ON THE LIFTING OF AUTOMORPHIC REPRESENTATIONS OF $\mathrm{PGL}_2(\mathbb{A})$ TO $\mathrm{Sp}_{2n}(\mathbb{A})$ OR $\widetilde{\mathrm{Sp}_{2n+1}}(\mathbb{A})$ OVER A TOTALLY REAL FIELD

TAMOTSU IKEDA

Abstract. We construct a lifting of automorphic representations on PGL_2 to symplectic or metaplectic groups over a totally real field.

Introduction

In this paper, we construct a lifting of automorphic representations on PGL_2 to symplectic or metaplectic groups over a totally real field. The main result of this paper can be considered as a generalization of [6] to an arbitrary totally real field under some mild local assumptions.

Let k be a totally real field. Fix a non-trivial additive character ψ of $\mathbb{A}/k$. We denote the set of all archimedean places of k by $\mathfrak{S}_\infty$. Let $\tau \simeq \otimes_v \tau_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. We assume that $\tau \simeq \otimes_v \tau_v$ satisfies the following conditions:

(A1) For $v \notin \mathfrak{S}_\infty$, τ_v is a principal series $\mathcal{B}(\mu_v, \mu_v^{-1})$.

(A2) For $v \in \mathfrak{S}_\infty$, τ_v is a discrete series representation with lowest weight $\pm 2\kappa_v$.

(A3) The root number $\varepsilon(1/2, \tau)$ is equal to 1.

Here, the (global) root number is defined as follows. The local root number $\varepsilon(1/2, \tau_v)$ is given by

$$\varepsilon(1/2, \tau_v) = \begin{cases} \mu_v(-1) & v \notin \mathfrak{S}_\infty, \\ (-1)^{\kappa_v} & v \in \mathfrak{S}_\infty. \end{cases}$$

Then the global root number $\varepsilon(1/2, \tau)$ is defined by

$$\varepsilon(1/2, \tau) = \prod_v \varepsilon(1/2, \tau_v).$$

For each place v of k, let $\widetilde{\mathrm{Sp}_n(k_v)} = \{(g, \zeta) \mid g \in \mathrm{Sp}_n(k_v), \zeta \in \{\pm 1\}\}$ be the metaplectic covering of the symplectic group $\mathrm{Sp}_n(k_v)$. The multiplication law of $\widetilde{\mathrm{Sp}_n(k_v)}$ is given by $(g_1, \zeta_1)\cdot(g_2, \zeta_2) = (g_1 g_2, c(g_1, g_2)\zeta_1\zeta_2)$, where $c(g_1, g_2)$ is Rao's 2-cocycle (cf. Rao [11]).

We denote the space of symmetric matrices of size n by $\mathcal{S}_n(k)$. For $A \in \mathrm{GL}_n(k)$ and $z \in \mathcal{S}_n(k)$, we define $\mathbf{m}(A), \mathbf{n}(z) \in \mathrm{Sp}_n(k)$ by

$$\mathbf{m}(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \qquad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Let

$$P_n = M_n N_n \subset \mathrm{Sp}_n$$

be a Siegel parabolic subgroup of Sp_n, where

$$M_n(k) = \{\mathbf{m}(A) \,|\, A \in \mathrm{GL}_n(k)\}, \quad N_n(k) = \{\mathbf{n}(z) \,|\, z \in \mathcal{S}_n(k)\}.$$

Let v be a place of k. For each $t \in k_v$, we denote by $\alpha_{\psi_v}(t)$ the Weil constant. Recall that the Weil constant $\alpha_{\psi_v}(t)$ satisfies the equation

$$\int_{k_v} \phi(x)\psi(tx^2)\,dx = \alpha_{\psi_v}(t)|2t|^{-1/2} \int_{k_v} \hat{\phi}(x)\psi_v(-t^{-1}x^2/4)\,dx$$

for any Schwartz function $\phi \in \mathcal{S}(k_v)$, where

$$\hat{\phi}(x) = \int_{k_v} \phi(y)\psi_v(xy)\,dy$$

is the Fourier transform of ϕ. Let $\langle\ ,\ \rangle_v$ be the Hilbert symbol for k_v. We set $\chi_t(x) = \langle t, x\rangle_v$ for $t, x \in k_v^\times$.

For $v \notin \mathfrak{S}_\infty$, we set

$$\Pi_{n,v} = \Pi(n, \tau_v) = \mathrm{Ind}_{\widetilde{P_n(k_v)}}^{\widetilde{\mathrm{Sp}_n(k_v)}}(\mu_v^{(n)}).$$

Here,

$$\mu_v^{(n)}((\mathbf{m}(A), \zeta)) = \zeta^n \left(\frac{\alpha_{\psi_v}(1)}{\alpha_{\psi_v}(\det A)}\right)^n \mu(\det A).$$

Note that $\Pi_{n,v}$ is a degenerate principal series induced from a character of the Siegel parabolic subgroup. Note that when n is even, we have

$$\mu_v^{(n)}((\mathbf{m}(A), \zeta)) = \chi_{(-1)}(\det A)^{n/2}\mu(\det A).$$

When $v \in \mathfrak{S}_\infty$, we let $\Pi_{n,v} = \Pi(n, \tau_v)$ be the lowest weight representation of $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ with lowest $\tilde{\mathrm{U}}(n)$-type $(\det)^{\kappa_v + (n/2)}$. Note that $\Pi_{n,v}$ is genuine if and only if n is odd.

Let $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ be the metaplectic double covering of the adele group $\mathrm{Sp}_n(\mathbb{A})$. We denote the space of cusp forms on $\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ by $\mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})})$. Then we have

$$\mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}) = \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k)\backslash\mathrm{Sp}_n(\mathbb{A})) \oplus \mathcal{A}_{\mathrm{cusp}}^{\mathrm{gen}}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}),$$

where $\mathcal{A}^{\rm gen}_{\rm cusp}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})})$ is the space of genuine cusp forms on $\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}$.

We consider the restricted tensor product

$$\Pi_n = \Pi(n,\tau) = \bigotimes_v{}' \, \Pi(n,\tau_v),$$

which we consider as a representation of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$. The multiplicity $m_{\rm auto}(\Pi_n)$ is defined by

$$m_{\rm auto}(\Pi_n) = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{Sp}_n(\mathbb{A})}}(\Pi_n, \mathcal{A}_{\rm cusp}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})})).$$

Then the main result of this paper is as follows.

Theorem 7.1. *Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the conditions (A1), (A2), and (A3). Then we have*

$$m_{\rm auto}(\Pi_n) = 1.$$

It is easy to show that if τ satisfies (A1), (A2), and if $\varepsilon(1/2,\tau) = -1$, then $m_{\rm auto}(\Pi_n) = 0$.

Note that Π_{2n} can be considered as a cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$. The standard L-function of Π_{2n} is given by

$$L(s,\Pi_{2n},\mathrm{st}) = \zeta_k(s)\prod_{i=1}^{2n} L(s+n-i+(1/2),\tau\otimes\chi_{(-1)^n}),$$

up to bad Euler factors. Here, $\zeta_k(s)$ is the Dedekind zeta function of k.

Assume that $k=\mathbb{Q}$ and that $\kappa \equiv n \bmod 2$. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and τ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Then we have

$$\varepsilon(1/2,\tau\otimes\chi_{(-1)^n}) = (-1)^{\kappa+n} = 1.$$

Let $F \in S_{\kappa+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ be the cusp form constructed in [6]. Then the automorphic representation generated by F is equal to $\Pi(2n,\tau\otimes\chi_{(-1)^n})$. Therefore Theorem 7.1 can be considered as a generalization of [6].

In fact, the method of the proof of this theorem is different from that of [6]. In [6], we used the Fourier coefficient formula of the Siegel Eisenstein series. In this paper, we use the theory of degenerate Whittaker models, and do not use the Eisenstein series.

In, [3], Ginzburg, Rallis and Soudry have constructed some CAP representations by means of the descent method. Our method is different from their method, and we can determine the multiplicity.

For the multiplicity of the Saito-Kurokawa lifting, Piatetski-Shapiro [10] proved that the Saito-Kurokawa lifting has multiplicity one as a representation of $\mathrm{PSp}_2(\mathbb{A})$. But it seems his result does not imply the multiplicity one as a representation of $\mathrm{Sp}_2(\mathbb{A})$. Recently, Heim proved the multiplicity one of the Saito-Kurokawa lifting as a representation of $\mathrm{Sp}_2(\mathbb{A}_{\mathbb{Q}})$ for the Saito-Kurokawa lifting for $S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$.

We explain the idea of the construction. Let $\mathcal{S}_n(k)^+$ be the set of elements $B \in \mathcal{S}_n(k)$ whose image in $\mathcal{S}_n(k_v)$ is positive definite for any real place v of k. Assume that $\Pi(n,\tau)$ is cuspidal automorphic. For $f \in \Pi(n,\tau)$, the corresponding automorphic form $F(g)$ has a Fourier expansion

$$\begin{aligned} F(g) &= \sum_{\mathcal{S}_n(k)^+} W_B(g) \\ W_B(g) &= \int_{z \in N(k)\backslash N(\mathbb{A})} F(\mathbf{n}(z)g)\overline{\psi_B(z)}\, dz. \end{aligned}$$

Note that $W_B(g) \neq 0$ unless $B \in \mathcal{S}_n(k)^+$ by the Köcher principle. The map $f \mapsto W_B(\mathbf{1}_{2n})$ can be considered as a Whittaker vector $w_B \in \mathrm{Wh}_B(\Pi_n) = \mathrm{Hom}_{N(\mathbb{A})}(\Pi_n, \psi_B)$. By standard local argument, one can show that $\mathrm{Wh}_B(\Pi_n)$ is one dimensional for any $B \in \mathcal{S}_n(k)^+$. Thus an embedding $\eta \in \mathrm{Hom}_{\widetilde{\mathrm{Sp}_n(\mathbb{A})}}(\Pi_n, \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}))$ gives rise to a family of Whittaker vectors

$$\{w_B\}_{B \in \mathcal{S}_n(k)^+} \in \prod_{B \in \mathcal{S}_n(k)^+} \mathrm{Wh}_B(\Pi(n,\tau)).$$

Conversely, we consider a family $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$, and investigate when such a family gives rise to an automorphic form. Clearly, it is necessary that $w_{B[A]} = w_B \circ \Pi_n(\mathbf{m}(A))$ for any $B \in \mathcal{S}_n(k)^+$ and $A \in \mathrm{GL}_n(k)$. Here, $B[A] = {}^tABA$, as usual. We call such a family a $\mathrm{GL}_n(k)$-family for Π_n.

We shall make use of the theory of Fourier-Jacobi expansion [2] and Jacobi forms [5]. Note that the normalization of the Weil representation in [5] is different from that of this paper. Let $0 < m < n$ be an integer and set $n' = n - m$. Put

$$\mathbf{v}(x,y,z) = \left(\begin{array}{cc|cc} \mathbf{1}_m & x & z & y \\ 0 & \mathbf{1}_{n'} & {}^ty & 0 \\ \hline 0 & 0 & \mathbf{1}_m & 0 \\ 0 & 0 & -{}^tx & \mathbf{1}_{n'} \end{array}\right) \in \mathrm{Sp}_n(k)$$

for $x, y \in \mathrm{Mat}(m, n'; k)$ and $z - x \cdot {}^t y \in \mathcal{S}_m(k)$. We set

$$\begin{aligned} V &= V_{n,m} = \left\{ \mathbf{v}(x,y,z) \,|\, x, y \in \mathrm{Mat}(m,n';k), z - x \cdot {}^t y \in \mathcal{S}_m(k) \right\}, \\ X &= X_{n,m} = \{ \mathbf{v}(x,0,0) \,|\, x \in \mathrm{Mat}(m,n';k) \}, \\ Z &= Z_m = \{ \mathbf{v}(0,0,z) \,|\, z \in \mathcal{S}_m(k) \}. \end{aligned}$$

We regard these groups as algebraic subgroups of Sp_n. Note that the quotient group

$$V / \{ \mathbf{v}(0,0,z) \in Z \,|\, \mathrm{tr}(Sz) = 0 \}$$

is a Heisenberg group. We regard $\widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$ as a subgroup of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ by the embedding

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \zeta \right) \mapsto \left(\left(\begin{array}{cc|cc} \mathbf{1}_m & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & \mathbf{1}_m & 0 \\ 0 & C & 0 & D \end{array} \right), \zeta \right).$$

Fix $S \in \mathcal{S}_m(k)^+$. Let ψ_S be the character of $Z(\mathbb{A})$ defined by $\mathbf{v}(0,0,z) \mapsto \psi(\mathrm{tr}(Sz))$. By Stone-von Neumann theorem, there is a unique irreducible admissible representation ω_S of $V(\mathbb{A})$ on which $Z(\mathbb{A})$ acts by ψ_S. The representation ω_S extends to the Weil representation of the group $\widetilde{J(\mathbb{A})} = V(\mathbb{A}) \rtimes \widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$, which we also denote by ω_S. The representation ω_S can be realized on the Schwartz space $\mathcal{S}(X(\mathbb{A}))$ on $X(\mathbb{A})$. Recall that for $\phi \in \mathcal{S}(X(\mathbb{A}))$, the theta function $\Theta_S^\phi(vg')$ is defined by

$$\Theta^\phi(\mathbf{v}(x,y,z)g') = \sum_{l \in X(k)} \psi_S(z + y \cdot {}^t x + 2l \cdot {}^t y) \omega_S(g') \phi(l + x)$$

for $v = \mathbf{v}(x,y,z) \in V(\mathbb{A})$ and $g' \in \widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$.

For a vector $f \in \Pi(n, \tau)$ and an $\mathrm{GL}_n(k)$-family $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$, set

$$\begin{aligned} F(g) &= \sum_{B \in \mathcal{S}_n(k)^+} W_B(g), \\ W(g) &= w_B(\Pi_n(g) f). \end{aligned}$$

We assume the Fourier series $F(g)$ is absolutely convergent. Then we have (Lemma 11.6)

$$\begin{aligned} &\int_{V(k) \backslash V(\mathbb{A})} F(vg') \overline{\Theta^\phi(vg')} \, dv \\ &= \sum_{B' \in \mathcal{S}_{n'}^+(k)} \int_{X(\mathbb{A})} W_{S \oplus B'} \left(\mathbf{v}(x,0,0) g' \right) \overline{\omega_S(g') \phi(x)} dx \end{aligned}$$

for any $\phi \in \mathcal{S}(X(\mathbb{A}))$. Note that if $F(g)$ is automorphic, then so is this integral.

Let $\Pi(n,\tau)^{\mathrm{lwt}}$ and $\mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}$ be the spaces of lowest weight vectors of $\Pi(n,\tau)$ and $\mathcal{S}(X(\mathbb{A}))$, respectively. We shall show that there exists a $V(\mathbb{A})$-invariant surjective map

$$\beta_S : \Pi(n,\tau)^{\mathrm{lwt}} \otimes \mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}} \longrightarrow \Pi(n', \tau \otimes \chi_S)^{\mathrm{lwt}}$$

and a map

$$\mathcal{FJ}_S = \mathcal{FJ}_{S,B'} : \mathrm{Wh}_{S\oplus B'}(\Pi(n,\tau)) \longrightarrow \mathrm{Wh}_{B'}(\Pi(n', \tau\otimes\chi_S))$$

such that the Whittaker function associated with $\beta_S(f\otimes\phi)$ and $\mathcal{FJ}_{S,B'}(w_{S\oplus B'})$ is equal to

$$\int_{X(\mathbb{A})} W_{S\oplus B'}\left(\mathbf{v}(x,0,0)g'\right)\overline{\omega_S(g')\phi(x)}dx.$$

Thus it is natural to define as follows. Let

$$\{w_B\}_{B\in\mathcal{S}_n(k)^+} \in \prod_{B\in\mathcal{S}_n(k)^+} \mathrm{Wh}_B(\Pi_n)$$

be a $\mathrm{GL}_n(k)$-family of Whittaker vectors for Π_n. We shall say that $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is a compatible family, if the following conditions are satisfied.

(1) When $n=1$, a family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is compatible if it comes from the Shimura correspondence of τ, i.e., for each $f\in\Pi_1$, the Fourier series

$$F(g) = \sum_{B\in\mathcal{S}_1(k)^+} W_B(g)$$

belongs to the space of the Shimura correspondence of τ.

(2) When $n\geq 2$, a family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is a compatible family, if the family

$$\{\mathcal{FJ}_\xi(w_{(\xi)\oplus B'})\}_{B'\in\mathcal{S}_{n-1}(k)^+}$$

is a compatible family for $\Pi(n-1,\tau\otimes\chi_\xi)$ for each $\xi\in k_+^\times$.

For precise definition, see Definition 11.7.

We shall show that the dimension of the space of compatible family of Whittaker vectors for $\Pi(n,\tau)$ is 1 (Proposition 11.15). To prove the main theorem, we also need to show that the Fourier series associated to a compatible family of Whittaker vector is absolutely convergent. We shall prove the absolute convergence in §12. The proof of the main theorem will be completed in §13. In §14, we discuss the relation to the Arthur conjecture. In §15, we discuss the case $k=\mathbb{Q}$.

Notation

Given a ring R, we denote by $\mathrm{Mat}_{mn}(R)$ or $\mathrm{Mat}(m,n;R)$ the set of matrices of size $m \times n$ with entries in R. When $m = n$, this is just denoted by $\mathrm{Mat}_n(R)$ or $\mathrm{Mat}(n;R)$. We denote by $\mathrm{GL}_n(R)$ the general linear group over R, and by $\mathrm{SL}_n(R)$ the special linear group over R. When there is no fear of confusion, $\mathrm{GL}_n(R)$ etc. are simply denoted by GL_n etc.

Let k be a local or global field of characteristic 0. When k is a global filed, we assume that k is totally real. For global field k, we denote the adele ring of k by $\mathbb{A}_k$ or $\mathbb{A}$. We denote the set of archimedean places of k by $\mathfrak{S}_\infty$. We set $k_\infty = \prod_{v \in \mathfrak{S}_\infty} k_v$. The subgroup of totally positive elements of $k^\times$ is denoted by $k_+^\times$. The set of symmetric matrices of size n over k is denoted by $\mathcal{S}_n(k)$. An element $B \in \mathcal{S}_n(k)$ is said to be totally positive definite, if the image of B in $\mathcal{S}_n(k_v)$ is totally positive for any $v \in \mathfrak{S}_\infty$. The subset of totally positive definite elements of $\mathcal{S}_n(k)$ is denoted by $\mathcal{S}_n(k)^+$. When $B \in \mathcal{S}_n(k)$ is considered as a quadratic form, we set $B[x] = {}^t x B x$ and $B(x,y) = {}^t x B y$ for $x, y \in k^n$. Thus $B[x+y] - B[x] - B[y] = 2B(x,y)$ for $x, y \in k^n$. When $x \in \mathrm{Mat}_{mn}(k)$, we also set $B[x] = {}^t x B x$. For $B_1 \in \mathcal{S}_m(k)$ and $B_2 \in \mathcal{S}_n(k)$, we say that B_1 is represented by B_2 if there exists $x \in \mathrm{Mat}_{nm}(k)$ such that $B_1 = B_2[x]$. We write $B_1 \hookrightarrow B_2$, if B_1 is represented by B_2.

1. **Metaplectic groups**

Recall that the symplectic group Sp_n is defined by

$$\mathrm{Sp}_n(k) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2n}(k) \;\middle|\; \begin{array}{l} A \cdot {}^t B = B \cdot {}^t A,\ C \cdot {}^t D = D \cdot {}^t C, \\ A \cdot {}^t D - B \cdot {}^t C = \mathbf{1}_n \end{array} \right\}.$$

For $A \in \mathrm{GL}_n$ and $z \in \mathcal{S}_n$, we define $\mathbf{m}(A), \mathbf{n}(z) \in \mathrm{Sp}_n$ by

$$\mathbf{m}(A) = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}, \qquad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Let

$$P_n = M_n N_n \subset \mathrm{Sp}_n$$

be a Siegel parabolic subgroup of Sp_n, where

$$M_n = \{ \mathbf{m}(A) \mid A \in \mathrm{GL}_n \}, \quad N_n = \{ \mathbf{n}(z) \mid z \in \mathcal{S}_n \}.$$

We sometimes identify $\mathcal{S}_n$ and N_n by $B \mapsto \mathbf{n}(B)$, if there is no fear of confusion.

Let k be a local field. We assume $k \not\simeq \mathbb{C}$. For each Schwartz function $\phi \in \mathcal{S}(k)$, the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = \int_k \phi(y)\psi(xy)\,dy.$$

Here, the Haar measure dy is the self-dual Haar measure for the Fourier transform. Let $\langle\ ,\ \rangle$ be the Hilbert symbol. For each $a \in k^\times$, we define a character $\chi_a : k^\times \to \mathbb{C}^\times$ by $\chi_a(t) = \langle a, t\rangle$. For each $a \in k^\times$, there exists a constant $\alpha_\psi(a)$ such that

$$\int_k \phi(x)\psi(ax^2)\,dx = \alpha_\psi(a)|2a|^{-1/2}\int_k \hat{\phi}(x)\psi(-a^{-1}x^2/4)\,dx$$

for any $\phi \in \mathcal{S}(k)$. The constant $\alpha_\psi(a)$ is called the Weil constant. The following properties of the Weil constants are well-known.

$$\begin{aligned} \alpha_\psi(-a) &= \overline{\alpha_\psi(a)}, \\ \alpha_\psi(a)^8 &= 1, \\ \frac{\alpha_\psi(a)\alpha_\psi(b)}{\alpha_\psi(1)\alpha_\psi(ab)} &= \langle a, b\rangle, \qquad a, b \in k^\times. \end{aligned}$$

In particular, $\alpha_\psi(a)/\alpha_\psi(1)$ is a 4-th root of unity for any $a \in k^\times$. For each quadratic form $Q \simeq \operatorname{diag}(q_1, \dots, q_m)$, we put

$$\alpha_Q(a) = \alpha_{\psi,Q}(a) = \alpha_\psi(q_1 a)\cdots\alpha_\psi(q_m a).$$

Set

$$d_Q = \det Q, \qquad \chi_Q(t) = \langle d_Q, t\rangle.$$

Then we have

$$\frac{\alpha_Q(1)}{\alpha_Q(t)} = \chi_Q(t)\left(\frac{\alpha_\psi(1)}{\alpha_\psi(t)}\right)^m.$$

The metaplectic group $\widetilde{\mathrm{Sp}_n(k)}$ is the unique topological double covering of $\mathrm{Sp}_n(k)$. It is defined by a Rao's 2-cocycle $c(g_1, g_2)$ of $\mathrm{Sp}_n(k)$ with values in $\{\pm 1\}$ (cf. Rao [11]). The group law of $\widetilde{\mathrm{Sp}_n(k)}$ is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, c(g_1, g_2)\zeta_1\zeta_2)$$

for $g_1, g_2 \in \mathrm{Sp}_n(k)$ and $\zeta_1, \zeta_2 \in \{\pm 1\}$. The double covering $\widetilde{\mathrm{Sp}_n(k)} \to \mathrm{Sp}_n(k)$ splits over the subgroup $N_n(k)$ by $\mathbf{n}(B) \mapsto (\mathbf{n}(B), 1)$. If k is a non-archimedean local field whose residual characteristic is not 2, there is a unique splitting

$$\begin{aligned} \mathrm{Sp}_n(\mathfrak{o}) &\to \widetilde{\mathrm{Sp}_n(k)} \\ g &\mapsto (g, s(g)). \end{aligned}$$

We identify $N_n(k)$ and $\mathrm{Sp}_n(\mathfrak{o})$ with the images of the splitting, if there is no fear of confusion.

