak{H}_2$ is the Siegel upper-half space of genus 2. It follows that

$$\begin{aligned}
\langle\varphi_1,\varphi_1\rangle &= \frac{\langle\mathcal{F},\mathcal{F}\rangle}{\xi(2)\xi(4)}, \\
\langle\varphi_0,\varphi_0\rangle &= \frac{\langle g,g\rangle^2}{2\xi(2)^2}, \\
\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} &= \frac{\xi(4)}{2\xi(2)}\frac{|\langle\mathcal{F}|_{\mathfrak{h}_1\times\mathfrak{h}_1}, g\times g\rangle|^2}{\langle\mathcal{F},\mathcal{F}\rangle\langle g,g\rangle^2}.
\end{aligned}$$

As noticed in the last section, it is well-known that $\langle f,f\rangle = 2^{-2\kappa}\Lambda(1,\mathrm{Ad}(\tau))$. By Kohnen-Skoruppa [39], we have

$$\frac{\langle\mathcal{F},\mathcal{F}\rangle}{\langle h,h\rangle} = 2^{\kappa-2}\pi^{-1}\xi(2)\Lambda(3/2,\tau).$$

(Note that there is a minor error in the unfolding argument of [39], p. 547. Since the action of the center of $\mathrm{Sp}_2(\mathbb{Z})$ on $\mathfrak{H}_2$ is trivial, the right hand side of the equation of [39] p. 547, line 23 must be multiplied by 2.) It follows that

$$\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle} = \pi\cdot\frac{\xi(4)}{\xi(2)}\cdot\frac{\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\tau))}.$$

It is easy to check that

$$\begin{aligned}
\Lambda(s,\pi_0) &= \Lambda(s,\mathrm{Ad}(\sigma))\xi(s), \\
\Lambda(s,\pi_1) &= \Lambda(s,\tau)\xi(s+(1/2))\xi(s-(1/2)), \\
\Lambda(s,\pi_0,\mathrm{Ad}) &= \Lambda(s,\mathrm{Ad}(\sigma))^2, \\
\Lambda(s,\pi_1,\mathrm{Ad}) &= \Lambda(s,\mathrm{Ad}(\tau))\Lambda(s+(1/2),\tau)\Lambda(s-(1/2),\tau) \\
&\qquad \times\xi(s+1)\xi(s)\xi(s-1).
\end{aligned}$$

From this, one can show that $\mathcal{P}_{\pi_1,\pi_0}(s)$ is equal to

$$\frac{\Lambda(s-(1/2),\mathrm{Ad}(\sigma))\Lambda(s,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(s+(3/2))\Lambda(s+1,\tau)\Lambda(s+(1/2),\mathrm{Ad}(\sigma))\Lambda(s+(1/2),\mathrm{Ad}(\tau))}.$$

It follows that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(1/2) =&\frac{\Lambda(0,\mathrm{Ad}(\sigma))\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\sigma))\Lambda(1,\mathrm{Ad}(\tau))}\\ =&\frac{\Lambda(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\xi(2)\Lambda(3/2,\tau)\Lambda(1,\mathrm{Ad}(\tau))}.\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2) =&\frac{\Gamma_{\mathbb{R}}(1)\Gamma_{\mathbb{C}}(\kappa)\cdot\Gamma_{\mathbb{C}}(\kappa)\Gamma_{\mathbb{C}}(2\kappa)\Gamma_{\mathbb{C}}(1)}{\Gamma_{\mathbb{R}}(2)\cdot\Gamma_{\mathbb{C}}(\kappa+1)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa+1)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2\kappa)}\\ =&4\kappa^{-2}\pi^4.\end{aligned}$$

Proposition 8.1. *Let $\pi_{1,\infty}$ be the irreducible holomorphic discrete series representation of* $\mathrm{SO}(3,2)$ *with lowest K-type* $(\det)^{\pm(\kappa+1)}$. *Let $\pi_{0,\infty}$ be the irreducible discrete series representation of* $\mathrm{SO}(2,2)$ *with lowest K-type $\pm(\kappa+1,\kappa+1)$. Choose lowest weight vectors $\varphi_{1,\infty}\in\pi_{1,\infty}$ and $\varphi_{0,\infty}\in\pi_{0,\infty}$ such that $\|\varphi_{1,\infty}\|=\|\varphi_{0,\infty}\|=1$. Then we have*

$$\begin{aligned}I(\varphi_{1,\infty},\varphi_{0,\infty}) =&16\kappa^{-2}\pi^2,\\ \alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty}) =&4\pi.\end{aligned}$$

The proof of Proposition 8.1 will be given in §11. Using Proposition 8.1, we have

$$\frac{|\langle\varphi_1|_{G_0},\varphi_0\rangle|^2}{\langle\varphi_1,\varphi_1\rangle\langle\varphi_0,\varphi_0\rangle}=\frac{1}{4}\Delta_{G_1}C_0\mathcal{P}_{\pi_1,\pi_0}(1/2)\cdot\frac{\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2\cdot\|\varphi_{0,\infty}\|^2}.$$

Therefore in this case, it seems Conjecture 3.2 holds with $2^\beta=1/4$.

Remark 8.2. Now choose another normalized Hecke eigenform $g'\in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ such that $g\neq g'$. Let σ' be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g'. Let φ_1 be as before and φ_0 the lifting of $g\times g'$ to $G_0(\mathbb{A})$. Then we have $\langle\varphi_1|_{G_0},\varphi_0\rangle=0$. Note that $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v}\otimes\bar\pi_{0,v},\mathbb{C})=\{0\}$ for some v (See e.g., [34] Proposition 3.1). After a little calculation, one can show the numerator of $\mathcal{P}_{\pi_1,\pi_0}(s)$ is equal to

$$\Lambda(s,\tau\times\sigma\times\sigma')\Lambda(s+(1/2),\sigma\times\sigma')\Lambda(s-(1/2),\sigma\times\sigma')$$

and the denominator is

$$\begin{aligned}&\Lambda(s+(1/2),\mathrm{Ad}(\tau))\Lambda(s+1,\tau)\Lambda(s,\tau)\\&\times\xi(s+(3/2))\xi(s+(1/2))\xi(s-(1/2))\\&\times\Lambda(s+(1/2),\mathrm{Ad}(\sigma))\Lambda(s+(1/2),\mathrm{Ad}(\sigma')).\end{aligned}$$

Note that as far as we know, any relation between $\mathrm{ord}_{s=1/2}\Lambda(s,\tau\times\sigma\times\sigma')$ and $\mathrm{ord}_{s=1/2}\Lambda(s,\tau)$ are not known. It seems this example suggest that there is no relation between the period $\langle\varphi_1|_{G_0},\varphi_0\rangle$ and the L-value $\mathcal{P}_{\pi_1,\pi_0}(1/2)$, when π_1 or π_0 are non-tempered and the condition $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v}\otimes\bar{\pi}_{0,v},\mathbb{C})\neq\{0\}$ fails.

9. Restriction of the hermitian Maass lift to $\mathfrak{H}_2$

Now we discuss the case $n=5$ and $k=\mathbb{Q}$. We put $G_0=\mathrm{SO}(3,2)\simeq\mathrm{PGSp}_2$. Let $\kappa>0$ be an odd integer and $f\in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$, $h\in S^+_{\kappa+(1/2)}(\Gamma_0(4))$, $\mathcal{F}\in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$, and τ be as in §8. Let

$$h(\tau)=\sum_{\substack{n>0\\-n\equiv0,1(4)}}c(n)q^n$$

be the Fourier expansion of $h(\tau)$.

Let K be an imaginary quadratic field with discriminant $-D$. We assume that $c(D)\neq0$. We denote by χ and w_K the associated Dirichlet character for $K/\mathbb{Q}$ and the number of units in K, respectively. We put $G_1=\mathrm{SO}(4,2)_{K/\mathbb{Q}}\simeq\mathrm{SU}(2,2)_{K/\mathbb{Q}}/\{\pm1\}$.

Now let $\Gamma_K=\mathrm{SU}(2,2)(\mathbb{Q})\cap\mathrm{GL}_4(\mathcal{O}_K)$ be the special hermitian modular group, where $\mathcal{O}_K$ is the integer ring of K.

By using the fact that the Tamagawa number of $\mathrm{SU}(2,2)$ is 1, one can show that the volume of the fundamental domain for Γ_K is equal to

$$\mathrm{Vol}(\Gamma_K\backslash\mathcal{H}_2)=2^{-3}D^{5/2}(4,w_K)\xi(2)\Lambda(3,\chi)\xi(4),$$

where $\mathcal{H}_2$ is the hermitian upper-half space of degree 2. Here, we have given an invariant measure on $\mathcal{H}_2$ as follows. Put $X=(Z+{}^t\bar{Z})/2$, $Y=(Z-{}^t\bar{Z})/(2\sqrt{-1})$ for $Z\in\mathcal{H}_2$. The measure dX on the space of hermitian matrices is defined by $dX=\prod_{i\le j}dX^{(r)}_{ij}\prod_{i<j}dX^{(i)}_{ij}$, where $X=X^{(r)}+\sqrt{-1}X^{(i)}$, $X^{(r)}_{ij},X^{(i)}_{ij}\in\mathbb{R}$. Then the invariant measure is given by $(\det Z)^{-4}dX\,dY$. This calculation will be carried out in the appendix to this section.

Let $g\in S_\kappa(\Gamma_0(D),\chi)$ be a primitive form and $\mathcal{G}\in S_{\kappa+1}(\Gamma_K)$ the hermitian Maass lift of g (cf. Kojima [41], Krieg [42], Ikeda [35]). We assume that $\mathcal{G} \neq 0$. Let ρ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g. By using Sugano [63], Corollary 8.3 and Ikeda [35] §15, we have

$$\langle \mathcal{G}, \mathcal{G} \rangle = 2^{-2\kappa-7} D^{\kappa+2} \pi^{-2} (4, w_K) \xi(2) \Lambda(2, \mathrm{Sym}^2(\rho)) \Lambda(1, \mathrm{Ad}(\rho)).$$

One can prove this formula using Raghavan-Sengupta [55]. The main theorem of Ichino and Ikeda [32] says

$$|c(D)|^2 \frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F} \rangle|^2}{\langle \mathcal{F}, \mathcal{F} \rangle^2} = 2^{-4\kappa-2} D^{2\kappa-1} \frac{\Lambda(1/2, \rho \times \rho \times \tau)}{\langle f, f \rangle^2}.$$

Combining these result and the Kohnen-Zagier formula [40]

$$|c(D)|^2 \frac{\langle f, f \rangle}{\langle h, h \rangle} = 2^{\kappa-1} D^{\kappa-(1/2)} \Lambda(1/2, \tau \otimes \chi),$$

we have

$$\begin{aligned} \frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F} \rangle|^2}{\langle \mathcal{G}, \mathcal{G} \rangle \langle \mathcal{F}, \mathcal{F} \rangle} =& 2\pi \cdot \mathrm{Vol}(\Gamma_K \backslash \mathcal{H}_2)^{-1} \xi(2) \Lambda(3, \chi) \xi(4) \\ & \times \frac{\Lambda(1/2, \mathrm{Sym}^2(\rho) \boxtimes \tau) \Lambda(3/2, \tau)}{\Lambda(2, \mathrm{Sym}^2(\rho)) \Lambda(1, \mathrm{Ad}(\rho)) \Lambda(1, \mathrm{Ad}(\tau))}. \end{aligned}$$

We translate these results to adelic language. Let φ_1 (resp. φ_0) be the automorphic form on $G_1(\mathbb{A})$ (resp. $G_0(\mathbb{A})$) corresponding to $\mathcal{G}$ (resp. $\mathcal{F}$). We put $S = S_{\mathrm{f}} \cup \{\infty\}$, where S_{f} is the set of primes which divide D. When $v < \infty$, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty^0 = G_0(\mathbb{R})^0 \cap \mathrm{SO}(5)$ be a maximal compact subgroup of $G_0(\mathbb{R})^0$. Let dk be the Haar measure of $\mathcal{K}_\infty^0$ with the total measure 1, and $dg_{0,\infty}$ the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(\det Y)^{-3} dX dY$ on $\mathfrak{H}_2 \simeq G_0(\mathbb{R})^0/\mathcal{K}_\infty^0$. Then we have $\mathrm{Vol}(\mathrm{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) = \mathrm{Vol}(\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2) = 2\xi(2)\xi(4)$. It follows that $C_0 = \xi(2)^{-1}\xi(4)^{-1} = 540\pi^{-3}$, since there is a bijection $G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}) \simeq (\mathrm{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) \times \prod_{p<\infty} \mathcal{K}_{0,p}$. Note also that $\Delta_{G_1} = \xi(2)\Lambda(3,\chi)\xi(4)$.

Let π_1 (resp. π_0) be the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_\mathbb{Q})$ (resp. $G_0(\mathbb{A}_\mathbb{Q})$) generated by φ_1 (resp. φ_0). Note that both π_1 and π_0 are non-tempered. It is easy to check that

$$\begin{aligned}\Lambda(s,\pi_1) &= \Lambda(s,\mathrm{Sym}^2(\rho))\xi(s+1)\xi(s)\xi(s-1),\\ \Lambda(s,\pi_0) &= \Lambda(s,\tau)\xi(s+(1/2))\xi(s-(1/2)),\\ \Lambda(s,\pi_1,\mathrm{Ad}) &= \Lambda(s+1,\mathrm{Sym}^2(\rho))\Lambda(s,\mathrm{Sym}^2(\rho))\Lambda(s-1,\mathrm{Sym}^2(\rho))\\ &\quad\times\Lambda(s,\mathrm{Ad}(\rho))\xi(s+1)\xi(s)\xi(s-1),\\ \Lambda(s,\pi_0,\mathrm{Ad}) &= \Lambda(s,\mathrm{Ad}(\tau))\Lambda(s+(1/2),\tau)\Lambda(s-(1/2),\tau)\\ &\quad\times\xi(s+1)\xi(s)\xi(s-1).\end{aligned}$$

It follows that $\mathcal{P}_{\pi_1,\pi_0}(s)=R(s)/Q(s)$, where

$$\begin{aligned}R(s) =&\Lambda(s,\mathrm{Sym}^2(\rho)\boxtimes\tau)\Lambda(s-1,\tau)\xi(s-(3/2)),\\ Q(s) =&\Lambda(s+(3/2),\mathrm{Sym}^2(\rho))\Lambda(s+(1/2),\mathrm{Ad}(\rho))\\ &\times\Lambda(s+(1/2),\mathrm{Ad}(\tau))\xi(s+(3/2)).\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(1/2) =&\frac{\Lambda(1/2,\mathrm{Sym}^2(\rho)\boxtimes\tau)\Lambda(-1/2,\tau)\xi(-1)}{\Lambda(2,\mathrm{Sym}^2(\rho))\Lambda(1,\mathrm{Ad}(\rho))\Lambda(1,\mathrm{Ad}(\tau))\xi(2)}\\ =&-\frac{\Lambda(1/2,\mathrm{Sym}^2(\rho)\boxtimes\tau)\Lambda(3/2,\tau)}{\Lambda(2,\mathrm{Sym}^2(\rho))\Lambda(1,\mathrm{Ad}(\rho))\Lambda(1,\mathrm{Ad}(\tau))}\end{aligned}$$

by the functional equations $\Lambda(1-s,\tau)=-\Lambda(s,\tau)$, $\xi(1-s)=\xi(s)$.

We consider the local factor $\alpha_v(\varphi_{1,v},\varphi_{0,v})$. For $v\notin S$, we may consider $\alpha_v(\varphi_{1,v},\varphi_{0,v})=1$. For $v\in S_{\mathrm{f}}$, the condition (U1) and (U2) of §1 fail. Instead, (U1) and (U2), we consider the following conditions:

(U1′) $G_{i,v}$ is quasi-split.
(U2′) $\mathcal{K}_{i,v}$ is a special maximal compact subgroup of $G_{i,v}$.

Lemma 9.1. *Assume $n=5$. Let v be a non-archimedean place such that the conditions* (U1′)*,* (U2′)*,* (U3)*,* (U4)*,* (U5)*, and* (U6)*. Then we have $I(\varphi_{1,v},\varphi_{0,v})=\Delta_{G_{1,v}}\mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)$, if it is convergent.*

The authors have verified this lemma by using computer calculation. By this lemma we may consider $\alpha_v(\varphi_{1,v},\varphi_{0,v})=1$ by "analytic continuation".

For $v=\infty$, one can easily see that $\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)$ is equal to

$$\begin{aligned}&\frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa)\Gamma_{\mathbb{C}}(2\kappa-1)\cdot\Gamma_{\mathbb{C}}(\kappa-1)\cdot\Gamma_{\mathbb{R}}(-1)}{\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa+1)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2\kappa)\cdot\Gamma_{\mathbb{R}}(2)}\\ &=-\frac{16\pi^7}{\kappa(\kappa-1)(2\kappa-1)}.\end{aligned}$$

Note that $\pi_{1,\infty}$ is a discrete series representation of $\mathrm{SO}(4,2)$, and the K-type of $\varphi_{1,\infty}$ is the lowest K-type. Similarly, $\pi_{0,\infty}$ is a discrete series representation of $\mathrm{SO}(3,2)$, and $\varphi_{0,\infty}$ is a lowest K-type vector. We may assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$.

Proposition 9.2. *We have*

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) =& \frac{64\pi^3}{\kappa(\kappa-1)(2\kappa-1)}, \\ \alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) =& -4\pi. \end{aligned}$$

A proof of Proposition 9.2 will be given in §11.

By Proposition 9.2, we have

$$\begin{aligned} \frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} &= \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \cdot \frac{\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2 \cdot \|\varphi_{0,\infty}\|^2} \\ &= \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2} \end{aligned}$$

under the assumption $c(D) \neq 0$. Therefore in this case, Conjecture 3.2 seems to hold with $2^\beta = 1/4$.

Appendix to §9: Calculation of the volume of the fundamental domain for $\Gamma_K \backslash \mathcal{H}_2$

In this appendix, we calculate the volume of the fundamental domain for the hermitian modular group. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$. We put $K_p = K \otimes \mathbb{Q}_p$ and $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$, where $\mathcal{O}_K$ is the integer ring of K.

