ON CRITICAL VALUES OF ADJOINT L-FUNCTIONS FOR $\mathrm{GSp}(4)$

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Abstract. We express Petersson norms of certain generic cusp forms on $\mathrm{GSp}(4)$ in terms of critical values of adjoint L-functions for $\mathrm{GSp}(4)$.

1. Introduction

Let $f \in S_k(\mathrm{SL}(2,\mathbb{Z}))$ be a normalized Hecke eigenform and π the irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ determined by f. Let $L(s,\pi,\mathrm{Ad})$ denote the adjoint L-function, which have been studied by Shimura [33] and Gelbart and Jacquet [9]. Then

$$\langle f, f\rangle = 2^{-k} L(1,\pi,\mathrm{Ad}),$$

where

$$\langle f, f\rangle = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} |f(\tau)|^2 \,\mathrm{Im}(\tau)^{k-2}\, d\tau.$$

The purpose of this paper is to prove a certain analogue for $\mathrm{GSp}(4)$.

We now give a precise description of our result. Let

$$\mathrm{GSp}(4) = \left\{ g \in \mathrm{GL}(4) \;\middle|\; g \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix},\ \nu(g) \in \mathbb{G}_m \right\}$$

be the symplectic similitude group in four variables. Let $\pi = \otimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ with trivial central character. We assume that

- π_p is unramified for all primes p,
- $\pi_\infty|_{\mathrm{Sp}(4,\mathbb{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)}$.

Here $D_{(\lambda_1,\lambda_2)}$ is the (limit of) discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$ with Blattner parameter (λ_1,λ_2). By Theorem 6.8.1 of [21], $D_{(\lambda_1,\lambda_2)}$ is large in the sense of Vogan [35, §6]. Hence we may assume that $1-\lambda_1 \le \lambda_2 \le 0$. By [6], [3], π has a functorial lift Π to $\mathrm{GL}(4,\mathbb{A}_\mathbb{Q})$. We say that π is stable (resp. endoscopic) if Π is cuspidal (resp. non-cuspidal).

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We consider a non-zero element $f = \otimes_v f_v \in \pi$ satisfying the following conditions:

$$f_p \text{ is } \mathrm{GSp}(4,\mathbb{Z}_p)\text{-invariant for all primes } p, \tag{1.1}$$

$$f_\infty \in \mathbb{C}v_0. \tag{1.2}$$

Here, with the notation of [30], [28], v_0 is the lowest weight vector of the minimal U(2)-type of $D_{(-\lambda_2,-\lambda_1)}$. Let $W = \otimes_v W_v$ be the Whittaker function of f. To be precise, let $\psi = \otimes_v \psi_v$ be the standard additive character of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$, so that $\psi_\infty(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{R}$. Let

$$U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in \mathrm{GSp}(4) \right\}$$

denote the standard maximal unipotent subgroup of GSp(4). By abuse of notation, we write ψ for the non-degenerate character of $U(\mathbb{A}_\mathbb{Q})$ given by

$$\begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \longmapsto \psi(-x-y).$$

Then

$$W(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})} f(ug)\overline{\psi(u)}\, du$$

for $g \in \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$. By explicit formulas for Whittaker functions [5], [30], [28], $W_v(1) \neq 0$ for all places v. We normalize f so that

$$W(1) = 1. \tag{1.3}$$

By [20], the multiplicity of π in the space of cusp forms on $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ is one. Hence the conditions (1.1)–(1.3) uniquely determine $f \in \pi$. Let

$$\langle f, f\rangle = \int_{\mathbb{A}_\mathbb{Q}^\times \mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})} |f(g)|^2\, dg$$

be the Petersson norm of f, where dg is the Tamagawa measure on $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$.

Our main results is as follows.