Let k be a global field. For non-archimedean place v of k, we put $\mathcal{K}_v = \mathrm{Sp}_n(\mathfrak{o})$. Let $\mathfrak{S}$ be a finite set of places of k, which contains all places above 2 and ∞. Put

$$\mathrm{Sp}_n(\mathbb{A})_{\mathfrak{S}} = \prod_{v \in \mathfrak{S}} \mathrm{Sp}_n(k_v) \times \prod_{v \notin \mathfrak{S}} \mathrm{Sp}_n(\mathfrak{o}_v).$$

Then the double covering $\widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}} \to \mathrm{Sp}_n(\mathbb{A})_{\mathfrak{S}}$ is defined by the 2-cocycle $\prod_{v \in \mathfrak{S}} c_v(g_{1,v}, g_{2,v})$. For $\mathfrak{S}_1 \subset \mathfrak{S}_2$, we define the embedding $\widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}_1} \to \widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}_2}$ by

$$((g_v)_v, \zeta) \mapsto ((g_v)_v, \zeta \prod_{\substack{v \in \mathfrak{S}_2 \\ v \notin \mathfrak{S}_1}} s_v(g_v)).$$

Here, $s_v : \mathrm{Sp}_n(\mathfrak{o}_v) \to \{\pm 1\}$ is the map which gives the splitting $\mathrm{Sp}_n(\mathfrak{o}_v) \to \widetilde{\mathrm{Sp}_n(k_v)}$. The global metaplectic group $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ is defined by the inductive limit

$$\widetilde{\mathrm{Sp}_n(\mathbb{A})} = \varinjlim_{\mathfrak{S}} \widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}},$$

where $\mathfrak{S}$ extends over all finite subsets of places of k. It is well-known that the covering $\widetilde{\mathrm{Sp}_n(\mathbb{A})} \to \mathrm{Sp}_n(\mathbb{A})$ splits over $\mathrm{Sp}_n(k)$ uniquely. We identify $\mathrm{Sp}_n(k)$ with the image of the splitting. Note that the image of $\gamma \in \mathrm{Sp}_n(k)$ is given by $(\gamma, 1) \in \widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}}$ for sufficiently large $\mathfrak{S}$.

Any representation of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$ considered in this paper is a restricted tensor product $\pi = \otimes'_v \pi_v$, where π_v is an irreducible admissible representation of $\widetilde{\mathrm{Sp}_n(k_v)}$ for each v. For almost all v, π_v is a class one representation of $\mathrm{Sp}_n(k_v)$ with a distinguished class one vector $\phi_v \in \pi_v$. In other words,

$$\pi = \varinjlim_{\mathfrak{S}} \bigotimes_{v \in \mathfrak{S}} \pi_v.$$

To describe the action of $g \in \widetilde{\mathrm{Sp}_n(\mathbb{A})}$, it is enough to write down the action of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}_{\mathfrak{S}}$ on $\otimes_{v \in \mathfrak{S}} \pi_v$. We write various formulae without explicitly mentioning $\mathfrak{S}$. This convention makes the formulae on the Weil representation simple. Note that in the expression $(g, \zeta) \in \widetilde{\mathrm{Sp}_n(\mathbb{A})}$, $\zeta \in \{\pm 1\}$ depends on the choice of $\mathfrak{S}$.

2. **Fourier-Jacobi modules**

In this section, k is a non-archimedean local field. The symbols $\mathrm{Sp}_n(k)$, $\mathrm{GL}_n(k)$ etc. will be simply denoted by Sp_n, GL_n, etc. in this section. We fix a non-trivial additive character ψ of k. For $\xi \in k^\times$, we put $\chi_\xi(t) = \langle \xi, t \rangle$, where $\langle\ ,\ \rangle$ is the Hilbert symbol.

The set $\mathcal{S}_n$ of symmetric matrices of degree n over k is identified with the set of quadratic forms over k. We denote the set of non-degenerate quadratic forms by $\mathcal{S}_n^{\mathrm{nd}}$.

If H is a subgroup of Sp_n, the inverse image of H by the covering $\widetilde{\mathrm{Sp}_n} \to \mathrm{Sp}_n$ is denoted by $\widetilde{H}$. For $A \in \mathrm{GL}_n$ and $\zeta \in \{\pm 1\}$, we put

$$(\mathbf{m}(A), \zeta) = \left(\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}, \zeta \right) \in \widetilde{M_n}.$$

For $B \in \mathcal{S}_n^{\mathrm{nd}}$, we define a character ψ_B of $\mathcal{S}_n$ by

$$\psi_B(z) = \psi(\mathrm{tr}(Bz)).$$

We also regard ψ_B as a character of N_n by the isomorphism $\mathcal{S}_n \simeq N_n$.

For a smooth representation π of $\widetilde{\mathrm{Sp}_n}$, we put

$$\mathrm{Wh}_B(\pi) = \mathrm{Hom}_{N_n}(\pi, \psi_B) \simeq \mathrm{Hom}_{\widetilde{\mathrm{Sp}_n}}(\pi, \mathrm{Ind}_{N_n}^{\widetilde{\mathrm{Sp}_n}} \psi_B).$$

$\mathrm{Wh}_B(\pi)$ is called the space of degenerate Whittaker vectors of π with respect to ψ_B. For $w_B \in \mathrm{Wh}_B(\pi)$ and $f \in \pi$, the function

$$g \mapsto w_B(\pi(g)f), \qquad g \in \mathrm{Sp}_n$$

is called a degenerate Whittaker function associated to w_B and f. The space of degenerate Whittaker functions is denoted by $\mathcal{W}_B(\pi)$.

We assume $0 < m \le n$ and $n = m + n'$. Let $S \in \mathcal{S}_m^{\mathrm{nd}}$ be a non-degenerate symmetric matrix of size m. Put

$$\mathbf{v}(x, y, z) = \left(\begin{array}{cc|cc} \mathbf{1}_m & x & z & y \\ 0 & \mathbf{1}_{n'} & {}^t y & 0 \\ \hline 0 & 0 & \mathbf{1}_m & 0 \\ 0 & 0 & -{}^t x & \mathbf{1}_{n'} \end{array} \right) \in \mathrm{Sp}_n$$

for $x, y \in \mathrm{Mat}(m, n'; k)$ and $z - x \cdot {}^t y \in \mathcal{S}_m$. We define

$$\begin{aligned} V = V_{n,m} &= \left\{ \mathbf{v}(x, y, z) \,|\, x, y \in \mathrm{Mat}(m, n'; k), z - x \cdot {}^t y \in \mathcal{S}_m \right\}, \\ X = X_{n,m} &= \left\{ \mathbf{v}(x, 0, 0) \,|\, x \in \mathrm{Mat}(m, n'; k) \right\}, \\ Y = Y_{n,m} &= \left\{ \mathbf{v}(0, y, 0) \,|\, y \in \mathrm{Mat}(m, n'; k) \right\}, \\ Z = Z_m &= \left\{ \mathbf{v}(0, 0, z) \,|\, z \in \mathcal{S}_m \right\}. \end{aligned}$$

Note that the quotient group

$$V/\{\mathbf{v}(0,0,z) \in Z \mid \mathrm{tr}(Sz) = 0\}$$

is a Heisenberg group. On the group X, we give the Haar measure $dx = \prod_{i,j} dx_{ij}$, where dx_{ij} is the self-dual Haar measure of the (i,j)-th cordinate of $X \simeq \mathrm{Mat}(m, n'; k)$. We regard $\widetilde{\mathrm{Sp}_{n'}}$ as a subgroup of $\widetilde{\mathrm{Sp}_n}$ by the embedding

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \zeta\right) \mapsto \left(\left(\begin{array}{cc|cc} \mathbf{1}_m & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & \mathbf{1}_m & 0 \\ 0 & C & 0 & D \end{array}\right), \zeta\right).$$

Let ψ_S be the character of Z defined by $\mathbf{v}(0,0,z) \mapsto \psi(\mathrm{tr}(Sz))$. By Stone-von Neumann theorem, there is a unique irreducible admissible representation ω_S of V on which Z acts by ψ_S. The representation ω_S extends to the Weil representation of the group $V \rtimes \widetilde{\mathrm{Sp}_{n'}}$, which we also denote by ω_S. The representation ω_S can be realized on the Schwartz space $\mathcal{S}(X)$. Set $J = J_{n,m} = V \rtimes \mathrm{Sp}_{n'}$ The action of $\tilde{J} = V \rtimes \widetilde{\mathrm{Sp}_{n'}}$ is given by

$$\begin{aligned}
\omega_S(\mathbf{v}(x,y,z))\phi(t) =& \phi(t+x)\psi(\mathrm{tr}(S(z + 2t \cdot {}^t y + x \cdot {}^t y))) \\
\omega_S\left((\mathbf{m}(A), \zeta)\right)\phi(t) =& \zeta^m \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^m \chi_S(\det A) |\det A|^{m/2} \phi(tA) \\
\omega_S\left((\mathbf{n}(z), \zeta)\right)\phi(t) =& \zeta^m \psi_S(tz \cdot {}^t t)\phi(t) \\
\omega_S\left((w_{n'}, \zeta)\right)\phi(t) =& \zeta^m \alpha_S(1)^{-n'} |\det 2S|^{n'/2} \int_X \phi(u) \overline{\psi(\mathrm{tr}(2St \cdot {}^t u))}\, du.
\end{aligned}$$

Here

$$w_{n'} = \begin{pmatrix} 0 & -\mathbf{1}_{n'} \\ \mathbf{1}_{n'} & 0 \end{pmatrix} \in \mathrm{Sp}_{n'}.$$

The Weil representation ω_S is unitary with respect to the inner product

$$(\phi_1, \phi_2) = \int_X \phi_1(t)\overline{\phi_2(t)}\, dt, \qquad \phi_1, \phi_2 \in \mathcal{S}(X).$$

Definition 2.1. For a smooth representation π of $\tilde{J}$, we put

$$\mathrm{FJ}_S(\pi) = (\pi \otimes \overline{\omega_S})_V\,.$$

Here, $(\ )_V$ means the maximal quotient on which V acts in trivial way. We call $\mathrm{FJ}_S(\pi)$ the Fourier-Jacobi module of π with index S.

Note that the functor $\pi \mapsto \mathrm{FJ}_S(\pi)$ is an exact functor from the category of smooth representations of $\widetilde{\mathrm{Sp}_n}$ to the category of smooth representations of $\tilde{J}$, since V is unipotent. The image of $f\otimes\bar{\phi}\in\pi\otimes\overline{\omega_S}$ on $\mathrm{FJ}_S(\pi)$ is denoted by $[f\otimes\bar{\phi}]$.

For a smooth representation π of $\widetilde{\mathrm{Sp}_n}$, we put $\mathrm{FJ}_S(\pi)=\mathrm{FJ}_S((\pi|_{\tilde{J}}))$, where $\pi|_{\tilde{J}}$ is the restriction of π to $\tilde{J}$. The isomorphism class of $\mathrm{FJ}_S(\pi)$ depends only on the equivalence class of $S\in\mathcal{S}_m^{\mathrm{nd}}$. In fact, the map such that

$$[f\otimes\bar{\phi}(x)]\mapsto[\pi(\mathbf{m}(A)\oplus\mathbb{1}_{n'})f\otimes\overline{\phi(Ax)}]$$

gives an isomorphism $\mathrm{FJ}_S(\pi)\simeq\mathrm{FJ}_{S[A]}(\pi)$.

Note that for $m=n$, we have

$$\mathrm{Wh}_B(\pi)=\mathrm{Hom}(\mathrm{FJ}_B(\pi),\mathbb{C})$$

for any $B\in\mathcal{S}_n^{\mathrm{nd}}$.

When $m=1$ and $S=(\xi)$, we write ψ_S, ω_S, and $\mathrm{FJ}_S(\pi)$ for ψ_ξ, ω_ξ, and $\mathrm{FJ}_\xi(\pi)$, respectively.

Lemma 2.2. *Assume that $n=m_1+m_2+n'$, $S_1\in\mathcal{S}_{m_1}^{\mathrm{nd}}$, and $S_2\in\mathcal{S}_{m_2}^{\mathrm{nd}}$. Then there exists a canonical isomorphism*

$$\rho_{S_1,S_2}:\mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi))\longrightarrow\mathrm{FJ}_{S_1\oplus S_2}(\pi).$$

Proof. Put $X_0=\mathrm{Mat}(m_1,m_2;k)$, $X_1=\mathrm{Mat}(m_1,n';k)$, and $X_2=\mathrm{Mat}(m_2,n';k)$. Then $X_{n,m_1}=X_0\oplus X_1$ and $X_{n,m_1+m_2}=X_1\oplus X_2$. Consider the multilinear map

$$\pi\times\overline{\mathcal{S}(X_0)}\times\overline{\mathcal{S}(X_1)}\times\overline{\mathcal{S}(X_2)}\longrightarrow\mathrm{FJ}_{S_1\oplus S_2}(\pi),$$

defined by

$$(f,\bar{\phi}_0,\bar{\phi}_1,\bar{\phi}_2)\mapsto[\pi(\bar{\phi}_0)f\otimes(\bar{\phi}_1\otimes\bar{\phi}_2)],$$

where $f\in\pi$, $\phi_0\in\mathcal{S}(X_0)$, $\phi_1\in\mathcal{S}(X_1)$, and $\phi_2\in\mathcal{S}(X_2)$. Since the induced map

$$\pi\otimes\overline{\mathcal{S}(X_0)}\otimes\overline{\mathcal{S}(X_1)}\otimes\overline{\mathcal{S}(X_2)}\longrightarrow\mathrm{FJ}_{S_1\oplus S_2}(\pi),$$

is V_{n,m_1}-invariant, we have a map

$$\mathrm{FJ}_{S_1}(\pi)\otimes\overline{\mathcal{S}(X_2)}\longrightarrow\mathrm{FJ}_{S_1\oplus S_2}(\pi).$$

One can easily show that this map is V_{n-m_1,m_2}-invariant. Hence we have a canonical map

$$\rho_{S_1,S_2}:\mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi))\longrightarrow\mathrm{FJ}_{S_1\oplus S_2}(\pi).$$

Conversely, consider the map

$$\pi\otimes\overline{\mathcal{S}(X_0)}\otimes\overline{\mathcal{S}(X_1)}\otimes\overline{\mathcal{S}(X_2)}\longrightarrow\mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi))$$

induced by

$$(f,\bar{\phi}_0,\bar{\phi}_1,\bar{\phi}_2)\mapsto[[f\otimes(\bar{\phi}_0\otimes\bar{\phi}_1)]\otimes\bar{\phi}_2].$$

since this map is invariant under the action of X_0, it factors through the map

$$\pi \otimes \overline{\mathcal{S}(X_0)} \otimes \overline{\mathcal{S}(X_1)} \otimes \overline{\mathcal{S}(X_2)} \longrightarrow \pi \otimes \overline{\mathcal{S}(X_1)} \otimes \overline{\mathcal{S}(X_2)}$$

such that

$$f \otimes \bar{\phi}_0 \otimes \bar{\phi}_1 \otimes \bar{\phi}_2 \mapsto \pi(\bar{\phi}_0) f \otimes \bar{\phi}_1 \otimes \bar{\phi}_2.$$

The induced map

$$\pi \otimes \overline{\mathcal{S}(X_1)} \otimes \overline{\mathcal{S}(X_2)} \longrightarrow \mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi))$$

is V_{n,m_1+m_2}-invariant. Therefore we have a map

$$\rho'_{S_1,S_2} : \mathrm{FJ}_{S_1 \oplus S_2}(\pi) \longrightarrow \mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi)).$$

Clearly ρ'_{S_1,S_2} is the inverse map of ρ_{S_1,S_2}. Hence the lemma. $\square$

Proposition 2.3. *Let π be a smooth representation of $\widetilde{\mathrm{Sp}_n}$. Assume $S \in \mathcal{S}_m^{\mathrm{nd}}$, $B' \in \mathcal{S}_{n'}^{\mathrm{nd}}$, $B = S \oplus B'$ and $w_B \in \mathrm{Wh}_B(\pi)$. Then*

(i) *The bilinear map $\pi \times \overline{\mathcal{S}(X)} \to \mathbb{C}$ defined by*

$$(f, \bar{\phi}) \mapsto w_B(\pi(\bar{\phi}) f).$$

is V-invariant.

(ii) *Let $\mathcal{FJ}_{S,B'}(w_B) : \mathrm{FJ}_S(\pi) \to \mathbb{C}$ be the map induced from the bilinear form given in (i). Then we have $\mathcal{FJ}_{S,B'}(w_B) \in \mathrm{Wh}_{B'}(\mathrm{FJ}_S(\pi))$.*

Proof. We prove (i). Note that

$$w_B(\pi(\bar{\phi}) f) = \int_{x \in X} \overline{\phi(x)} w_B(\pi(\mathbf{v}(x,0,0) f))\, dx.$$

It is easy to show that this map is $X \oplus Z$-invariant. It is enough to show the map is Y-invariant. Since

$$\mathbf{v}(x,0,0) \cdot \mathbf{v}(0,y,0) = \mathbf{v}(0, y, x \cdot {}^t y + y \cdot {}^t x) \cdot \mathbf{v}(x,0,0),$$

we have

$$\begin{aligned} &w_B(\pi(\omega_S(\mathbf{v}(0,y,0)\bar{\phi})\pi(\mathbf{v}(0,y,0) f) \\ &= \int_{x \in X} \overline{\phi(x)\psi(\mathrm{tr}(2Sy \cdot {}^t x))} w_B(\pi(\mathbf{v}(x,0,0)\mathbf{v}(0,y,0) f))\, dx \\ &= \int_{x \in X} \overline{\phi(x)\psi(\mathrm{tr}(2Sy \cdot {}^t x))} w_B(\pi(\mathbf{v}(0,y,x \cdot {}^t y + y \cdot {}^t x) \cdot \mathbf{v}(x,0,0) f))\, dx \end{aligned}$$

Since $\mathbf{v}(0, y, x \cdot {}^t y + y \cdot {}^t x) \in N_n$, we have

$$w_B(\pi(\mathbf{v}(0,y,x \cdot {}^t y + y \cdot {}^t x) \cdot \mathbf{v}(x,0,0) f)) = \psi(\mathrm{tr}(2Sy \cdot {}^t x)) w_B(\pi(\mathbf{v}(x,0,0) f)).$$

Hence the desired V-invariance follows. Now we prove (ii). Note that

$$\omega_S(\mathbf{n}(z) g')\phi(x) = \psi_S(xz \cdot {}^t x)\omega_S(g')\phi(x)$$

for $z \in \mathcal{S}_{n'} \subset \mathrm{Sp}_{n'}$. On the other hand, we have

$$\begin{aligned}\mathbf{v}(x,0,0)\mathbf{n}\begin{pmatrix}0&0\\0&z\end{pmatrix} =&\mathbf{m}\begin{pmatrix}1&x\\0&1\end{pmatrix}\mathbf{n}\begin{pmatrix}0&0\\0&z\end{pmatrix}\\ =&\mathbf{n}\begin{pmatrix}xz\cdot{}^t x & xz\\ z\cdot{}^t x & z\end{pmatrix}\mathbf{v}(x,0,0)\end{aligned}$$

It follows that

$$w_B\left(\pi\left(\mathbf{v}(x,0,0)\mathbf{n}\begin{pmatrix}0&0\\0&z\end{pmatrix}\right)f\right) = \psi_S(xz\cdot{}^t x)\psi_{B'}(z)w_B(\pi(\mathbf{v}(x,0,0))f).$$

Hence we have (ii). □

By Proposition 2.3, there exists a map

$$\mathcal{FJ}_{S,B'} : \mathrm{Wh}_B(\pi) \to \mathrm{Wh}_{B'}(\mathrm{FJ}_S(\pi))$$

such that

$$\mathcal{FJ}_{S,B'}(w_B)([f\otimes\bar{\phi}]) = w_B(\pi(\bar{\phi})f).$$

If B' is clear from the context, $\mathcal{FJ}_{S,B'}$ is simply denoted by $\mathcal{FJ}_S$. The following proposition is a restatement of Proposition 2.3 in terms of Whittaker functions.

Proposition 2.4. *Let $B = S \oplus B'$ be as in Proposition 2.3. If W_B is the Whittaker function associated to $f \in \pi$ and $w_B \in \mathrm{Wh}_B(\pi)$, then the Whittaker function associated to $[f\otimes\bar{\phi}] \in \mathrm{FJ}_S(\pi)$ and $\mathcal{FJ}_S(w_B)$ is given by*

$$\int_{x\in X} W_B(\mathbf{v}(x,0,0)g')\overline{\omega_S(g')\phi(x)}\,dx, \qquad g' \in \mathrm{Sp}_{n'}.$$

Lemma 2.5. *Suppose that $S_1 \in \mathcal{S}^{\mathrm{nd}}_{m_1}(k)$, $S_2 \in \mathcal{S}^{\mathrm{nd}}_{m_2}(k)$, $B = S_1 \oplus S_2 \oplus B' \in \mathcal{S}^{\mathrm{nd}}_n(k)$, and $w_B \in \mathrm{Wh}_B(\pi)$. Let $\rho_{S_1,S_2} : \mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\pi)) \to \mathrm{FJ}_{S_1\oplus S_2}(\pi)$ be the canonical isomorphism defined in Lemma 2.2. Then we have*

$$\mathcal{FJ}_{S_2,B'}(\mathcal{FJ}_{S_1,S_2\oplus B'}(w_B)) = \mathcal{FJ}_{S_1\oplus S_2,B'}(w_B)\circ\rho_{S_1,S_2}.$$

Proof. Let f, $\phi_0 \in \mathcal{S}(X_0)$, $\phi_1 \in \mathcal{S}(X_1)$, and $\phi_2 \in \mathcal{S}(X_2)$ as in the proof of Lemma 2.2. We calculate the Whittaker function associated to $\mathcal{FJ}_{S_2}(\mathcal{FJ}_{S_1}(w_B))$ and $[[f\otimes\overline{\phi_0\otimes\phi_1}]\otimes\bar{\phi}_2]$. For $g' \in \mathrm{Sp}_{n'}$, the Whittaker function is equal to

$$\begin{aligned}
&\int_{x_0\in X_0}\int_{x_1\in X_1}\int_{x_2\in X_2} W_B\left(\mathbf{m}\left(\begin{smallmatrix}1&x_0&x_1\\0&1&0\\0&0&1\end{smallmatrix}\right)\mathbf{m}\left(\begin{smallmatrix}1&0&0\\0&1&x_2\\0&0&1\end{smallmatrix}\right)g'\right)\\
&\qquad\times\overline{\omega_{S_1}(\mathbf{m}\left(\begin{smallmatrix}1&x_2\\0&1\end{smallmatrix}\right)g')(\phi_0\otimes\phi_1)(x_0,x_1)\omega_{S_2}(g')\phi_2(x_2)}\,dx_0\,dx_1\,dx_2\\
=&\int_{x_0\in X_0}\int_{x_1\in X_1}\int_{x_2\in X_2} W_B\left(\mathbf{m}\left(\begin{smallmatrix}1&x_0&x_1+x_0x_2\\0&1&x_2\\0&0&1\end{smallmatrix}\right)g'\right)\\
&\qquad\times\overline{\omega_{S_1}(g')(\phi_0\otimes\phi_1)(x_0,x_1+x_0x_2)\omega_{S_2}(g')\phi_2(x_2)}\,dx_0\,dx_1\,dx_2\\
=&\int_{x_0\in X_0}\int_{x_1\in X_1}\int_{x_2\in X_2} W_B\left(\mathbf{m}\left(\begin{smallmatrix}1&x_0&x_1\\0&1&x_2\\0&0&1\end{smallmatrix}\right)g'\right)\\
&\qquad\times\overline{\omega_{S_1}(g')(\phi_0\otimes\phi_1)(x_0,x_1)\omega_{S_2}(g')\phi_2(x_2)}\,dx_0\,dx_1\,dx_2\\
=&\int_{x_0\in X_0}\int_{x_1\in X_1}\int_{x_2\in X_2} W_B\left(\mathbf{m}\left(\begin{smallmatrix}1&0&x_1\\0&1&x_2\\0&0&1\end{smallmatrix}\right)g'\mathbf{m}\left(\begin{smallmatrix}1&x_0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)\right)\overline{\phi_0(x_0)}\\
&\qquad\times\overline{\omega_{S_1\oplus S_2}(g')(\phi_1\otimes\phi_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}}\,dx_0\,dx_1\,dx_2,
\end{aligned}$$

which is the Whittaker function associated to $\mathcal{FJ}_{S_1\oplus S_2}(w_B)$ and $[\pi(\bar{\phi}_0)f\otimes(\overline{\phi_1\otimes\phi_2})]$. Hence the lemma. $\square$

Lemma 2.6. *The homomorphism $\mathcal{FJ}_S:\mathrm{Wh}_{S\oplus B'}(\pi)\to\mathrm{Wh}_{B'}(\mathrm{FJ}_S(\pi))$ is an isomorphism.*

Proof. By definition, $\mathrm{Wh}_B(\pi)=\mathrm{Hom}(\mathrm{FJ}_B(\pi),\mathbb{C})$. It follows that

$$\mathrm{Wh}_{B'}(\mathrm{FJ}_S(\pi))=\mathrm{Hom}(\mathrm{FJ}_{B'}(\mathrm{FJ}_S(\pi)),\mathbb{C})\simeq\mathrm{Hom}(\mathrm{FJ}_B(\pi),\mathbb{C})=\mathrm{Wh}_B(\pi).$$

$\square$

3. Degenerate principal series

Let $\mu:k^\times\to\mathbb{C}^\times$ be a (quasi-) character, and $\tau=\mathcal{B}(\mu,\mu^{-1})$ be a principal series of PGL_2. We assume τ is unitary. Note that $q^{-1/2}<|\mu(\varpi)|<q^{1/2}$.