Let $\Gamma_K^{(n)} = \mathrm{SU}(n,n)(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathcal{O}_K)$ be the special hermitian modular group. By using the fact that the Tamagawa number of $\mathrm{SU}(n,n)$ is 1, we shall show that

$$\mathrm{Vol}(\Gamma_K^{(n)} \backslash \mathcal{H}_n) = 2^{-n^2+1} D^{(2n^2-n-1)/2} (2n, w_K) \prod_{i=2}^{2n} \Lambda(i, \chi^i),$$

where $\mathcal{H}_n$ is the hermitian upper half space of degree n.

Put $\mathfrak{G} = \mathrm{SU}(n,n)$. Then

$$\mathrm{Lie}(\mathfrak{G}) = \left\{ X \in \mathrm{M}_{2n}(K) \,\middle|\, XJ + J \cdot {}^t\bar{X} = 0,\ \mathrm{tr}(X) = 0 \right\},$$

where $J = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$. We choose a basis of the $\mathrm{Lie}(\mathfrak{G})$ as follows. Let $E[i,j] \in \mathrm{M}_n(\mathbb{Z})$ be the (i,j)-elementary matrix of size n. Set

$$\begin{aligned} S[i,j] &= \begin{cases} E[i,i] & (i = j), \\ E[i,j] + E[j,i] & (i \neq j) \end{cases} \\ A[i,j] &= E[i,j] - E[j,i]. \end{aligned}$$

Put

$$
\begin{aligned}
X_{ij} &= \begin{pmatrix} E[i,j] & 0 \\ 0 & -E[j,i] \end{pmatrix}, \\
Y_{ij} &= \begin{pmatrix} 0 & S[i,j] \\ 0 & 0 \end{pmatrix}, \qquad Y'_{ij} = \begin{pmatrix} 0 & 0 \\ S[i,j] & 0 \end{pmatrix}, \\
V_{ij} &= \sqrt{-D}\begin{pmatrix} 0 & A[i,j] \\ 0 & 0 \end{pmatrix}, \quad V'_{ij} = -\sqrt{-D}\begin{pmatrix} 0 & 0 \\ A[i,j] & 0 \end{pmatrix}, \\
W_{ij} &= \sqrt{-D}\begin{pmatrix} E[i,j] & 0 \\ 0 & E[j,i] \end{pmatrix}, \\
W'_{i} &= \sqrt{-D}\begin{pmatrix} E[i,i]-E[i+1,i+1] & 0 \\ 0 & E[i,i]-E[i+1,i+1] \end{pmatrix}.
\end{aligned}
$$

The following vectors make up a basis of $\mathrm{Lie}(\mathfrak{G})$.

$$
\begin{array}{rl}
X_{ij} & (1 \le i,j \le n), \\
Y_{ij} & (1 \le i \le j \le n), \\
Y'_{ij} & (1 \le i \le j \le n), \\
V_{ij} & (1 \le i < j \le n), \\
V'_{ij} & (1 \le i < j \le n), \\
W_{ij} & (1 \le i < j \le n), \\
W'_{i} & (1 \le i < n).
\end{array}
$$

Let $\mathfrak{L} \subset \mathrm{Lie}(\mathfrak{G})$ be the lattice generated by this basis. This basis determines a Haar measure dg_v on $\mathfrak{G}(\mathbb{Q}_v)$ for each place v, and the product measure $\prod_v dg_v$ is the Tamagawa measure on $\mathfrak{G}(\mathbb{A})$. For each prime p, we define a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_p}$ of $\mathfrak{G}(\mathbb{Q}_p)$ by $\mathcal{K}_{\mathfrak{G}_p} = \mathfrak{G}(\mathbb{Q}_p) \cap \mathrm{GL}_{2n}(\mathcal{O}_p)$. Since $[\mathcal{O}_p : \mathbb{Z}_p + \sqrt{-D}\mathbb{Z}_p] = (2,p)$, we have

$$[\mathrm{Lie}(\mathfrak{G})(\mathbb{Q}_p) \cap \mathrm{M}_{2n}(\mathcal{O}_p) \;:\; \mathfrak{L} \otimes \mathbb{Z}_p] = (2,p)^{2n^2-n-1}.$$

It follows that the volume of $\mathcal{K}_{\mathfrak{G}_p}$ is equal to $(2,p)^{2n^2-n-1}\prod_{i=2}^{2n} L(i,\chi_p^i)^{-1}$.

For the real place, the vectors

$$
\begin{array}{rl}
X_{ij} - X_{ji} & (1 \le i < j \le n), \\
Y_{ij} - Y'_{ij} & (1 \le i \le j \le n), \\
V_{ij} + V'_{ij} & (1 \le i < j \le n), \\
W_{ij} + W_{ji} & (1 \le i < j \le n), \\
W'_{i} & (1 \le i < n)
\end{array}
$$

generate the Lie algebra of a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ of $\mathfrak{G}(\mathbb{R})$. The maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ is isomorphic to

$$\{(u_1, u_2) \in \mathrm{U}(n) \times \mathrm{U}(n) \,|\, \det u_1 \cdot \det u_2 = 1\}.$$

This isomorphism is explicitly given by $\mathrm{Ad}(A) : \kappa \mapsto A\kappa A^{-1}$, where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_n & -\sqrt{-1} \cdot \mathbf{1}_n \\ \mathbf{1}_n & \sqrt{-1} \cdot \mathbf{1}_n \end{pmatrix}.$$

Note that

$$\begin{aligned} \mathrm{Ad}(A)(X_{ij} - X_{ji}) &= \begin{pmatrix} A[i,j] & 0 \\ 0 & A[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(Y_{ij} - Y'_{ij}) &= \sqrt{-1} \begin{pmatrix} S[i,j] & 0 \\ 0 & -S[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(V_{ij} + V'_{ij}) &= \sqrt{D} \begin{pmatrix} -A[i,j] & 0 \\ 0 & A[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(W_{ij} + W_{ji}) &= \sqrt{-D} \begin{pmatrix} S[i,j] & 0 \\ 0 & S[i,j] \end{pmatrix}, \\ \mathrm{Ad}(A)(W'_i) &= W'_i. \end{aligned}$$

Let dk_∞ be the Haar measure on $\mathcal{K}_{\mathfrak{G}_\infty}$ determined by these vectors. By Macdonald [49], the volume of $\mathrm{U}(n)$ is equal to $(2\pi)^{n(n+1)/2} \prod_{i=1}^n \Gamma(i)^{-1}$, if the Haar measure is normalized by a Chevalley basis of $\mathrm{Lie}(\mathrm{U}(n)) \otimes \mathbb{C}$. Using this, we have

$$\mathrm{Vol}(\mathcal{K}_{\mathfrak{G}_\infty}; dk_\infty) = D^{(-n^2+1)/2} 2^{-n^2+2n} \pi^{n^2+n-1} \prod_{i=1}^n \Gamma(i)^{-2}.$$

We now consider the invariant measure on the hermitian upper half space $\mathcal{H}_n$. We define an invariant measure on $\mathcal{H}_n$ as follows. Let $\mathrm{Her}_n(\mathbb{C}/\mathbb{R})$ be the space of hermitian matrix of size n. Then the Haar measures dX and dY on $\mathrm{Her}_n(\mathbb{C}/\mathbb{R})$ are such that the covolume of the lattice $\mathrm{Her}_n(\mathbb{C}/\mathbb{R}) \cap \mathrm{M}_n(\mathbb{Z}[\sqrt{-1}])$ is 1. Then the measure $(\det Y)^{-2n} dX\, dY$ is invariant under the action of $\mathfrak{G}(\mathbb{R}) = \mathrm{SU}(n,n)(\mathbb{R})$.

Note that $\mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. We claim that dg_∞/dk_∞ is equal to $2^{-n} D^{-(n^2-n)/2} (\det Y)^{-2n} dX\, dY$. To prove this, we consider the Iwasawa decomposition $\mathfrak{G}(\mathbb{R}) = A_{\mathfrak{G}_\infty} N_{\mathfrak{G}_\infty} \mathcal{K}_{\mathfrak{G}_\infty}$, where $A_{\mathfrak{G}_\infty}$ and $N_{\mathfrak{G}_\infty}$ are Lie subgroup of $\mathfrak{G}(\mathbb{R})$ corresponding to the Lie algebras generated by

$$\{X_{ii} \,|\, 1 \le i \le n\}$$

and

$$\{X_{ij}, V_{ij}, W_{ij} \,|\, 1 \le i < j \le n\} \cup \{Y_{ij} \,|\, 1 \le i \le j \le n\},$$

respectively. Then it is easy to check the left invariant Haar measure determined by these basis induces $2^{-n}D^{-(n^2-n)/2}(\det Y)^{-2n}dX\,dY$ on $\mathcal{H}_n$, which implies the claim.

Now we consider the adele space $\mathfrak{G}(\mathbb{A})$. Let $\mathfrak{X}$ be a fundamental domain for $\Gamma_K^{(n)}\backslash\mathcal{H}_n$. We regard $\mathfrak{X}$ as a subset of $A_{\mathfrak{G}_\infty}N_{\mathfrak{G}_\infty}$ by the bijection $A_{\mathfrak{G}_\infty}N_{\mathfrak{G}_\infty} \simeq \mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. Then each fibre of the map

$$(\prod_p \mathcal{K}_{\mathfrak{G}_p}) \times \mathfrak{X} \times \mathcal{K}_{\mathfrak{G}_\infty} \to \mathfrak{G}(\mathbb{Q})\backslash\mathfrak{G}(\mathbb{A})$$

has exactly $\sharp Z(\Gamma_K^{(n)})$ elements, where $Z(\Gamma_K^{(n)})$ is the center of $\Gamma_K^{(n)}$. Note that $\sharp Z(\Gamma_K^{(n)}) = (2n, w_K)$. It follows that

$$(2n, w_K)^{-1} \cdot 2^{2n^2-n-1} \prod_{i=2}^{2n} L(i,\chi^i)^{-1} \cdot D^{(-n^2+1)/2} 2^{-n^2+2n} \pi^{n^2+n-1} \prod_{i=1}^{n} \Gamma(i)^{-2}$$
$$\times\, 2^{-n} D^{-(n^2-n)/2} \mathrm{Vol}(\mathfrak{X}) = 1.$$

It follows that

$$\mathrm{Vol}(\Gamma_K^{(n)}\backslash\mathcal{H}_n) = 2^{-n^2+1} D^{(2n^2-n-1)/2} (2n, w_K) \prod_{i=2}^{2n} \Lambda(i,\chi^i),$$

as desired.

10. The trivial representation

Let k be a totally real field and S the set of archimedean places of k. The discriminant of k is denoted by D_k. Recall that the completed Dedekind zeta function $\xi_k(s)$ satisfies the functional equation $\xi_k(1-s) = D_k^{s-(1/2)}\xi_k(s)$. Put $d = [k:\mathbb{Q}]$. We assume the following conditions:

(a) Both G_1 and G_0 are unramified over k_v for each $v \notin S$.
(b) $G_{0,v}$ is compact for each $v \in S$.

Note that such a pair $G_0 \subset G_1$ exists, then the following (i), (ii), and (iii) hold:

(i) The discriminant field K is unramified over k.
(ii) K is totally real if $n \equiv 0 \bmod 4$, and totally imaginary if $n \equiv 2 \bmod 4$.
(iii) d is even if $n \equiv 3, 4, 5, 6 \bmod 8$.

Let $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v}$ be a maximal compact subgroup of $G_0(\mathbb{A})$. We assume $\mathcal{K}_{0,v}$ is a hyperspecial maximal compact subgroup for $v \notin S$. For $v \notin S$, we give the standard Haar measure $dg_{0,v}$ on $G_{0,v}$. For $v \in S$, we give the Haar measure $dg_{0,v}$ with total volume 1 on $\mathcal{K}_{0,v} = G_{0,v}$. The constant C_0 can be calculated directly, but here we make use of the mass formula. There exists a finite subset $\mathfrak{B} \subset G_0(\mathbb{A})$ such that $G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} G_0(k)x\mathcal{K}_0$. For each $x \in \mathfrak{B}$, the group $\Gamma^x = x^{-1}G_0(k)x \cap \mathcal{K}_0$ is a finite group. The left coset $G_0(k)\backslash G_0(\mathbb{A})$ is decomposed into a disjoint union

$$G_0(k)\backslash G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} x \cdot (\Gamma^x \backslash \mathcal{K}_0).$$

Let e_x be the order of the group Γ^x. The mass M is defined by $M = \sum_{x \in \mathfrak{B}} e_x^{-1}$. Then the mass formula (See Shimura [61], p.27, Theorem 5.8) says that

$$M = 2D_k^{m^2-(m/2)} \left[(2\pi)^{-m}\Gamma(m)\right]^d L(m,\chi) \prod_{j=1}^{m-1} \left\{ \left[(2\pi)^{-2j}\Gamma(2j)\right]^d \zeta_k(2j) \right\}$$

if $n = 2m$ is even, and that

$$M = 2^{1-md} D_k^{m^2+(m/2)} \prod_{j=1}^{m} \left\{ \left[(2\pi)^{-2j}\Gamma(2j)\right]^d \zeta_k(2j) \right\}$$

if $n = 2m+1$ is odd.

Then we have

$$\int_{G_0(k)\backslash G_0(\mathbb{A})} \prod_v dg_{0,v} = M.$$

Since the Tamagawa number of G_0 is 2, we have $C_0 = 2M^{-1}$.

By definition, we have

$$\Delta_{G_1} = \begin{cases} \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m \text{ is even,} \\ \Lambda(m+1,\chi) \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

We now put $\varphi_1 = 1$ and $\varphi_0 = 1$. Then π_i is the trivial representation of $G_i(\mathbb{A})$. Obviously, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 1.$$

The L-function of the trivial representation of G_0 is given by

$$\Lambda(s,\pi_0) = \begin{cases} \Lambda(s,\chi) \prod_{j=1}^{2m-1} \xi_k(s-m+j) & \text{if } n = 2m \text{ is even,} \\ \prod_{j=1}^{2m} \xi_k(s-m+j-(1/2)) & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Similarly, we have

$$\Lambda(s,\pi_1) = \begin{cases} \prod_{j=1}^{2m} \xi_k(s-m+j-(1/2)) & \text{if } n = 2m \text{ is even,} \\ \Lambda(s,\chi) \prod_{j=1}^{2m+1} \xi_k(s-m+j-1) & \text{if } n = 2m+1 \text{ is odd} \end{cases}$$

When $n = 2m$ is even, we have

$$\begin{aligned}\Lambda(s, \pi_1 \boxtimes \pi_0) &= \prod_{i=1}^{2m} \Lambda(s - m + i - (1/2), \chi) \\ &\quad \times \prod_{i=1}^{2m} \prod_{j=1}^{2m-1} \xi_k(s - 2m + i + j - (1/2))\end{aligned}$$

$$\begin{aligned}\Lambda(s, \pi_0, \mathrm{Ad}) &= \prod_{i=1}^{2m-1} \Lambda(s - m + i, \chi) \\ &\quad \times \prod_{1 \le i < j \le 2m-1} \xi_k(s - 2m + i + j) \\ \Lambda(s, \pi_1, \mathrm{Ad}) &= \prod_{1 \le i \le j \le 2m} \xi_k(s - 2m + i + j - 1).\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(s) &= \frac{\Lambda(s - m + (1/2), \chi)}{\xi_k(s + 2m - (1/2))} \prod_{j=1}^{m-1} \frac{\xi_k(s - 2j + (1/2))}{\xi_k(s + 2j - (1/2))}, \\ \mathcal{P}_{\pi_1,\pi_0}(1/2) &= \frac{\Lambda(1 - m, \chi)}{\xi_k(2m)} \prod_{j=1}^{m-1} \frac{\xi_k(-2j + 1)}{\xi_k(2j)} \\ &= D_k^{m^2 - (m/2)} \frac{\Lambda(m, \chi)}{\xi_k(2m)}\end{aligned}$$

if $n = 2m$ is even. A similar calculation shows that

$$\begin{aligned}\mathcal{P}_{\pi_1,\pi_0}(s) &= \Lambda(s + m + (1/2), \chi)^{-1} \prod_{j=1}^{m} \frac{\xi_k(s - 2j + (1/2))}{\xi_k(s + 2j - (1/2))}, \\ \mathcal{P}_{\pi_1,\pi_0}(1/2) &= \Lambda(m + 1, \chi)^{-1} \prod_{j=1}^{m} \frac{\xi_k(-2j + 1)}{\xi_k(2j)} \\ &= D_k^{m^2 + (m/2)} \Lambda(m + 1, \chi)^{-1},\end{aligned}$$

if $n = 2m + 1$ is odd. When $v \in S$, the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is clearly equal to 1. It follows that

$$\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1}$$
$$= \begin{cases} \Gamma_{\mathbb{R}}(1-m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m \equiv 0 \bmod 4, \\ \Gamma_{\mathbb{R}}(2-m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m \equiv 2 \bmod 4, \\ \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(-2j+1)^{-1} & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Therefore we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = 2^{\beta} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v \in S} \alpha_v(\varphi_{1,v}, \varphi_{0,v}),$$

where

$$\beta = \begin{cases} -md & \text{if } n = 2m \text{ is even,} \\ -2md & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Note that the integer β depends on the number of bad places.

11. Calculation for the real place

In this section, we carry out the calculation of the archimedean local integrals which appeared in §§7-9. Every algebraic group is defined over $\mathbb{R}$ in this section.