For each integer n, define a character $\mu^{(n)}$ of $\widetilde{M_n}$ by

$$\mu^{(n)}((\mathbf{m}(A),\zeta))=\zeta^m\left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^n\mu(\det A).$$

We sometimes regard $\mu^{(n)}$ as a character of $\widetilde{P_n}$. We define an irreducible admissible representation $\Pi_n=\Pi(n,\tau)$ of $G_n=\widetilde{\mathrm{Sp}_n}$ by

$$\Pi(n,\tau)=\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}}(\mu^{(n)}).$$

The representation $\Pi(n,\tau)$ is simply denoted by Π_n, if there is no fear of confusion.

Assume $n = m + n'$ and $S \in \mathcal{S}_m^{\mathrm{nd}}$. Put $J = V \cdot \mathrm{Sp}_{n'}$ and

$$\eta_0 = \left(\begin{array}{cc|cc} 0 & 0 & -\mathbf{1}_m & 0 \\ 0 & \mathbf{1}_{n'} & 0 & 0 \\ \hline \mathbf{1}_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n'} \end{array}\right).$$

Then $P_n \eta_0 J$ is an open subset of Sp_n. Let $\mathcal{X}_0$ be the subspace of $\Pi(n,\tau)$ which consists of all element $f \in \Pi(n,\tau)$ such that $\mathrm{Supp}(f) \subset \tilde{P}_n \eta_0 \tilde{J}$. For $f \in \mathcal{X}_0$, $\phi \in \mathcal{S}(X)$, and $g' \in \widetilde{\mathrm{Sp}}_{n'}$, we define an integral

$$\mathcal{R}(g'; f, \phi) = \int_X \int_Z f(\eta_0 \mathbf{v}(x,0,z) g') \overline{\omega_S(g')\phi(x)\psi_S(z)}\, dz\, dx.$$

It is proved in [5] that $\mathcal{R}(g'; f, \phi) \in \Pi(n', \tau \otimes \chi_S)$.

Lemma 3.1. *The map*

$$\mathcal{X}_0 \otimes \overline{\omega_S} \longrightarrow \Pi(n', \tau \otimes \chi_S)$$
$$f \otimes \bar{\phi} \mapsto \mathcal{R}(*; f, \phi)$$

can be extended to a V-invariant surjective map $\Pi(n,\tau) \otimes \overline{\omega_S} \to \Pi(n', \tau \otimes \chi_S)$.

Proof. We regard $\widetilde{\mathrm{Sp}}_m$ as a subgroup of $\widetilde{\mathrm{Sp}}_n$ by the embedding

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \zeta \right) \mapsto \left(\left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & \mathbf{1}_m & 0 & 0 \\ \hline C & 0 & D & 0 \\ 0 & 0 & 0 & \mathbf{1}_m \end{array}\right), \zeta \right).$$

Then the pullback of $f \in \Pi(n,\tau)$ can be considered as an element of the induced representation

$$\mathrm{Ind}_{\widetilde{P_m}}^{\widetilde{\mathrm{Sp}_m}} \mu^{(n)} |\det|^{n'/2}.$$

The integral

$$\int_Z f(\eta_0 \mathbf{v}(0,0,z) g') \overline{\psi_S(z)}\, dz$$

is can be considered as a Whittaker integral for the element of

$$\mathrm{Ind}_{\widetilde{P_m}}^{\widetilde{\mathrm{Sp}_m}} \mu^{(n)} |\det|^{n'/2}.$$

It is well-known that the Whittaker integral is absolutely convergent for the induced representation

$$\mathrm{Ind}_{\widetilde{P_m}}^{\widetilde{\mathrm{Sp}_m}} \mu^{(n)} |\det|^{n'/2}.$$

for $\mathrm{Re}(s) \gg 0$ and can be analytically continued to whole $s \in \mathbb{C}$. Therefore the integral

$$\mathcal{R}(g'; f, \phi) = \int_X \left[\int_Z f(\eta_0 \mathbf{v}(x, 0, z) g') \overline{\psi_S(z)}\, dz \right] \overline{\omega_S(g')\phi(x)}\, dx$$

is well-defined. It is easy to see that the extended integral $\mathcal{R}(*; f, \phi)$ belongs to $\Pi(n', \tau \otimes \chi_S)$, whenever the integral is well-defined. □

Proposition 3.2. *Assume that $n = m + n'$ and $S \in \mathcal{S}_m^{\mathrm{nd}}$. Then there exists an isomorphism*

$$\mathrm{FJ}_S(\Pi(n, \tau)) \simeq \Pi(n', \tau \otimes \chi_S)$$

In particular, we have an isomorphism

$$\mathrm{FJ}_\xi(\Pi(n, \tau)) \simeq \Pi(n - 1, \tau \otimes \chi_\xi)$$

for $\xi \in k^\times$.

Proof. It is enough to consider the case $m = 1$ and $S = (\xi)$. Put $J = V \cdot \mathrm{Sp}_{n-1}$. The double coset $P_n \backslash \mathrm{Sp}_n / J$ has a complete set of representatives $\{\mathbf{1}_{2n}, \eta_0\}$, where

$$\eta_0 = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & \mathbf{1}_{n-1} & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-1} \end{array} \right).$$

Put

$$\begin{aligned} \mathcal{X}_0 &= \left\{ f \in \Pi(n, \tau) \,\middle|\, \mathrm{Supp}(f) \subset \widetilde{P_n} \eta_0 \tilde{J} \right\}, \\ \mathcal{X}_1 &= \Pi(n, \tau). \end{aligned}$$

Then we have

$$\mathcal{X}_1 / \mathcal{X}_0 \simeq \mathrm{Ind}_{\tilde{J} \cap \widetilde{P_n}}^{\tilde{J}} (\mu^{(n)})$$

as $\tilde{J}$-module. Since Z acts on $\mathcal{X}_1/\mathcal{X}_0$ trivially, we have $((\mathcal{X}_1/\mathcal{X}_0) \otimes \overline{\omega_\xi})_V = (0)$. By the exactness of the functor FJ_W, it is enough to prove $(\mathcal{X}_0 \otimes \overline{\omega_\xi})_V = \Pi(n-1, \tau \otimes \chi_\xi)$. Note that $f \in \mathcal{X}_0$ is determined by the restriction of f to $\tilde{J}$. For $f \in \mathcal{X}_0$, we put

$$f_{\psi_\xi}(vg') = \int_Z f(zvg') \psi_\xi(z)^{-1}\, dz \qquad (v \in V, g' \in \widetilde{\mathrm{Sp}_n}).$$

Then for each $g' \in \widetilde{\mathrm{Sp}_n}$, the function $v \mapsto f_{\psi_\xi}(vg')$ belongs to $\mathrm{Ind}_{YZ}^V \psi$. Since $f_{\psi_\xi}(vg')$ is right invariant by some open compact subgroup of $\tilde{J}$,

$$\int_{Z \backslash V} f_{\psi_\xi}(ug'v) \varphi(v^{-1})\, dv = f_{\psi_\xi}(ug')$$

for some $\varphi \in C(V;\psi)$. For $\phi_1, \phi_2 \in \mathcal{S}(X)$, set

$$\Phi_{\phi_1,\phi_2}(v) = (\omega_\xi(v)\phi_1, \phi_2).$$

Then for $u \in V$, we have

$$\begin{aligned}
&\int_{Z\backslash V} f_{\psi_\xi}(ug'v)\Phi_{\phi_1,\phi_2}(v^{-1})\,dv \\
=&\int_{Z\backslash V} f_{\psi_\xi}(vg')\Phi_{\phi_1,\phi_2}({g'}^{-1}v^{-1}ug')\,dv \\
=&\int_{Z\backslash V} f_{\psi_\xi}(vg')\cdot(\omega_\xi({g'}^{-1}v^{-1}ug')\phi_1, \phi_2)\,dv \\
=&\int_{Z\backslash V} f_{\psi_\xi}(vg')\cdot(\omega_\xi(v^{-1}ug')\phi_1, \omega_\xi(g')\phi_2)\,dv \\
=&\int_{Z\backslash V} f_{\psi_\xi}(vg')\Phi_{\omega_\xi(ug')\phi_1,\omega_\xi(g')\phi_2}(v^{-1})\,dv \\
=&\int_X f_{\psi_\xi}(\mathbf{v}(x,0,0)g')\overline{\omega_\xi(g')\phi_2(x)}\,dx\cdot\omega_\xi(ug')\phi_1(0) \\
=&\mathcal{R}(g';f,\phi)\cdot\omega_\xi(ug')\phi_1(0).
\end{aligned}$$

By Lemma 3.1, we have $\mathcal{R}(*;f,\phi) \in \Pi(n-1,\tau\otimes\chi_\xi)$. Therefore, as a representation of $\tilde{J}$, we have

$$(\mathcal{X}_0)_{\psi_\xi} \subset \Pi(n-1,\tau\otimes\chi_\xi)\otimes\omega_\xi.$$

It is easy to see $\mathcal{R}(*;f,\phi)\not\equiv 0$ for some $f \in \mathcal{X}_0$, $\phi \in \mathcal{S}(X)$. It follows that $\mathrm{FJ}_\xi(\Pi(n,\tau))$ is a non-zero subspace of $\Pi(n-1,\tau\otimes\chi_\xi)$. Since $\Pi(n-1,\tau\otimes\chi_\xi)$ is irreducible, we have $\mathrm{FJ}_\xi(\Pi(n,\tau)) \simeq \Pi(n-1,\tau\otimes\chi_\xi)$. Hence the lemma. □

Proposition 3.3. *For each $B \in \mathcal{S}_n^{\mathrm{nd}}$, we have*

$$\dim_{\mathbb{C}} \mathrm{Wh}_B(\Pi_n) = 1.$$

Proof. We apply Proposition 3.2 to the case $n = m$. Then we have $\mathrm{Wh}_B(\Pi_n) = \mathrm{Hom}(\mathrm{FJ}_B(\Pi_n), \mathbb{C}) \simeq \mathbb{C}$. □

Proposition 3.3 was proved by Karel [7] for degenerate principal series of Sp_n. It is also easy to generalise the proof of [7] for metaplectic groups.

We fix an isomorphism $\mathrm{FJ}_S(\Pi(n,\tau)) \simeq \Pi(n',\tau\otimes\chi_S)$. The we get a V-invariant surjective map

$$\beta_S : \Pi(n,\tau)\otimes\overline{\mathcal{S}(X)} \to \Pi(n',\tau\otimes\chi_S).$$

If $B = S \oplus B'$, $B' \in \mathcal{S}_{n'}^{\mathrm{nd}}$, and $w_B \in \mathrm{Wh}_B(\Pi(n,\tau))$, then one can regard $\mathcal{FJ}_S(w_B)$ as a Whittaker vector for $\Pi(n', \tau \otimes \chi_S)$. Then we have

$$\mathcal{FJ}_S(w_B)(\beta_S(f \otimes \bar{\phi})) = w_B(\Pi_n(\bar{\phi})f).$$

for $f \in \Pi_n = \Pi(n,\tau)$ and $\phi \in \mathcal{S}(X)$. If $S = S_1 \oplus S_2$, the surjective map $\rho_{S_1,S_2} : \mathrm{FJ}_{S_2}(\mathrm{FJ}_{S_1}(\Pi_n)) \to \mathrm{FJ}_S(\Pi_n)$ can be considered as an automorphism of $\Pi(n', \tau \otimes \chi_S)$. Since $\Pi(n', \tau)$ is irreducible, ρ_{S_1,S_2} is a non-zero scalar. Therefore we have

$$\mathcal{FJ}_{S_2,B'} \circ \mathcal{FJ}_{S_1,S_2 \oplus B'} = \rho_{S_1,S_2} \cdot \mathcal{FJ}_{S,B'},$$

where $\rho_{S_1,S_2} \in \mathbb{C}^\times$ is a constant which does not depend on $B' \in \mathcal{S}_{n'}^{\mathrm{nd}}$.

Lemma 3.4. *Suppose that $S \in \mathcal{S}_m^{\mathrm{nd}}(k)$, $B = S \oplus B' \in \mathcal{S}_n^{\mathrm{nd}}(k)$, and $w_B \in \mathrm{Wh}_B(\Pi_n)$. Set $\Pi_n = \Pi(n,\tau)$ and $\Pi_{n'} = \Pi(n', \tau \otimes \chi_S)$. Then we have*

$$\begin{aligned}
&\mathcal{FJ}_S(w_B \circ \Pi_n((\mathbf{1}_m \oplus \mathbf{m}(A), \zeta))) \\
&\quad = \zeta^m \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^m \chi_S(\det A) |\det A|^{m/2} \mathcal{FJ}_S(w_B) \circ \Pi_{n'}((\mathbf{m}(A), \zeta)).
\end{aligned}$$

Proof. Note that $w_B \circ \Pi_n((\mathbf{m}(\mathbf{1}_m \oplus A), \zeta)) \in \mathrm{Wh}_{B[1_m \oplus A]}(\Pi_n) = \mathrm{Wh}_{S \oplus B'[A]}(\Pi_n)$. Let $W_B(g) = W_B(g; w_B, f)$ be the Whittaker function associated to w_B and $f \in \Pi_n$. Then the Whittaker function associated to $\mathcal{FJ}_S(w_B \circ \Pi_n((\mathbf{m}(A), \zeta)))$ and $\beta_S(f \otimes \bar{\phi})$ is equal to

$$\begin{aligned}
&\int_{x \in X} W_B((\mathbf{m}(\mathbf{1}_m \oplus A), \zeta)\mathbf{v}(x,0,0)g') \overline{\omega_S(g')\phi(x)}\, dx \\
=&\int_{x \in X} W_B(\mathbf{v}(xA^{-1},0,0) \cdot (\mathbf{m}(A), \zeta)g') \overline{\omega_S(g')\phi(x)}\, dx \\
=&|\det A|^m \int_{x \in X} W_B(\mathbf{v}(x,0,0) \cdot (\mathbf{m}(A), \zeta)g') \overline{\omega_S(g')\phi(xA)}\, dx \\
=&\zeta^m \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^m \chi_S(\det A) |\det A|^{m/2} \\
&\quad \times \int_{x \in X} W_B(\mathbf{v}(x,0,0) \cdot (\mathbf{m}(A), \zeta)g') \overline{\omega_S((\mathbf{m}(A), \zeta)g')\phi(x)}\, dx.
\end{aligned}$$

Hence the lemma. $\square$

Lemma 3.5. *Assume that $A \in \mathrm{GL}_n$, $B \in \mathcal{S}_n^{\mathrm{nd}}$ and $B[A] = B$. Then we have*

$$w_B \circ \Pi_n((\mathbf{m}(A), \zeta)) = \mu^{(n)}((\mathbf{m}(A), \zeta)) \cdot w_B.$$

for any $w_B \in \mathrm{Wh}_B(\Pi_n)$.

Proof. By Proposition 3.3, $w_B \circ \Pi_n((\mathbf{m}(A),\zeta)) = \alpha \cdot w_B$ for some α. Let $P_n w P_n \subset \mathrm{Sp}_n$ be the unique open coset. If $\mathrm{Supp}(f) \subset \widetilde{P_n} w \widetilde{P_n}$, we may assume

$$w_B(f) = \int_{N_n} f(w \cdot \mathbf{n}(x)) \overline{\psi_B(x)}\, dx.$$

Then we have $\alpha = \mu^{(n)}((\mathbf{m}(A),\zeta))$ by change of variables. □

Proposition 3.6. *Let $\mathcal{C} \subset \widetilde{\mathrm{Sp}_n}$ be a compact set. Then there exists $W \in \mathcal{W}_B(\Pi_n)$ such that $W(g) \neq 0$ for any $g \in \mathcal{C}$.*

Proof. Since $\mathrm{Wh}_B(\Pi_n) \neq (0)$, there exists a Whittaker function W_0 such that $W_0(\mathbf{1}) \neq 0$. Let U be an open subgroup of $\widetilde{\mathrm{Sp}_n}$ such that W_0 is invariant under right translation by gUg^{-1} for any $g \in \mathcal{C}$. Choose a finite subset $g_1, \dots, g_m$ such that $\mathcal{C} \subset \bigcup_{i=1}^m g_i U$. Put $W_i(g) = W_0(g g_i^{-1})$ for $i = 1, \dots m$. Then W_i is invariant under right translation by U and $W_i(g) \neq 0$ for $g \in g_i U$. Put $X = \langle W_i \,|\, i = 1, \dots m\rangle$. Then X is a finite dimensional vector space over $\mathbb{C}$ and $\{W \in X \mid W|_{g_i U} \neq 0\} \subsetneq X$. Hence the proposition. □

Put $\nu(z) = [z\mathfrak{o}^n + \mathfrak{o}^n : \mathfrak{o}^n]$ for $z \in \mathcal{S}_n$. If z has elementary divisors z_1, ..., z_n, then $\nu(z) = \prod_{i=1}^n \max(1, |z_i|)$. The following lemma is obvious from the definition of $\nu(z)$.

Lemma 3.7. *For each open subgroup U of $\mathcal{S}_n$, there exists a positive constant L such that $\psi_B|_U \not\equiv 1$ for any $B \in \mathcal{S}_n^{\mathrm{nd}}$ with $\nu(B) > L$.*

For each $g \in \widetilde{\mathrm{Sp}_n}$ with Iwasawa decomposition

$$g = (\mathbf{m}(A),\zeta)\mathbf{n}(z)u, \qquad u \in \widetilde{\mathrm{Sp}_n(\mathfrak{o})},$$

we put $H_B(g) = \nu(B[A])$.

Proposition 3.8. *Let $W \in \mathcal{W}_B(\Pi_n)$ be a Whittaker function. Then there exists a constant $L > 0$ such that $W(g) = 0$ for any $g \in \widetilde{\mathrm{Sp}_n}$ such that $H_B(g) > L$.*

Proof. Since W is right-finite by the action of the maximal compact subgroup of $\widetilde{\mathrm{Sp}_n}$, it is enough to prove the case $g \in \widetilde{M_n}$.

Note that W is invariant under the right translation by some open compact subgroup of N_n. On the other hand, we have

$$\begin{aligned} W((\mathbf{m}(A),\zeta)\mathbf{n}(z)) =& \psi_B(Az \cdot {}^t A) W((\mathbf{m}(A),\zeta)) \\ =& \psi_{B[A]}(z) W((\mathbf{m}(A),\zeta)). \end{aligned}$$

The Proposition follows from Lemma 3.7. □

Proposition 3.9. *Let $W \in \mathcal{W}_B(\Pi_n)$ be a Whittaker function. Then there exists a constant $C > 0$ such that*

$$|W(g)| < C|\det A|^{-n-1-\varepsilon}.$$

for any $g \in \widetilde{\mathrm{Sp}_n}$ with Iwasawa decomposition

$$g = (\mathbf{m}(A), \zeta)\mathbf{n}(z)u, \qquad u \in \widetilde{\mathrm{Sp}_n(\mathfrak{o})}.$$

Proof. As in the last proposition, it is enough to prove the estimate

$$|W((\mathbf{m}(A), \zeta))| < C|\det A|^{-n-1-\varepsilon}.$$

We may assume $\mu(x) = \mu_0(x)|x|^{s_0}$ for some unitary character μ_0 and $-1/2 < s_0 < 1/2$. Let $HS(\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^s)$ be the space of functions $f^{(s)}(g)$ on $\widetilde{\mathrm{Sp}_n} \times \mathbb{C}$ which satisfy the following conditions (1), (2), and (3).

(1) For each $s \in \mathbb{C}$, $f^{(s)} \in \mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^s$.
(2) For each $g \in \widetilde{\mathrm{Sp}_n}$, the function $s \mapsto f^{(s)}(g)$ belongs to $\mathbb{C}[q^s, q^{-s}]$.
(3) There exists an open compact subgroup $U \subset \widetilde{\mathrm{Sp}_n}$ such that $f^{(s)}(gu) = f^{(s)}(g)$ for any $s \in \mathbb{C}$, $g \in \widetilde{\mathrm{Sp}_n}$, $u \in U$.

For $f^{(s)}(g) \in HS(\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^s)$, put

$$w_B(f^{(s)}) = \int_{N_n} f^{(s)}\left(w_n \mathbf{n}(z)\right) \overline{\psi_B(z)}\, dz.$$

Then $w_B(f^{(s)})$ is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$, and can be analytically continued to whole s-plane. In particular, the map $f^{(s_0)} \mapsto w_B(f^{(s_0)})$ gives an element of $\mathrm{Wh}(\Pi_n)$. For $\mathrm{Re}(s) > (n+1)/2$, we have

$$\begin{aligned} |w_B(f^{(s)}((\mathbf{m}(A), \zeta)))| &\leq \int_{N_n} |\, f^{(s)}\left(w_n \mathbf{n}(z)(\mathbf{m}(A), \zeta)\right) |\, dz \\ &= \int_{N_n} |f^{(s)}\left((\mathbf{m}({}^tA^{-1}), \zeta) w_n \mathbf{n}(Az \cdot {}^tA^{-1})\right) |\, dz \\ &= |\det A|^{-s-((n+1)/2)} \int_{N_n} |f^{(s)}\left(w_n \mathbf{n}(z)\right) |\, dz. \end{aligned}$$

Since the integral

$$\int_{N_n} |f^{(s)}\left(w_n \mathbf{n}(z)\right) |\, dz$$

is bounded on the vertical line $\mathrm{Re}(s) = ((n+1)/2) + \varepsilon$, there exists a constant $C > 0$ such that

$$|w_B(f^{(s)}((\mathbf{m}(A), \zeta)))| < C|\det A|^{-n-1-\varepsilon}$$

for $\mathrm{Re}(s) = ((n+1)/2) + \varepsilon$.

Put

$$M_{w_n}(f^{(s)})(g) = \int_{N_n} f^{(s)}\left(w_n \mathbf{n}(z)\right)\, dz.$$

Then $M_{w_n}(f^{(s)})$ is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$ and $M_{w_n}(f^{(s)}) \in \mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \nu_0^{(n)} |\det|^{-s}$ for some character ν_0. It is well-known that there exists a polynomial $\gamma(s)$ whose zeros lie on the points of reducibility of $\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^{-s}$ such that

$$\gamma(s) M_{w_n}(f^{(-s)}) \in HS(\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \nu_0^{(n)} |\det|^{s}).$$

Note that the points of reducibility of $\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^{s}$ lie on the vertical strip $1/2 \leq |\mathrm{Re}(s)| \leq (n+1)/2$. It is also well-known that there exists a function $\delta(s) \in \mathbb{C}(q^s)$ whose poles lie on the points of reducibility of $\mathrm{Ind}_{\widetilde{P_n}}^{\widetilde{\mathrm{Sp}_n}} \mu_0^{(n)} |\det|^{-s}$ such that

$$w_B(M_{w_n}(f^{(s)})) = \delta(s) w_B(f^{(s)}).$$

It follows that there exists a constant $C' > 0$ such that

$$|w_B(f^{(s)}((\mathbf{m}(A), \zeta)))| < C' |\det A|^{-n-1-\varepsilon}$$

for $\mathrm{Re}(s) = -((n+1)/2) - \varepsilon$. Hence the proposition.

$\square$

4. **Siegel series and its functional equation**

As before, let k be a non-archimedean local field. We assume that the additive character ψ is of order 0.

For $B \in \mathcal{S}_n^{\mathrm{nd}}(k)$, put

$$\begin{aligned} D_B =& (-4)^{[n/2]} \det(B), \\ \xi_B =& \begin{cases} \langle D_B, \varpi \rangle & \text{if } k(\sqrt{\chi_{D_B}})/k \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\mathfrak{d}_B$ be the conductor of the extension $k(\sqrt{D_B})/k$. We set

$$\delta_B = (\mathrm{ord} D_B - \mathrm{ord} \mathfrak{d}_B)/2,$$

where ord is the valuation of k.