We first consider the case $G_0 = \mathrm{SO}(2,1) \simeq \mathrm{PGL}_2(\mathbb{R})$. The (topological) identity component of G_0 is denoted by $G_0(\mathbb{R})^0$. Note that $G_0(\mathbb{R})^0 \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. The image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{R})$ is denoted by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The maximal compact subgroup $\mathrm{O}(2)/\{\pm 1\} \subset \mathrm{PGL}_2(\mathbb{R})$ is denoted by $\mathcal{K}$. Put $\mathcal{K}^0 = \mathrm{SO}(2)/\{\pm 1\} \subset \mathcal{K}$. The Haar measure dk on $\mathcal{K}^0$ is such that the total measure is 1. By Iwasawa decomposition, an element $g \in G_0(\mathbb{R})^0$ can be uniquely written as

$$g = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} k,$$

$t, n \in \mathbb{R}$, $k \in \mathcal{K}^0$. We choose a Haar measure dg on $G_0(\mathbb{R})^0$ such that dg/dk induces the measure $y^{-2} dx\, dy$ on the upper half plane $\mathfrak{H}_1 \simeq G_0(\mathbb{R})/\mathcal{K}^0$. Note that $dg = 2dt\, dn\, dk$. The Haar measure dg can be naturally extended to $G_0(\mathbb{R})$. We put

$$A^+ = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \,\middle|\, t \geq 0 \right\}.$$

We consider the map

$$\begin{array}{rcl} \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 & \to & G_0(\mathbb{R})^0 \\ \left(k, \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, k'\right) & \mapsto & k \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} k'. \end{array}$$

By Cartan decomposition, this map is bijective outside the boundary of A^+. It is well-known (e.g., [27], Theorem 5.8) that

$$dg = C \cdot \sinh(2t) dk\, dt\, dk'$$

for some constant $C > 0$. Let $A(T)$ be the area of the small disc with radius T and center $\sqrt{-1} \in \mathfrak{H}_1$. Then we have $A(T) \sim C \int_0^{T/2} \sinh(2t) dt$ when $T \to 0$, and so we have $C = 4\pi$.

Let τ_j be the (limit of) discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with minimal weight $\pm\kappa_j$. Let Φ_j be the matrix coefficient of $\tau_{j,\infty}$ with respect to the lowest weight vector with norm 1. Then the support of Φ_j is contained in $G_0(\mathbb{R})^0$ and

$$\Phi_j\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\right) = \cosh(t)^{-\kappa_j}.$$

Proof of Proposition 7.1. Let $\varphi_{1,\infty}$ and $\varphi_{0,\infty}$ be as in Proposition 7.1. Then we have

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) =& 4\pi \int_0^\infty \cosh(t)^{-2\kappa_3} \sinh(2t) dt \\ =& 4\pi(\kappa_3 - 1)^{-1}. \end{aligned}$$

For the latter part of the proposition,

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)^{-1} I(\varphi_{1,\infty}, \varphi_{0,\infty}) = 2.$$

□

Next, we consider the case $G_0 = \mathrm{SO}(2,2)$. Put

$$\mathrm{GL}_2^{(2)} = \{(h_1, h_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det h_1 = \det h_2\}.$$

Then, we have $\mathrm{SO}(2,2) \simeq \mathrm{GL}_2^{(2)}(\mathbb{R})/\mathbb{R}^\times$. We denote the image of $(h_1, h_2) \in \mathrm{GL}_2^{(2)}(\mathbb{R})$ in $\mathrm{SO}(2,2)$ by $[h_1, h_2]$. Put

$$\begin{aligned} A =& \left\{ \left[\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right] \middle| \, t_1, t_2 \in \mathbb{R} \right\}, \\ N =& \left\{ \left[\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} \right] \middle| \, n_1, n_2 \in \mathbb{R} \right\}, \\ \mathcal{K} =& \{[k_1, k_2] \mid k_1, k_2 \in \mathrm{O}(2), \det k_1 = \det k_2\}. \end{aligned}$$

For each $(t_1, t_2) \in \mathbb{R}^2$, we put

$$m(t_1, t_2) = \left[\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right].$$

The connected component $\mathrm{SO}(2,2)^0$ is equal to the image of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Put $\mathcal{K}^0 = \mathcal{K} \cap \mathrm{SO}(2,2)^0$. Then we have an Iwasawa decomposition $\mathrm{SO}(2,2)^0 = AN\mathcal{K}^0$. Then $\mathrm{SO}(2,2)^0/\mathcal{K}^0$ can be identified with $\mathfrak{H}_1 \times \mathfrak{H}_1$. The Haar measure dk on $\mathcal{K}^0$ is the Haar measure such that the total volume are 1. We choose a Haar measure dg on $\mathrm{SO}(2,2)^0$ such that the induced measure dg/dk on $\mathfrak{H}_1 \times \mathfrak{H}_1$ is equal to $y_1^{-2} y_2^{-2} dx_1\, dx_2\, dy_1\, dy_2$. Then $dg = 4 dt_1\, dt_2\, dn_1\, dn_2\, dk$. The Haar measure dg can be naturally extended to $G_0(\mathbb{R}) = \mathrm{SO}(2,2)$. Put $A^+ = \{m(t_1, t_2) \,|\, t_1, t_2 \geq 0\}$. Consider the map

$$\begin{aligned} \lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\to \mathrm{SO}(2,2)^0 \\ (k, m(t_1, t_2), k') &\mapsto k \cdot m(t_1, t_2) \cdot k'. \end{aligned}$$

Let ∂A^+ be the boundary of A^+. If $g \in G_0(\mathbb{R})^0$ is not in the image of ∂A^+, then $\lambda^{-1}(g)$ consists of two elements. In terms of the map λ, we have

$$\begin{aligned} &\int_{G_0(\mathbb{R})^0} f(g) dg \\ =&16\pi^2 \int_{\mathcal{K}^0 \times A^+ \times \mathcal{K}^0} f(\lambda(k, m(t_1, t_2), k')) \sinh(2t_1) \sinh(2t_2) dk\, dt_1\, dt_2\, dk' \end{aligned}$$

for any integrable function f on $G_0(\mathbb{R})^0$.

Proof of Proposition 8.1. We need to calculate the matrix coefficient of $\varphi_{1,\infty} \in \pi_{1,\infty}$. In fact, it is enough to consider the pullback of the matrix coefficient by the map $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2,2) \subset \mathrm{SO}(3,2)$, since A^+ is contained in the image of this map. Note that the image of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is contained in the identity component $\mathrm{SO}(3,2)^0 = \mathrm{Sp}_2(\mathbb{R})/\{\pm 1\}$. The restriction of $\pi_{1,\infty}$ is a direct sum of a holomorphic discrete series and an anti-holomorphic discrete series. Since the holomorphic discrete series is a lowest weight representation, its pullback to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a direct sum of lowest weight representations. We denote τ_λ the holomorphic discrete series of $\mathrm{SL}_2(\mathbb{R})$ with lowest weight λ. Since the lowest weight $(\kappa+1, \kappa+1)$ occurs with multiplicity one, the summand contains $\tau_{\kappa+1} \boxtimes \tau_{\kappa+1}$ exactly once, and the other summands are of the form $\tau_{\lambda_1} \boxtimes \tau_{\lambda_2}$, where $\lambda_1, \lambda_2 \geq \kappa+1$ and $(\lambda_1, \lambda_2) \neq (\kappa+1, \kappa+1)$. (In fact, the precise decomposition of the restriction is known in this case.) Therefore the value of the matrix coefficient at $m(t_1, t_2) \in A^+$ is equal to $\cosh(t_1)^{-\kappa-1} \cosh(t_2)^{-\kappa-1}$.

It follow that

$$\begin{aligned}
I(\varphi_{1,\infty},\varphi_{0,\infty}) &= 16\pi^2\left(\int_0^\infty \cosh(t)^{-2\kappa-2}\sinh(2t)dt\right)^2 \\
&= 16\pi^2/\kappa^2, \\
\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty}) &= \Delta_{G_1,\infty}^{-1}\mathcal{P}_{\pi_{1,\infty},\pi_{0,\infty}}(1/2)^{-1}I(\varphi_{1,\infty},\varphi_{0,\infty}) \\
&= 4\pi.
\end{aligned}$$

□

Now, we consider the case $G_0 = \mathrm{SO}(3,2) = \mathrm{GSp}_2(\mathbb{R})/\mathbb{R}^\times$. we denote the image of $\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{GSp}_2(\mathbb{R})$ in $G_0(\mathbb{R})$ by $\begin{bmatrix} A & B \\ C & D\end{bmatrix}$. Put

$$A = \left\{ \left[\begin{array}{cc|cc} e^{t_1} & 0 & \multicolumn{2}{c}{0} \\ 0 & e^{t_2} & & \\ \hline \multicolumn{2}{c|}{0} & e^{-t_1} & 0 \\ & & 0 & e^{-t_2} \end{array}\right] \middle| \; t_1, t_2 \in \mathbb{R} \right\},$$

$$N' = \left\{ \left[\begin{array}{cc|cc} 1 & n'_1 & \multicolumn{2}{c}{0} \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 \\ & & -n'_1 & 1 \end{array}\right] \middle| \; n'_1 \in \mathbb{R} \right\},$$

$$N'' = \left\{ \left[\begin{array}{c|cc} \mathbf{1}_2 & n''_{11} & n''_{12} \\ & n''_{12} & n''_{22} \\ \hline 0 & \multicolumn{2}{c}{\mathbf{1}_2} \end{array}\right] \middle| \; n''_{11}, n''_{12}, n''_{22} \in \mathbb{R} \right\},$$

$$\mathcal{K}^0 = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \middle| \; A + \sqrt{-1}B \in \mathrm{U}(2) \right\}.$$

Then the topological identity component $G_0(\mathbb{R})^0 = \mathrm{SO}(3,2)^0$ has an Iwasawa decomposition $G_0(\mathbb{R})^0 = AN\mathcal{K}^0$, where $N = N'N''$. Note that $G_0(\mathbb{R})^0/\mathcal{K}^0$ can be identified with $\mathfrak{H}_2$. We take the Haar measures dk on $\mathcal{K}^0$ with the total volume 1. We choose the Haar measure dg of $G_0(\mathbb{R})^0$ such that the induced measure dg/dk is equal to $(\det Y)^{-3}dXdY$. Then we have

$$dg = 4dt_1\, dt_2\, dn'_1\, dn''_{11}\, dn''_{12}\, dn''_{22}\, dk.$$

The Haar measure dg can be naturally extended to $G_0(\mathbb{R})$. Put $\mathfrak{a} = \mathrm{Lie}(A)$. Then $\mathfrak{a}$ can be identified with $\mathbb{R}^2$ and we put

$$m(t_1,t_2) = \left[\begin{array}{cc|cc} e^{t_1} & 0 & \multicolumn{2}{c}{0} \\ 0 & e^{t_2} & & \\ \hline \multicolumn{2}{c|}{0} & e^{-t_1} & 0 \\ & & 0 & e^{-t_2} \end{array}\right]$$

for each $(t_1, t_2) \in \mathbb{R}^2 \simeq \mathfrak{a}$. The positive chamber A^+ is defined by $A^+ = \{m(t_1, t_2) \in A \,|\, t_1 \geq t_2 \geq 0\}$. Then the map

$$\begin{aligned}\lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\to \mathrm{SO}(3,2)^0 \\ (k, m(t_1, t_2), k') &\mapsto k \cdot m(t_1, t_2) \cdot k'\end{aligned}$$

is a double covering outside the boundary of A^+. In terms of this map, we have (cf. [27], Theorem 5.8)

$$dg = C \sinh(2t_1) \sinh(2t_2) \sinh(t_1 - t_2) \sinh(t_1 + t_2) dk dt_1 dt_2 dk'.$$

for some positive constant $C > 0$.

The constant C can be calculated as follows. We recall the argument of [27], Ch.I, Theorem 5.8. We shall calculate the Jacobian of the induced map

$$\bar{\lambda} : \mathcal{K}^0 \times A^+ \to G_0(\mathbb{R})^0/\mathcal{K}^0 \simeq AN$$

at $(k, m(t_1, t_2)) \in \mathcal{K}^0 \times A^+$. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(\mathrm{SO}(3,2)^0)$. Then the tangent space of $\mathcal{K}^0 \times A^+$ at $(k, m(t_1, t_2))$ can be identified with $\mathfrak{k} + \mathfrak{a}$ by left translation. Let Σ^+ be the set of positive roots for $(G_0(\mathbb{R})^0, A)$. Then for each $\alpha \in \Sigma^+$, we put

$$\mathfrak{k}_\alpha = \left\{ T \in \mathfrak{k} \,|\, \mathrm{ad}((x_1, x_2))^2 T = \alpha((x_1, x_2))^2 T \quad \text{for all } (x_1, x_2) \in \mathfrak{a} \right\}.$$

Then $\dim \mathfrak{a}_\alpha = 1$ for any $\alpha \in \Sigma^+$. Choose a non-zero vector $T_\alpha \in \mathfrak{k}_\alpha$ for each $\alpha \in \Sigma^+$. For example, we can choose

$$T_{\varepsilon_1 - \varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right), \qquad T_{2\varepsilon_1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$T_{\varepsilon_1 + \varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right), \qquad T_{2\varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right).$$

For each $\alpha \in \Sigma^+$,

$$U_\alpha = \alpha((t_1, t_2))^{-1} \mathrm{ad}((t_1, t_2))(T_\alpha)$$

belongs to $\mathfrak{p}$, and does not depend on $(t_1, t_2) \in \mathfrak{a}$. Note that

$$U_{\varepsilon_1-\varepsilon_2} = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \end{array}\right), \qquad U_{2\varepsilon_1} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 0 \\ \hline 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \end{array}\right),$$

$$U_{\varepsilon_1+\varepsilon_2} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right), \qquad U_{2\varepsilon_2} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 1 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 1 & & \end{array}\right).$$

Then

$$T_\alpha \ (\alpha \in \Sigma^+), \qquad (1,0), (0,1) \in \mathfrak{a}$$

make up a basis of $\mathfrak{k} + \mathfrak{a}$, and

$$U_\alpha \ (\alpha \in \Sigma^+), \qquad (1,0), (0,1) \in \mathfrak{a}$$

make up a basis of $\mathfrak{p}$. By the proof of [27], Ch.I, Theorem 5.8,

$$|\det(d\bar{\lambda}_{(k,m(t_1,t_2))})| = \prod_{\alpha \in \Sigma^+} (\sinh(\alpha(t_1, t_2)))$$

with respect to these basis.

Let ω_α $(\alpha \in \Sigma^+)$ be the basis of the space of left invariant 1-forms on $\mathcal{K}^0$ dual to T_α $(\alpha \in \Sigma^+)$. Then it is easy to check that

$$\int_{\mathcal{K}^0} |\bigwedge_{\alpha \in \Sigma^+} \omega_\alpha| = 2\pi^3.$$

On the other hand, the dual basis of

$$(1,0), (0,1) \in \mathfrak{a}, \qquad U_\alpha \quad (\alpha \in \Sigma^+)$$

induces

$$\frac{1}{16} dt_1\, dt_2\, dn'_1\, dn''_{11}\, dn''_{12}\, dn''_{22}$$

on $AN \simeq G_0(\mathbb{R})^0/\mathcal{K}^0$. It follows that $C = 64\pi^3$.

Proof of Proposition 9.2. As in the proof of Proposition 8.1, the value of the matrix coefficient $\langle \pi_{1,\infty}(g_0)\varphi_{1,\infty}, \varphi_{1,\infty}\rangle$ at $g_0 = m(t_1, t_2)$ is equal to $\cosh(t_1)^{-\kappa-1}\cosh(t_2)^{-\kappa-1}$. It follows that

$$\begin{aligned} I(\varphi_{1,\infty},\varphi_{0,\infty}) =& 64\pi^3 \int_{t_1\geq t_2\geq 0} \cosh(t_1)^{-2\kappa-2}\cosh(t_2)^{-2\kappa-2} \\ & \times \sinh(2t_1)\sinh(2t_2)\sinh(t_1+t_2)\sinh(t_1-t_2)dt_1dt_2 \\ =& 64\pi^3 \int_0^\infty\int_0^\infty \cosh(x+y)^{-2\kappa-2}\cosh(y)^{-2\kappa-2} \\ & \times \sinh(2x+2y)\sinh(2y)\sinh(x+2y)\sinh(x)dxdy. \end{aligned}$$

By using the formulas

$$\sinh(2a) = 2\sinh(a)\cosh(a),$$
$$\sinh(a+b)\sinh(a-b) = \cosh^2(a)-\cosh^2(b),$$

one can show that the integral $I(\varphi_{1,\infty},\varphi_{0,\infty})$ is equal to

$$\begin{aligned} & 256\pi^3\int_0^\infty \cosh(y)^{-2\kappa-1}\sinh(y) \\ & \quad \times \int_0^\infty \cosh(x+y)^{-2\kappa-1}\sinh(x+y)[\cosh^2(x+y)-\cosh^2(y)]dxdy \\ =& 256\pi^3\int_0^\infty \cosh(y)^{-2\kappa-1}\sinh(y) \\ & \quad \times \left\{ \left[-\frac{u^{-2\kappa+2}}{2\kappa-2}\right]_{u=\cosh(y)}^\infty - \cosh^2(y)\left[-\frac{u^{-2\kappa}}{2\kappa}\right]_{u=\cosh(y)}^\infty \right\} dy \\ =& \frac{128\pi^3}{\kappa(\kappa-1)}\int_0^\infty \cosh(y)^{-4\kappa+1}\sinh(y)dy \\ =& \frac{64\pi^3}{\kappa(\kappa-1)(2\kappa-1)}. \end{aligned}$$

Since $\Delta_{G_{1,\infty}} = \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{R}}(4)^2 = \pi^{-5}$, we have $\alpha_\infty(\varphi_{1,\infty},\varphi_{0,\infty}) = -4\pi$. □

Part IV. Examples over function fields

12. Basic definitions

We give several examples over a function field k. We assume k is a function field of genus 1 over a finite field $\mathbb{F}_q$ with odd characteristic. The zeta function of k is denoted by $\zeta(s) = \zeta_k(s)$. The residue of $\zeta(s)$ at $s=1$ is denoted by ρ_k. For each place v of k, let k_v, $\mathfrak{o}_v$, and $\mathfrak{k}_v$ denote the completion of k at v, the maximal order of k_v, and the residue field of k_v. The order of $\mathfrak{k}_v$ is denoted by q_v. Put $\hat{\mathfrak{o}} = \prod_v \mathfrak{o}_v$ and $\hat{\mathfrak{o}}^\times = \prod_v \mathfrak{o}_v^\times$. The subgroup of square elements of $\mathbb{A}^\times$ is denoted by $\mathbb{A}^{\times 2}$. When (a_{ij}) is a matrix, we denote $f(a_{ij})$, rather than $f((a_{ij}))$, to avoid notational complexity.

Lemma 12.1. *There exists an additive character* $\psi = \prod_v \psi_v$ *of* $k\backslash\mathbb{A}$ *such that the order of* ψ_v *is* 0 *for any* v.

Proof. We fix a faithful character ψ_0 of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let Ω be a non-zero invariant differential form defined over $\mathbb{F}_q$. Put

$$\psi_v(x) = \psi_0(\mathrm{Tr}_{\mathfrak{k}_v/\mathbb{F}_p}(\mathrm{Res}_v(x\Omega))$$

for $x \in k_v$. Then ψ_v has order 0 for any v. By the residue theorem, $\psi = \prod_v \psi_v$ induces an additive character of $k\backslash\mathbb{A}$. $\square$

We fix such an additive character ψ once for all. The Haar measure on algebraic groups will be given as follows. For a unipotent group N, we give the Tamagawa measure dn on $N(\mathbb{A})$. For any algebraic group $G \subset \mathrm{GL}_n$ defined over k, we put $\mathcal{K}_G = G(\mathbb{A}) \cap \mathrm{GL}_n(\hat{\mathfrak{o}})$. When G is a connected reductive algebraic group defined over k, we denote by dg the Tamagawa measure. If G is unramified over k_v, then dg_v denotes the Haar measure such that the volume of the hyperspecial maximal compact subgroup is 1. For locally unramified algebraic group G, we define the constant C_G by $dg = C_G \prod_v dg_v$. For example, $C_{\mathbb{G}_m} = \rho_k^{-1}$, $C_{\mathrm{SL}_2} = \zeta(2)^{-1}$, and $C_{\mathrm{Sp}_2} = \zeta(2)^{-1}\zeta(4)^{-1}$. The Haar measure of the adele groups of non-connected algebraic groups are defined as follows. Let $\mathcal{G}$ be an algebraic group defined over k such that the connected component $\mathcal{G}^0$ is a reductive group. Let $\mathcal{Z}^0$ be the split component of the connected component of the center of $\mathcal{G}^0$. We assume that $\mathcal{Z}^0$ is contained in the center of $\mathcal{G}$ and that every coset for $\mathcal{G}/\mathcal{G}^0$ has a k-valued point. Consider the exact sequence

$$1 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}^0 \longrightarrow 1.$$

We give the Tamagawa measures on $\mathcal{G}^0(\mathbb{A})$ and $\mathcal{Z}^0(\mathbb{A})$. On the group $\mathcal{G}(\mathbb{A})/\mathcal{G}^0(\mathbb{A}) = \prod_v (\mathcal{G}/\mathcal{G}^0)$, we give the Haar measure such that the total volume is 1. In particular, if $\mathcal{Z}^0 = \{1\}$, then $\mathrm{Vol}(\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A})) = [\mathcal{G} : \mathcal{G}^0]^{-1}\mathrm{Tam}(\mathcal{G}^0)$, where $\mathrm{Tam}(\mathcal{G}^0)$ is the Tamagawa number of $\mathcal{G}^0$. For a unitary character ω of $\mathcal{Z}^0(\mathbb{A})/\mathcal{Z}^0(k)$, we define $L^2(\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A});\omega)$ to be the space of functions h on $\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A})$ such that $h(zg) = \omega(z)h(g)$ for $z \in \mathcal{Z}^0(\mathbb{A})$ and $|h| \in L^2(\mathcal{G}(k)\mathcal{Z}^0(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$. We put

$$\langle h_1, h_2\rangle_{\mathcal{G}} = \int_{\mathcal{G}(k)\mathcal{Z}^0(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})} h_1(g)\overline{h_2(g)}\,dg$$

for $h_1, h_2 \in L^2(\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A}); \omega)$, where dg is the Haar measure defined as above. For example,

$$\begin{aligned} \langle 1,1\rangle_{\mathrm{SO}(n)} &= 2\langle 1,1\rangle_{\mathrm{O}(n)} = 2 && \text{if } n \geq 3, \\ \langle 1,1\rangle_{\mathrm{SO}(n)} &= 2\langle 1,1\rangle_{\mathrm{O}(n)} = 2 && \text{if } n = 2,\ \mathrm{SO}(2) \not\simeq \mathbb{G}_m, \\ \langle 1,1\rangle_{\mathrm{SO}(n)} &= 2\langle 1,1\rangle_{\mathrm{O}(n)} = 1 && \text{if } n = 1. \end{aligned}$$

The metaplectic cover $\widetilde{\mathrm{Sp}_n(k_v)} \to \mathrm{Sp}_n(k_v)$ is given by Ranga Rao's 2-cocycle $c_v(g_1, g_2)$ (cf. Ranga Rao [57]). Thus an element of $\widetilde{\mathrm{Sp}_n(k_v)}$ is represented by $(g, \zeta) \in \mathrm{Sp}_n(k_v) \times \{\pm 1\}$ and the multiplication is given by $(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1 g_2, c_v(g_1, g_2)\zeta_1\zeta_2)$. There is unique splitting over the maximal compact subgroup $\mathrm{Sp}_n(\mathfrak{o}_v)$, which is denoted by $g \mapsto (g, s_v(g))$ for $g \in \mathrm{Sp}_n(\mathfrak{o}_v)$.

For each finite set S of places of k, put $\mathbb{A}_S = \left(\prod_{v\in S} k_v\right) \times \left(\prod_{v\notin S} \mathfrak{o}_v\right)$. Then we have

$$\mathrm{Sp}_n(\mathbb{A}_S) = \left(\prod_{v\in S} \mathrm{Sp}_n(k_v)\right) \times \left(\prod_{v\notin S} \mathrm{Sp}_n(\mathfrak{o}_v)\right).$$

The double covering $\widetilde{\mathrm{Sp}_n(\mathbb{A}_S)}$ of $\mathrm{Sp}_n(\mathbb{A}_S)$ is given by the 2-cocycle $\prod_{v\in S} c_v(g_{1,v}, g_{2,v})$. For $S_1 \subset S_2$, there is a natural inclusion map $\widetilde{\mathrm{Sp}_n(\mathbb{A}_{S_1})} \to \widetilde{\mathrm{Sp}_n(\mathbb{A}_{S_2})}$ such that the image of $((g_v)_v, \zeta) \in \widetilde{\mathrm{Sp}_n(\mathbb{A})}_{S_1}$ is equal to

$$((g_v)_v, \prod_{\substack{v\in S_2 \\ v\notin S_1}} s_v(g_v) \cdot \zeta).$$

The metaplectic covering $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ of the adele groups $\mathrm{Sp}_n(\mathbb{A})$ is given by the inductive limit, i.e.,

$$\widetilde{\mathrm{Sp}_n(\mathbb{A})} = \varinjlim_S \widetilde{\mathrm{Sp}_n(\mathbb{A}_S)}.$$

It is well-known that the covering $\widetilde{\mathrm{Sp}_n(\mathbb{A})} \to \mathrm{Sp}_n(\mathbb{A})$ splits over $\mathrm{Sp}_n(k)$ uniquely. We identify $\mathrm{Sp}_n(k)$ with the image of the splitting.

For square-integrable genuine automorphic forms h_1, h_2 on the metaplectic group $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$, we put

$$\langle h_1, h_2\rangle_{\mathrm{Sp}_n} = \int_{\mathrm{Sp}_n(k)\backslash\mathrm{Sp}_n(\mathbb{A})} h_1(g)\overline{h_2(g)}\, dg,$$

where dg is the Tamagawa measure on $\mathrm{Sp}_n(\mathbb{A})$.

For each place v of k, we denote the Weil constant associated to ψ_v by $\gamma_{\psi_v}(t)$ for $t \in k_v^\times$. We adopt the normalization such that

$$\begin{aligned}\int_{k_v} \psi_v(tx^2/2)\phi(x)\,dx =& \gamma_{\psi_v}(t)|t|_v^{-1/2}\int_{k_v} \psi_v(-x^2/(2t))\hat{\phi}(x)\,dx,\\ \hat{\phi}(x) =& \int_{k_v} \phi(y)\psi_v(xy)\,dy\end{aligned}$$

for any $\phi \in \mathcal{S}(k_v)$. Here, dx and dy are the self-dual Haar measures with respect to ψ_v. Note that $\gamma_{\psi_v}(t) = 1$ for $t \in \mathfrak{o}_v$, since ψ_v is of order 0.

Let $Q = {}^tQ$ be a non-degenerate symmetric matrix of size m. For simplicity, we assume $Q \in \mathrm{GL}_m(\mathbb{F}_q)$. In particular, the quadratic form associated to Q is locally unramified. For $t \in k_v$, the Weil constant $\gamma_{Q,v}(t)$ associated to Q is defined by $\gamma_{Q,v}(t) = \prod_{i=1}^m \gamma_{\psi_v}(q_i t)$, if Q is equivalent to $\mathrm{diag}(q_1, \dots, q_m)$. For an idele $t = (t_v)_v \in \mathbb{A}^\times$, we put $\gamma_Q(t) = \prod_v \gamma_{Q,v}(t_v)$.

Let S be a finite set of places k. The Schwartz space $\mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))$ is equal to the inductive limit of

$$\mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))_S = \left(\otimes_{v\in S}\mathcal{S}(\mathrm{M}_{m\,n}(k_v))\right) \otimes \left(\otimes_{v\notin S}\Phi_{0,v}\right),$$

where $\Phi_{0,v}$ is the characteristic function of $\mathrm{M}_{m\,n}(\mathfrak{o}_v)$. The Weil representation ω_ψ of $\widetilde{\mathrm{Sp}_n(\mathbb{A}_S)} \times \mathrm{O}_Q(\mathbb{A}_S)$ on $\mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))_S$ is given by

$$\begin{aligned}\omega_\psi\left(\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix}, \zeta\right)\Phi(x) =& \zeta^m \frac{1}{\gamma_Q(\det A)} |\det A|^{m/2}\Phi(xA),\\ \omega_\psi\left(\begin{pmatrix} \mathbf{1}_n & B \\ 0 & \mathbf{1}_n\end{pmatrix}, \zeta\right)\Phi(x) =& \zeta^m \Phi(x)\psi(\mathrm{tr}({}^txQxB)/2),\\ \omega_\psi\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0\end{pmatrix}, \zeta\right)\Phi(x) =& \zeta^m \int_{\mathrm{M}_{m\,n}(\mathbb{A})} \Phi(y)\overline{\psi(\mathrm{tr}({}^tyQx))}\,dy\\ \omega_\psi(h)\Phi(x) =& \Phi(h^{-1}x)\end{aligned}$$

for $A \in \mathrm{GL}_n(\mathbb{A}_S)$, $B = {}^tB \in \mathrm{M}_n(\mathbb{A}_S)$, $\zeta \in \{\pm 1\}$, $h \in \mathrm{O}_Q(\mathbb{A}_S)$, and $\Phi \in \mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))_S$. This action is compatible with the inductive limits, and so we obtain the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ on the Schwartz space $\mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))$.

For $\Phi \in \mathcal{S}(\mathrm{M}_{m\,n}(\mathbb{A}))$, the associated theta function is given by

$$\Theta(g,h;\Phi) = \sum_{x\in \mathrm{M}_{m\,n}(k)} \omega_\psi(g,h)\Phi(x)$$

for $(g,h) \in \widetilde{\mathrm{Sp}_n(\mathbb{A})} \times \mathrm{O}_Q(\mathbb{A})$. When Φ is the characteristic function of $\mathrm{M}_{mn}(\hat{\mathfrak{o}})$, we usually drop Φ from the notation.

When m is even, we regard ω_ψ as a representation of $\mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}_Q(\mathbb{A})$. In this case, the Weil representation ω_ψ can be extended to a representation of

$$G(\mathrm{Sp}_n \times \mathrm{O}_Q) = \{(g,h) \in \mathrm{GSp}_n \times \mathrm{GO}_Q \mid \nu(g) = \nu(h)\},$$

where ν is the multiplier character (cf. [25] §3). Then action is given by

$$\omega_\psi(g,h)\Phi(x) = |\nu(h)|^{-mn/4}\,\omega_\psi(d(\nu(g))^{-1}g)\Phi(h^{-1}x)$$

for $(g,h) \in G(\mathrm{Sp}_n \times \mathrm{O}_Q)(\mathbb{A})$ and $\Phi \in \mathcal{S}(\mathrm{M}_{mn}(\mathbb{A}))$, where

$$d(\nu) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \nu \cdot \mathbf{1}_n \end{pmatrix}, \qquad \nu \in \mathbb{A}^\times.$$

For $\Phi \in \mathcal{S}(\mathrm{M}_{mn}(\mathbb{A}))$, the associated theta function is given by

$$\Theta(g,h;\Phi) = \sum_{x \in \mathrm{M}_{mn}(k)} \omega_\psi(g,h)\Phi(x)$$

for $(g,h) \in G(\mathrm{Sp}_n \times \mathrm{O}_Q)(\mathbb{A})$.

The Siegel-Eisenstein series $E(s,g)$ on $\mathrm{Sp}_n(\mathbb{A})$ is defined by

$$E(s,g) = \sum_{\gamma \in P_n(k)\backslash \mathrm{Sp}_n(k)} \Psi(s,\gamma g),$$

where P_n is the Siegel parabolic subgroup of Sp_n and $\Psi(s,g)$ is the unique right $\mathcal{K}_{\mathrm{Sp}_n}$-invariant function such that

$$\Psi\left(s, \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}\right) = |\det A|^{s+((n+1)/2)}.$$

When χ is a locally unramified character of $\mathbb{A}^\times/k^\times$, we define $E(s,\chi,g)$ by

$$E(s,\chi,g) = \sum_{\gamma \in P_n(k)\backslash \mathrm{Sp}_n(k)} \Psi(s,\chi,\gamma g),$$

where $\Psi(s,\chi,g)$ is the unique right $\mathcal{K}_{\mathrm{Sp}_n}$-invariant function such that

$$\Psi\left(s,\chi, \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}\right) = \chi(\det A)|\det A|^{s+((n+1)/2)}.$$

Similarly, the metaplectic Siegel-Eisenstein series $\tilde{E}(s,g)$ on $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ is defined by

$$\tilde{E}(s,g) = \sum_{\gamma \in P_n(k)\backslash \mathrm{Sp}_n(k)} \tilde{\Psi}(s,\gamma g),$$

where $\tilde{\Psi}(s,g)$ is the unique right $\mathcal{K}_{\mathrm{Sp}_n}$-invariant function such that

$$\tilde{\Psi}\left(s,\left(\begin{pmatrix} A & B \\ 0 & {}^tA^{-1}\end{pmatrix},\zeta\right)\right)=\zeta\frac{1}{\gamma_\psi(\det A)}|\det A|^{s+((n+1)/2)}$$

for $\left(\begin{pmatrix} A & B \\ 0 & {}^tA^{-1}\end{pmatrix},\zeta\right)\in\widetilde{\mathrm{Sp}_n(\mathbb{A}_S)}$. Here, $\mathcal{K}_{\mathrm{Sp}_n}$ is identified with the image of the canonical splitting $\mathcal{K}_{\mathrm{Sp}_n}\to\widetilde{\mathrm{Sp}_n(\mathbb{A})}$.

13. Unramified tempered examples over a function field

The main results of this section are Proposition 13.7, Proposition 13.10, and Proposition 13.13. Let τ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that τ_v is unramified for any v. We denote the central character of τ by ω_τ. We assume that ω_τ is unitary. We shall say that an automorphic form $f\in\tau$ is a primitive form if the Whittaker function

$$W_f(g)=\int_{k\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}g\right)\overline{\psi(x)}dx$$

is identically 1 on $\mathcal{K}_{\mathrm{GL}_2}$. Note that the primitive form f is uniquely determined by τ. In particular, $\bar{f}$ is the primitive form of the contragredient $\tilde{\tau}$.

Proposition 13.1. *Let $f\in\tau$ be a primitive form of a unitary irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$. Then we have*

$$\langle f,f\rangle_{\mathrm{PGL}_2}=2\zeta(2)^{-1}L(1,\tau,\mathrm{Ad}).$$

Proof. By Rankin-Selberg method, we have

$$L(s,\tau\times\tilde{\tau})=C_{\mathrm{PGL}_2}^{-1}\zeta(2s)\int_{\mathrm{PGL}_2(k)\backslash\mathrm{PGL}_2(\mathbb{A})}E(s-(1/2),g)|f(g)|^2\,dg.$$

Note that

$$\begin{aligned}C_{\mathrm{PGL}_2}=&\zeta(2)^{-1}\\ L(s,\tau\times\tilde{\tau})=&\zeta(s)L(s,\tau,\mathrm{Ad})\\ \mathrm{Res}_{s=1/2}E(s,g)=&2^{-1}\zeta(2)^{-1}\mathrm{Res}_{s=1}\zeta(s).\end{aligned}$$

Hence the proposition. □

Let τ be a unitary irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. The restriction of the primitive form $f\in\tau$ to $\mathrm{SL}_2(\mathbb{A})$ is denoted by f^0. The irreducible automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ generated by f^0 is denoted by τ^0. Then τ_v^0 is the unique irreducible constituent of the restriction of τ to $\mathrm{SL}_2(k_v)$ which has an $\mathrm{SL}_2(\mathfrak{o}_v)$-fixed vector.

We put

$$\mathfrak{X}_\tau = \{\omega \in \mathrm{Hom}(\mathbb{A}^\times/k^\times, \mathbb{C}^\times) \mid \tau \otimes \omega \simeq \tau\}.$$

Note that each element of $\mathfrak{X}_\tau$ is unramified. By the result of Labesse-Langlands [46], the group $\mathfrak{X}_\tau$ is an elementary 2-abelian group with order $|\mathfrak{X}_\tau|$ at most 4.