We recall the theory of Siegel series (cf. Shimura [15], [16]). For $B \in \mathcal{S}_n^{\mathrm{nd}}(k)$, we define a polynomial $\gamma(B, X) \in \mathbb{Z}[X]$ by

$$\gamma(B, X) = \begin{cases} (1-X)(1-q^{n/2}\xi_B X)^{-1} \prod_{i=1}^{n/2} (1-q^{2i}X^2) & \text{if } n \text{ is even,} \\ (1-X) \prod_{i=1}^{(n-1)/2} (1-q^{2i}X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Let $f_0^{(s)}$ be the function on $\mathrm{Sp}_n(k)$ defined by

$$f_0^{(s)}(g) = |\det A|^{s+((n+1)/2)},$$

for

$$g = \mathbf{m}(A)\mathbf{n}(z)u, \qquad A \in \mathrm{GL}_n(k), z \in \mathcal{S}_n, u \in \mathrm{Sp}_n(\mathfrak{o}).$$

Then $f_0^{(s)}$ is a class one vector for $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n} |\det|^s$. Set

$$w_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

Consider the integral

$$b(B,s) = \int_{N_n(k)} f_0^{(s-((n+1)/2))}(w_n \mathbf{n}(z)) \overline{\psi_B(z)}\, dz.$$

This integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$. Moreover, there exists a polynomial $F(B,X) \in \mathbb{Z}[X]$ such that

$$b(B,s) = \gamma(B, q^{-s}) F(B; q^{-s}).$$

For a proof of this fact, see [16]. Let $\mathcal{S}_n(\mathfrak{o})^\sharp$ be the dual lattice of $\mathcal{S}_n(\mathfrak{o})$. It is known that $F(B,X) = 0$ unless $B \in \mathcal{S}_n(\mathfrak{o})^\sharp$. Moreover, if $B \in \mathcal{S}_n(\mathfrak{o})^\sharp$, then $F(B,0) = 1$.

Proposition 4.1. *The following functional equations hold.*

(1) *If n is even, then*

$$F(B, q^{-n-1}X^{-1}) = (q^{(n+1)/2}X)^{-2\delta_B} F(B,X).$$

(2) *If n is odd, then*

$$F(B, q^{-n-1}X^{-1}) = \zeta_B (q^{(n+1)/2}X)^{-\mathrm{ord}(D_B)} F(B,X).$$

Here, the sign $\zeta_B \in \{\pm 1\}$ of the functional equation is equal to 1 if and only if the quadratic form B has split rank $[n/2]$. In other words,

$$\zeta_B = \begin{cases} 1 & \textit{if } B \simeq (D_B) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus [n/2]}, \\ -1 & \textit{otherwise.} \end{cases}$$

Katsurada [8] proved this proposition for $k = \mathbb{Q}_p$. Watanabe pointed out that the functional equation can be proved by using representation theory (see Remark after Proposition 3.1 of [8]). Here, we give a proof along this line.

We prove only (1), since we do not need (2). It is possible to prove (2) in a similar way. For the rest of this section, we assume n is even.

To prove Proposition 4.1, we need several lemmas. For each Schwartz function $\Phi \in \mathcal{S}(\mathcal{S}_n(k))$, we define the Fourier transform $\widehat{\Phi}$ of Φ by

$$\widehat{\Phi}(x) = \int_{y \in \mathcal{S}_n(k)} \Phi(y)\psi(\mathrm{tr}(xy))\, dy.$$

Note that the product measure $dx = \prod_{i \le j} dx_{ij}$ is not the self-dual measure for this Fourier transform. In fact, we have

$$\widehat{\widehat{\Phi}}(x) = |2|^{-n(n-1)/2}\Phi(-x).$$

It is well-known that there exists some functional equation for the local zeta integrals for a prehomogeneous vector space. Sweet [17] the "gamma matrix" for the prehomogeneous vector space $\mathcal{S}_n$ for a non-archimedean local field. Note that Sweet treated the case when n is odd as well, although we treat only the case n is even.

For $\eta \in k^\times$, we set

$$\mathcal{O}_\eta = \{x \in \mathcal{S}_n(k) \mid D_x \equiv (-1)^{n/2}\eta \bmod (k^\times)^2\}.$$

(The set $\mathcal{O}_\eta$ is not a single orbit under the action of GL_n for $n > 1$.)

If ω is a quasi-character of $k^\times$, then we set

$$\varepsilon'(s, \omega, \psi) = \varepsilon(s, \omega, \psi)\frac{L(1-s, \omega^{-1})}{L(s, \omega)},$$

Lemma 4.2 (Sweet)**.** *Assume that n is even. We have a functional equation*

$$\int_{\mathcal{S}_n(k)} \Phi(x)|\det x|^{s-((n+1)/2)}\, dx = \sum_{\eta \in k^\times/(k^\times)^2} c(s;\eta) \int_{x \in \mathcal{O}_\eta} \widehat{\Phi}(x)|\det x|^{-s}\, dx$$

where the function $c(s;\eta)$ is defined by

$$\begin{aligned} c(s;\eta) =& |2|^{-ns}\alpha_\psi(\eta)\alpha_\psi(1)^{-1}\varepsilon'(s+(1/2), \chi_\eta, \psi) \\ & \times \varepsilon'(s-((n-1)/2), \mathbf{1}, \psi)^{-1}\prod_{r=1}^{n/2}\varepsilon'(2s-n+2r, \mathbf{1}, \psi)^{-1}. \end{aligned}$$

Proof. See Sweet [17]. □

We set $I(s) = \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}|\det|^s$. For $f(g) \in I(s)$ and $B \in \mathcal{S}_n^{\mathrm{nd}}(k)$, put

$$M(s)f(g) = \int_{\mathcal{S}_n(k)} f(w_n\mathbf{n}(x)g)\, dx.$$

$$\mathrm{Wh}_B(s)f(g) = \int_{\mathcal{S}_n(k)} f(w_n\mathbf{n}(x)g)\overline{\psi(\mathrm{tr}Bx)}\, dx.$$

The integrals $M(s)$ and $\mathrm{Wh}_B(s)$ are absolutely convergent for $\mathrm{Re}(s) \gg 0$ and can be meromorphically continued to the whole complex plane. If s is not a pole of $M(s)$, then $M(s)f(g) \in I(-s)$. Moreover, it is known that $\mathrm{Wh}_B(s)$ is entire. It is also well-known that

$$M(s)f_0^{(s)} = \frac{L(s-((n-1)/2), \mathbf{1})}{L(s+((n+1)/2), \mathbf{1})} \prod_{i=1}^{[n/2]} \frac{L(2s-n+2i, \mathbf{1})}{L(2s+n+1-2i, \mathbf{1})} f_0^{(-s)}.$$

Lemma 4.3. *The following functional equation holds:*

$$\mathrm{Wh}_B(-s) \circ M(s) = |\det B|^{-s} c(s; \eta_0) \mathrm{Wh}_B(s).$$

Here, $\eta_0 = (-1)^{n/2} \det B$.

Proof. Let m be a sufficiently large integer. We assume

$$B + \mathfrak{p}^m \mathcal{S}_n(\mathfrak{o}) \subset \{x \in \mathcal{O}_{\eta_0} \mid |\det x| = |\det B|\}.$$

Let $\Phi \in \mathcal{S}(\mathcal{S}_n(k))$ be the characteristic function of $B + \mathfrak{p}^m \mathcal{S}_n(\mathfrak{o})$. We define $f_\Phi \in I(s)$ such that

- $\mathrm{Supp}(f_\Phi) \subset P_n(k) w N_n(k)$.
- $f(w_n \mathbf{n}(x)) = \widehat{\Phi}(x)$ for $x \in \mathcal{S}_n(k)$.

Then, we have $\mathrm{Wh}(s) f_\Phi = \widehat{\widehat{\Phi}}(-B) \neq 0$. On the other hand, $M(s) f_\Phi(w\mathbf{n}(x))$ is equal to

$$\begin{aligned}\int_{y \in \mathcal{S}_n(k)} f_\Phi(w_n \mathbf{n}(y) w_n \mathbf{n}(x))\, dx &= \int_{y \in \mathcal{S}_n(k)} |\det y|^{-s-((n+1)/2)} \widehat{\Phi}(x - y^{-1})\, dy \\ &= \int_{y \in \mathcal{S}_n(k)} |\det y|^{s-((n+1)/2)} \widehat{\Phi}(x - y)\, dy.\end{aligned}$$

By Lemma 4.2, this is equal to

$$\begin{aligned}&\sum_{\eta \in k^\times/(k^\times)^2} c(s; \eta) \int_{y \in \mathcal{O}_\eta} \Phi(y) \psi(\mathrm{tr}(xy)) |\det y|^{-s}\, dy \\ =& c(s; \eta_0) |\det B|^{-s} \int_{y \in \mathcal{S}_n(k)} \Phi(y) \psi(\mathrm{tr}(xy))\, dy \\ =& c(s; \eta_0) |\det B|^{-s} \widehat{\Phi}(x).\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Wh}(-s) M(s) f_\Phi =& \int_{x \in \mathcal{S}_n(k)} M(s) f_\Phi(w_n \mathbf{n}(x)) \overline{\psi_B(x)}\, dx \\ =& c(s; \eta_0) |\det B|^{-s} \widehat{\widehat{\Phi}}(-B).\end{aligned}$$

Hence the lemma. $\square$

Proof of Proposition 4.1 (1). Note that $c(s;\eta_0)$ is equal to

$$|2|^{-ns}|\mathfrak{d}_B|^s \frac{L(-s+(1/2),\chi_{D_B})}{L(s+(1/2),\chi_{D_B})} \frac{L(s-((n-1)/2),\mathbf{1})}{L(-s+((n+1)/2),\mathbf{1})} \times \prod_{i=1}^{n/2} \frac{L(2s-n+2i,\mathbf{1})}{L(-2s+n+1-2i,\mathbf{1})}.$$

By Lemma 4.3, we have

$$\begin{aligned}
&\gamma(B,q^{s-((n+1)/2)})F(B;q^{s-((n+1)/2)})\frac{L(s-((n-1)/2),\mathbf{1})}{L(s+((n+1)/2),\mathbf{1})} \\
&\times \prod_{i=1}^{n/2} \frac{L(2s-n+2i,\mathbf{1})}{L(2s+n+1-2i,\mathbf{1})} \\
=&c(s;\eta_0)|\det B|^{-s}\gamma(B,q^{-s-((n+1)/2)})F(B;q^{-s-((n+1)/2)}).
\end{aligned}$$

Since

$$\begin{aligned}
\gamma(B,q^{s-((n+1)/2)}) =&L(-s+((n+1)/2),\mathbf{1})^{-1}L(-s+(1/2),\chi_{D_B}) \\
&\times \prod_{i=1}^{n/2} L(-2s+n+1-2i,\mathbf{1})^{-1}, \\
\gamma(B,q^{-s-((n+1)/2)}) =&L(s+((n+1)/2),\mathbf{1})^{-1}L(s+(1/2),\chi_{D_B}) \\
&\times \prod_{i=1}^{n/2} L(2s+n+1-2i,\mathbf{1})^{-1},
\end{aligned}$$

the functional equation of $F(B;X)$ follows. □

Proposition 4.4. *Assume n is even. Set*

$$\tilde{F}(B,X) = X^{-\delta_B}F(B,q^{-(n+1)/2}X).$$

Then we have $\tilde{F}(B,X^{-1}) = \tilde{F}(B,X)$.

Proof. This is a restatement of Proposition 4.1 (1). □

5. The unramified Whittaker functions

In this section, we assume $2 \nmid q$. Let $\tau \simeq \mathcal{B}(\mu,\mu^{-1})$ be an unramified unitary principal series with Satake parameter $\alpha = \mu(\varpi)$. Set $\epsilon = \langle -1,\varpi\rangle$. For $B \in \mathcal{S}_n^{\mathrm{nd}}$, we define the function $W^{\mathrm{ur}}(g)$ on $\widetilde{\mathrm{Sp}}_n$ as follows. Consider the Iwasawa decomposition

$$g = \mathbf{n}(z)(\mathbf{m}(A),\zeta)u, \qquad \mathbf{n}(z) \in N_n,\ A \in \mathrm{GL}_n,\ u \in \mathrm{Sp}_n(\mathfrak{o})$$

of $g \in \widetilde{\mathrm{Sp}_n}$. Then we set

$$W_B^{\mathrm{ur}}(g) = \psi_B(z) |\det(B[A])|^{(n+1)/4} \times \begin{cases} \tilde{F}(B[A], \epsilon^{n/2}\alpha) & \text{if } 2 \mid n, \, , \\ \zeta \alpha_\psi(\det A) \alpha_\psi(1)^{-1} \tilde{F}((1) \oplus B[A], \epsilon^{(n+1)/2}\alpha) & \text{if } 2 \nmid n. \end{cases}$$

Here, $\tilde{F}(B, X)$ is as in Proposition 4.4.

Proposition 5.1. *The function $W_B^{\mathrm{ur}}(g)$ is a class one Whittaker function of $\Pi(n, \tau)$.*

Proof. We first consider the case when n is even. For $B \in \mathcal{S}_n^{\mathrm{nd}}$ and $\mathrm{Re}(s) \gg 0$,

$$\psi_B(z) |\det(B[A])|^{(n+1)/4} \tilde{F}(B[A], \epsilon^{n/2}\alpha) = \gamma(B, q^{-s-((n+1)/2)})^{-1} q^{\delta_B \cdot s} \times \int_{N_n(k)} f_0^{(s)}(w_n \mathbf{n}(z) g) \overline{\psi_B(z)} \, dz$$

is a class one Whittaker function for $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n} |\det|^s$. By analytic continuation, $W_B^{\mathrm{ur}}(g)$ is a class one Whittaker function for $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n} \mu^{(n)}$. This proves the proposition for the case n is even.

When n is odd, put $\widehat{\Pi} = \Pi(n+1, \tau)$. Then $\mathrm{FJ}_\psi(\widehat{\Pi}) = \Pi(n, \tau)$. Then one can easily show

$$W_B^{\mathrm{ur}}(g) = |\det B|^{-1/4} \int_X W_{(1)\oplus B}^{\mathrm{ur}}(\mathbf{v}(x, 0, 0)g) \overline{\omega_\psi(g)\phi^0(x)} \, dx.$$

Here, $W_{(1)\oplus B}^{\mathrm{ur}}$ is the normalized Whittaker function for $\widehat{\Pi}$ and $\phi^0 \in \mathcal{S}(X) = \mathcal{S}(k^n)$ is the characteristic function of $\mathfrak{o}^n$. It follows that $W_B^{\mathrm{ur}}(g)$ is a class one Whittaker function for $\Pi(n, \tau)$ by Proposition 2.4. $\square$

We call W_B^{ur} the normalized Whittaker function. Fix a class one vector $f^0 \in \Pi(n, \tau)$. The Whittaker vector $w_B^0 \in \mathrm{Wh}_B(\Pi_n)$ such that $W_B^{\mathrm{ur}}(g) = w_B^0(\Pi_n(g) f^0)$ is called the normalized Whittaker vector. It is easily seen that

$$w_B^0 \circ \Pi_n((\mathbf{m}(A), \zeta)) = \begin{cases} w_{B[A]}^0 & \text{if } n \text{ is even,} \\ \zeta \alpha_\psi(\det A) \alpha_\psi(1)^{-1} w_{B[A]}^0 & \text{if } n \text{ is odd} \end{cases}$$

for $A \in \mathrm{GL}_n$ and $\zeta \in \{\pm 1\}$.

Recall that $B \in \mathcal{S}_n^{nd}(\mathfrak{o})$ is called maximal, if for $A \in \mathrm{Mat}_n(k)$, $B[A] \in \mathcal{S}_n(\mathfrak{o})$ implies $A \in \mathrm{Mat}_n(\mathfrak{o})$. Note that $0 \leq \delta_B \leq 1$ if B is maximal.

Lemma 5.2. *Assume n is even and $B \in \mathcal{S}_n^{\mathrm{nd}}(\mathfrak{o})$ is maximal. Then*

$$F(B,X) = \begin{cases} 1 & \text{if } \delta_B = 0, \\ 1 - \xi_B(q^{n/2} + q^{(n/2)+1})X + q^{n+1}X^2 & \text{if } \delta_B = 1. \end{cases}$$

In other word, we have

$$\tilde{F}(B,X) = \begin{cases} 1 & \text{if } \delta_B = 0, \\ -\xi_B(q^{-1/2} + q^{1/2}) + X + X^{-1} & \text{if } \delta_B = 1. \end{cases}$$

Proof. We denote the Minkowski-Hasse invariant of B by η_B. It is enough to prove $F(B, \xi_B q^{-n/2}) = 0$ for $\delta_B = 1$, by the functional equation of $F(B,X)$. Assume B is maximal and $\delta_B = 1$. For an integer m and a non-degenerate $B_1 \in \mathcal{S}_n(\mathfrak{o})$, we denote by $N_m(B_1, B)$ the number of $x \in \mathrm{Mat}_{2n}(\mathfrak{o})/\mathfrak{p}^m\mathrm{Mat}_{2n}(\mathfrak{o})$ such that ${}^t x B_1 x \equiv B \mod \mathfrak{p}^m$. Let $B_1 \in \mathcal{S}_n(\mathfrak{o}) \cap \mathrm{GL}_n(\mathfrak{o})$ be an element such that $\det B_1 \equiv \det B \mod (k^\times)^2$. Then we have $\xi_{B_1} = \xi_B$, and $\eta_{B_1} = -\eta_B$. Then Lemma 14. 8 of Shimura [16] implies

$$\gamma(B, \xi_B q^m) F(B, \xi_B q^m) = q^{-mn(n-1)/2} N_m(B_1, B)$$

for sufficiently large m. Note that $\gamma(B, \xi_B q^m) \neq 0$. For sufficiently large m, we have $N_m(B_1, B) = 0$, since $B' \equiv B \mod \mathfrak{p}^m$, ${}^t B' = B'$ implies $\eta_B = \eta'_B$. $\square$

Lemma 5.3. *Let m and n' be even non-negative integers. Assume that $n = m + n'$, $B = S \oplus B'$, $S \in \mathcal{S}_m \cap \mathrm{GL}_m(\mathfrak{o})$, and $B' \in \mathcal{S}_{n'}^{\mathrm{nd}}$. Then we have*

$$\begin{aligned} F(B,X) =& F(B', \xi_S X), \\ \tilde{F}(B,X) =& \xi_S^{\delta_B} \tilde{F}(B', \xi_S X). \end{aligned}$$

Proof. We shall prove the second identity. Let $W_B^{\mathrm{ur}}(g)$ be the normalized Whittaker function for $\Pi(n,\tau)$. We calculate the integral

$$W'(g) = |\det B|^{-m/4} \int_{x \in X} W_B^{\mathrm{ur}}(\mathbf{v}(x,0,0)g) \overline{\omega_S(g)\phi^0(x)}\, dx$$

for $g \in \mathrm{Sp}_{n'}$. By Proposition 2.4, W' is a class one element of $\mathcal{W}_{B'}(\Pi(n', \tau \otimes \chi_S))$. It follows that $W'(g) = u \cdot W^{\mathrm{ur}}_{B'}(g)$ for some $u \in \mathbb{C}^\times$. We have

$$\begin{aligned}
W'(\mathbf{m}(A)) =& |\det B|^{-m/4} \int_X W^{\mathrm{ur}}_B(\mathbf{v}(x,0,0)\mathbf{m}(A))\overline{\omega_S(\mathbf{m}(A))\phi^0(x)}\, dx \\
=&\chi_S(\det A) \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^n |\det B|^{-m/4} |\det A|^{m/2} \\
&\times \int_X W^{\mathrm{ur}}_B(\mathbf{m}(A)\mathbf{v}(xA,0,0))\overline{\phi^0(xA)}\, dx \\
=&\xi_S^{\mathrm{ord}(\det A)} |\det(B'[A])|^{-m/4} \int_X W^{\mathrm{ur}}_B(\mathbf{m}(A)\mathbf{v}(x,0,0))\overline{\phi^0(x)}\, dx \\
=&\xi_S^{\mathrm{ord}(\det A)} |\det(B'[A])|^{-m/4} W^{\mathrm{ur}}_B(\mathbf{m}(A)) \\
=&\xi_S^{\mathrm{ord}(\det A)} |\det(B'[A])|^{(n+1)/4} \tilde{F}(S \oplus B'[A], \xi_S \epsilon^{n'/2}\alpha).
\end{aligned}$$

It follows that there exists a constant $u \in \mathbb{C}^\times$ such that

$$\xi_S^{\mathrm{ord}(\det A)} \tilde{F}(S \oplus B'[A], \xi_S \epsilon^{n'/2}\alpha) = u\tilde{F}(B'[A], \epsilon^{n'/2}\alpha)$$

for any $A \in \mathrm{GL}_n$. Choosing $A \in \mathrm{GL}_n$ such that $B'[A]$ is maximal, we have $u = \xi_S^{\delta_B}$ by Lemma 5.2. Hence the second identity. The first identity follows immediately from the second identity. □

Note that Kohnen [9] has proved a special case of Lemma 5.3 by different method.

Proposition 5.4. *Assume that $n = m + n'$, $B = S \oplus B'$, $S \in \mathcal{S}_m \cap \mathrm{GL}_m(\mathfrak{o})$, and $B' \in \mathcal{S}^{\mathrm{nd}}_{n'}$. Let $\phi^0 \in \mathcal{S}(X)$ be the characteristic function of $X(\mathfrak{o}) = \mathrm{Mat}_{mn'}(\mathfrak{o})$. Then we have*

$$\int_{x \in X} W^{\mathrm{ur}}_B(\mathbf{v}(x,0,0)g)\overline{\omega_S(g)\phi^0(x)}\, dx = |\det B'|^{m/4} (\epsilon^{mn}\xi_S)^{\delta_B} W^{\mathrm{ur}}_{B'}(g)$$

for $g \in \widetilde{\mathrm{Sp}}_{n'}$. Here, W^{ur}_B and $W^{\mathrm{ur}}_{B'}$ are the normalized Whittaker functions for $\Pi(n, \tau)$ and $\Pi(n', \tau \otimes \chi_S)$, respectively.

Proof. Both sides are unramified Whittaker functions for $\Pi(n', \tau \otimes \chi_S)$. Therefore the left hand side is equal to $uW^{\mathrm{ur}}_{B'}(g)$ for some $u \in \mathbb{C}^\times$. By Lemma 5.3, one can easily show that $u = (\epsilon^{mn}\xi_S)^{\delta_B}$. □

Suppose that $n = m + n'$, $B = S \oplus B'$, $S \in \mathcal{S}_m \cap \mathrm{GL}_m(\mathfrak{o})$, and $B' \in \mathcal{S}^{\mathrm{nd}}_{n'}$. Let $f^0 \in \Pi(n, \tau)$ and $f'^0 \in \Pi(n', \tau \otimes \chi_S)$ be the distinguished class one vectors. We normalize $\beta_S : \Pi(n, \tau) \otimes \overline{\mathcal{S}(X)} \to \Pi(n', \tau \otimes \chi_S)$ by $\beta_S(f^0 \otimes \bar{\phi}^0) = f'^0$. For $B = S \oplus B'$, $B' \in \mathcal{S}^{\mathrm{nd}}_{n'}$, we consider the map $\mathcal{FJ}_S : \mathrm{Wh}_B(\Pi(n, \tau)) \to \mathrm{Wh}_{B'}(\Pi(n', \tau \otimes \chi_S)$, which is given by

$$\mathcal{FJ}_S(w_B)(\beta_S(f \otimes \bar{\phi})) = w_B(\Pi_n(\bar{\phi})f).$$

Then we have

$$\mathcal{FJ}_S(w_B^0) = |\det B|^{m/4}(\epsilon^{mn}\xi_S)^{\delta_B} w_{B'}^0.$$

If $S = S_1 \oplus S_2 \in \mathcal{S}_m^{\rm nd} \cap \mathrm{GL}_m(\mathfrak{o})$, $B = S \oplus B'$, and $B' \in \mathcal{S}_{n'}^{\rm nd}$, then

$$\mathcal{FJ}_{S_2} \circ \mathcal{FJ}_{S_1} = \mathcal{FJ}_S.$$

Lemma 5.5. *There exists a positive constant M depending only on n such that*

$$|W_B^{\rm ur}(\mathbf{1}_{2n})| < |\det B|^{-M}.$$

Proof. As in [6], one can show that the coeffiicients of $F(B,X)$ is at most $q^{M'\mathrm{ord}(D_B)}$ for some constant $M' > 0$ which depends only on n. Since $q^{-1/2} < \alpha < q^{1/2}$, we have

$$\begin{aligned} W_B^{\rm ur}(\mathbf{1}_{2n})| =& |\det B|^{-1/4}|\tilde{F}(B,\alpha)| \\ <& |\det B|^{-1/4}(\mathrm{deg}F(B,X)+1)q^{M'\mathrm{ord}(\det B)} \cdot q^{(1/4)\mathrm{deg}F(B,X)} \\ <& |\det B|^{-M'-2}. \end{aligned}$$

□

6. **Archimedean local theory**

In this section, we consider the case $k = \mathbb{R}$. We assume the additive character ψ of $\mathbb{R}$ is of the form $\psi(x) = \mathbf{e}(ax) = \exp(2\pi a\sqrt{-1}x)$ for $a > 0$. It is well-known that the Weil constant $\alpha_\psi(t)$ is equal to $\mathbf{e}(1/4)$ if $t > 0$, and $\mathbf{e}(-1/4)$, if $t < 0$. Recall that the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acts on the Siegel upper half space $\mathfrak{H}_n$ by

$$g(Z) = (AZ+B)(CZ+D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The automorphy factor $j(g,Z)$ is defined by $j(g,Z) = \det(CZ+D)$. The stabilizer of $\mathbf{i} = \sqrt{-1}\cdot\mathbf{1}_n \in \mathfrak{H}_n$ can be identified with the unitary group $\mathrm{U}(n)$ by the isomorphism $C\mathbf{i}+D \mapsto \left(\begin{smallmatrix} D & C \\ -C & D \end{smallmatrix}\right)$.

The real metaplectic group $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ acts on the Siegel upper half space $\mathfrak{H}_n$ through $\mathrm{Sp}_n(\mathbb{R})$. The inverse image of $\mathrm{U}(n)$ in $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ is denoted by $\tilde{\mathrm{U}}(n)$. There exists a unique automorphy factor $j(\tilde{g},Z)^{1/2}$ such that $(j(\tilde{g},Z)^{1/2})^2 = j(g,Z)$, where g is the image of $\tilde{g} \in \widetilde{\mathrm{Sp}_n(\mathbb{R})}$ in $\mathrm{Sp}_n(\mathbb{R})$. For $u \in \mathrm{U}(n)$, put $\det u = e^{\sqrt{-1}\theta}$, $-\pi \le \theta < \pi$. Then we have

$$j((u,\zeta),\mathbf{i})^{-1/2} = \zeta e^{-\sqrt{-1}\theta/2}.$$

We denote the irreducible lowest weight representation of $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ with lowest K-type $(\det)^\lambda$ by $\mathcal{D}_\lambda^{(n)}$.

Let τ be a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with minimal weight $\pm 2\kappa$. We set $\Pi_n = \Pi(n,\tau) = \mathcal{D}^{(n)}_{\kappa+(n/2)}$. For $B \in \mathcal{S}_n(\mathbb{R})^+$, it is known that $\dim_{\mathbb{C}} \mathrm{Wh}_B(\Pi_n) = 1$. (See Yamashita [20]). Note that Π_n is a genuine representation of $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ if and only if n is odd.

We define a function W^0_B on $\widetilde{\mathrm{Sp}_n(\mathbb{R})}$ by

$$\begin{aligned} &W^0_B(g) \\ &= \mathbf{e}(Bz)\det(B[A])^{(2\kappa+n)/4}\exp(-2\pi a\cdot \mathrm{tr}(B[A]))j(\tilde{u},\mathbf{i})^{-(2\kappa+n)/2}, \end{aligned}$$

for $g = \mathbf{n}(z)(\mathbf{m}(A),1)\tilde{u}$, $z = {}^t z$, $A \in \mathrm{GL}_n(\mathbb{R})^+$, $\tilde{u} \in \tilde{\mathrm{U}}(n)$. It is well-known that W^0_B generates a representation isomorphic to Π_n.

Lemma 6.1. *For $B \in \mathcal{S}_n(\mathbb{R})^+$ and $A \in \mathrm{GL}_n(\mathbb{R})$, we have*

$$W^0_B((\mathbf{m}(A),\zeta)g) = \zeta^n \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^{2\kappa+n} W^0_{B[A]}(g).$$

Proof. If $\det A > 0$, then one can easily show the equality. Assume now $\det A < 0$. Choose $U \in \mathrm{O(n)}$ such that $\det U = -1$ and $B[A][U] = B[A]$. Then we have

$$\begin{aligned} W^0_B((\mathbf{m}(A),\zeta)g) =&W^0_B((\mathbf{m}(AU^{-1}),1)(\mathbf{m}(U),\zeta)g) \\ =&W^0_{B[A]}(\mathbf{m}(U),\zeta)g). \end{aligned}$$

Observe that $(\mathbf{m}(U),\zeta) \in \tilde{\mathrm{U}}(n)$ and

$$j((\mathbf{m}(U),\zeta))^{-(2\kappa+n)/2} = \zeta^n \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^{2\kappa+n}.$$

Hence the lemma. □

We denote the space of lowest weight vectors of $\Pi(n,\tau)$ by $\Pi(n,\tau)^{\mathrm{lwt}}$. We fix a distinguished vector $f^0 \in \Pi(n,\tau)^{\mathrm{lwt}}$. Then there exists a Whittaker vector $w^0_B \in \mathrm{Wh}_B(\Pi_n)$ such that the Whittaker function associated to w^0_B and $f^0 \in \Pi_n$ is equal to $W^0_B(g)$. Then by Lemma 6.1, we have the following lemma.

Lemma 6.2. *Let $w^0_B \in \mathrm{Wh}_B(\Pi_n)$ be as above. Then we have*

$$w^0_B \circ \Pi_n((\mathbf{m}(A),\zeta)) = \zeta^n \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^{2\kappa+n} w^0_{B[A]}$$

for any $B \in \mathcal{S}_n(\mathbb{R})^+$ and $A \in \mathrm{GL}_n(\mathbb{R})$. In particular, if $B[A] = B$, then we have

$$w^0_B \circ \Pi_n((\mathbf{m}(A),\zeta)) = \zeta^n(\det A)^\kappa \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)}\right)^n w^0_B.$$

Assume $n = m + n'$, $S \in \mathcal{S}_m(\mathbb{R})^+$, $B' \in \mathcal{S}_{n'}(\mathbb{R})^+$, and $B = S \oplus B'$. Put $X = \mathrm{Mat}_{mn'}(\mathbb{R})$. We define $\phi_S^0 \in \mathcal{S}(X)$ by $\phi_S^0(x) = e^{-2\pi a \cdot \mathrm{tr}(S[x])}$. Note that ϕ_S^0 is a lowest weight vector of $\mathcal{S}(X)$ as a representation of $\widetilde{\mathrm{Sp}}_{n'}$. We set $\mathcal{S}(X)^{\mathrm{lwt}} = \mathbb{C} \cdot \phi_S^0$. The following lemma can be proved by direct calculation.

Lemma 6.3. *Assume $n = m + n'$, $S \in \mathcal{S}_m(\mathbb{R})^+$, $B' \in \mathcal{S}_{n'}(\mathbb{R})^+$, and $B = S \oplus B'$. Then we have*

$$\int_{x \in X} W_B^0(\mathbf{v}(x,0,0)g')\overline{\omega_S(g')\phi_S^0(x)}\, dx = |\det B|^{m/4} e^{-2\pi a \mathrm{tr}(S)} W_{B'}^0(g').$$

We also fix a distinguished vector $f'^0 \in \Pi(n', \tau)^{\mathrm{lwt}}$ and obtain a Whittaker vector $w_{B'}^0 \in \mathrm{Wh}_{B'}(\Pi_{n'})$. We define a $\mathbb{C}$-linear map

$$\beta_S : \Pi(n,\tau)^{\mathrm{lwt}} \otimes \mathcal{S}(X)^{\mathrm{lwt}} \longrightarrow \Pi(n',\tau)^{\mathrm{lwt}}$$

by

$$\beta_S(f^0 \otimes \phi_S^0) = f'^0.$$

For $B = S \oplus B'$, $B' \in \mathcal{S}_{n'}(\mathbb{R})^+$, we define the map

$$\mathcal{FJ}_S : \mathrm{Wh}_B(\Pi(n,\tau)) \to \mathrm{Wh}_{B'}(\Pi(n', \tau \otimes \chi_S)$$

such that the Whittaker function $W_{B'}(g')$ associated to $\mathcal{FJ}_S(w_B)$ and $\beta_S(f \otimes \bar{\phi})$ is given by

$$W_{B'}(g') = \int_X W_B(\mathbf{v}(x,0,0)g')\overline{\omega_S(g')\phi(x)}\, dx$$

for $w_B \in \mathrm{Wh}_B(\Pi(n,\tau))$, $f \in \Pi(n,\tau)^{\mathrm{lwt}}$, and $\phi \in \mathcal{S}(X)^{\mathrm{lwt}}$. Here, W_B is the Whittaker function associated to f and w_B.

Lemma 6.4. *Suppose that $S \in \mathcal{S}_m(\mathbb{R})^+$, $B' \in \mathcal{S}_{n'}(\mathbb{R})^+$, and $B = S \oplus B' \in \mathcal{S}_n(\mathbb{R})^+$.*

(1) *We have*

$$\mathcal{FJ}_S(w_B^0) = |\det B|^{m/4} e^{-2\pi a \mathrm{tr}(S)} w_{B'}^0.$$

(2) *If $S = S_1 \oplus S_2 \in \mathcal{S}_m(\mathbb{R})^+$, $B = S \oplus B'$, and $B' \in \mathcal{S}_{n'}(\mathbb{R})^+$, then*

$$\mathcal{FJ}_{S_2} \circ \mathcal{FJ}_{S_1} = (\det S_1)^{m_2/4} \mathcal{FJ}_S.$$

(3) *For $A \in \mathrm{GL}_{n'}(\mathbb{R})$ and $w_B \in \mathrm{Wh}_B(\Pi_n)$, we have*

$$\begin{aligned} &\mathcal{FJ}_S(w_B \circ \Pi_n((\mathbf{m}(\mathbf{1}_m \oplus A), \zeta))) \\ &\quad = \zeta^m \left(\frac{\alpha_\psi(1)}{\alpha_\psi(\det A)} \right)^m \chi_S(\det A) |\det A|^{m/2} \mathcal{FJ}_S(w_B) \circ \Pi_{n'}((\mathbf{m}(A), \zeta)). \end{aligned}$$

Proof. (1) follows from the definition of $\mathcal{FJ}_S$. (2) follows from (1). (3) can be proved in the same way as Lemma 3.4. $\square$

Remark 6.5. Yamashita's theorem [20] is valid for lowest weight modules of $\widetilde{\mathrm{Sp}_n}(\mathbb{R})$. It is also possible to avoid his theorem by replacing $\mathrm{Wh}_B(\Pi_n)$ by $\mathbb{C} \cdot w_B^0$.

7. **Statement of the main theorem**

From now on, k is a totally real number field with $[k : \mathbb{Q}] = d$. The subset of totally positive elements of k is denoted by $k_+^\times$. We fix an additive character ψ of $\mathbb{A}/k$. We assume $\psi_v(x) = \exp(2\pi\sqrt{-1}a_v x)$ for some $a_v > 0$ for each $v \in \mathfrak{S}_\infty$.

Let $\tau \simeq \otimes_v \tau_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. The local root number $\varepsilon(1/2, \tau_v)$ is given by

$$\varepsilon(1/2, \tau_v) = \begin{cases} \mu_v(-1) & v \notin \mathfrak{S}_\infty, \\ (-1)^{\kappa_v} & v \in \mathfrak{S}_\infty. \end{cases}$$

The global root number $\varepsilon(1/2, \tau)$ is defined by

$$\varepsilon(1/2, \tau) = \prod_v \varepsilon(1/2, \tau_v).$$

We assume that τ satisfies the following conditions (A1), (A2), and (A3).

(A1) For each $v \notin \mathfrak{S}_\infty$, τ_v is a principal series $\mathcal{B}(\mu_v, \mu_v^{-1})$.
(A2) For each $v \in \mathfrak{S}_\infty$, τ_v is a discrete series representation with lowest weight $\pm 2\kappa_v$.
(A3) $\varepsilon(1/2, \tau) = 1$.

Recall that we set

$$\Pi_{n,v} = \Pi(n, \tau_v) = \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\mu_v^{(n)})$$

for each $v \notin \mathfrak{S}_\infty$. When $v \in \mathfrak{S}_\infty$, we let $\Pi_{n,v} = \Pi(n, \tau_v)$ be the lowest weight representation of $\widetilde{\mathrm{Sp}_n}(\mathbb{R})$ with lowest $\tilde{\mathrm{U}}(n)$-type $(\det)^{\kappa_v + (n/2)}$.

We consider the restricted tensor product

$$\Pi_n = \Pi(n, \tau) = \bigotimes_v{}' \, \Pi(n, \tau_v).$$

We define the multiplicity $m_{\mathrm{auto}}(\Pi_n)$ by

$$m_{\mathrm{auto}}(\Pi_n) = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{Sp}_n(\mathbb{A})}}(\Pi_n, \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k) \backslash \widetilde{\mathrm{Sp}_n(\mathbb{A})})).$$

Here, $\mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k) \backslash \widetilde{\mathrm{Sp}_n(\mathbb{A})})$ is the space of cusp forms on $\mathrm{Sp}_n(k) \backslash \widetilde{\mathrm{Sp}_n(\mathbb{A})}$.

Then the main result of this paper is as follows.

Theorem 7.1. *Let τ be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *which satisfies the conditions (A1), (A2), and (A3). Then we have*

$$m_{\mathrm{auto}}(\Pi_n) = 1.$$

Remark 7.2. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the conditions (A1) and (A2). It is easy to show that $m_{\mathrm{auto}}(\Pi_n) = 0$ if $\varepsilon(1/2, \tau) = -1$. See the remark after Lemma 11.4.

8. The Shimura correspondence: the case $n = 1$

The correspondence between modular forms of integral weight and those of half-integral weight was first considered by Shimura [14]. Waldspurger ([18], [19]) treated the Shimura correspondence in terms of automorphic representations. In this section, we review Waldspurger's theory of the Shimura correspondence.

Let $\mathcal{A}_0$ be the space of genuine cusp forms of $\mathrm{SL}_2(k)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$. The space of cusp forms orthogonal to the Weil representations associated to one-dimensional quadratic forms is denoted by $\mathcal{A}_{00}$. Then the multiplicity of an irreducible genuine cuspidal automorphic representation in $\mathcal{A}_{00}$ is one ([19], Theorem 3).

Let σ be an irreducible genuine cuspidal automorphic representation in $\mathcal{A}_{00}$. A non-trivial additive character ψ_ξ of $\mathbb{A}/k$ is called a missing character of σ, if

$$\int_{k\backslash\mathbb{A}} f(\mathbf{n}(x)g)\overline{\psi_\xi(x)}\, dx = 0$$

for any $f \in \sigma$ and $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$. We denote by $\theta(\sigma, \psi)$ the theta correspondence of σ for the dual pair $\mathrm{SL}_2 \times \mathrm{PGL}_2$. Then the theta correspondence $\theta(\sigma, \psi_\xi^{-1}) = 0$ if and only if ψ_ξ is a missing character ([18], Proposition 26). Moreover, if ψ_ξ is not a missing character, then $\theta(\sigma, \psi_\xi^{-1}) \otimes \chi_\xi$ does not depend on the choice of $\xi \in k^\times$ ([18], Proposition 28).

Put $\mathrm{Wd}(\sigma, \psi) = \theta(\sigma, \psi_\xi^{-1}) \otimes \chi_\xi$. Denote by $L(s, \sigma; \psi)$ the L-function $L(s, \mathrm{Wd}(\sigma, \psi))$. Then ψ_ξ is not a missing character for σ if and only if the following conditions (1) and (2) hold ([19], Proposition 21):

(1) $\mathrm{Wh}_{\psi_{\xi,v}}(\sigma_v) \neq (0)$ for any v.
(2) $L(1/2, \sigma; \psi_\xi) \neq 0$.

We also need the following theorem.

Theorem 8.1 (Waldspurger [19], Theorem 4)**.** *Let τ be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *such that* $\varepsilon(1/2,\tau)=1$*. Let Σ be a finite set of places of k and $\delta>0$ a positive number. Then there exists an element $\xi\in k^\times$ such that the following conditions (1) and (2) hold:*

(1) $|\xi-1|_v<\delta$ *for each* $v\in\Sigma$.
(2) $L(1/2,\tau\otimes\chi_\xi)\neq 0$.

Now let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies (A1), (A2), and (A3). By Theorem 8.1, there exists an element $\xi\in k_+^\times$ such that $L(1/2,\tau\otimes\chi_\xi)\neq 0$. Put $\sigma=\theta(\tau\otimes\chi_\xi,\psi_\xi)$. Then σ is isomorphic to $\Pi(1,\tau)$. Thus we obtain the following proposition.

Proposition 8.2. *Let τ be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *which satisfies (A1) and (A2), and (A3). Then we have $m_{\mathrm{auto}}(\Pi_1)=1$. Moreover, for $\xi\in k_+^\times$, ψ_ξ is a missing character of Π_1 if and only if $L(1/2,\tau\otimes\chi_\xi)\neq 0$.*

9. **The Saito-Kurokawa lift: the case** $n=2$

Recall that the simplectic similitude group GSp_2 is defined by

$$\mathrm{GSp}_2(k)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Mat}_{2n}(k)\;\middle|\;\begin{array}{l}A\cdot{}^tB=B\cdot{}^tA,\ C\cdot{}^tD=D\cdot{}^tC,\\ A\cdot{}^tD-B\cdot{}^tC=m\mathbf{1}_n, m\in k^\times\end{array}\right\}.$$

For $t\in k^\times$, we set

$$\mathbf{d}(t)=\begin{pmatrix}t\cdot\mathbf{1}_2 & 0\\ 0 & \mathbf{1}_2\end{pmatrix}.$$

The Siegel parabolic subgroup $\check{P}_2(k)$ is given by

$$\begin{aligned}\check{P}_2(k)&=\check{M}_2(k)N_2(k),\\ \check{M}_2(k)&=\{\mathbf{d}(t)\mathbf{m}(A)\,|\,t\in k^\times, A\in\mathrm{GL}_2(k)\}.\end{aligned}$$

There exists a exact sequence

$$1\to k^\times\to\mathrm{GSp}_2(k)\to\mathrm{SO}(3,2)(k)\to 1.$$

By this exact sequence, we identify $\mathrm{PGSp}_2(k)=\mathrm{GSp}_2(k)/k^\times$ with $\mathrm{SO}(3,2)(k)$.

Now we consider the theta correspondence between SL_2 and $\mathrm{SO}(3,2)$. Let σ be an irreducible genuine cuspidal automorphic representation in $\mathcal{A}_{00}$. We denote by $\Theta(\sigma,\psi)$ the theta correspondence of σ for the dual pair $\widetilde{\mathrm{SL}_2}\times\mathrm{SO}(3,2)$.

The following theorem is due to Piatetski-Shapiro (cf. [10], Theorem 5.1, Theorem 5.2, Theorem 6.1, and Theorem 6.2)

Theorem 9.1 (Piatetski-Shapiro [10]). *$\Theta(\sigma, \psi)$ is always a non-zero representation. $\Theta(\sigma, \psi)$ is cuspidal if and only if ψ^{-1} is a missing character of σ. In this case, $\Theta(\sigma, \psi)$ is an irreducible cuspidal automorphic representation of $\mathrm{PGSp}_2(\mathbb{A})$.*

Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies (A1), (A2), and (A3). By Proposition 8.2, there exists an irreducible cuspidal automorphic representation σ of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ such that $\sigma \simeq \Pi(1, \tau)$. Then we have $\mathrm{Wd}(\sigma, \psi^{-1}) \simeq \tau \otimes \chi_{-1}$. Note that ψ^{-1} is a missing character for σ, since $\mathrm{Wh}_{\psi_v^{-1}}(\sigma) = (0)$ for $v \in \mathfrak{S}_\infty$. Put $\check{\Pi} = \Theta(\sigma, \psi)$. Then $\check{\Pi}$ is an irreducible cuspidal automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ with trivial central character by Theorem 9.1.

If $v < \infty$, then $\check{\Pi}_v$ is isomorphic to the representation Σ induced from the character

$$\mathbf{d}(t)\mathbf{m}(A) \mapsto \mu_v^{(2)}(t)^{-1}\mu_v^{(2)}(\det A)$$

of the Siegel parabolic subgroup $\check{P}_2(k_v)$ (see [13], p.236). It is known that the pullback of $\check{\Pi}_v$ to $\mathrm{Sp}_2(k_v)$ is isomorphic to $\Pi(2, \tau_v)$ (see [12] Proposition 5.4).

For $v \in \mathfrak{S}_\infty$, the pullback of $\check{\Pi}_v$ to $\mathrm{Sp}_2(k_v)$ is not irreducible. It is isomorphic to the direct sum of $\mathcal{D}^{(2)}_{\kappa_v+1}$ and its contragredient $\bar{\mathcal{D}}^{(2)}_{\kappa_v+1}$ (see [13] Lemma 4.1).

It follows that the restriction of Π to $\mathrm{Sp}_2(\mathbb{A})$ is given by

$$\Pi|_{\mathrm{Sp}_2(\mathbb{A})} = \left(\bigotimes_{v \notin \mathfrak{S}_\infty} \Pi(2, \tau)_v\right) \bigotimes \left(\bigotimes_{v \in \mathfrak{S}_\infty} \left(\mathcal{D}^{(2)}_{\kappa_v+1} \oplus \bar{\mathcal{D}}^{(2)}_{\kappa_v+1}\right)\right).$$

We claim each irreducible component Π' is an automorphic representation of $\mathrm{Sp}_n(\mathbb{A})$. Put

$$T = \bigcup_{f \in \Pi'} \mathrm{Supp}(f)$$

Then it is enough to prove $\mathrm{Sp}_2(\mathbb{A}) \subset T$. In fact, T is right-invariant under $\prod_{v \notin \mathfrak{S}_\infty} \mathrm{GSp}_2(k_v) \times \prod_{v \in \mathfrak{S}_\infty} \mathrm{GSp}_2(\mathbb{R})^+$ and left-invariant under $\mathrm{GSp}_2(k)$. Therefore $T = \mathrm{GSp}_2(\mathbb{A})$ by weak approximation.

Thus we obtain the following proposition.

Proposition 9.2. *Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, which satisfies (A1), (A2), and (A3). Then $\Pi(2, \tau)$ is a cuspidal automorphic representation.*

The following lemma will be used later.

Lemma 9.3. (1) *Let v be a finite place of k. If $t \cdot B[A] = B$ for $t \in k_v^\times$, $B \in \mathcal{S}_2^{\mathrm{nd}}(k_v)$, and $A \in \mathrm{GL}_2(k_v)$, then we have*

$$w_B \circ \check{\Pi}_v(\mathbf{d}(t)\mathbf{m}(A)) = \mu_v^{(2)}(t \det A) w_B.$$

for any $w_B \in \mathrm{Wh}_B(\check{\Pi}_v)$.

(2) *Let v be an infinite place of k. If $t \cdot B[A] = B$ for $t \in \mathbb{R}_+^\times$, $B \in \mathcal{S}_2(\mathbb{R})^+$, and $A \in \mathrm{GL}_2(\mathbb{R})$, then we have*

$$w_B \circ \check{\Pi}_v(\mathbf{d}(t)\mathbf{m}(A)) = \left(\frac{\det A}{|\det A|}\right)^{\kappa_v+1} w_B.$$

for any $w_B \in \mathrm{Wh}_B(\check{\Pi}_v)$.

(3) *If $v \notin \mathfrak{S}$, then we have*

$$w^0_{t\cdot B[A]} = w^0_B \circ \check{\Pi}_v(\mathbf{d}(t)\mathbf{m}(A))$$

for any $B \in \mathcal{S}_2^{\mathrm{nd}}(k_v)$, $t \in k_v^\times$, and $A \in \mathrm{GL}_2(k_v)$.