Proposition 13.2. *Let $f \in \tau$ be a primitive form of a unitary irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$. Then we have*

$$\langle f^0, f^0 \rangle_{\mathrm{SL}_2} = |\mathfrak{X}_\tau| \cdot \zeta(2)^{-1} L(1, \tau, \mathrm{Ad}).$$

Proof. Let $\{\omega_1, \ldots, \omega_r\}$ be the set of characters of $\mathbb{A}^\times/k^\times\mathbb{A}^{\times 2}\hat{\mathfrak{o}}^\times$. Put $(f \otimes \omega_i)(g) = \omega_i(\det g) f(g)$ for $1 \le i \le r$. Then $f \otimes \omega_i$ is the primitive form associated to $\tau \otimes \omega_i$. In particular, $f \otimes \omega_i = f \otimes \omega_j$ if and only if $\omega_i\omega_j \in \mathfrak{X}_\tau$. Put

$$\phi(g) = \sum_{i=1}^{r} (f \otimes \omega_i)(g).$$

Then the support of ϕ is contained in $\{g \in \mathrm{GL}_2(\mathbb{A}) \mid \det g \in k^\times\mathbb{A}^{\times 2}\hat{\mathfrak{o}}^\times\}$. It follows that

$$\phi(\gamma z g \kappa) = [\mathbb{A}^\times : k^\times\mathbb{A}^{\times 2}\hat{\mathfrak{o}}^\times]\omega_\tau(z) f^0(g)$$

for $\gamma \in \mathrm{GL}_2(k)$, $z \in \mathbb{A}^\times$, $g \in \mathrm{SL}_2(\mathbb{A})$, and $\kappa \in \mathcal{K}_{\mathrm{GL}_2}$. Therefore we have

$$\langle \phi, \phi \rangle_{\mathrm{PGL}_2} = 2[\mathbb{A}^\times : k^\times\mathbb{A}^{\times 2}\hat{\mathfrak{o}}^\times]\, \langle f^0, f^0 \rangle_{\mathrm{SL}_2}.$$

On the other hand, we have

$$\langle \phi, \phi \rangle_{\mathrm{PGL}_2} = |\mathfrak{X}_\tau| \cdot [\mathbb{A}^\times : k^\times\mathbb{A}^{\times 2}\hat{\mathfrak{o}}^\times]\, \langle f, f \rangle_{\mathrm{PGL}_2}.$$

Hence the proposition. ☐

The proof above is essentially due to Hiraga and Saito [28] §4. We thank Prof. Hiraga for explaining their result to us.

Now we assume $\omega_\tau = 1$. Then we have $\tilde{\tau} \simeq \bar{\tau} \simeq \tau$ and $\bar{f} = f$, where f is the primitive form of τ. Recall that the group PGL_2 is isomorphic to $\mathrm{SO}(2,1)$, where $\mathrm{SO}(2,1)$ is the special orthogonal group associated with the quadratic form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can extend f to a $\mathcal{K}_{\mathrm{O}(2,1)}$-invariant automorphic form on $\mathrm{O}(2,1)(\mathbb{A})$. Then we have

$$\langle f, f \rangle_{\mathrm{O}(2,1)} = \zeta(2)^{-1} L(1, \tau, \mathrm{Ad}).$$

Let ω_ψ be the Weil representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})} \times \mathrm{O}(2,1)(\mathbb{A})$, acting on the Schwartz-Bruhat space $\mathcal{S}(\mathbb{A}^3)$. The theta function $\Theta(g_1, g_2)$ is defined by

$$\Theta(g_1, g_2) = \sum_{x \in k^3} \omega_\psi(g_1, g_2)\Phi_0(x),$$

where $g_1 \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, $g_2 \in \mathrm{O}(2,1)(\mathbb{A})$, and Φ_0 is the characteristic function of $\hat{\mathfrak{o}}^3$. The theta lift $\theta(f)$ of f to $\widetilde{\mathrm{SL}_2}$ is defined by

$$\theta(f)(g_1) = \int_{\mathrm{O}(2,1)(k)\backslash \mathrm{O}(2,1)(\mathbb{A})} \Theta(g_1, g_2) f(g_2) dg_2.$$

Then $\theta(f)$ is an automorphic form on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. The automorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

induces an automorphism of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$, which we denote by $g_1 \mapsto g_1^\iota$ as well. Note that

$$\theta(f)(g_1^\iota) = \overline{\theta(f)(g_1)}$$

Consider

$$\theta(\overline{\theta(f)})(g_2) = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \Theta(g_1, g_2)\overline{\theta(f)(g_1)} dg_1.$$

Then $\theta(\overline{\theta(f)})$ is an automorphic form on $\mathrm{O}(2,1)(\mathbb{A})$.

Proposition 13.3. *Let f be a primitive form of an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. Then we have*

$$\theta(\overline{\theta(f)}) = 2^{-1}\zeta(2)^{-2} L(1/2, \tau) \cdot f.$$

The Petersson norm of $\theta(f)$ is given by

$$\langle \theta(f), \theta(f) \rangle_{\mathrm{SL}_2} = 2^{-1}\zeta(2)^{-3} L(1/2, \tau) L(1, \tau, \mathrm{Ad}).$$

Proof. Note that $\theta(\overline{\theta(f)})$ is a $\mathcal{K}_{\mathrm{O}(2,1)}$-fixed vector of $\tilde{\tau} \simeq \tau$. It follows that $\theta(\overline{\theta(f)}) = cf$ for some $c \in \mathbb{C}$. Consider the seesaw

$$\begin{array}{ccccccc} \theta(f) & : & \widetilde{\mathrm{SL}_2} & & \mathrm{O}(2,1) & : & cf \\ & & | & \times & | & & \\ \overline{\theta(f)} & : & \widetilde{\mathrm{SL}_2} & & \mathrm{O}(2,1) & : & f \end{array}$$

Then by the seesaw dual identity, we have

$$\langle \theta(f), \theta(f) \rangle_{\mathrm{SL}_2} = c \langle f, f \rangle_{\mathrm{O}(2,1)}.$$

In particular, $c > 0$. Now we consider the seesaw

$$\begin{array}{ccccccc}
\frac{1}{2}\tilde{E}(0) & : & \widetilde{\mathrm{Sp}_2} & & \mathrm{O}(2,1)\times\mathrm{O}(2,1) & : & cf \times cf \\
& & | & \times & | & & \\
\theta(f)\times\overline{\theta(f)} & : & \widetilde{\mathrm{SL}_2}\times\widetilde{\mathrm{SL}_2} & & \mathrm{O}(2,1) & : & 1
\end{array}$$

Here, $\mathrm{SL}_2 \times \mathrm{SL}_2$ is embedded into Sp_2 by

$$\iota : \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & -b_2 \\ \hline c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{array}\right).$$

Then we have a seesaw dual identity

$$\begin{aligned} & c^2\langle f, f\rangle_{\mathrm{O}(2,1)} \\ & \quad = \frac{1}{2}\int_{(\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A}))^2} \tilde{E}(0, \iota(g,g'))\cdot\theta(f)(g)\cdot\overline{\theta(f)(g')}dg\,dg'. \end{aligned}$$

Here, we have used the regularized Siegel-Weil formula (See [45], [43], §3)

$$\oint_{\mathrm{O}(2,1)(k)\backslash\mathrm{O}(2,1)(\mathbb{A})} \Theta(g,h)\,dh = \frac{1}{2}\tilde{E}(0,g),$$

where the left hand side is a regularized theta integral ([45], [29]). The Rankin-Selberg identity says

$$\begin{aligned} & \int_{(\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A}))^2} \tilde{E}(s, \iota(g,g'))\cdot\theta(f)(g)\cdot\overline{\theta(f)(g')}dg\,dg' \\ & \qquad = C_{\mathrm{SL}_2} d(s+(1/2))^{-1}L(s+(1/2),\tau)\langle\theta(f),\theta(f)\rangle_{\mathrm{SL}_2}. \end{aligned}$$

Here, the normalization factor $d(s)$ for the Eisenstein series is equal to $\zeta(2s+1)$. For the calculation of Rankin-Selberg integral for metaplectic groups, see e.g., [48], §4. From these identities, we have

$$c^2\langle f,f\rangle_{\mathrm{O}(2,1)} = 2^{-1}\zeta(2)^{-2}L(1/2,\tau)\langle\theta(f),\theta(f)\rangle_{\mathrm{SL}_2}.$$

Hence the proposition. □

Next, we consider the groups $\mathrm{O}(2,2)$ and $\mathrm{GO}(2,2)$ associated to the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We define two homomorphisms $j_1, j_2 : \mathrm{GL}_2 \to \mathrm{GO}(2,2)$ by

$$j_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{0} \\ c & d & & \\ \hline \multicolumn{2}{c|}{0} & a & -b \\ & & -c & d \end{array}\right),$$

$$j_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{-1} \left(\begin{array}{cc|cc} a & 0 & -b & 0 \\ 0 & a & 0 & b \\ \hline -c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right).$$

Let $\langle \mathbf{t} \rangle$ be a group of order 2 which acts on $\mathrm{GL}_2 \times \mathrm{GL}_2$ by $\mathbf{t}(g_1, g_2) = ((\det g_2^{-1})g_2, (\det g_1^{-1})g_1)$. Then there is an exact sequence

$$1 \to \mathbb{G}_m \to (\mathrm{GL}_2 \times \mathrm{GL}_2) \rtimes \langle \mathbf{t} \rangle \to \mathrm{GO}(2,2) \to 1,$$

where $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GO}(2,2)$ is given by $j_1 \times j_2$ and the embedding $\mathbb{G}_m \to \mathrm{GL}_2 \times \mathrm{GL}_2$ is given by $z \mapsto (z \cdot \mathbf{1}_2, z \cdot \mathbf{1}_2)$. The image of $\mathrm{GL}_2 \times \mathrm{GL}_2$ in $\mathrm{GO}(2,2)$ is denoted by $\mathrm{GSO}(2,2)$. The center $\mathcal{Z}$ of $\mathrm{GSO}(2,2)$ is equal to the image of $\{(z \cdot \mathbf{1}_2, \mathbf{1}_2) \mid z \in \mathbb{G}_m\}$. Put

$$\mathrm{GL}_2^{(2)} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det g_1 = \det g_2\}.$$

Then $\mathrm{SO}(2,2) \simeq \mathrm{GL}_2^{(2)}/\mathbb{G}_m$ and $\mathrm{O}(2,2) \simeq (\mathrm{GL}_2^{(2)} \rtimes \langle \mathbf{t} \rangle)/\mathbb{G}_m$. Note that $j_1 \times j_2 : \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}(2,2)$ is an isogeny.

Let $f_1 \in \tau_1$ and $f_2 \in \tau_2$ be primitive forms on $\mathrm{GL}_2(\mathbb{A})$. We assume $\omega_{\tau_1}\omega_{\tau_2} = 1$. The function $f_1(g_1)f_2(g_2)$ for $(g_1, g_2) \in \mathrm{GL}_2^{(2)}(\mathbb{A})$ can be considered as an automorphic form on $\mathrm{SO}(2,2)(\mathbb{A})$, which we denote by $\varphi^{\mathrm{SO}}_{f_1,f_2}$. Note that $\varphi^{\mathrm{SO}}_{f_1,f_2}$ is an $\mathbb{R}$-valued function. When $\tau_1 \simeq \tilde{\tau}_2$, we write $\varphi^{\mathrm{SO}}_{f_1} = \varphi^{\mathrm{SO}}_{f_1,f_2}$, for simplicity.

Similarly, the function $f_1(g_1)f_2(g_2)$ for $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ can be considered as an automorphic form on $\mathrm{GSO}(2,2)(\mathbb{A})$, which we denote by $\varphi^{\mathrm{GSO}}_{f_1,f_2}$. We denote the automorphic representation of $\mathrm{GSO}(2,2)(\mathbb{A})$ generated by $\varphi^{\mathrm{GSO}}_{f_1,f_2}$ by $\tau_1 \boxtimes \tau_2$. The function $(\varphi^{\mathrm{GSO}}_{f_1,f_2} + \varphi^{\mathrm{GSO}}_{f_1,f_2} \circ \mathbf{t})/2$ can be extended to a right $\mathcal{K}_{\mathrm{GO}(2,2)}$-invariant automorphic form on $\mathrm{GO}(2,2)(\mathbb{A})$, which we denote by $\varphi^{\mathrm{GO}}_{f_1,f_2}$. Note that $\varphi^{\mathrm{GSO}}_{f_1,f_2} \circ \mathbf{t} = \varphi^{\mathrm{GSO}}_{f_1,f_2}$ if and only if $\tau_1 \simeq \tilde{\tau}_2$.

Let $\mathfrak{X}_{\tau_1,\tau_2}$ be the group of characters $\omega \in \mathrm{Hom}(\mathbb{A}^\times/k^\times, \mathbb{C}^\times)$ such that

$$\omega(\det g_1 g_2^{-1}) f_1(g_1) f_2(g_2) = f_1(g_1) f_2(g_2)$$

for any $g_1, g_2 \in \mathrm{GL}_2(\mathbb{A})$. Then it is easy to see $\mathfrak{X}_{\tau_1,\tau_2} = \mathfrak{X}_{\tau_1} \cap \mathfrak{X}_{\tau_2}$. In particular, we have $\mathfrak{X}_{\tau,\tilde{\tau}} = \mathfrak{X}_\tau$. As in Proposition 13.2, one can prove the following proposition.

Proposition 13.4. *Let $f_1 \in \tau_1$ and $f_2 \in \tau_2$ be primitive forms on* $\mathrm{GL}_2(\mathbb{A})$ *such that $\omega_{\tau_1}\omega_{\tau_2} = 1$. Then we have*

$$\langle \varphi^{\mathrm{SO}}_{f_1,f_2}, \varphi^{\mathrm{SO}}_{f_1,f_2} \rangle_{\mathrm{SO}(2,2)} = 2|\mathfrak{X}_{\tau_1,\tau_2}|\zeta(2)^{-2} L(1,\tau_1,\mathrm{Ad}) L(1,\tau_2,\mathrm{Ad}),$$
$$\langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{f_1,f_2} \rangle_{\mathrm{GO}(2,2)} = \begin{cases} \zeta(2)^{-2} L(1,\tau_1,\mathrm{Ad}) L(1,\tau_2,\mathrm{Ad}) & \textit{if } \tau_1 \not\simeq \tilde{\tau}_2, \\ 2\zeta(2)^{-2} L(1,\tau_1,\mathrm{Ad})^2 & \textit{if } \tau_1 \simeq \tilde{\tau}_2. \end{cases}$$

Let ω_ψ be the Weil representation of $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}(2,2)(\mathbb{A})$, acting on the Schwartz space $\mathcal{S}(\mathbb{A}^4)$. The theta function $\Theta(g,h)$ is defined by

$$\Theta(g,h) = \sum_{x \in k^4} \omega_\psi(g,h)\Phi_0(x),$$

where Φ_0 is the characteristic function of $\hat{\mathfrak{o}}^4$. In fact, we use a different model for the Weil representation. The Weil representation ω_ψ can be also realized on the Schwartz space $\mathcal{S}(\mathrm{M}_2(\mathbb{A}))$. Consider the partial Fourier transform $\mathcal{S}(\mathbb{A}^4) \to \mathcal{S}(\mathrm{M}_2(\mathbb{A}))$ given by

$$\hat{\Phi}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \int_{\mathbb{A}^2} \Phi\begin{pmatrix} y_1 \\ y_2 \\ x_4 \\ x_3 \end{pmatrix} \overline{\psi(x_1y_1 + x_2y_2)}\, dy_3\, dy_4.$$

Then the new action $\hat{\omega}_\psi$ on $\mathcal{S}(\mathrm{M}_2(\mathbb{A}))$ is given by $\hat{\omega}_\psi(g,h)\hat{\Phi} = (\omega_\psi(g,h)\Phi)\hat{}$. It is easily seen that

$$\begin{aligned} \hat{\omega}_\psi(\mathbf{1}_2, j_1(g))\Phi(x) &= \Phi(xg), \\ \hat{\omega}_\psi\left(\mathbf{1}_2, j_2\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right)\Phi(x) &= \Phi(x)\psi(b \cdot \det x), \\ \hat{\omega}_\psi(g, \mathbf{1}_4)\Phi(x) &= \Phi(g^{-1}x), \\ \hat{\omega}_\psi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a\cdot\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{smallmatrix}\right)\right)\Phi(x) &= |a|\Phi\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)x\right) \end{aligned}$$

for $g \in \mathrm{SL}_2(\mathbb{A})$ and $\Phi \in \mathcal{S}(\mathrm{M}_2(\mathbb{A}))$. Let $\Phi_0 \in \mathcal{S}(\mathrm{M}_2(\mathbb{A}))$ be the characteristic function of $\mathrm{M}_2(\hat{\mathfrak{o}})$. Then the theta function $\Theta(g,h)$ is equal to

$$\sum_{x \in \mathrm{M}_2(k)} \hat{\omega}_\psi(g,h)\Phi_0(x).$$

Let $\theta(f)$ be the automorphic form on $\mathrm{GO}(2,2)(\mathbb{A})$ obtained by theta correspondence

$$\theta(f)(h) = \int_{\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})} \Theta(g \cdot d(\nu(h)), h)\, f(g \cdot d(\nu(h)))\, dg,$$

where $d(\nu) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right)$.

Proposition 13.5. *Let $f \in \tau$ be a primitive form on* $\mathrm{GL}_2(\mathbb{A})$*. Then we have*

$$\theta(f) = \zeta(2)^{-1}\varphi_f^{\mathrm{GO}}.$$

Proof. Since the right-hand side is $\mathcal{K}_{\mathrm{GO}(2,2)}$-invariant, it is enough to prove the equation as a function on $\mathrm{GSO}(2,2)(\mathbb{A})$. It is well-known that the automorphic representation generated by $\theta(f)$ is isomorphic to $\tau \boxtimes \tilde{\tau}$. Therefore there is a constant $c \in \mathbb{C}$ such that $\theta(f) = c\varphi_f^{\mathrm{GO}}$.