Proof. One can prove (1) and (2) as in the proof of Lemma 3.5 and Lemma 6.2, respectively. Now we prove (3). Define a function $\check{W}_B^{\mathrm{ur}}(g)$ on $\mathrm{GSp}_2(k_v)$ by

$$\check{W}_B^{\mathrm{ur}}(\mathbf{n}(z)\mathbf{d}(t_1)\mathbf{m}(A_1)u) = \psi_B(z)|\det(t_1 \cdot B[A_1])|^{3/4}\tilde{F}(t_1 \cdot B[A_1], \epsilon\alpha).$$

for

$$\mathbf{n}(z) \in N_2(k_v),\ t_1 \in k_v^\times, A_1 \in \mathrm{GL}_2(k_v),\ u \in \mathrm{GSp}_2(\mathfrak{o}_v).$$

here, ϵ and α are as in §5. Then as in the proof of Proposition 5.1, one can show that $\check{W}_B^{\mathrm{ur}}(g)$ is a Whittaker function for $\check{\Pi}_v$. Then (3) follows from the equation

$$\check{W}_B^{\mathrm{ur}}(\mathbf{d}(t)\mathbf{m}(A)g) = \check{W}_{t\cdot B[A]}^{\mathrm{ur}}(g).$$

□

10. **Some results on quadratic forms**

Let B be a non-degenerate quadratic form defined over k. Recall that the Minkowski-Hasse theorem says that B represents an element $\xi \in k$ over k if and only if B represents ξ over k_v for any place v of k.

It is well-known that an isotropic quadratic form represents any element. If B is an anisotropic quadratic form of rank 4 over a non-archimedean local field k_v, then any element $\xi \in k_v^\times$ is represented by B. If B is an anisotropic quadratic form of rank 3 over a non-archimedean local field k_v, then $\xi \in k_v^\times$ is represented by B if and only if $\xi \notin -(\det B) \cdot (k_v^\times)^2$.

Lemma 10.1. *Assume that* $n \geq 3$*. Suppose that* B_1*,* B_2*,* $B_3 \in \mathcal{S}_n(k)^+$*. Then there exists an element* $\xi \in k_+^\times$ *such that* $\xi \hookrightarrow B_1, B_2, B_3$*.*

Proof. Let S be the set of non-archimedean places v such that at least one of B_1, B_2 and B_3 are anisotropic over k_v. Then S is a finite set of non-archimedean places. For each $v \in S$, there exists an element $\xi_v \in k_v^\times$ which is represented by B_1, B_2, and B_3 over k_v, since $[k_v^\times : (k_v^\times)^2] \geq 4$. By the independence of the valuations, we can choose a totally positive element ξ such that $\xi \hookrightarrow B_1, B_2, B_3$ over k_v, $v \in S$. □

Definition 10.2. The set of $B \in \mathcal{S}_n(k)^+$ such that $L(1/2, \tau \otimes \chi_B) \neq 0$ is denoted by $\mathcal{S}_n(k)_\tau^+$.

Remark 10.3. Suppose that $n = m+n'$, $S \in \mathcal{S}_m(k)^+$, and $B' \in \mathcal{S}_{n'}(k)^+$. Then $S \oplus B' \in \mathcal{S}_n(k)_\tau^+$ if and only if $B' \in \mathcal{S}_{n'}(k)_{\tau\otimes\chi_S}^+$.

Lemma 10.4. *Assume that* $n \geq 3$*. Suppose that* $\xi_1, \xi_2 \hookrightarrow B \in \mathcal{S}_n^+(k)_\tau$*. Then there exist* $\eta \in k_+^\times$*,* S_1*,* $S_2 \in \mathcal{S}_2(k)^+$ *and* $T \in \mathcal{S}_n(k)_\tau^+$ *satisfying the following conditions (K1), (K2), (K3), and (K4).*

- (K1) $\xi_1, \eta \hookrightarrow S_1$.
- (K2) $\xi_2, \eta \hookrightarrow S_2$.
- (K3) $S_1, S_2 \hookrightarrow B$.
- (K4) $S_1, S_2, (\xi_1) \oplus (\xi_2) \hookrightarrow T$.

Proof. We first consider the case $n = 3$. Choose vectors $x, y \in k^3$ such that $B[x] = \xi_1$, $B[y] = \xi_2$. If $B(x, y) = 0$, then B is equivalent to $(\xi_1) \oplus (\xi_2) \oplus (\xi_3)$ for some $\xi_3 \in k_+^\times$. In this case, we can put $\eta = \xi_3$, $S_1 = (\xi_1) \oplus (\xi_3)$, $S_2 = (\xi_2) \oplus (\xi_3)$, and $T = B$.

Assume that $B(x, y) \neq 0$. For a vector $z \in k^3$, we set

$$\begin{aligned}
\eta =& B(z, z), \\
S_1 =& \begin{pmatrix} B(x,x) & B(x,z) \\ B(x,z) & B(z,z) \end{pmatrix} \\
S_2 =& \begin{pmatrix} B(z,z) & B(y,z) \\ B(z,y) & B(y,y) \end{pmatrix}, \\
T =& \begin{pmatrix} B(x,x) & B(x,z) & 0 \\ B(x,z) & B(z,z) & B(y,z) \\ 0 & B(z,y) & B(y,y) \end{pmatrix}.
\end{aligned}$$

Then the condition (K1), (K2), (K3), and (K4) are satisfied. It remains to prove that $T \in \mathcal{S}_3(k)_\tau^\times$ for some $z \in k^3$. As a function of $z \in k^3$,

$$\det T = B(x,x)B(y,y)B(z,z) - B(x,x)B(y,z)^2 - B(y,y)B(x,z)^2$$

is a quadratic form of 3 variables, which we denote by Q. Then we have $\det T = Q[z]$. By direct calculation, one can show that $\det Q =$

$-\det B \cdot B(y,y)^2 B(x,x)^2 B(x,y)^2 \neq 0$. Let $\mathfrak{S}_Q$ be the set of places of k such that Q is anisotropic. Then the quadratic form Q represents any element $t \in k_+^\times$ such that $t \notin (\det B)\cdot(k_v^\times)^2$ for $v \in \mathfrak{S}_Q$. Note that $\mathfrak{S}_Q$ does not contain real places, since $-\det Q \in k_+^\times$ and $\xi_1, \xi_2 \hookrightarrow T$. Therefore the lemma in the case $n=3$ follows from Theorem 8.1.

For $n>3$, take any 3-dimensional quadratic subspace $B_0 \hookrightarrow B$, such that $\xi_1, \xi_2 \hookrightarrow B_0$. Then $B \simeq B_0 \oplus B'$ for some $B' \in \mathcal{S}_{n-3}(k)^+$. We apply the lemma in the case $n=3$ for ξ_1, ξ_2 and $B_0 \in \mathcal{S}_3(k)^\times_{\tau\otimes\chi_{B'}}$. Then we obtain η, S_1, S_2, and $T_0 \in \mathcal{S}_3(k)^+_{\tau\otimes\chi_{B'}}$ satisfying the condition (K1), (K2), (K3), and (K4) for $\tau\otimes\chi_{B'}$. Set $T = T_0 \oplus B'$. Then η, S_1, S_2, and $T \in \mathcal{S}_n(k)^+_\tau$ satisfies the condition (K1), (K2), (K3), and (K4) for τ. □

Definition 10.5. For $B_1, B_2 \in \mathcal{S}_n(k)^+_\tau$, an admissible sequence between B_1 and B_2 is a sequence $(T_0, T_1, \dots, T_r; \xi_1, \dots, \xi_r)$ such that

(1) $T_0, T_1, \dots, T_r \in \mathcal{S}_n(k)^+_\tau$.
(2) $\xi_1, \dots, \xi_r \in k_+^\times$.
(3) $B_1 = T_0$ and $B_2 = T_r$.
(4) $\xi_i \hookrightarrow T_{i-1}, T_i$ for $i = 1, \dots, r$.

Lemma 10.6. *Assume $n \geq 2$. For B_1, $B_2 \in \mathcal{S}_n(k)^+_\tau$, there exists an admissible sequence between B_1 and B_2.*

Proof. If $n \geq 3$, then there exists $\xi \in k_+^\times$ which is represented by B_1 and B_2 by Lemma 10.1. Then $(B_1, B_2; \xi)$ is an admissible sequence.

Now assume $n=2$. Set $\tilde{B}_1 = (1)\oplus B_1$ and $\tilde{B}_2 = (1)\oplus B_2$. Choose an element $\eta \in k_+^\times$, which is represented by $\tilde{B}_1$ and $\tilde{B}_2$. We may assume that $\eta \notin (k^\times)^2$. We apply Lemma 10.4 to 1, η and B_1. Then there exist $S_1 \in \mathcal{S}_2(k)^+$ and $\tilde{T}_1 \in \mathcal{S}_3(k)^+_\tau$ satisfying the following (a), (b), and (c).

(a) $1 \hookrightarrow S_1$.
(b) $S_1 \hookrightarrow \tilde{B}_1$.
(c) $S_1, (1)\oplus(\eta) \hookrightarrow \tilde{T}_1$.

By (a), $S_1 \simeq (1)\oplus(\xi_1)$, where $\xi_1 = \det S \in k_+^\times$. By (c), there exists $T_1 \in \mathcal{S}_2(k)^+_\tau$ such that $\tilde{T}_1 \simeq (1)\oplus T_1$. Then we have $\xi_1 \hookrightarrow B_1$ and $\xi_1 \hookrightarrow T_1$ by (b) and (c). Moreover, $\eta \hookrightarrow T_1$ by (c).

By a similar argument, we find $T_2 \in \mathcal{S}_2(k)^+_\tau$ and $\xi_2 \in k_+^\times$ such that $\xi_2 \hookrightarrow B_2$ and $\xi_2 \hookrightarrow T_2$, and $\eta \hookrightarrow T_2$. It follows that $(B_1, T_1, T_2, B_2; \xi_1, \eta, \xi_2)$ is an admissible sequence between B_1 and B_2. □

11. **Compatible family of Whittaker vectors**

Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ satisfying (A1) and (A2). Assume $n = m + n'$ and $S \in \mathcal{S}_m(k)^+$, and

$B = S \oplus B'$. For each non-archimedean place v of k, we have homomorphisms

$$\beta_{v,S} : \Pi_v(n, \tau_v) \otimes \overline{\mathcal{S}(X_v)} \to \Pi_v(n', \tau_v \otimes \chi_{v,d_S}),$$
$$\mathcal{FJ}_{v,S} : \mathrm{Wh}_B(\Pi_v(n, \tau_v)) \to \mathrm{Wh}_{B'}(\Pi_v(n', \tau_v \otimes \chi_{v,d_S}))$$

such that the Whittaker function $W_{B'}(g')$ associated to $\mathcal{FJ}_{v,S}(w_B)$ and $\beta_{v,S}(f \otimes \bar{\phi})$ is given by

$$W_{B'}(g') = \int_{X_v} W_B(\mathbf{v}(x, 0, 0)g')\overline{\omega_{S,v}(g')\phi(x)}\, dx$$

for $w_B \in \mathrm{Wh}_B(\Pi_v(n, \tau_v))$, $f \in \Pi_v(n, \tau_v)$, and $\phi \in \mathcal{S}(X_v)$. Here, $W_B(g)$ is the Whittaker function associated to w_B and f. If $S_1 \in \mathcal{S}_{m_1}(k)^+$, $S_2 \in \mathcal{S}_{m_2}(k)^+$ and $S = S_1 \oplus S_2$, then we have

$$\mathcal{FJ}_{v,S_2} \circ \mathcal{FJ}_{v,S_1} = \rho_{v,S_1,S_2} \cdot \mathcal{FJ}_{v,S},$$

where ρ_{v,S_1,S_2} is a constant which does not depend on $B' \in \mathcal{S}_{n'}(k)^+$.

If v is a good prime, the we have distinguiched vectors $f_v^0 \in \Pi_v(n, \tau_v)$, $w_B^0 \in \mathrm{Wh}_B(\Pi(n, \tau_v))$, etc. Then we have

$$w_B^0(f_v^0) = 1, \qquad \beta_{v,S}(f_v^0 \otimes \phi_v^0) = {f'}_v^0, \qquad \mathcal{FJ}_{v,S}(w_B^0) = w_{B'}^0.$$

Moreover, we have $\rho_{v,S_1,S_2} = 1$.

Similarly, for real place v of k, we have homomorphisms

$$\beta_{v,S} : \Pi_v(n, \tau_v)^{\mathrm{lwt}} \otimes \overline{\mathcal{S}(X_v)}^{\mathrm{lwt}} \to \Pi_v(n', \tau_v \otimes \chi_{v,d_S})^{\mathrm{lwt}},$$
$$\mathcal{FJ}_{v,S} : \mathrm{Wh}_B(\Pi_v(n, \tau_v)) \to \mathrm{Wh}_{B'}(\Pi_v(n', \tau_v \otimes \chi_{v,d_S}))$$

such that the Whittaker function $W_{B'}(g')$ associated to $\mathrm{FJ}_{v,S}(w_B)$ and $\beta_{v,S}(f \otimes \bar{\phi})$ is given by

$$W_{B'}(g') = \int_{X_v} W_B(\mathbf{v}(x, 0, 0)g')\overline{\omega_{S,v}(g')\phi(x)}\, dx$$

for $w_B \in \mathrm{Wh}_B(\Pi_v(n, \tau_v))$, $f \in \Pi_v(n, \tau_v)^{\mathrm{lwt}}$, and $\phi \in \mathcal{S}(X_v)$. Here, $W_B(g)$ is the Whittaker function associated to w_B and f. If $S_1 \in \mathcal{S}_{m_1}(k)^+$, $S_2 \in \mathcal{S}_{m_2}(k)^+$ and $S = S_1 \oplus S_2$, then we have

$$\mathcal{FJ}_{v,S_2} \circ \mathcal{FJ}_{v,S_1} = \rho_{v,S_1,S_2} \cdot \mathcal{FJ}_{v,S},$$

where ρ_{v,S_1,S_2} is a constant which does not depend on $B' \in \mathcal{S}_{n'}(k)^+$.

Put

$$\Pi_n^{\mathrm{lwt}} = \Pi(n, \tau)^{\mathrm{lwt}} = \bigotimes_{v \notin \mathfrak{S}_\infty} \Pi_v(n, \tau_v) \bigotimes \bigotimes_{v \in \mathfrak{S}_\infty} \Pi_v(n, \tau_v)^{\mathrm{lwt}},$$
$$\mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}} = \bigotimes_{v \notin \mathfrak{S}_\infty} \mathcal{S}(X_v) \bigotimes \bigotimes_{v \in \mathfrak{S}_\infty} \mathcal{S}(X_v)^{\mathrm{lwt}}.$$

Then we have a homomorphism

$$\beta_S = \otimes_v \beta_{v,S} : \Pi(n,\tau)^{\mathrm{lwt}} \otimes \overline{\mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}} \longrightarrow \Pi(n',\tau\otimes\chi_S)^{\mathrm{lwt}}$$

and a homomorphism

$$\mathcal{FJ}_{S,B'} = \otimes_v \mathcal{FJ}_{v,S,B'} : \mathrm{Wh}_{S\oplus B'}(\Pi(n,\tau)) \to \mathrm{Wh}_{B'}(\Pi(n',\tau\otimes\chi_S)).$$

for each $B' \in \mathcal{S}_{n'}(k)^+$. By what we have explained as above, the following two propositions hold.

Proposition 11.1. *Let $W_B(g)$ be the Whittaker function associated to $w_B \in \mathrm{Wh}_{S\oplus B'}(\Pi(n,\tau))$ and $f \in \Pi(n,\tau)^{\mathrm{lwt}}$, and $W_{B'}(g')$ be the Whittaker function associated to $\mathcal{FJ}_S(w_B)$ and $\beta_S(f\otimes\bar{\phi})$, where $\phi \in \mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}$. Then we have*

$$W_{B'}(g') = \int_{x\in X(\mathbb{A})} W_B(\mathbf{v}(x,0,0)g')\overline{\omega_S(g')\phi(x)}dx.$$

Proposition 11.2. *The homomorphism*

$$\mathcal{FJ}_{S,B'} : \mathrm{Wh}_{S\oplus B'}(\Pi(n,\tau)) \to \mathrm{Wh}_{B'}(\Pi(n',\tau\otimes\chi_S))$$

is an isomorphism. If $n = m_1 + m_2 + n'$, $S_1 \in \mathcal{S}_{m_1}(k)^+$, $S_2 \in \mathcal{S}_{m_2}(k)^+$, and $S = S_1 \oplus S_2$, then we have

$$\mathcal{FJ}_{S_2,B'} \circ \mathcal{FJ}_{S_1,S_2\oplus B'} = \rho_{S_1,S_2} \cdot \mathcal{FJ}_{S,B'}$$

for any $B' \in \mathcal{S}_{n'}(k)^+$. Here, ρ_{S_1,S_2} is a constant which does not depend on B'.

When there is no fear of confusion, $\mathcal{FJ}_{S,B'}$ is simply denoted by $\mathcal{FJ}_S$.

Definition 11.3. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the conditions (A1) and (A2). A family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is called an $\mathrm{GL}_n(k)$-family of Whittaker vectors for Π_n, if

$$w_{B[A]} = w_B \circ \Pi_n(\mathbf{m}(A))$$

for any $B \in \mathcal{S}_n(k)^+$ and $A \in \mathrm{GL}_n(k)$.

Lemma 11.4. *Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the conditions (A1) and (A2). Suppose that $B \in \mathcal{S}_n(k)^+$, $A \in \mathrm{GL}_n(k)$, $B[A] = B$, and $w_B \in \mathrm{Wh}_B(\Pi_n)$. Then we have*

$$w_B \circ \Pi_n(\mathbf{m}(A)) = \begin{cases} w_B & \text{if } \det A = 1, \\ \varepsilon(1/2,\tau)w_B & \text{if } \det A = -1. \end{cases}$$

Proof. By Lemma 3.5 and Lemma 6.2, we have

$$w_B \circ \Pi_n(\mathbf{m}(A)) = \prod_{v \notin \mathfrak{S}_\infty} \mu_v(\det A) \left(\frac{\alpha_{\psi_v}(1)}{\alpha_{\psi_v}(\det A)} \right)^n \times \prod_{v \in \mathfrak{S}_\infty} (\det A)^{\kappa_v} \left(\frac{\alpha_{\psi_v}(1)}{\alpha_{\psi_v}(\det A)} \right)^n \cdot w_B.$$

By the property of the Weil constant, we have

$$\prod_v \alpha_{\psi_v}(1) = \prod_v \alpha_{\psi_v}(\det A) = 1.$$

Note that $\det A = \pm 1$. If $\det A = -1$, then

$$\prod_{v \notin \mathfrak{S}_\infty} \mu_v(\det A) \prod_{v \in \mathfrak{S}_\infty} (\det A)^{\kappa_v} = \varepsilon(1/2, \tau).$$

Hence the lemma. □

By Lemma 11.4, a non-trivial $\mathrm{GL}_n(k)$-family of Whittaker vectors exists if and only if $\varepsilon(1/2, \tau) = 1$. In particular, $\Pi(n, \tau)$ is not automorphic if $\varepsilon(1/2, \tau) = -1$.

Hereafter, we assume τ satisfies (A1), (A2), and (A3). Fix $S \in \mathcal{S}_m(k)^+$ $(0 < m < n)$ and set $n' = n - m$.

Lemma 11.5. *Let $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$ be a $\mathrm{GL}_n(k)$-family of Whittaker vectors for $\Pi(n, \tau)$. Then the family*

$$\{\mathcal{F}\mathcal{J}_{S,B'}(w_{S \oplus B'})\}_{B' \in \mathcal{S}_{n'}(k)^+}$$

is a $\mathrm{GL}_{n'}(k)$-family of Whittaker vectors for $\Pi(n', \tau \otimes_S)$.

Proof. The lemma follows from Lemma 3.4 and Lemma 6.4. □

Let $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$ be a family of Whittaker vectors for Π_n. Let $W_B(g)$ be the Whittaker function associated to $f \in \Pi_n$ and $w_B \in \mathrm{Wh}_B(\Pi_n)$ for each $B \in \mathcal{S}_n(k)$. We consider the Fourier series

$$F(g) = \sum_{B \in \mathcal{S}_n(k)^+} W_B(g).$$

We do not discuss the convergence of $F(g)$ and assume the convergence of $F(g)$ in this section. By definition, the Fourier series $F(g)$ is left $P_n(k)$-invariant if and only if $\{w_B\}$ is a $\mathrm{GL}_n(k)$-family.

Recall that for $\phi \in \mathcal{S}(X(\mathbb{A}))$, the theta function $\Theta_S^\phi(vg')$ is defined by

$$\Theta^\phi(\mathbf{v}(x, y, z)g') = \sum_{l \in X(k)} \psi_S(z + x \cdot {}^t y + 2l \cdot {}^t y)\omega_S(g')\phi(l + x)$$

for $v = \mathbf{v}(x, y, z) \in V_{n,m}(\mathbb{A})$ and $g' \in \widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$.

Lemma 11.6. *We assume that the Fourier series $F(g)$ is absolutely convergent. Let $\{w_B\}_{B\in\mathcal{S}_n^+(k)}$ be a $\mathrm{GL}_n(k)$-family of Whittaker vectors for Π_n. Then we have*

$$\int_{V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta^\phi(vg')}\,dv = \sum_{B'\in\mathcal{S}_{n'}^+(k)} \int_{X(\mathbb{A})} W_{S\oplus B'}\left(\mathbf{v}(x,0,0)g'\right)\overline{\omega_S(g')\phi(x)}dx$$

for any $\phi \in \mathcal{S}(X(\mathbb{A}))$.

Proof. The contribution of $B \in \mathcal{S}_n(k)^+$ in

$$\int_{V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta^\phi(vg')}\,dv = \int_{V(k)\backslash V(\mathbb{A})} \sum_{B\in\mathcal{S}_n^+(k)} W_B(vg')\overline{\Theta^\phi(vg')}dv$$

vanishes unless the upper left $m\times m$ block of B is equal to S. In this case,

$$B = \begin{pmatrix} S & 0 \\ 0 & B' \end{pmatrix}\left[\begin{pmatrix} \mathbf{1} & \lambda \\ 0 & \mathbf{1} \end{pmatrix}\right]$$

for some $\lambda \in X(k)$ and $B' \in \mathcal{S}_{n'}(k)^+$. Note that

$$\mathbf{v}(x,y,z) = \mathbf{v}(0,y,z+y\cdot {}^t x)\cdot \mathbf{v}(x,0,0),$$

It follows that if $B = \begin{pmatrix} S & 0 \\ 0 & B' \end{pmatrix}\left[\begin{pmatrix} \mathbf{1} & \lambda \\ 0 & \mathbf{1} \end{pmatrix}\right]$, then

$$\begin{aligned} W_B(\mathbf{v}(x,y,z)g') =& W_B(\mathbf{v}(0,y,z+y\cdot {}^t x)\cdot \mathbf{v}(x,0,0)g') \\ =& \psi_S(z+y\cdot {}^t x)\psi_S(2\lambda\cdot {}^t y)W_B(\mathbf{v}(x,0,0)g') \\ =& \psi_S(z+y\cdot {}^t x)\psi_S(2\lambda\cdot {}^t y)W_{S\oplus B'}(\mathbf{v}(\lambda+x,0,0)g'). \end{aligned}$$

We have

$$\begin{aligned} &\int_{V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta^\phi(vg')}\,dv \\ =& \int_{V(k)\backslash V(\mathbb{A})} \sum_{B\in\mathcal{S}_n^+(k)} W_B(vg')\overline{\Theta^\phi(vg')}dv \\ =& \int_{x\in X(k)\backslash X(\mathbb{A})}\int_{y\in Y(k)\backslash Y(\mathbb{A})}\int_{z\in Z(k)\backslash Z(\mathbb{A})} \sum_{B'\in\mathcal{S}_{n'}(k)^+} W_{S\oplus B'}(\mathbf{v}(\lambda+x,0,0)g') \\ &\times \sum_{\lambda\in X(k)}\sum_{l\in X(k)} \psi_S(2(\lambda-l){}^t y)\overline{\omega_S(g')\phi(l+x)}dz\,dy\,dx. \end{aligned}$$

In this integral, only $l=\lambda$ contributes. It follows that

$$\int_{V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta^{\phi}(vg')}\,dv$$
$$=\int_{x\in X(k)\backslash X(\mathbb{A})}\sum_{B'\in\mathcal{S}_{n'}(k)^+}\sum_{\lambda\in X(k)} W_{S\oplus B'}(\mathbf{v}(\lambda+x,0,0)g')\overline{\omega_S(g')\phi(\lambda+x)}dx$$
$$=\int_{x\in X(\mathbb{A})}\sum_{B'\in\mathcal{S}_{n'}(k)^+} W_{S\oplus B'}(\mathbf{v}(x,0,0)g')\overline{\omega_S(g')\phi(x)}dx.$$

Hence the lemma.