By easy calculation (cf [32], Lemma 5.1, [65], II.2), we have

$$\begin{aligned}
&\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}} \theta(f)\left(j_1\left(\begin{smallmatrix}1 & b_1\\ 0 & 1\end{smallmatrix}\right)\times j_2\left(\begin{smallmatrix}1 & b_2\\ 0 & 1\end{smallmatrix}\right)\right)\overline{\psi(b_1+b_2)}\,db_1\,db_2\\
=&\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\int_{\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})} \Theta\left(g, j_1\left(\begin{smallmatrix}1 & b_1\\ 0 & 1\end{smallmatrix}\right)\times j_2\left(\begin{smallmatrix}1 & b_2\\ 0 & 1\end{smallmatrix}\right)\right)\\
&\quad\times f(g)\overline{\psi(b_1+b_2)}\,dg\,db_1\,db_2\\
=&\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\int_{\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})}\sum_{x\in\mathrm{M}_2(k)} \hat{\omega}_\psi\left(\mathbf{1}_2, j_1\left(\begin{smallmatrix}1 & b_1\\ 0 & 1\end{smallmatrix}\right)\times j_2\left(\begin{smallmatrix}1 & b_2\\ 0 & 1\end{smallmatrix}\right)\right)\Phi_0(g^{-1}x)\\
&\quad\times f(g)\overline{\psi(b_1+b_2)}\,dg\,db_1\,db_2\\
=&\int_{k\backslash\mathbb{A}}\int_{\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})}\sum_{x\in\mathrm{SL}_2(k)} \hat{\omega}_\psi\left(\mathbf{1}_2, j_1\left(\begin{smallmatrix}1 & b_1\\ 0 & 1\end{smallmatrix}\right)\right)\Phi_0(g^{-1}x)\\
&\quad\times f(g)\overline{\psi(b_1)}\,dg\,db_1\\
=&\int_{k\backslash\mathbb{A}}\int_{\mathrm{SL}_2(\mathbb{A})} \hat{\omega}_\psi\left(\mathbf{1}_2, j_1\left(\begin{smallmatrix}1 & b_1\\ 0 & 1\end{smallmatrix}\right)\right)\Phi_0(g^{-1})f(g)\overline{\psi(b_1)}\,dg\,db_1\\
=&\int_{\mathrm{SL}_2(\mathbb{A})} W_f(g)\Phi_0(g^{-1})\,dg\\
=&\zeta(2)^{-1}.
\end{aligned}$$

Hence the proposition. ∎

Let $\mathrm{GL}_2^{(3)}$ and $\mathrm{GO}(2,2)^{(3)}$ be the groups defined by

$$\begin{aligned}
\mathrm{GL}_2^{(3)} &= \{(g_1,g_2,g_3)\in(\mathrm{GL}_2)^3 \mid \det g_1 = \det g_2 = \det g_3\},\\
\mathrm{GO}(2,2)^{(3)} &= \{(g_1,g_2,g_3)\in\mathrm{GO}(2,2)^3 \mid \nu(g_1)=\nu(g_2)=\nu(g_3)\},
\end{aligned}$$

where $\nu(g)$ is a multiplier of $g \in \mathrm{GO}(2,2)$. We define the embedding $\iota^{(3)} : \mathrm{GL}_2^{(3)} \to \mathrm{GSp}_3$ by

$$\iota^{(3)}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) = \left(\begin{array}{ccc|ccc} a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ \hline c_1 & 0 & 0 & d_1 & 0 & 0 \\ 0 & c_2 & 0 & 0 & d_2 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \end{array}\right).$$

Proposition 13.6. *Let $f_1 \in \tau_1$, $f_2 \in \tau_2$, and $f_3 \in \tau_3$ be primitive forms on $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3} = 1$. Then we have*

$$\left|\int_{\mathrm{PGL}_2(k)\backslash \mathrm{PGL}_2(\mathbb{A})} f_1(g) f_2(g) f_3(g)\, dg\right|^2 = \zeta(2)^{-2} L(1/2, \tau_1 \times \tau_2 \times \tau_3).$$

Proof. Since $\mathrm{GL}_2^{(3)}/\mathbb{G}_m$ is isogenous to $(\mathrm{SL}_2)^3$, we have $C_{\mathrm{GL}_2^{(3)}/\mathbb{G}_m} = \zeta(2)^{-3}$. By the integral representation of the triple L-function ([11], [53]), we have

$$\begin{aligned} d(s) \int_{\mathrm{GL}_2^{(3)}(k)\mathbb{A}^\times \backslash \mathrm{GL}_2^{(3)}(\mathbb{A})} & E(s, \iota^{(3)}(g_1, g_2, g_3)) \\ & \times f_1(g_1) f_2(g_2) f_3(g_3)\, dg_1\, dg_2\, dg_3 \\ & \quad = \zeta(2)^{-3} L(s + (1/2), \tau_1 \times \tau_2 \times \tau_3), \end{aligned}$$

where $d(s) = \zeta(2s+2)\zeta(4s+2)$.

Consider the seesaw

$$\begin{array}{ccccccc} \frac{1}{2}E(0) & : & \mathrm{GSp}_3 & & \mathrm{GO}(2,2)^{(3)} & : & \zeta(2)^{-3}\varphi_{f_1}^{\mathrm{GO}} \times \varphi_{f_2}^{\mathrm{GO}} \times \varphi_{f_3}^{\mathrm{GO}} \\ & & | & \times & | & & \\ f_1 \times f_2 \times f_3 & : & \mathrm{GL}_2^{(3)} & & \mathrm{GO}(2,2) & : & 1. \end{array}$$

Here, we have used the regularized Siegel-Weil formula ([45], [25], §4). The left hand side of the seesaw identity is equal to

$$\begin{aligned} & \frac{1}{2}\int_{\mathrm{GL}_2^{(3)}(k)\mathbb{A}^\times \backslash \mathrm{GL}_2^{(3)}(\mathbb{A})} E(0, \iota^{(3)}(g_1, g_2, g_3)) f_1(g_1) f_2(g_2) f_3(g_3)\, dg_1\, dg_2\, dg_3 \\ = \; & \frac{1}{2}\zeta(2)^{-5} L(1/2, \tau_1 \times \tau_2 \times \tau_3). \end{aligned}$$

The right hand side of the seesaw identity is equal to

$$\begin{aligned}
&\zeta(2)^{-3}\int_{\mathrm{GO}(2,2)(k)\mathbb{A}^\times\backslash\mathrm{GO}(2,2)(\mathbb{A})}\varphi_{f_1}^{\mathrm{GO}}(g)\varphi_{f_2}^{\mathrm{GO}}(g)\varphi_{f_3}^{\mathrm{GO}}(g)\,dg\\
=&\frac{1}{2}\zeta(2)^{-3}\int_{\mathrm{GSO}(2,2)(k)\mathbb{A}^\times\backslash\mathrm{GSO}(2,2)(\mathbb{A})}\varphi_{f_1}^{\mathrm{GSO}}(g)\varphi_{f_2}^{\mathrm{GSO}}(g)\varphi_{f_3}^{\mathrm{GSO}}(g)\,dg\\
=&\frac{1}{2}\zeta(2)^{-3}\left[\int_{\mathrm{PGL}_2(k)\backslash\mathrm{PGL}_2(\mathbb{A})}f_1(g)f_2(g)f_3(g)\,dg\right]\\
&\qquad\times\left[\int_{\mathrm{PGL}_2(k)\backslash\mathrm{PGL}_2(\mathbb{A})}\overline{f_1(g)f_2(g)f_3(g)}\,dg\right].
\end{aligned}$$

Hence the proposition. □

Now put $G_1 = \mathrm{SO}(2,2)$ and $G_0 = \mathrm{SO}(2,1)$. The following proposition is an analogue of the results of §7. The following proposition follows from Proposition 13.1, Proposition 13.4 and Proposition 13.6 easily.

Proposition 13.7. *Let $f_1 \in \tau_1$, $f_2 \in \tau_2$, and $f_3 \in \tau_3$ be primitive forms such that $\omega_{\tau_1}\omega_{\tau_2} = \omega_{\tau_3} = 1$. Put $G_1 = \mathrm{SO}(2,2)$ and $G_0 = \mathrm{SO}(2,1) \simeq \mathrm{PGL}_2$. Then we have*

$$\frac{|\langle \varphi_{f_1,f_2}^{\mathrm{SO}}|_{G_0}, f_3\rangle|^2}{\langle \varphi_{f_1,f_2}^{\mathrm{SO}}, \varphi_{f_1,f_2}^{\mathrm{SO}}\rangle\langle f_3, f_3\rangle} = \frac{\zeta(2)}{4|\mathfrak{X}_{\tau_1,\tau_2}|}\frac{L(1/2,\tau_1\times\tau_2\times\tau_3)}{\prod_{j=1}^3 L(1,\tau_j,\mathrm{Ad})}.$$

Note that $|\mathcal{S}_{\psi_1}|$ and $|\mathcal{S}_{\psi_0}|$ should be $2|\mathfrak{X}_{\tau_1,\tau_2}|$ and 2, respectively. Therefore, it seems Conjecture 2.1 holds in this case.

We next consider the orthogonal group $\mathrm{O}(3,1)$. Let $K = \mathbb{F}_{q^2}\cdot k$ be the quadratic constant field extension of k. Choose a non-square element $\delta \in \mathbb{F}_q^\times$. The group $\mathrm{O}(3,1)$ is the orthogonal group for the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 2 & 0 & 0\\ 0 & 0 & -2\delta & 0\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

Recall that there exists an exact sequence

$$1 \to \mathbb{G}_m \to H \to \mathrm{SO}(3,1) \to 1,$$

where

$$H = \{g \in \mathrm{GL}_2(K) \mid \det g \in k^\times\}.$$

Let h be a primitive form of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_K)$ such that $\omega|_{\mathbb{A}^\times} = 1$. The restriction of h to $H(\mathbb{A})$ can be considered an automorphic form on $\mathrm{SO}(3,1)(\mathbb{A})$, which we denote by h^H. The proof of the following propositions are the same as Proposition 13.4 and Proposition 13.6. We omit the detail.

Proposition 13.8. *Let h be a primitive form of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_K)$ such that $\omega|_{\mathbb{A}^\times} = 1$. Then we have*

$$\langle h^H, h^H \rangle_{\mathrm{SO}(3,1)} = 2|\mathfrak{X}_{K/k,\sigma}| \frac{L(1, \sigma, \mathrm{Ad})}{\zeta(2) L(2, \chi_{K/k})}.$$

Here,

$$\mathfrak{X}_{K/k,\sigma} = \{\omega \in \mathrm{Hom}(\mathbb{A}_K^\times / K^\times, \mathbb{C}^\times) \,|\, \omega|_{\mathbb{A}^\times} = 1, \sigma \otimes \omega \simeq \sigma\}.$$

Proposition 13.9. *Let f be a primitive form of an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$. We assume $\omega_\tau \cdot \omega_\sigma|_{\mathbb{A}^\times} = 1$. Then we have*

$$\left| \int_{\mathrm{PGL}_2(k) \backslash \mathrm{PGL}_2(\mathbb{A})} h(g) f(g) \, dg \right|^2 = \zeta(2)^{-2} L(1/2, \mathrm{Asai}(\sigma) \times \tau).$$

Here, $L(s, \mathrm{Asai}(\sigma))$ is the Asai L-function of σ.

Put $G_1 = \mathrm{SO}(3,1)$ and $G_0 = \mathrm{SO}(2,1)$. The following proposition follows from Proposition 13.1, Proposition 13.8, and Proposition 13.9.

Proposition 13.10. *Let h be a primitive form of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_K)$ such that $\omega_\sigma|_{\mathbb{A}^\times} = 1$. Let f be a primitive form of an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. Then we have*

$$\frac{|\langle h^H|_{G_0}, f \rangle|^2}{\langle h^H, h^H \rangle \langle f, f \rangle} = \frac{L(2, \chi_{K/k})}{4|\mathfrak{X}_{K/k,\sigma}|} \frac{L(1/2, \mathrm{Asai}(\sigma) \times \tau)}{L(1, \sigma, \mathrm{Ad}) L(1, \tau, \mathrm{Ad})}.$$

Note that $|\mathcal{S}_{\psi_1}|$ and $|\mathcal{S}_{\psi_0}|$ should be $2|\mathfrak{X}_{K/k,\sigma}|$ and 2, respectively. Therefore, it seems Conjecture 2.1 holds in this case.

Next, we consider an analogue of the Yoshida lifting [67].

Let τ_1 and τ_2 be irreducible unramified cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_{\tau_1} = \omega_{\tau_2} = 1$. We assume that $\tau_1 \not\simeq \tilde{\tau}_2$. Let $f_1 \in \tau_1$ and $f_2 \in \tau_2$ be primitive forms. We think of $\varphi^{\mathrm{GO}}_{f_1,f_2}$ as a function on $\mathrm{GO}(2,2)(k) \backslash \mathrm{GO}(2,2)(\mathbb{A})$ or a function on $\mathrm{GO}(2,2)(k)\mathbb{A}^\times \backslash \mathrm{GO}(2,2)(\mathbb{A})$. Note that $\varphi^{\mathrm{GO}}_{f_1,f_2}$ is $\mathbb{R}$-valued. Let $Y_{f_1,f_2} = \theta(\varphi^{\mathrm{GO}}_{f_1,f_2})$ be the theta lift of $\varphi^{\mathrm{GO}}_{f_1,f_2}$ to the similitude group GSp_2. Since Y_{f_1,f_2} is invariant by the center of $\mathrm{GSp}_2(\mathbb{A})$, Y_{f_1,f_2} can be considered as an automorphic form on $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$. Note that Y_{f_1,f_2} is a cusp form, since $\varphi^{\mathrm{GO}}_{f_1,f_2}$ does not come from the theta correspondence for $\mathrm{GL}_2 \times \mathrm{GO}(2,2)$. One can think of Y_{f_1,f_2} as an analogue of the Yoshida lifting [67] for function field.

We consider the theta lift $\theta(\bar{Y}_{f_1,f_2})$ of $\bar{Y}_{f_1,f_2}$ to $\mathrm{GO}(2,2)$. There exists a constant c such that $\theta(\bar{Y}_{f_1,f_2}) = c\varphi^{\mathrm{GO}}_{f_1,f_2}$. The seesaw identity for

$$\begin{array}{ccccccc} Y_{f_1,f_2} & : & \mathrm{GSp}_2 & & \mathrm{GO}(2,2) & : & c\varphi^{\mathrm{GO}}_{f_1,f_2} \\ & & | & \times & | & & \\ \bar{Y}_{f_1,f_2} & : & \mathrm{GSp}_2 & & \mathrm{GO}(2,2) & : & \varphi^{\mathrm{GO}}_{f_1,f_2} \end{array}$$

implies that

$$\langle Y_{f_1,f_2}, Y_{f_1,f_2}\rangle_{\mathrm{PGSp}_2} = c\langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{f_1,f_2}\rangle_{\mathrm{GO}(2,2)}.$$

Hence we have

$$\theta(Y_{f_1,f_2}) = \frac{\langle Y_{f_1,f_2}, Y_{f_1,f_2}\rangle_{\mathrm{PGSp}_2}}{\langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{f_1,f_2}\rangle_{\mathrm{GO}(2,2)}} \cdot \varphi^{\mathrm{GO}}_{f_1,f_2}.$$

Now we compute $\langle Y_{f_1,f_2}, Y_{f_1,f_2}\rangle_{\mathrm{PGSp}_2}$. Put

$$\begin{aligned} \mathrm{GO}(2,2)^{(2)} =& \{(g_1,g_2) \in \mathrm{GO}(2,2) \times \mathrm{GO}(2,2)\,|\,\nu(g_1) = \nu(g_2)\}, \\ \mathrm{GSp}_2^{(2)} =& \{(g_1,g_2) \in \mathrm{GSp}_2 \times \mathrm{GSp}_2\,|\,\nu(g_1) = \nu(g_2)\}. \end{aligned}$$

The regularized Siegel-Weil formula [45], [29] and [33], Theorem 9.7 says that

$$\mathrm{Res}_{s=1/2} E(s) = \frac{\rho_k \zeta(2)}{\zeta(3)\zeta(4)} \theta(1).$$

Here $E(s)$ is the Siegel-Eisenstein series on GSp_4 and $\theta(1)$ is the regularized theta integral associated to the dual pair $\mathrm{GO}(2,2) \times \mathrm{GSp}_4$. We use the seesaw

$$\begin{array}{ccccccc} \theta(1) & : & \mathrm{GSp}_4 & & \mathrm{GO}(2,2)^{(2)} & : & \theta(Y_{f_1,f_2}) \times \theta(Y_{f_1,f_2}) \\ & & | & \times & | & & \\ Y_{f_1,f_2} \times \bar{Y}_{f_1,f_2} & : & \mathrm{GSp}_2^{(2)} & & \mathrm{GO}(2,2) & : & 1 \end{array}$$

Here, $\mathrm{GSp}_2^{(2)}$ is embedded into GSp_4 by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \times \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & -B_2 \\ \hline C_1 & 0 & D_1 & 0 \\ 0 & -C_2 & 0 & D_2 \end{array}\right).$$

By Rallis' inner product formula, we have

$$\frac{\rho_k \zeta(2)}{\zeta(3)\zeta(4)} \langle \theta(Y_{f_1,f_2}), \theta(Y_{f_1,f_2}) \rangle_{\mathrm{GO}(2,2)}$$
$$= C_{\mathrm{PGSp}_2} \mathrm{Res}_{s=1/2} \frac{L(s+(1/2), Y_{f_1,f_2}, \mathrm{st})}{d(s)} \langle Y_{f_1,f_2}, Y_{f_1,f_2} \rangle_{\mathrm{PGSp}_2},$$

where

$$\begin{aligned} d(s) =& \zeta(s+(5/2))\zeta(2s+1)\zeta(2s+3), \\ L(s, Y_{f_1,f_2}, \mathrm{st}) =& \zeta(s)L(s, \tau_1 \times \tau_2). \end{aligned}$$

This shows that

$$\frac{\rho_k \zeta(2)}{\zeta(3)\zeta(4)} \frac{\langle Y_{f_1,f_2}, Y_{f_1,f_2} \rangle^2_{\mathrm{PGSp}_2}}{\langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{f_1,f_2} \rangle_{\mathrm{GO}(2,2)}} = \frac{\rho_k L(1, \tau_1 \times \tau_2)}{\zeta(2)^2 \zeta(3)\zeta(4)^2} \langle Y_{f_1,f_2}, Y_{f_1,f_2} \rangle_{\mathrm{PGSp}_2}.$$

Thus we have

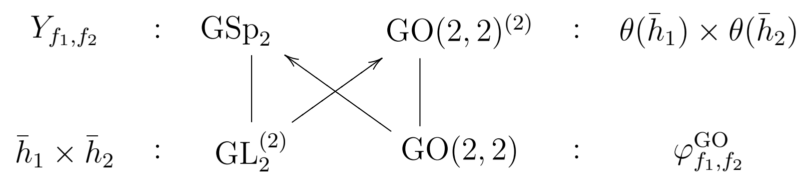
$$\begin{aligned} \langle Y_{f_1,f_2}, Y_{f_1,f_2} \rangle_{\mathrm{PGSp}_2} =& \frac{L(1, \tau_1 \times \tau_2)}{\zeta(2)^3 \zeta(4)} \langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{f_1,f_2} \rangle_{\mathrm{GO}(2,2)} \\ =& \frac{L(1, \tau_1 \times \tau_2)}{\zeta(2)^5 \zeta(4)} L(1, \tau_1, \mathrm{Ad}) L(1, \tau_2, \mathrm{Ad}). \end{aligned}$$

We state this result as a proposition.

Proposition 13.11. *We have*

$$\langle Y_{f_1,f_2}, Y_{f_1,f_2} \rangle_{\mathrm{PGSp}_2} = \frac{L(1, \tau_1 \times \tau_2) L(1, \tau_1, \mathrm{Ad}) L(1, \tau_2, \mathrm{Ad})}{\zeta(2)^5 \zeta(4)}.$$

We consider the restriction of Y_{f_1,f_2} to $\mathrm{GL}_2^{(2)}(\mathbb{A})$. Let σ_1 and σ_2 be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_{\sigma_1}\omega_{\sigma_2} = 1$. Let $h_1 \in \sigma_1$ and $h_2 \in \sigma_2$ be primitive forms. We consider the seesaw



Note that $\theta(\bar{h}_1) = \zeta(2)^{-1}\varphi^{\mathrm{GO}}_{h_1}$ and $\theta(\bar{h}_2) = \zeta(2)^{-1}\varphi^{\mathrm{GO}}_{h_2}$. We get the seesaw identity

$$\begin{aligned} \langle Y_{f_1,f_2}|_{\mathrm{SO}(2,2)}, \varphi^{\mathrm{SO}}_{h_1,h_2} \rangle_{\mathrm{SO}(2,2)} =& \zeta(2)^{-2} \langle \varphi^{\mathrm{GO}}_{f_1,f_2}, \varphi^{\mathrm{GO}}_{h_1} \varphi^{\mathrm{GO}}_{h_2} \rangle_{\mathrm{GO}(2,2)} \\ =& \frac{1}{2} \zeta(2)^{-2} \langle f_1, h_1 h_2 \rangle_{\mathrm{PGL}_2} \langle f_2, h_1 h_2 \rangle_{\mathrm{PGL}_2}. \end{aligned}$$

Let $\mathrm{SO}(3,2)$ be the special orthogonal group for the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$, which is compatible with $\mathrm{GL}_2^{(2)}/\mathbb{G}_m \simeq \mathrm{SO}(2,2)$. Thus we obtain the following proposition, which can be regarded as an analogue of [4] for function field.

Proposition 13.12. *We have*

$$|\langle Y_{f_1,f_2}|_{\mathrm{SO}(2,2)}, \varphi_{h_1,h_2}\rangle|^2 = \frac{1}{4}\zeta(2)^{-8} L(1/2, \tau_1 \times \sigma_1 \times \sigma_2) L(1/2, \tau_2 \times \sigma_1 \times \sigma_2).$$

Combining Proposition 13.4, Proposition 13.11, and Proposition 13.12, we obain the following proposition.

Proposition 13.13. *Let τ_1, τ_2, σ_1, and σ_2 be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\tau_1 \not\simeq \tilde{\tau}_2$, $\omega_{\tau_1} = \omega_{\tau_2} = \omega_{\sigma_1}\omega_{\sigma_2} = 1$. We think of Y_{f_1,f_2} as a cusp form on $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$. Then we have*

$$\frac{|\langle Y_{f_1,f_2}|_{\mathrm{SO}(2,2)}, \varphi_{h_1,h_2}\rangle|^2}{\langle Y_{f_1,f_2}, Y_{f_1,f_2}\rangle \langle \varphi_{h_1,h_2}, \varphi_{h_1,h_2}\rangle} = \frac{1}{8|\mathfrak{X}_{\sigma_1,\sigma_2}|} \frac{\zeta(4)}{\zeta(2)} \frac{L(1/2, \tau_1 \times \sigma_1 \times \sigma_2) L(1/2, \tau_2 \times \sigma_1 \times \sigma_2)}{L(1, \tau_1 \times \tau_2) \prod_{i=1}^2 L(1, \tau_i, \mathrm{Ad}) L(1, \sigma_i, \mathrm{Ad})}.$$

Note that $|\mathcal{S}_{\psi_1}|$ and $|\mathcal{S}_{\psi_0}|$ should be 4 and $2|\mathfrak{X}_{\sigma_1,\sigma_2}|$, respectively. Thus, it seems Conjecture 2.1 holds in this case.

Remark 13.14. The examples in this section are tempered by the result of Lafforgue [47].

14. Unramified non-tempered examples over a function field

We first consider an analogue of the results of §8. Let $f \in \tau$ be a primitive form on $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. If the multiplicity formula for the Saito-Kurokawa packet (cf. Piatetski-Shapiro [52], Cogdell and Piatetski-Shapiro [7]) of $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$ holds for the function field k, then the Saito-Kurokawa packet consists of only one element, since τ is locally unramified. Moreover, the automorphic representation in the Saito-Kurokawa packet is cuspidal if and only if $L(1/2, \tau) = 0$.

To get a cusp form, we need to twist the theta correspondence by a quadratic character, which complicates the situation considerably. Therefore we consider the residual automorphic representation in the Saito-Kurokawa packet.

Let $\theta(f)$ be the theta lift of f to $\widetilde{\mathrm{SL}_2}$. We assume $\theta(f) \neq 0$. This is the case if and only if $L(1/2,\tau) \neq 0$. Put

$$F(h) = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \Theta(g,h)\overline{\theta(f)(g)}\, dg,$$

where $\Theta(g,h)$ is the theta function of $\widetilde{\mathrm{SL}_2(\mathbb{A})} \times \mathrm{O}(3,2)(\mathbb{A})$ associated with the characteristic function of $\hat{\mathfrak{o}}^5$. Then F can be considered as a theta lift of $\overline{\theta(f)}$ to $\mathrm{O}(3,2) \simeq \mathrm{PGSp}_2 \times \{\pm 1\}$, although F is not a cusp form. We denote F by $\mathrm{SK}(f)$ and call it the Saito-Kurokawa lift of f.

We shall show that $\mathrm{SK}(f)$ is a residue of some Eisenstein series. Let $P = MN$ be a parabolic subgroup of $\mathrm{SO}(3,2)$ with $M = \mathbb{G}_m \times \mathrm{SO}(2,1)$. Note that the unipotent radical N of P is abelian. Let $h = h_M h_N h_{\mathcal{K}}$ be an Iwasawa decomposition of an element $h \in \mathrm{SO}(3,2)(\mathbb{A})$, where $h_M \in M(\mathbb{A})$, $h_N \in N(\mathbb{A})$, and $h_{\mathcal{K}} \in \mathcal{K}_{\mathrm{SO}(3,2)}$. We define a function $\mathfrak{f}^{(s)}$ on $\mathrm{SO}(3,2)(\mathbb{A})$ by

$$\mathfrak{f}^{(s)}(h) = f(m)|t|^{s+(3/2)},$$

where $h_M = (t,m) \in \mathbb{A}^\times \times \mathrm{SO}(2,1)(\mathbb{A})$. Set

$$\mathcal{E}^{(s)}(h;f) = \sum_{\gamma \in P(k)\backslash \mathrm{SO}(3,2)(k)} \mathfrak{f}^{(s)}(\gamma h),$$

$$\mathcal{E}(h;f) = \mathrm{Res}_{s=1/2}\left[\mathcal{E}^{(s)}(h;f)\right].$$

Lemma 14.1. *Let $f \in \tau$ be a primitive form of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. Assume that $L(1/2,\tau) \neq 0$. Then we have*

$$\mathcal{E}(f) = \frac{\rho_k \zeta(2)}{L(3/2,\tau)} \mathrm{SK}(f).$$

In particular, $\mathrm{SK}(f)$ *is an* $\mathbb{R}$*-valued function.*

Proof. One can easily prove that the representation generated by $\mathcal{E}(f)$ is isomorphic to the representation generated by $\mathrm{SK}(f)$. One can also show that the constant terms of $\mathcal{E}(f)_{P'}$ and $\mathrm{SK}(f)_{P'}$ vanish, where P' is a proper parabolic subgroup other than P. For the constant terms along P, we have

$$\mathcal{E}(f)_P = \mathrm{Res}_{s=1/2}\mathcal{M}(s)\mathfrak{f}^{(s)} = \frac{\rho_k L(1/2,\tau)}{2\zeta(2)L(3/2,\tau)}\mathfrak{f}^{(-1/2)},$$

$$\mathrm{SK}(f)_P = \frac{L(1/2,\tau)}{2\zeta(2)^2}\mathfrak{f}^{(-1/2)},$$

where $\mathcal{M}(s)$ is the intertwining operator. The latter equation follows from Rallis [56] §1 and Proposition 13.3. It follows that $\mathcal{E}(f) - \rho_k\zeta(2)L(3/2,\tau)^{-1}\mathrm{SK}(f)$ is a cusp form belonging to the Saito-Kurokawa packet. Since τ_v is a principal series for any place v, the results of Piatetski-Shapiro [52] implies that there are no cuspidal automorphic Saito-Kurokawa representations. Hence the lemma. □

Proposition 14.2. *The Petersson norm of $\mathcal{E}(f)$ is given by*

$$\langle \mathcal{E}(f), \mathcal{E}(f)\rangle_{\mathrm{SO}(3,2)} = \rho_k^2\zeta(2)^{-2}\zeta(4)^{-1}L(1/2,\tau)L(3/2,\tau)^{-1}L(1,\tau,\mathrm{Ad}).$$

Proof. We use the inner product formula of the truncated Eisenstein series (see Arthur [2], Morris [51] p.143):

$$\begin{aligned}&C_M C_{\mathrm{SO}(3,2)}^{-1}\cdot\langle \wedge^T\mathcal{E}^{(s)}(f), \wedge^T\mathcal{E}^{(s)}(f)\rangle\\ =&\frac{e^{(s+\bar{s})T}}{s+\bar{s}}(\mathfrak{f}^{(s)},\mathfrak{f}^{(s)}) + \frac{e^{(s-\bar{s})T}}{s-\bar{s}}(\mathfrak{f}^{(s)},\mathcal{M}(s)\mathfrak{f}^{(s)})\\ &+\frac{e^{(-s+\bar{s})T}}{-s+\bar{s}}(\mathcal{M}(s)\mathfrak{f}^{(s)},\mathfrak{f}^{(s)}) + \frac{e^{(-s-\bar{s})T}}{-s-\bar{s}}(\mathcal{M}(s)\mathfrak{f}^{(s)},\mathcal{M}(s)\mathfrak{f}^{(s)}).\end{aligned}$$

Here, $\wedge^T$ is the truncation operator (cf. [2]) and

$$(\mathfrak{f}_1,\mathfrak{f}_2) = \int_{\mathcal{K}_{\mathrm{SO}(3,2)}}\int_{M(k)\backslash M(\mathbb{A})^1}\mathfrak{f}_1(h_M h_{\mathcal{K}})\overline{\mathfrak{f}_2(h_M h_{\mathcal{K}})}\,dh_M\,dh_{\mathcal{K}}.$$

Note that $C_M C_{\mathrm{SO}(3,2)}^{-1} = C_{\mathbb{G}_m}C_{\mathrm{SO}(2,1)}C_{\mathrm{SO}(3,2)}^{-1} = \rho_k^{-1}\zeta(4)$. We put $s = (1/2)+\sqrt{-1}t$, and let t tend to 0, and then let T tend to $+\infty$, we have

$$\langle\mathcal{E}(f),\mathcal{E}(f)\rangle = \frac{\rho_k}{\zeta(4)}(\mathfrak{f}^{(1/2)}, \mathrm{Res}_{s=1/2}\mathcal{M}(s)\mathfrak{f}^{(s)}).$$

Since

$$\mathcal{M}(s)\mathfrak{f}^{(s)} = \frac{\zeta(2s)L(s,\tau)}{\zeta(2s+1)L(s+1,\tau)}\mathfrak{f}^{(-s)},$$

we have

$$\begin{aligned}\langle\mathcal{E}(f),\mathcal{E}(f)\rangle =&\frac{\rho_k}{\zeta(4)}\frac{\rho_k L(1/2,\tau)}{2\zeta(2)L(3/2,\tau)}(\mathfrak{f}^{(1/2)},\mathfrak{f}^{(-1/2)})\\ =&\frac{\rho_k}{\zeta(4)}\frac{\rho_k L(1/2,\tau)}{2\zeta(2)L(3/2,\tau)}2\zeta(2)^{-1}L(1,\tau,\mathrm{Ad}).\end{aligned}$$

Hence the proposition. □

Using Lemma 14.1 and Proposition 14.2, we have the following formula for the Petersson norm of the Saito-Kurokawa lift $\mathrm{SK}(f)$.

Proposition 14.3. *Let $f \in \tau$ be a primitive form on $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. Then we have*

$$\langle \mathrm{SK}(f), \mathrm{SK}(f)\rangle_{\mathrm{SO}(3,2)} = \zeta(2)^{-4}\zeta(4)^{-1} L(1/2,\tau)L(3/2,\tau)L(1,\tau,\mathrm{Ad}).$$

Let $h_1 \in \sigma_1$ and $h_2 \in \sigma_2$ be primitive forms on $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_{\sigma_1}\omega_{\sigma_2} = 1$. Put $G_1 = \mathrm{SO}(3,2)$ and $G_0 = \mathrm{SO}(2,2)$. The main results of Ginzburg, Piatetski-Shapiro, and Rallis [14] imply

$$\begin{aligned} &C_{\mathrm{SO}(2,2)}^{-1}\langle \mathcal{E}^{(s)}(f)|_{G_0}, \varphi_{h_1,h_2}^{\mathrm{SO}}\rangle_{G_0} \\ =&C_{\mathrm{SO}(2,1)}^{-1}\frac{L(s+(1/2),\sigma_1\times\sigma_2)}{\zeta(2s+1)L(s+1,\tau)}\overline{\langle \varphi_{h_1,h_2}^{\mathrm{SO}}|_{\mathrm{SO}(2,1)}, f\rangle}_{\mathrm{SO}(2,1)}. \end{aligned}$$

In particular, $\langle \mathcal{E}(f)|_{G_0}, \varphi_{h_1,h_2}^{\mathrm{SO}}\rangle_{G_0} = 0$ unless $\sigma_1 \simeq \tilde{\sigma}_2$. We consider the case $\sigma = \sigma_1 \simeq \tilde{\sigma}_2$. Let h be the primitive form of σ. Then taking the residue, we have

$$\langle \mathcal{E}(f)|_{G_0}, \varphi_h^{\mathrm{SO}}\rangle_{G_0} = \frac{\rho_k L(1,\sigma,\mathrm{Ad})}{\zeta(2)^2 L(3/2,\tau)}\langle h\bar{h}, f\rangle_{\mathrm{PGL}_2}.$$

By Proposition 13.6, we obtain the following proposition.

Proposition 14.4. *We have*

$$|\langle \mathcal{E}(f)|_{G_0}, \varphi_h^{\mathrm{SO}}\rangle|^2 = \frac{\rho_k^2 L(1,\sigma,\mathrm{Ad})^2 L(1/2,\tau)L(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{\zeta(2)^6 L(3/2,\tau)^2}.$$

Note that

$$\frac{|\langle \mathrm{SK}(f)|_{G_0}, \varphi_h^{\mathrm{SO}}\rangle_{G_0}|^2}{\langle \mathrm{SK}(f), \mathrm{SK}(f)\rangle\langle \varphi_h^{\mathrm{SO}}, \varphi_h^{\mathrm{SO}}\rangle} = \frac{|\langle \mathcal{E}(f)|_{G_0}, \varphi_h^{\mathrm{SO}}\rangle_{G_0}|^2}{\langle \mathcal{E}(f), \mathcal{E}(f)\rangle\langle \varphi_h^{\mathrm{SO}}, \varphi_h^{\mathrm{SO}}\rangle}.$$

Therefore, we obtain the following proposition by combining Proposition 13.4, Proposition 14.2, and Proposition 14.4.

Proposition 14.5. *Put $G_1 = \mathrm{SO}(3,2)$ and $G_0 = \mathrm{SO}(2,2)$. Let $f \in \tau$ and $h \in \sigma$ be primitive forms on $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$. We assume $L(1/2,\tau) \neq 0$. Let $\mathrm{SK}(f)$ be the Saito-Kurokawa lift of f. Then we have*

$$\frac{|\langle \mathrm{SK}(f)|_{G_0}, \varphi_h^{\mathrm{SO}}\rangle|^2}{\langle \mathrm{SK}(f), \mathrm{SK}(f)\rangle\langle \varphi_h^{\mathrm{SO}}, \varphi_h^{\mathrm{SO}}\rangle} = \frac{1}{2|\mathfrak{X}_\sigma|}\frac{\zeta(4)}{\zeta(2)^2}\frac{L(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau)}{L(3/2,\tau)L(1,\tau,\mathrm{Ad})}.$$

In this case, it seems Conjecture 3.2 holds with $2^\beta = 1/(2|\mathfrak{X}_\sigma|)$.

Next, we consider an analogue of the results of §9. Let $\delta \in \mathbb{F}_q$ be a non-square element. Let Q be a non-degenerate quadratic form of rank

6 defined by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\delta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The anisotropic kernel of Q is isomorphic to the norm form of $\mathbb{F}_{q^2}$. We denote the orthogonal group for Q by $\mathrm{O}(4,2)$. The quadratic character associated to Q is denoted by $\chi = \chi_Q$, which is the unramified character of order two associated to the quadratic constant field extension of k. Put $G_0 = \mathrm{SO}(3,2)$ and $G_1 = \mathrm{SO}(4,2)$. The symmetric matrix defining $\mathrm{SO}(3,2)$ can be considered as a submatrix of Q by deleting 4-th column and row. We regard G_0 as a sugroup of G_1 by this embedding. Let $P_{0,\min} = A_{0,\min}N_{0,\min} \subset G_0$ and $P_{1,\min} = T_{1,\min}N_{1,\min} \subset G_1$ be Borel subgroups such that $P_{0,\min} \subset P_{1,\min}$ and $A_{0,\min} \subset T_{1,\min}$. Note that $A_{0,\min} \simeq (k^\times)^2$ is a maximal split subtorus of G_1. We may assume the set of simple roots $\{\alpha_1, \alpha_2\}$ of $A_{0,\min}$ corresponding to the Borel subgroup $P_{0,\min}$ is given by $\alpha_1(t_1,t_2) = t_1t_2^{-1}$ and $\alpha_2(t_1,t_2) = t_2$ for $(t_1,t_2) \in A_{0,\min}$. Fix a place v_0 of k. Put

$$A_{0,\min}(k_{v_0};T) = \{(t_1,t_2) \in A_{0,\min}(k_{v_0}) \mid |t_1t_2^{-1}|_{v_0}, |t_2|_{v_0} \geq T\}.$$

Recall that Siegel domains $\mathfrak{S}_0 = \mathfrak{S}_0(\Omega_0,T) \subset G_0(\mathbb{A})$, $\mathfrak{S}_1 = \mathfrak{S}_1(\Omega_1,T) \subset G_1(\mathbb{A})$ are subsets of the form

$$\begin{aligned} \mathfrak{S}_0(\Omega_0,T) =& \Omega_0 \cdot A_{0,\min}(k_{v_0};T) \cdot \mathcal{K}_{G_0} \\ \mathfrak{S}_1(\Omega_1,T) =& \Omega_1 \cdot A_{0,\min}(k_{v_0};T) \cdot \mathcal{K}_{G_1}, \end{aligned}$$

for some compact subsets $\Omega_0 \subset P_{0,\min}(\mathbb{A})$, $\Omega_1 \subset P_{1,\min}(\mathbb{A})$ and a constant $T > 0$. We have $G_i(k)\mathfrak{S}_i(\Omega_i,T) = G_i(\mathbb{A})$ for some Ω_i and $T > 0$.