□

By Lemma 11.6, if $F(g)$ is the Fourier series obtained from $f\in\Pi(n,\tau)$ and the family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$, the the Fourier series obtained from $\beta_S(f\otimes\bar{\phi})\in\Pi(n',\tau\otimes\chi_S)$ and the family $\{\mathcal{FJ}_S(w_{S\oplus B'})\}_{B'\in\mathcal{S}_{n'}(k)^+}$ is equal to

$$\int_{V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta^{\phi}(vg')}\,dv.$$

Definition 11.7. Let

$$\{w_B\}_{B\in\mathcal{S}_n(k)^+}\in\prod_{B\in\mathcal{S}_n(k)^+}\mathrm{Wh}_B(\Pi_n)$$

be a $\mathrm{GL}_n(k)$-family of Whittaker vectors for $\Pi(n,\tau)$. We shall say that $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is a compatible family, if the following conditions are satisfied.

(1) When $n=1$, a family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is compatible if it comes from the Shimura correspondence of τ, i.e., for each $f\in\Pi_1$, the Fourier series
$$F(g)=\sum_{B\in\mathcal{S}_1(k)^+}W_B(g)$$
belongs to the space of the Shimura correspondence of τ.

(2) When $n\geq 2$, a family $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is a compatible family, if the family
$$\{\mathcal{FJ}_\xi(w_{(\xi)\oplus B'})\}_{B'\in\mathcal{S}_{n-1}(k)^+}$$
is a compatible family for $\Pi(n-1,\tau\otimes\chi_\xi)$ for each $\xi\in k_+^\times$.

The following lemma follows immediately from the definition.

Lemma 11.8. *Let $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ be a $\mathrm{GL}_n(k)$-family of Whittaker vectors for $\Pi(n,\tau)$. Then $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ is a compatible family of Whittaker vectors for $\Pi(n,\tau)$, if and only if $\{\mathrm{FJ}_S(w_{S\oplus B'})\}_{B'\in\mathcal{S}_{n-m}(k)^+}$ is* *a compatible family for* $\Pi(n-m, \tau \otimes \chi_S)$ *for any* $m < n$ *and any* $S \in \mathcal{S}_m(k)^+$.

Assume that $\Pi(n,\tau)$ is isomorphic to an automorphic representation of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$. Then there exists an embedding

$$\iota : \Pi(n,\tau) \to \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_n(k)\backslash\widetilde{\mathrm{Sp}_n(\mathbb{A})}).$$

For each $B \in \mathcal{S}_n(k)^+$ and $f \in \Pi(n,\tau)^{\mathrm{lwt}}$, put

$$w_B(f) = \int_{N_n(k)\backslash N_n(\mathbb{A})} \iota(f)(\mathbf{n}(x))\overline{\psi_B(x)}\,dx.$$

Then we have

$$\iota(f)(g) = \sum_{B\in\mathcal{S}_n(k)^+} w_B(\Pi_n(g)f)$$

by Köcher principle. Put

$$\varphi_{f,\phi}(g') = \int_{V(k)\backslash V(\mathbb{A})} \iota(f)(vg')\overline{\Theta_S^{\phi}(vg')}\,dv,$$

for $g' \in \widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$ and $\phi \in \mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}$. Then we have

$$\int_{N_{n'}(k)\backslash N_{n'}(\mathbb{A})} \varphi_{f,\phi}(\mathbf{n}(x)g')\overline{\psi_{B'}(x)}\,dx = \mathcal{FJ}_S(w_B)(\Pi_{n'}(g')(\beta_S(f\otimes\bar{\phi})))$$

by Lemma 11.6.

Lemma 11.9. *The family* $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ *arising from an embedding* ι *is a non-zero compatible family.*

Proof. The automorphic representation generated by

$$\int_{V(k)\backslash V(\mathbb{A})} \iota(f)(vg')\overline{\Theta_S^{\phi}(vg')}\,dv$$

for $f \in \Pi(n,\tau)^{\mathrm{lwt}}$ and $\phi \in \mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}$ is isomorphic to $\Pi(n', \tau\otimes\chi_S)$. Therefore $\{w_B\}$ is a compatible family by induction. □

Recall that (see Definition 10.2)

$$\mathcal{S}_n(k)^+_\tau = \{B \in \mathcal{S}_n(k)^+ \mid L(1/2, \tau\otimes\chi_B) \neq 0\}.$$

Lemma 11.10. *Assume that* $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ *is a compatible family. If* $B \notin \mathcal{S}_n(k)^+_\tau$, *then* $w_B = 0$.

Proof. If $n = 1$, then the lemma follows from Proposition 8.2.

Assume $n \geq 2$ and $w_B \neq 0$ for $B \in \mathcal{S}_n(k)^+$. By replacing B by $B[A]$ for some $A \in \mathrm{GL}_n(k)$, we may assume $B = (\xi)\oplus B'$, $\xi \in k^\times_+$,

$B' \in \mathcal{S}_{n-1}(k)^+$. Then $\mathcal{FJ}_\xi(w_B) \neq 0$, since $\mathcal{FJ}_\xi$ is injective. By the induction hypothesis, we have

$$L(1/2, (\tau \otimes \chi_\xi) \otimes \chi_{B'}) = L(1/2, \tau \otimes \chi_B) \neq 0.$$

□

We denote the dimension of compatible family of Whittaker vectors for $\Pi(n,\tau)$ by $m_{\mathrm{comp}}(\Pi(n,\tau))$. By definition, we have $m_{\mathrm{comp}}(\Pi(n,\tau)) = 1$ for $n=1$. We are going to prove $m_{\mathrm{comp}}(\Pi(n,\tau)) = 1$. By induction hypothesis, we assume

(H1) For any $n' < n$ and any cuspidal automorphic representation τ' of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies (A1), (A2), and (A3), we have $m_{\mathrm{comp}}(\Pi(n',\tau')) = 1$. Moreover, if $\{w_{B'}\}$ is a non-trivial compatible family for $\Pi(n',\tau')$, then $w_{B'} \neq 0$ for any $B' \in \mathcal{S}_{n'}(k)^+_{\tau'}$.

From now on, and until the end of this section, we assume (H1).

Suppose $B_1, B_2 \in \mathcal{S}_n(k)^+_\tau$. Let $S \in \mathcal{S}_m(k)^+$ be an element which is represented by B_1 and B_2. Then we define a linear map

$$U^S_{B_1,B_2} : \mathrm{Wh}_{B_1}(\Pi(n,\tau)) \longrightarrow \mathrm{Wh}_{B_2}(\Pi(n,\tau))$$

as follows. First assume that there exist $B'_1, B'_2 \in \mathcal{S}^+_{n'}(k)$ such that

$$B_1 = \begin{pmatrix} S & 0 \\ 0 & B'_1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} S & 0 \\ 0 & B'_2 \end{pmatrix}.$$

In this case, we set $U^S_{B_1,B_2}(w_{B_1}) = w_{B_2}$ if and only if there exists a compatible family $\{w'_{B'}\}_{B' \in \mathcal{S}^+_{n'}(k)}$ of $\Pi(n', \tau \otimes \chi_S)$ such that $w'_{B'_1} = \mathcal{FJ}_S(w_{B_1})$ and $w'_{B'_2} = \mathcal{FJ}_S(w_{B_2})$. Note that $U^S_{B_1,B_2}$ is an isomorphism by (H1).

In general, there exists $A_1, A_2 \in \mathrm{GL}_n(k)$ and $B'_1, B'_2 \in \mathcal{S}^+_{n'}(k)$ such that

$$B_1[A_1] = \begin{pmatrix} S & 0 \\ 0 & B'_1 \end{pmatrix}, \qquad B_2[A_2] = \begin{pmatrix} S & 0 \\ 0 & B'_2 \end{pmatrix}.$$

In this case, we put

$$U^S_{B_1,B_2} = \Pi_n(\mathbf{m}(A_2))^{-1} \circ U^S_{B_1[A_1],B_2[A_2]} \circ \Pi_n(\mathbf{m}(A_1)).$$

The right hand side does not depend on the choice of $A_1, A_2 \in \mathrm{GL}_n(k)$ by Lemma 3.5, Lemma 3.4, Lemma 6.1, and Lemma 6.4. Moreover $U^S_{B_1,B_2}$ depends only on the equivalence class of $S \in \mathcal{S}_m(k)^+$. When $S = (\xi) \in \mathcal{S}_1(k)^+$, we simply denote $U^{(\xi)}_{B_1,B_2}$ by $U^\xi_{B_1,B_2}$. Note that if $S \hookrightarrow B_1, B_2, B_3$, then we have $U^S_{B_2,B_3} \circ U^S_{B_1,B_2} = U^S_{B_1,B_3}$ by definition.

Lemma 11.11. *Suppose that $S \in \mathcal{S}_{m_1}(k)^+$, $T \in \mathcal{S}_{m_2}(k)^+$, $S \hookrightarrow T$, $T \hookrightarrow B_1, B_2 \in \mathcal{S}_n(k)^+_\tau$. When we have $U^S_{B_1,B_2} = U^T_{B_1,B_2}$.*

Proof. Since $S \hookrightarrow T$, we may assume $T = S \oplus S'$ for some $S' \in \mathcal{S}_{m_2-m_1}(k)^+$. We may also assume $B_1 = T \oplus B_1'$ and $B_2 = T \oplus B_2'$. Then we have

$$\begin{aligned}\mathcal{FJ}_{T,B_1'} =&\rho_{S,S'}\mathcal{FJ}_{S',B_1'} \circ \mathcal{FJ}_{S,S'\oplus B_1'},\\ \mathcal{FJ}_{T,B_2'} =&\rho_{S,S'}\mathcal{FJ}_{S',B_2'} \circ \mathcal{FJ}_{S,S'\oplus B_2'}\end{aligned}$$

by Proposition 11.2. By the definition of the compatible family, $\mathcal{FJ}_S(w_{B_1})$ and $\mathcal{FJ}_S(w_{B_2})$ belongs to a compatible family if and only if $\mathcal{FJ}_T(w_{B_1})$ and $\mathcal{FJ}_T(w_{B_2})$ belongs to a compatible family. Hence the lemma. □

Lemma 11.12. *Suppose that $\xi_1, \xi_2 \in k_+^\times$ are represented by $B_1, B_2 \in \mathcal{S}_n(k)_\tau^+$. Assume that there exists $S \in \mathcal{S}_2^+(k)$ such that*

(1) *$S \hookrightarrow B_1, B_2$.*
(2) *$\xi_1, \xi_2 \hookrightarrow S$.*

Then we have $U_{B_1,B_2}^{\xi_1} = U_{B_1,B_2}^{\xi_2}$.

Proof. By Lemma 11.11, we have $U_{B_1,B_2}^{\xi_1} = U_{B_1,B_2}^{S} = U_{B_1,B_2}^{\xi_2}$. Hence the lemma. □

Lemma 11.13. *Assume $n \geq 3$. Suppose that $\xi_1, \xi_2 \in k_+^\times$ are represented by $B_1, B_2 \in \mathcal{S}_n^+(k)_\tau$. Then we have $U_{B_1,B_2}^{\xi_1} = U_{B_1,B_2}^{\xi_2}$.*

Proof. We first show that there exists $T_1 \in \mathcal{S}_n(k)_\tau^+$ such that $(\xi_1) \oplus (\xi_2) \hookrightarrow T_1$ and $U_{B_1,T_1}^{\xi_1} = U_{B_1,T_1}^{\xi_2}$. In fact, by Lemma 10.4, there exist $\eta \in k_+^\times$, S_1, $S_2 \in \mathcal{S}_2(k)^+$ and $T_1 \in \mathcal{S}_n(k)_\tau^+$ satisfying the following conditions (a), (b), (c), and (d).

(a) $\xi_1, \eta \hookrightarrow S_1$.
(b) $\xi_2, \eta \hookrightarrow S_2$.
(c) $S_1, S_2 \hookrightarrow B_1$.
(d) $S_1, S_2, (\xi_1) \oplus (\xi_2) \hookrightarrow T_1$.

Then (a), (c), and (d) implies $U_{B_1,T_1}^{\xi_1} = U_{B_1,T_1}^{\eta}$ by Lemma 11.12. Similarly, (b), (c), and (d) implies $U_{B_1,T_1}^{\xi_2} = U_{B_1,T_1}^{\eta}$. It follows that $U_{B_1,T_1}^{\xi_1} = U_{B_1,T_1}^{\eta} = U_{B_2,T_1}^{\xi_1}$.

By a similar argument, there exists $T_2 \in \mathcal{S}_n(k)_\tau^+$ such that $(\xi_1) \oplus (\xi_2) \hookrightarrow T_2$ and $U_{T_2,B_2}^{\xi_1} = U_{T_2,B_2}^{\xi_1}$. Then we have $U_{T_1,T_2}^{\xi_1} = U_{T_1,T_2}^{\xi_2}$, since $(\xi_1) \oplus (\xi_2) \hookrightarrow T_1, T_2$. It follows that

$$U_{B_1,B_2}^{\xi_1} = U_{B_2,T_2}^{\xi_1} \circ U_{T_1,T_2}^{\xi_1} \circ U_{B_1,T_1}^{\xi_1} = U_{B_2,T_2}^{\xi_2} \circ U_{T_1,T_2}^{\xi_2} \circ U_{B_1,T_1}^{\xi_2} = U_{B_1,B_2}^{\xi_2}.$$

Hence the Lemma. □

If $\mathcal{R} = (T_0, T_1, \ldots, T_r, ; \xi_1, \xi_2, \ldots, \xi_r)$ is an admissible sequence between B_1 and B_2, then we set

$$U^{\mathcal{R}}_{B_1,B_2} = U^{\xi_r}_{T_{r-1},T_r} \circ \cdots U^{\xi_2}_{T_2,T_1} \circ U^{\xi_1}_{T_0,T_1}.$$

Then $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$ is a compatible family if and only if $U^{\mathcal{R}}_{B_1,B_2}(w_{B_1}) = w_{B_2}$ for any $B_1, B_2 \in \mathcal{S}_n(k)^+_\tau$ and admissible sequence $\mathcal{R}$ between B_1 and B_2.

Lemma 11.14. *Assume that $n \geq 3$. If $\mathcal{R}$ and $\mathcal{R}'$ are admissible sequences between $B_1 \in \mathcal{S}_n(k)^+_\tau$ and $B_2 \in \mathcal{S}_n(k)^+_\tau$, then we have $U^{\mathcal{R}}_{B_1,B_2} = U^{\mathcal{R}'}_{B_1,B_2}$.*

Proof. It is enough to prove that if $\mathcal{R} = (T_0, T_1, \ldots, T_r, ; \xi_1, \xi_2, \ldots, \xi_r)$ is an admissible sequence such that $T_0 = T_r$, then we have $U^{\mathcal{R}}_{T_0,T_r} = \mathrm{id}$.

There is nothing to prove for $r = 1$. The case $r = 2$ is Lemma 11.13. For $r \geq 3$, there exists $\eta \in k^\times_+$ such that $\eta \hookrightarrow T_0, T_1, T_2$ by Lemma 10.1. By Lemma 11.13, we have $U^{\xi_2}_{T_1,T_2} = U^{\eta}_{T_1,T_2}$ and $U^{\xi_1}_{T_0,T_1} = U^{\eta}_{T_0,T_1}$. Then by induction, we have

$$\begin{aligned} U^{\mathcal{R}}_{T_0,T_r} =& U^{\xi_r}_{T_{r-1},T_0} \circ \cdots \circ U^{\xi_3}_{T_2,T_3} \circ U^{\eta}_{T_1,T_2} \circ U^{\eta}_{T_0,T_1} \\ =& U^{\xi_r}_{T_{r-1},T_0} \circ \cdots \circ U^{\xi_3}_{T_2,T_3} \circ U^{\eta}_{T_0,T_2} \\ =& \mathrm{id}. \end{aligned}$$

Hence the lemma. □

Proposition 11.15. *We have $m_{\mathrm{comp}}(\Pi(n,\tau)) = 1$. Moreover, if $\{w_B\}$ is a non-trivial compatible family for $\Pi(n,\tau)$, then $w_B \neq 0$ for any $B \in \mathcal{S}_n(k)^+_\tau$.*

Proof. We first prove that $m_{\mathrm{comp}}(\Pi(n,\tau)) \leq 1$. We may assume $n \geq 2$. If the family $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$ is non-zero, there exists $B_o \in \mathcal{S}_n(k)^+$ such that $w_{B_0} \neq 0$. By Lemma 11.10, we have $B_0 \in \mathcal{S}_n(k)^+_\tau$. By Lemma 10.6, there exists an admissible sequence $\mathcal{R} = (T_0, \ldots, T_r ; \xi_1, \ldots, \xi_r)$ for B_0 and B. Then we have $w_B = U^{\mathcal{R}}_{B_0,B}(w_B)$. It follows that $m_{\mathrm{comp}}(\Pi(n,\tau)) \leq 1$. Moreover, if $\{w_B\}$ is a non-trivial compatible family, then we have $w_B \neq 0$ for any $B \in \mathcal{S}_n(\tau)_\tau$, since $U^{\mathcal{R}}_{B_0,B}$ is an isomorphism.

Now we prove $m_{\mathrm{comp}}(\Pi(n,\tau)) \geq 1$. We may assume $n \geq 3$. Choose any $B_0 \in \mathcal{S}_n(k)^+_\tau$ and $w_{B_0} \in \mathrm{Wh}_{B_0}(\Pi(n,\tau))$, $w_{B_0} \neq 0$. For each $B \in \mathcal{S}_n(k)^+_\tau$, there exists an admissible sequence $\mathcal{R}$ between B_0 and B. set $w_B = U^{\mathcal{R}}_{B_0,B}(w_{B_0})$. By Lemma 11.4, w_B does not depend on the choice of $\mathcal{R}$. Then $\{w_B\}$ is a non-trivial compatible family. Thus Proposition 11.15 is proved. □

12. Convergence of the Fourier series

Let $\{w_B\}_{B\in\mathcal{S}_n^+(k)}$ be a compatible family of Whittaker vectors for $\Pi = \Pi(n,\tau)$. We fix a vector $f \in \Pi(n,\tau)^{\mathrm{lwt}}$. Let W_B be the Whittaker function associated to f and w_B. By definition, we have

$$W_B(g) = w_B(\Pi(g)f)$$

for each $B \in \mathcal{S}_n(k)^+$.

We consider the sum

$$F(g) = \sum_{B\in\mathcal{S}_n^+(k)} W_B(g).$$

We are now going to prove that the sum is convergent absolutely and uniformly on any compact subset of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$. By translating f from the right, it is enough to consider the convergence of $F(g)$ on some neighbourhood of $\widetilde{\mathrm{Sp}_n(k_\infty)}$. Since f is a smooth vector, it is enough to consider the convergence of $F(g)$ on $\widetilde{\mathrm{Sp}_n(k_\infty)}$.

Note that $F(g)$ satisfies the equation $F(gu_v) = j(u_v,\mathbf{i})^{\kappa_v+(n/2)}F(g)$ for any $u_v \in \tilde{\mathrm{U}}(n)$. It follows that one can define a function $F^\natural(Z)$ on $\mathfrak{H}^{\mathfrak{S}_\infty}$ by

$$F^\natural(Z) = F(g)\prod_{v\in\mathfrak{S}_\infty} j(g_v,\mathbf{i})^{\kappa_v+(n/2)},$$

where $g \in \widetilde{\mathrm{Sp}_n(k_\infty)}$ satisfies $g_v(\mathbf{i}) = Z_v$ for any $v \in \mathfrak{S}_\infty$.

We decompose the Whittaker function W_B as a product of the finite part and the infinite part as

$$W_B = W_{B,\mathrm{fin}} \times \prod_{v\in\mathfrak{S}_\infty} W^0_{B,v}.$$

Then we have

$$F^\natural(Z) = \sum_{B\in\mathcal{S}_n^+(k)}\left(\prod_{v\in\mathfrak{S}_\infty} |\det B|_v^{\kappa_v+(n/2)}\right) W_{B,\mathrm{fin}}(\mathbf{1}_{2n})\mathbf{e}_B(Z).$$

Since $f \in \Pi(n,\tau)$ is a smooth vector, f is invariant from the right by

$$\{\mathbf{n}(z) \,|\, z \in \mathcal{L}_v\}$$

for some lattice $\mathcal{L}_v \subset \mathcal{S}_n(k_v)$ for any finite place v. We may assume $\mathcal{L}_v = \mathcal{S}_n(\mathfrak{o}_v)$ for almost all v. Put

$$L = \left\{ B \in \mathcal{S}_n(k) \,\middle|\, \psi_B \text{ is trivial on } \prod_{v\notin\mathfrak{S}_\infty} \mathcal{L}_v \right\}.$$

Then L is a lattice in $\mathcal{S}_n(k)$. Since $W_{B,\mathrm{fin}}(\mathbf{1}_{2n})=0$ unless $B\notin L$, we have

$$F^\natural(Z)=\sum_{B\in L\cap\mathcal{S}_n^+(k)}\left(\prod_{v\in\mathfrak{S}_\infty}|\det B|_v^{\kappa_v+(n/2)}\right)W_{B,\mathrm{fin}}(\mathbf{1}_{2n})\mathbf{e}_B(Z).$$

Let $\mathfrak{S}$ be a finite set of places of k. We assume $\mathfrak{S}$ contains all places above 2 and ∞. We also assume $\mathfrak{S}$ contains all places where τ_v is ramified.

Definition 12.1. $B_1,B_2\in\mathcal{S}_n(k)^+$ are $\mathfrak{S}$-equivalent if B_1 and B_2 are equivalent over k_v for any $v\in\mathfrak{S}$.

Definition 12.2. Let B be an element of $\mathcal{S}_n^{\mathrm{nd}}(k_v)$. For w_B, $w'_B\in\mathrm{Wh}_B(\Pi_v)$, we denote $w_B\approx w'_B$ if $w_B=uw'_B$ for some $u\in\mathbb{C}^\times$, $|u|=1$. Similarly, for $B\in\mathcal{S}_n(k)^+$, $w_B,w'_B\in\mathrm{Wh}_B(\Pi)$, we denote $w_B\approx w'_B$ if $w_B=uw'_B$ for some $u\in\mathbb{C}^\times$, $|u|=1$.

For $v\notin\mathfrak{S}$, there exists a distinguished vector $w^0_{B,v}\in\mathrm{Wh}_B(\Pi_v)$ for each $B\in\mathcal{S}_n^{\mathrm{nd}}(k_v)$. For $v\in\mathfrak{S}_\infty$, we have also chosen a distinguished vector $w^0_{B,v}\in\mathrm{Wh}_B(\Pi_v)$ for each $B\in\mathcal{S}_n(k_v)^+$. These distinguished vectors satisfy the condition

$$w^0_{B[A],v}\approx w^0_{B,v}\circ\Pi_v((\mathbf{m}(A),1))$$

for any $A\in\mathrm{GL}_n(k_v)$.

For $v\in\mathfrak{S}$, $v\notin\mathfrak{S}_\infty$, we choose any family of non-zero vectors $\{w^0_{B,v}\}_{B\in\mathcal{S}_n^{\mathrm{nd}}(k_v)}$, $w^0_{B,v}\in\mathrm{Wh}_B(\Pi_v)$ such that

$$w^0_{B[A],v}\approx w^0_{B,v}\circ\Pi_v((\mathbf{m}(A),1))$$

for any $A\in\mathrm{GL}_n(k_v)$. Such a family exists by Lemma 3.5.

For $B\in\mathcal{S}_n(k)^+$, we put

$$w^0_B=\prod_v w^0_{B,v}\in\mathrm{Wh}_B(\Pi).$$

Note that

$$w^0_{B[A]}\approx w^0_B\circ\Pi(\mathbf{m}(A)).$$

for any $B\in\mathcal{S}_n(k)^+$, $A\in\mathrm{GL}_n(k)$.

Let $\{w_B\}_{B\in\mathcal{S}_n(k)^+}$ be a compatible family. For each $B\in\mathcal{S}_n(k)^+$, there exists a non-negative number $c_B\geq 0$ such that

$$w_B\approx c_B\cdot w^0_B.$$

Proposition 12.3. *If B_1 and B_2 are $\mathfrak{S}$-equivalent and* $\det B_1\equiv\det B_2$ *mod $(k^\times)^2$, then we have $c_{B_1}=c_{B_2}$.*

Proof. Note that the proposition is valid for some choice of $\{w^0_{v,B}\}$, then it is valid for any choice of $\{w^0_{v,B}\}$.

We may assume $\det B_1 = \det B_2$. In particular, the proposition holds for $n = 1$. If $n = 2$, then the representation $\Pi_2 = \Pi(2,\tau)$ can be extended to a cuspidal automorphic representation $\check{\Pi}_2$ of $\mathrm{PGSp}_2(\mathbb{A})$ with trivial central character. Since $\det B_1 = \det B_2$, there exist $t \in k^\times_+$ and $A \in \mathrm{GL}_2(k)$ such that $B_2 = tB_1[A]$. Then we have

$$w_{B_2} = w_{B_1} \circ \check{\Pi}_2(\mathbf{d}(t)\mathbf{m}(A))$$

since $\check{\Pi}_2$ is an automorphic representation on $\mathrm{GSp}_2(\mathbb{A})$. It is enough to prove

$$w^0_{B_2,v} \approx w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(A)).$$

In fact, for $v \notin \mathfrak{S}$, we have

$$w^0_{B_2,v} = w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(A))$$

by Lemma 9.3 (3). For $v \in \mathfrak{S}_\infty$, we have $w^0_{B_2,v} \approx w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(A))$ by Lemma 9.3 (2). For $v \in \mathfrak{S}$, $v \notin \mathfrak{S}_\infty$, there exists $A' \in \mathrm{GL}_2(k_v)$ such that $B_2 = B_1[A'^{-1}]$, since B_1 and B_2 are $\mathfrak{S}$-equivalent. Then we have $B_1 = tB_1[AA']$. Therefore $\mathbf{d}(t)\mathbf{m}(AA')$ and $\mathbf{n}(B_1)$ commute and $t \det AA' = \pm 1$. It follows that

$$\begin{aligned} w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(AA')) =& \mu^{(2)}_v(t \det AA') w^0_{B_1,v} \\ \approx& w^0_{B_1,v} \end{aligned}$$

by Lemma 9.3 (1). Therefore we have

$$\begin{aligned} w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(A)) =& w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{d}(t)\mathbf{m}(AA')) \circ \check{\Pi}_{2,v}(\mathbf{m}(A'^{-1})) \\ \approx& w^0_{B_1,v} \circ \check{\Pi}_{2,v}(\mathbf{m}(A'^{-1})) \\ \approx& w^0_{B_2,v}. \end{aligned}$$

Now assume $n \geq 3$. We may assume $\det B_1 = \det B_2$. Let $\mathfrak{T}$ be the set of places v where B_1 and B_2 are not equivalent. Let $\xi \in k^\times_+$ be an element such that $\xi \hookrightarrow B_1, B_2$. We may assume $\xi \in \mathfrak{o}^\times_v$ for $v \in \mathfrak{T}$, since either B_1 or B_2 is isotropic over k_v. Note that $\mathfrak{T}$ is a finite set and $\mathfrak{T} \cap (\mathfrak{S} \cup \mathfrak{S}') = \emptyset$, where $\mathfrak{S}' = \{v \,|\, v \notin \mathfrak{S}, \mathrm{ord}_v \xi \neq 0\}$. By replacing B_1 and B_2 by equivalent forms, we may assume

$$B_1 = (\xi) \oplus B'_1, \qquad B_2 = (\xi) \oplus B'_2,$$

where $B'_1, B'_2 \in \mathcal{S}_{n-1}(k)^+$. Consider $\mathcal{FJ}_\xi(w_{B_1})$ and $\mathcal{FJ}_\xi(w_{B_2})$. Since w_{B_1} and w_{B_2} belong to a compatible family, $\mathcal{FJ}_\xi(w_{B_1})$ and $\mathcal{FJ}_\xi(w_{B_2})$

also belong to a compatible family for $\Pi_{n-1} = \Pi(n-1, \tau \otimes \chi_\xi)$. Note that

$$\mathcal{FJ}_\xi(w_{B_i}) \approx c_{B_i} \prod_{v \notin \mathfrak{S} \cup \mathfrak{S}'} \mathcal{FJ}_\xi(w^0_{B_i,v}) \prod_{v \in \mathfrak{S}'} \mathcal{FJ}_\xi(w^0_{B_i,v}) \prod_{v \in \mathfrak{S}} \mathcal{FJ}_\xi(w^0_{B_i,v}),$$
$$\approx c_{B_i} \prod_{v \notin \mathfrak{S} \cup \mathfrak{S}'} |D_{B_i}|_v^{1/4} w^0_{B'_i,v} \prod_{v \in \mathfrak{S}'} \mathcal{FJ}_\xi(w^0_{B_i,v}) \prod_{v \in \mathfrak{S}} \mathcal{FJ}_\xi(w^0_{B_i,v})$$

for $i = 1, 2$. For $v \in \mathfrak{S} \cup \mathfrak{S}'$, there exists $A_v \in \mathrm{GL}_{n-1}(k_v)$ such that $B'_2 = B'_1[A_v]$, since B_1 and B_2 are $\mathfrak{S} \cup \mathfrak{S}'$-equivalent. Note that $\det A_v = \pm 1$, since $\det B_1 = \det B_2$. By Lemma 3.4 and Lemma 6.4 (3), we have

$$\mathcal{FJ}_\xi(w^0_{B_2,v}) \approx \mathcal{FJ}_\xi(w^0_{B_1,v}) \circ \Pi_{n-1}((\mathbf{m}(A_v), 1))$$

for $v \in \mathfrak{S} \cup \mathfrak{S}'$. Note that B'_1 and B'_2 are $\mathfrak{S} \cup \mathfrak{S}'$-equivalent and $\det B'_1 = \det B'_2$. By the induction hypothesis, we have $c_{B_1} = c_{B_2}$. $\square$

Lemma 12.4. *Put*

$$Q_{\mathfrak{S}} = \left\{ x \in k^\times \,\middle|\, \begin{array}{ll} 0 \le \mathrm{ord}_v x \le 1 & \text{for } v \notin \mathfrak{S}, \\ 0 \le \mathrm{ord}_v x < 2h_k & \text{for } v \in \mathfrak{S}_{\mathrm{fin}} \end{array} \right\},$$

where h_k is the class number of k. If $\mathfrak{S}$ is sufficiently large, then we have $k^\times = Q_{\mathfrak{S}} \cdot (k^\times)^2$.

Proof. We may assume $\{\mathfrak{p}_v \,|\, v \in \mathfrak{S}, v \notin \mathfrak{S}_\infty\}$ generate the ideal class of k. Consider the map

$$k^\times \to \bigoplus_{v \notin \mathfrak{S}} \mathbb{Z}$$

given by $x \mapsto (\mathrm{ord}_v x)_{v \notin \mathfrak{S}}$. Then this map is surjective. For each $x \in k^\times$, there exists an element $y \in k^\times$ such that $0 \le \mathrm{ord}_v(xy^2) \le 1$ for any $v \notin \mathfrak{S}$. Moreover, we may assume $0 \le \mathrm{ord}_v(xy^2) < 2h_k$ for $v \in \mathfrak{S}_{\mathrm{fin}}$, since $\mathfrak{p}_v^{h_k}$ is a principal ideal for any v. Hence the lemma. $\square$

Lemma 12.5. *Let $\{w_B\}_{B \in \mathcal{S}(k)^+}$ and c_B be as in Proposition 12.3. Then there exist constants $A, M \in \mathbb{R}$, $A > 0$ such that*

$$c_B < A|\mathfrak{d}_B|^M.$$

Here, $\mathfrak{d}_B$ is the conductor of $k(\sqrt{D_B})/k$, as in §4.

Proof. Let $\mathfrak{S}$ be a set of bad places, which satisfies the condition of Lemma 12.4. Since there are only finitely many $\mathfrak{S}$-equivalence classes, it is enough to consider an $\mathfrak{S}$-equivalence class. Fix $S \in \mathcal{S}^+_{n-1}$. By enlarging $\mathfrak{S}$, if necessary, we may assume $S \in \mathcal{S}_{n-1}(\mathfrak{o}_v)$ for $v \notin \mathfrak{S}$. By Proposition 12.3, it is enough to prove that there exist $A, M \in \mathbb{R}$, $A > 0$ such that

$$c_{\xi \oplus S} < A|\mathfrak{d}_{\xi \oplus S}|^M$$

for any $\xi \in k_+^\times$. Put $\mathcal{N}(\xi) = \prod_{v \in \mathfrak{S}_\infty} |\xi|_v$. By Lemma 12.4, we may assume $\xi \in Q_{\mathfrak{S}}$. Note that there exists $A' > 0$ such that

$$|\mathfrak{d}_{\xi \oplus S}| < A' \cdot \mathcal{N}(\xi),$$

for any $\xi \in Q_{\mathfrak{S}}$. As in the proof of Proposition 12.3, we have

$$\mathcal{F}\mathcal{J}_S(w_{\xi \oplus S}) \approx c_{\xi \oplus S} \prod_{v \notin \mathfrak{S}} |\xi|_v^{(n-1)/4} w_{v,\xi}^{\mathrm{ur}} \prod_{v \in \mathfrak{S}} \mathcal{F}\mathcal{J}_S(w_{v,\xi \oplus S}^0).$$

Therefore the problem is reduced to the case $n = 1$. Consider the Fourier series

$$F^\natural(Z) = \sum_{\xi \in k_+^\times} \left(\prod_{v \in \mathfrak{S}_\infty} |\det B|_v^{\kappa_v + (1/2)} \right) W_{\xi,\mathrm{fin}}(\mathbf{1}_2) \mathbf{e}(\xi Z).$$

Since $\xi \in Q_{\mathfrak{S}}$, we have $0 \leq \mathrm{ord}_v \xi \leq 1$ for any $v \notin \mathfrak{S}$. It follows that

$$F^\natural(Z) = \sum_{\xi \in k_+^\times} c_\xi u_\xi \left(\prod_{v \in \mathfrak{S}_\infty} |\det B|_v^{\kappa_v + (1/2)} \right) \prod_{v \in \mathfrak{S}} W_{v,\xi}(\mathbf{1}_2) \mathbf{e}(\xi Z).$$

for some $|u_\xi| = 1$. By Proposition 3.6, for each $v \in \mathfrak{S}_{\mathrm{fin}}$, one can choose a vector $f_v \in \Pi_{1,v}$ such that $|W_{v,\xi}(\mathbf{1}_2)| > C$ for some constant $C > 0$. Since the Fourier coefficients of a Hilbert modular form is slowly increasing, we obtain the proposition. □

Proposition 12.6. *Let $\{w_B\}_{B \in \mathcal{S}(k)^+}$ be a compatible family. For each $f \in \Pi_n^{\mathrm{lwt}} = \Pi(n,\tau)^{\mathrm{lwt}}$, the Fourier series*

$$F(g) = \sum_{B \in \mathcal{S}_n(k)^+} w_B(\Pi_n(g) f)$$

converges absolutely and uniformly on each compact subset of $\widetilde{\mathrm{Sp}_n(\mathbb{A})}$.

Proof. The proposition follows from Proposition 3.9, Lemma 5.5, and Lemma 12.5. □

13. **End of the proof of Theorem 7.1.**

Let $\{w_B\}_{B \in \mathcal{S}_n(k)^+}$ be a compatible family of Whittaker vectors for $\Pi_n = \Pi(n,\tau)$. For each $B \in \mathcal{S}_n(k)^+$, let W_B be a Whittaker function associated to w_B and $f \in \Pi_n^{\mathrm{lwt}}$. To prove Theorem 7.1, it is enough to show that the convergent Fourier series

$$F(g) = \sum_{B \in \mathcal{S}_n(k)^+} W_B(g)$$

is left $\mathrm{Sp}_n(k)$-invariant. Since $\{w_B\}$ is an $\mathrm{GL}_n(k)$-family, $F(g)$ is left $P_n(k)$-invariant. We are going to show that $F(g)$ is left $J_{n,m}(k)$-invariant.

Assume $n = m+n'$ and $S \in \mathcal{S}_n(k)^+$. Consider the "S-th Fourier-Jacobi coefficient"

$$F_S(g) = \int_{Z_m(k)\backslash Z_m(\mathbb{A})} F(zg)\overline{\psi_S(z)}\,dz = \sum_{B=\left(\begin{smallmatrix} S & \lambda \\ {}^t\lambda & N \end{smallmatrix}\right)} W_B(g).$$

Let C be a compact open subgroup of $\widetilde{\mathrm{Sp}_n(\mathbb{A}_{\mathrm{fin}})}$ such that $F(g)$ is right C-invariant. Put $C' = C \cap \widetilde{J_{n,m}(\mathbb{A}_{\mathrm{fin}})}$ and set

$$\mathcal{V} = \{\phi \in \mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}} \mid \omega_S(C')\phi = \phi\}.$$

Then $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{S}(X(\mathbb{A}))^{\mathrm{lwt}}$. Let $\phi_1, \ldots, \phi_r$ be an orthonormal basis of $\mathcal{V}$. Then we have

$$F_S(vg'c) = \sum_{i=1}^{r} \Theta_S^{\phi_i}(vg') \int_{v\in V(k)\backslash V(\mathbb{A})} F(vg')\overline{\Theta_S^{\phi_i}(vg')}\,dv.$$

for any $g' \in \widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$ and $c \in C'$ by Proposition 1.3 of [5]. By Lemma 11.6, we have

$$\begin{aligned}\int_{V(k)\backslash V(\mathbb{A})} & F(vg')\overline{\Theta^{\phi}(vg')}\,dv \\ &= \sum_{B'\in\mathcal{S}^+_{n'}(k)} \int_{X(\mathbb{A})} W_{S\oplus B'}\left(\mathbf{v}(x,0,0)g'\right)\overline{\omega_S(g')\phi(x)}dx,\end{aligned}$$

which is an automorphic form on $\widetilde{\mathrm{Sp}_{n'}(\mathbb{A})}$ by induction hypothesis. Therefore $F_S(g)$ is left $J_{n,m}(k)$-invariant. Since $\mathrm{Sp}_n(k)$ is generated by $P_n(k)$ and $J_{n,m}(k)$, the Fourier series $F(g)$ is left $\mathrm{Sp}_n(k)$-invariant, as desired.

14. Relation to the Arthur conjecture

In this section, we discuss the relation to the Arthur conjecture. Since the Arthur conjecture is not formulated for metaplectic group, we assume the degree is even in this section. We also assume that κ is sufficiently large so that the representation $\mathcal{D}^{(2n)}_{\kappa+n}$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ is a holomorphic discrete series representation. Let $\mathcal{L}_k$ be the (hypothetical) Langlands group over k. Hypothetically, there is a one-to-one correspondence between the set of all equivalence classes of r-dimensional irreducible representations of $\mathcal{L}_k$ and set of all irreducible cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A})$. Let τ be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character. We assume the conditions (A1), (A2), and (A3) hold for τ. Let ρ_τ be the corresponding 2-dimensional irreducible representation of $\mathcal{L}_k$. Note that

$\mathrm{Im}(\rho_\tau) \subset \mathrm{SL}_2(\mathbb{C})$. Let χ be a quadratic Hecke character of $\mathbb{A}_k^\times/k^\times$. Then we have $\mathrm{Im}(\rho_{\tau\otimes\chi}) = \mathrm{Im}(\rho_\tau \otimes \chi) \subset \mathrm{SL}_2(\mathbb{C})$.

Set $G = \mathrm{Sp}_{2n}$. The dual group $\hat{G}$ of G is $\mathrm{SO}_{4n+1}(\mathbb{C})$. Let sym_{2n-1} be the irreducible $2n$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. There is an non-degenerate $\mathrm{SL}_2(\mathbb{C})$-invariant alternating form on sym_{2n-1}. It follows that the $4n$-dimensional representation $\rho_{\tau\otimes\chi} \boxtimes \mathrm{sym_{2n-1}}$ of $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C})$ is orthogonal. It follows that there exists a non-degenerate $\mathcal{L} \times \mathrm{SL}_2(\mathbb{C})$-invariant symmetric bilinear form. Therefore $\rho_{\tau\otimes\chi} \boxtimes \mathrm{sym_{2n-1}}$ gives rise to a homomorphism $\mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C})$. Embedding $\mathrm{SO}_{4n}(\mathbb{C})$ into $\hat{G} = \mathrm{SO}_{4n+1}(\mathbb{C})$, we get a homomorphism

$$(\rho_{\tau\otimes\chi} \boxtimes \mathrm{sym_{2n-1}}) \boxplus 1 : \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}.$$

If we admit the Arthur conjecture, then $\Pi(2n,\tau)$ belongs to the A-packet for $(\rho_{\tau\otimes\chi_{(-1)^n}} \boxtimes \mathrm{sym_{2n-1}}) \boxplus 1$.

Let $\mathrm{st} : \hat{G} = \mathrm{SO}_{4n+1}(\mathbb{C}) \to \mathrm{GL}_{4n+1}(\mathbb{C})$ be the standard representation of $\hat{G}$. Then we have

$$L(s, \Pi(2n,\tau), \mathrm{st}) = \zeta_k(s) \prod_{i=1}^{2n} L(s+n-i+(1/2), \tau \otimes \chi_{(-1)^n}),$$

up to bad Euler factors.

The Arthur conjecture suggest that $m_{\mathrm{auto}}(\Pi(2n,\tau)) = 1$ if and only if

$$\varepsilon(1/2, \tau\otimes\chi_{(-1)^n}) = \prod_{v\in\mathfrak{S}_\infty} (-1)^n = (-1)^{n[k:\mathbb{Q}]}.$$

We claim that

$$\varepsilon(1/2, \tau\otimes\chi_{(-1)^n}) = \varepsilon(1/2,\tau)\cdot(-1)^{n[k:\mathbb{Q}]}$$

under the assumptions (A1) and (A2). In fact,

$$\varepsilon(1/2, (\tau\otimes\chi_{(-1)^n v})_v) = \begin{cases} \varepsilon(1/2,\tau_v)\cdot\langle -1,-1\rangle_v^n & \text{if } v \notin \mathfrak{S}_\infty, \\ \varepsilon(1/2,\tau_v) & \text{if } v \in \mathfrak{S}_\infty. \end{cases}$$

It follows that

$$\begin{aligned} \varepsilon(1/2, \tau\otimes\chi_{(-1)^n}) =& \varepsilon(1/2,\tau) \prod_{v\notin\mathfrak{S}_\infty} \langle -1,-1\rangle_v^n \\ =& \varepsilon(1/2,\tau) \prod_{v\in\mathfrak{S}_\infty} \langle -1,-1\rangle_v^n \\ =& \varepsilon(1/2,\tau)\cdot(-1)^{n[k:\mathbb{Q}]}. \end{aligned}$$

Therefore Theorem 7.1 is compatible with the Arthur conjecture.

15. The case $k = \mathbb{Q}$.

Let

$$f = \sum_{n=1}^{\infty} a(n, f) q^n \in S_{2\kappa}(\Gamma_0(N))$$

be a primitive form of level N, and τ be the irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Then the condition (A2) is satisfied.

We shall explain the condition (A1) and (A3) in terms of classical modular forms. The root number $\varepsilon(1/2, \tau)$ is equal to the sign of the functional equation. It follows that the condition (A3) holds if and only if the L-function $L(s, f) = \sum_n a(n, f) n^{-s}$ of f has a functional equation

$$L(s, f) = N^{2\kappa - 2s} L(2\kappa - s, f).$$

Next, we explain the condition (A1). Recall that for each Dirichlet character η mod M, there exists a primitive form

$$f_\eta = \sum_{n=1}^{\infty} a(n, f_\eta) q^n \in S_{2\kappa}(\Gamma_0(N'), (\eta^2)_0)$$

with the following properties (1) and (2):

(1) $N' | NM^2$.
(2) $a(n, f_\eta) = \eta(n) a(n, f)$ for $(n, NM) = 1$.

Here, $(\eta^2)_0$ be the primitive Dirichlet character equivalent to η^2. Then exactly one of the following three conditions holds:

(a) The condition (A1) holds at $v = p$. In other words, τ_p is a principal series.
(b) τ_p is a quadratic twist of the Steinberg representation.
(c) τ_p is a supercuspidal representation.

If $p \nmid N$, then the condition (A1) holds for $v = p$. The condition (b) holds if and only if $p \,||N'$ for some quadratic Dirichlet character η. The condition (c) holds if and only if $a(p, f_\eta) = 0$ for any Dirichlet character η.

If f satisfy the condition (A1), (A2), and (A3), then Theorem 7.1 implies there exists a Siegel modular form

$$F \in S_{\kappa + (n/2)}(\Gamma)$$

for some congruence subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{Z})$, which is a common eigenform for Hecke operators for $\Gamma \backslash \mathrm{Sp}_n(\mathbb{Z}[1/p]) / \Gamma$ for almost all p. The

Hecke eigenvalue can be calculated by Satake isomorphism (in principle). If $n = 2r$ is even, the standard L-function of F is equal to

$$L(s, F, \mathrm{st}) = \begin{cases} \zeta(s)\prod_{i=1}^{n} L(s + r + \kappa - i, f), & \text{if } r \equiv 0 \bmod 2, \\ \zeta(s)\prod_{i=1}^{n} L(s + r + \kappa - i, f_{\chi(-1)}), & \text{if } r \equiv 1 \bmod 2, \end{cases}$$

up to bad Euler factors. Here, $\chi_{(-1)}$ is the odd primitive Dirichlet character mod 4.

Now let n be an integer such that $\kappa \equiv n \bmod 2$. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and τ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Then, we have

$$\varepsilon(1/2, \tau \otimes \chi_{(-1)^n}) = (-1)^{\kappa+n} = 1.$$

By the result of [6], there exists a Hecke eigenform $F \in S_{\kappa+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$, whose standard L-function is $\zeta(s)\prod_{i=1}^{2n} L(s + n + \kappa - i, f)$. Then one can easily show that the automorphic representation generated by F is isomorphic to $\Pi(2n, \tau \otimes \chi_{(-1)^n})$. By Theorem 7.1, we have $m_{\mathrm{auto}}(\Pi(2n, \tau \otimes \chi_{(-1)^n})) = 1$. It follows that $\Pi(2n, \tau \otimes \chi_{(-1)^n})$ is generated by F. Therefore Theorem 7.1 can be considered as a generalization of [6]. The half-integral weight analogue of [6] was considered by Hayashida [4].

References

[1] J. Arthur, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71.

[2] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55** Birkhäuser Boston, Inc., Boston, Mass. 1985.

[3] D. Ginzburg, S. Rallis, and D. Soudry, *Construction of CAP representations for symplectic groups using the descent method*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin (2005) 193–224,

[4] S. Hayashida, *Fourier-Jacobi expansion and Ikeda lifting*, Automorphic representations, L-functions, and periods (Kyoto, 2006) Su-rikaisekikenkyu-sho Ko-kyu-roku, **1523** (2006) 44–52.

[5] T. Ikeda, *On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series*, J. Math. Kyoto Univ. **34** (1994), 615–636.

[6] ———, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. Math. **154** (2001), 641–682.

[7] M. Karel, *Functional equations of Whittaker functions on p-adic groups*, Amer. J. of Math. **101** (1979), 1303–1325.

[8] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. of Math. **121** (1999), 415–452.

[9] W. Kohnen, *Linear relations between Fourier coefficients of special Siegel modular forms*, Nagoya Math. J. **173** (2004), 153–161.

[10] I. I. Piatetski-Shapiro, *On the Saito-Kurokawa lifting*, Inv. Math. **71** (1983), 309-338.

[11] R. Rao, *On some explicit formulas in the theory of Weil representation*, Pacific J. Math. **157** (1993) 335-371.

[12] P. Sally and M. Tadic, *Induced representations and classification for $GSp(2, F)$ and $Sp(2, F)$*, Mémoires de la S.M. F. 2^e série, **52** (1993) 75–133.

[13] R. Schmidt, *The Saito-Kurokawa lifting and functoriality*, Amer. J. Math. **127** (2005), 209–240.

[14] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[15] ———, *Euler products and Fourier coefficients of automorphic forms on symplectic groups*, Inv. Math. **116** (1994), 531–576.

[16] ———, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, **93** (1997), AMS.

[17] W. J. Sweet, Jr. *A computation of the gamma matrix of a family of p-adic zeta integrals*, J. Number Theory, **55** (1995) 222–260.

[18] J.-L. Waldspurger, *Correspondances de Shimura*, J. Math. pure et appl. **59** (1980), 1–133.

[19] ———, *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), 219–307.

[20] H. Yamashita, *Cayley transform and generalized Whittaker models for irreducible highest weight modules*, Astérisque **273** (2001), 81–137.

Graduate school of mathematics, Kyoto University, Kitashirakawa, Kyoto, 606-8502, Japan

E-mail address: `ikeda@math.kyoto-u.ac.jp`