Let g be a primitive form of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$. We assume σ does not come from the unramified quadratic extension $K = \mathbb{F}_{q^2} \cdot k$. We consider the theta lift $\mathcal{M}(g) := \theta(g^0)$ of g^0 from SL_2 to $\mathrm{O}(4,2)$. This is an analogue of the hermitian Maass lift considered in §9. We call $\mathcal{M}(g)$ the hermitian Maass lift of g.

Proposition 14.6. *We have*

$$\langle \mathcal{M}(g), \mathcal{M}(g)\rangle_{\mathrm{SO}(4,2)} = 2|\mathfrak{X}_\sigma|\frac{L(1,\mathrm{Ad}(\sigma))L(2,\mathrm{Ad}(\sigma)\otimes\chi)}{\zeta(2)^2L(3,\chi)\zeta(4)}.$$

Proof. We make use of the seesaw

$$\begin{array}{ccccccc} \mathcal{M}(g)\times\overline{\mathcal{M}(g)} & : & \mathrm{O}(4,2)\times\mathrm{O}(4,2) & & \mathrm{Sp}_2 & : & E(3/2,\chi) \\ & & | & \times & | & & \\ 1 & : & \mathrm{O}(4,2) & & \mathrm{SL}_2\times\mathrm{SL}_2 & : & g^0\times g^0 \end{array}$$

where $E(s,\chi,g)$ is the Siegel-Eisenstein series with character χ. Here, we have used the Siegel-Weil formula (see [44])

$$\int_{\mathrm{O}(4,2)(k)\backslash\mathrm{O}(4,2)(\mathbb{A})}\Theta(g,h)\,dh = E(3/2,\chi,g).$$

Let $\iota:\mathrm{SL}_2\times\mathrm{SL}_2\to\mathrm{Sp}_2$ be as in the proof of Proposition 13.3. Then the seesaw dual identity says

$$\begin{aligned}&\langle\mathcal{M}(g),\mathcal{M}(g)\rangle_{\mathrm{O}(4,2)}\\ =&\int_{(\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A}))^2}E(3/2,\chi,\iota(x_1,x_2))g(x_1)g(x_2)\,dx_1\,dx_2\\ =&C_{\mathrm{SL}_2}\frac{L(2,\mathrm{Ad}(\sigma)\otimes\chi)}{L(3,\chi)\zeta(4)}\langle g^0,g^0\rangle_{\mathrm{SL}_2}.\end{aligned}$$

By Proposition 13.2, we have

$$\langle\mathcal{M}(g),\mathcal{M}(g)\rangle_{\mathrm{O}(4,2)} = |\mathfrak{X}_\sigma|\frac{L(1,\mathrm{Ad}(\sigma))L(2,\mathrm{Ad}(\sigma)\otimes\chi)}{\zeta(2)^2L(3,\chi)\zeta(4)}.$$

Hence the proposition. □

Recall that there exists an exact sequence

$$1\to\mathbb{G}_m\to H\to\mathrm{SO}(3,1)\to 1$$

where

$$H=\{x\in\mathrm{GL}_2(K)\mid\det x\in k^\times\}.$$

Let $\det:H\to k^\times$ be the restriction of the determinant map $\mathrm{GL}_2(K)\to K^\times$. Now let $g\in\sigma$ be a primitive form on $\mathrm{GL}_2(\mathbb{A})$. We assume that σ does not come from a Hecke character of $\mathbb{A}_K^\times$. The base change lift $\mathcal{BC}(\sigma)$ of σ to $\mathrm{GL}_2(\mathbb{A}_K)$ has a central character $\omega_\sigma\circ\mathcal{N}_{K/k}$, where $\mathcal{N}_{K/k}:\mathbb{A}_K^\times\to\mathbb{A}_k^\times$ is the norm map. Let $\mathcal{BC}(g)$ be the primitive form of $\mathcal{BC}(\sigma)$. Note that $\mathcal{BC}(g)$ is a cusp form by our assumption. The twist of the restriction of $\mathcal{BC}(g)$ to $H(\mathbb{A})$ by $\omega_\sigma^{-1}\circ\det$ can be regarded as an automorphic form on $\mathrm{SO}(3,1)(\mathbb{A})$. We denote this cusp form by $\mathcal{BC}(g)^0$. It can be extended to a right $\mathcal{K}_{\mathrm{O}(3,1)}$-invariant automorphic form on $\mathrm{O}(3,1)(\mathbb{A})$, which we also denote by $\mathcal{BC}(g)^0$. Note that $\mathcal{BC}(g)^0$ is an $\mathbb{R}$-valued function. As in Ichino [30] §10, one can prove the following lemma.

Lemma 14.7. *The theta lift of* g^0 *to* $\mathrm{O}(3,1)$ *is equal to* $\zeta(2)^{-1}\mathcal{BC}(g)^0$.

Next, we consider theta correspondence between $\mathrm{O}(3,2)$ and $\widetilde{\mathrm{SL}_2}$.

Lemma 14.8. *Let*

$$\Theta(g,h) = \sum_{x \in k^5} \omega_\psi(g,h)\Phi_0(x)$$

be the theta function associated to the Weil representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})} \times \mathrm{O}(3,2)(\mathbb{A})$, *where* Φ_0 *is the characteristic function on* $\hat{\mathfrak{o}}^5$. *Let* $\mathfrak{S}_0 = \mathfrak{S}_0(\Omega_0, T) \subset \mathrm{SO}(3,2)(\mathbb{A})$ *be a Siegel domain and* $X \subset \widetilde{\mathrm{SL}_2(\mathbb{A})}$ *be a compact set. Then there exists a constant* $C > 0$ *such that*

$$|\Theta(x, pt\kappa)| < C|t_1t_2|_{v_0}$$

for any $x \in X$, $p \in \Omega_0$, $t = (t_1, t_2) \in A_{0,\min}(k_{v_0} : T)$, *and* $\kappa \in \mathcal{K}_{\mathrm{SO}(3,2)}$.

Proof. Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}_5(\mathbb{A})}$ on $\mathcal{S}(\mathbb{A}^5)$. Take a Siegel domain $\mathfrak{S} = \Omega A^{\mathrm{Sp}_5}_{\min}(k_{v_0}, T')\mathcal{K}_{\mathrm{Sp}_5} \subset \mathrm{Sp}_5(\mathbb{A})$. Let $\tilde{\Omega}$ and $\tilde{A}^{\mathrm{Sp}_5}_{\min}(k_{v_0}, T')$ be the pullbacks of Ω and $A^{\mathrm{Sp}_5}_{\min}(k_{v_0}, T')$ in $\widetilde{\mathrm{Sp}_5(\mathbb{A})}$, respectively. Then there exists a constant $C' > 0$ such that

$$|\Theta(p' \cdot (t', \zeta) \cdot \kappa')| < C'|t'_1t'_2t'_3t'_4t'_5|^{1/2}_{v_0}$$

for any $p' \in \tilde{\Omega}$, $\kappa' \in \mathcal{K}_{\mathrm{Sp}_5}$, and

$$(t', \zeta) = (\mathrm{diag}(t'_1, t'_2, t'_3, t'_4, t'_5, {t'_1}^{-1}, {t'_2}^{-1}, {t'_3}^{-1}, {t'_4}^{-1}, {t'_5}^{-1}), \zeta) \in \tilde{A}^{\mathrm{Sp}_5}_{\min}(k_{v_0}; T').$$

Here $\Theta(g)$ is the theta function on $\widetilde{\mathrm{Sp}_5(\mathbb{A})}$ associated to the characteristic function of $\hat{\mathfrak{o}}^5$. The desired estimate is obtained by taking a pullback by the embedding $\widetilde{\mathrm{SL}_2(\mathbb{A})} \times \mathrm{SO}(3,2)(\mathbb{A}) \to \widetilde{\mathrm{Sp}_5(\mathbb{A})}$. Note that the image of $\mathrm{diag}(t_1, t_2, 1, t_2^{-1}, t_1^{-1}) \in \mathrm{SO}(3,2)$ is equivalent to

$$\mathrm{diag}(t_1, t_1, t_2, t_2, 1, 1, t_1^{-1}, t_1^{-1}, t_2^{-1}, t_2^{-1}) \in \mathrm{Sp}_5$$

under the action of the Weyl group. □

Proposition 14.9. *Let* f *be a primitive form of an irreducible cuspidal automorphic representation* τ *of* $\mathrm{GL}_2(\mathbb{A})$ *such that* $\omega_\tau = 1$, $L(1/2, \tau) \neq 0$. *Consider the theta lift* $\theta(\mathrm{SK}(f))$ *of* $\mathrm{SK}(f)$ *from* $\mathrm{O}(3,2)$ *to* $\widetilde{\mathrm{SL}_2}$. *Then we have*

$$\theta(\mathrm{SK}(f)) = \zeta(2)^{-1}\zeta(4)^{-1}L(3/2, \tau) \cdot \overline{\theta(f)}.$$

Proof. First we note that the theta lift $\theta(\mathrm{SK}(f))$ is well-defined. Fix a Siegel domain $\mathfrak{S}_0 = \mathfrak{S}_0(\Omega_0, T) \subset G_0(\mathbb{A})$. By Mœglin-Waldspurger [50] Lemma I.4.1, there exists a constant C'' such that

$$|\mathrm{SK}(f)(pt\kappa)| < C''|t_1|_{v_0}$$

for $p \in \Omega_0$, $t = (t_1, t_2) \in A_{0,\min}(k_{v_0}; T)$, $\kappa \in \mathcal{K}_{G_0}$. By Lemma 14.8 and the estimate as above, one can easily show that the theta integral $\theta(\mathrm{SK}(f))$ is absolutely convergent.

We use the seesaw

$$\begin{array}{ccccccc} \mathrm{SK}(f) & : & \mathrm{O}(3,2) & & \widetilde{\mathrm{SL}_2} & : & \theta(\mathrm{SK}(f)) \\ & & \Big| & \times & \Big| & & \\ \mathrm{SK}(f) & : & \mathrm{O}(3,2) & & \widetilde{\mathrm{SL}_2} & : & \theta(f) \end{array}$$

It is well-known from the theory of dual paring that the representation generated by $\theta(\mathrm{SK}(f))$ is isomorphic to the contragredient of $\theta(\tau)$. By the multiplicity one theorem of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$, there exist a constant c such that $\theta(\mathrm{SK}(f)) = c\overline{\theta(f)}$. By the seesaw identity, we have

$$\langle \mathrm{SK}(f), \mathrm{SK}(f) \rangle_{\mathrm{O}(3,2)} = c \langle \theta(f), \theta(f) \rangle_{\mathrm{SL}_2}.$$

By Proposition 13.3 and Proposition 14.3, the proposition follows. $\square$

Now we consider the restriction of $\mathcal{M}(g)$ to $\mathrm{SO}(3,2)(\mathbb{A})$. We first show that the inner product $\langle \mathcal{M}(g)|_{G_0}, \mathrm{SK}(f) \rangle$ is absolutely convergent. By Mœglin-Waldspurger [50] Lemma I.4.1, there exists a constant $C''' > 0$ such that

$$|\mathcal{M}(g)(pt\kappa)| < C''' |t_1|_{v_0}$$

for $p \cdot (t_1, t_2) \cdot \kappa \in \Omega_1 \cdot A_{0,\min}(k_{v_0}; T) \cdot \mathcal{K}_{G_1} = \mathfrak{S}_1$. Then the inner product $|\langle \mathcal{M}(g)|_{G_0}, \mathrm{SK}(f) \rangle|$ can be estimated by

$$\begin{aligned} & C'' C''' \int_{\mathfrak{S}_0(\Omega_0, T)} |t_1|_{v_0}^2 \, dg \\ & \leq C'' C''' \int_{p \in \Omega_0} \int_{(t_1,t_2) \in A_{0,\min}(k_{v_0}; T)} \int_{\kappa \in \mathcal{K}_{G_0}} \delta_{P_{0,\min}}^{-1}(t_1, t_2) \cdot |t_1|_{v_0}^2 \, dp \, d^\times t_1 \, d^\times t_2 \, d\kappa \\ & < \infty. \end{aligned}$$

Here, $\delta_{P_{0,\min}}(t_1, t_2) = |t_1|_{v_0}^3 |t_2|_{v_0}$ is the modulus character of $A_{0,\min}(k_{v_0})$. It follows that $|\langle \mathcal{M}(g)|_{G_0}, \mathrm{SK}(f) \rangle|$ is convergent. Now we consider the seesaw

$$\begin{array}{ccccccc} \mathcal{M}(g) & : & \mathrm{O}(4,2) & & \widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} & : & \frac{1}{2}\Theta_\chi \times \theta(\mathrm{SK}(f)) \\ & & \Big| & \times & \Big| & & \\ 1 \times \mathrm{SK}(f) & : & \mathrm{O}(1) \times \mathrm{O}(3,2) & & \mathrm{SL}_2 & : & g^0 \end{array}$$

Here, $\mathrm{O}(1) \simeq \{\pm 1\}$ is the orthogonal group associated to (-2δ), and Θ_χ is a theta function associated with the characteristic function of $\hat{\mathfrak{o}}$

with respect to the Weil representation $\omega_{\psi_{2\delta}}$, where $\psi_{2\delta}(x) = \psi(2\delta x)$. By the seesaw dual identity and Proposition 14.9, we have

$$|\langle \mathcal{M}(g)|_{\mathrm{SO}(3,2)}, \mathrm{SK}(f)\rangle_{\mathrm{SO}(3,2)}|^2 = \frac{4L(3/2,\tau)^2}{\zeta(2)^2\zeta(4)^2}|\langle \Theta_\chi \overline{\theta(f)}, \bar{g}^0\rangle_{\mathrm{SL}_2}|^2.$$

Next, we calculate $\langle \Theta_\chi \overline{\theta(f)}, \bar{g}^0\rangle_{\mathrm{SL}_2}$. We use the seesaw:

$$\begin{array}{ccccccc} \zeta(2)^{-1}\mathcal{BC}(g)^0 & : & \mathrm{O}(3,1) & & \widetilde{\mathrm{SL}_2}\times\widetilde{\mathrm{SL}_2} & : & \frac{1}{2}\Theta_\chi \times \overline{\theta(f)} \\ & & \Big| & & \Big| & & \\ 1\times f & : & \mathrm{O}(1)\times \mathrm{O}(2,1) & & \mathrm{SL}_2 & : & g^0. \end{array}$$

By seesaw dual identity, we have

$$\langle \Theta_\chi \overline{\theta(f)}, \bar{g}^0\rangle_{\mathrm{SL}_2} = \frac{1}{2\zeta(2)}\langle \mathcal{BC}(g)^0|_{\mathrm{PGL}_2}, f\rangle_{\mathrm{PGL}_2}.$$

By Proposition 13.9, we have

$$|\langle \mathcal{BC}(g)^0|_{\mathrm{PGL}_2}, f\rangle_{\mathrm{PGL}_2}|^2 = \zeta(2)^{-2}L(1/2,\tau)L(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau\otimes\chi).$$

Therefore we have

$$\begin{aligned} &|\langle \mathcal{M}(g)|_{\mathrm{SO}(3,2)}, \mathrm{SK}(f)\rangle_{\mathrm{SO}(3,2)}|^2 \\ &\quad= \frac{L(3/2,\tau)^2}{\zeta(2)^6\zeta(4)^2}L(1/2,\tau)L(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau\otimes\chi). \end{aligned}$$

Combining these results, we obtain the following proposition.

Proposition 14.10. *Let f be a primitive form of an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$ such that $\omega_\tau = 1$, $L(1/2,\tau) \neq 0$. We denote the Saito-Kurokawa lift of f by $\mathrm{SK}(f)$. Let g be a primitive form of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$, which does not come from the quadratic constant field extension K. We denote the hermitian Maass lift of g by $\mathcal{M}(g)$. Put $G_1 = \mathrm{SO}(4,2)$ and $G_0 = \mathrm{SO}(3,2)$. Then we have*

$$\begin{aligned} &\frac{|\langle \mathcal{M}(g)|_{G_0}, \mathrm{SK}(f)\rangle|^2}{\langle \mathcal{M}(g),\mathcal{M}(g)\rangle\langle \mathrm{SK}(f),\mathrm{SK}(f)\rangle} \\ &\quad= \frac{1}{2|\mathfrak{X}_\sigma|}\frac{L(3,\chi)L(3/2,\tau)L(1/2,\mathrm{Ad}(\sigma)\boxtimes\tau\otimes\chi)}{L(1,\mathrm{Ad}(\tau))L(1,\mathrm{Ad}(\sigma))L(2,\mathrm{Ad}(\sigma)\otimes\chi)}. \end{aligned}$$

It seems Conjecture 3.2 holds with $2^\beta = 1/(2|\mathfrak{X}_\sigma|)$ in this case.

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Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail address: ichino@sci.osaka-cu.ac.jp

Graduate school of mathematics, Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan
E-mail address: ikeda@math.kyoto-u.ac.jp