Theorem 1.1. *There exists a constant $C_\infty \in \mathbb{C}^\times$ which depends only on π_∞ such that*

$$\langle f, f\rangle = 2^c C_\infty L(1,\pi,\mathrm{Ad}).$$

Here

$$c = \begin{cases} 1 & \textit{if } \pi \textit{ is stable,} \\ 2 & \textit{if } \pi \textit{ is endoscopic,} \end{cases}$$

and $\mathrm{Ad} : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{GL}(10, \mathbb{C})$ *is the adjoint representation.*

Remark 1.2. Let $\mathcal{L}_{\mathbb{Q}}$ be the hypothetical Langlands group of $\mathbb{Q}$. Let $\psi_\pi : \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GSp}(4, \mathbb{C})$ be the (conjectural) Arthur parameter associated to π and $\mathcal{S}_{\psi_\pi}$ the centralizer of the image of ψ_π in $\mathrm{GSp}(4, \mathbb{C})$. Then Arthur's conjecture [1], [2] asserts that

$$|\mathcal{S}_{\psi_\pi}| = \begin{cases} 2 & \text{if } \pi \text{ is stable,} \\ 4 & \text{if } \pi \text{ is endoscopic.} \end{cases}$$

Hence the constant 2^c is related to the theory of endoscopy. (See also [16].)

Remark 1.3. It seems difficult to compute the constant C_∞ which is expressed in terms of local zeta integrals. (See (4.2) and (4.3) for the definition of C_∞.)

Remark 1.4. Theorem 1.1 is compatible with the following speculation about the transcendental part of critical values of adjoint L-functions for GSp(4) in terms of Deligne's conjecture [7]. Let f_{hol} be a Siegel cusp form of degree 2 and of weight k with respect to $\mathrm{Sp}(4, \mathbb{Z})$. We assume that f_{hol} is a Hecke eigenform and is not a Saito-Kurokawa lift. Let π_{hol} be the irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ determined by f_{hol}. Since π_{hol} is unramified at every finite place and is expected to be tempered, Arthur's conjecture [1], [2] predicts that there exists an irreducible generic cuspidal automorphic representation π_{gen} of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that $\Pi = \{\pi_{\mathrm{hol}}, \pi_{\mathrm{gen}}\}$ is an L-packet. Let M be the (hypothetical) motive attached to the spinor L-function $L(s, \Pi)$. Then M would be of rank 4 and of pure weight $2k-3$. Moreover, $H_{\mathrm{DR}}(M)$ would have the Hodge decomposition

$$H_{\mathrm{DR}}(M) \otimes \mathbb{C} \cong H^{2k-3,0} \oplus H^{k-1,k-2} \oplus H^{k-2,k-1} \oplus H^{0,2k-3}$$

with a basis

$$\{f_{\mathrm{hol}}, f_{\mathrm{gen}}, \overline{f_{\mathrm{gen}}}, \overline{f_{\mathrm{hol}}}\}$$

over $\mathbb{C}$. Here f_{gen} is an element of π_{gen} and would also play an important role in Deligne's conjecture [29], [12], [14], [10, §12]. By Yoshida's formula [37, (4.15)],

$$c^+(\mathrm{Sym}^2(M)) = \left(2\pi\sqrt{-1}\right)^{12-6k} c^+(M)c^-(M)\langle f_{\mathrm{hol}}, f_{\mathrm{hol}}\rangle,$$

where $c^+(\mathrm{Sym}^2(M))$ is Deligne's period of $\mathrm{Sym}^2(M)$, etc. Moreover, the (conjectural) relative trace formula of Furusawa and Shalika [8] suggests that

$$\frac{|B_D(1)|^2}{\langle f_{\mathrm{hol}}, f_{\mathrm{hol}}\rangle} \sim L(\tfrac{1}{2}, \Pi) L(\tfrac{1}{2}, \Pi \otimes \chi_D) \frac{|W(1)|^2}{\langle f_{\mathrm{gen}}, f_{\mathrm{gen}}\rangle}.$$

Here $D < 0$ is a fundamental discriminant, χ_D is the Dirichlet character associated to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$, B_D is the D-th Bessel function of f_{hol}, and W is the Whittaker function of f_{gen}. This leads to speculation that

$$c^+(\mathrm{Sym}^2(M)) \sim \frac{\langle f_{\mathrm{gen}}, f_{\mathrm{gen}}\rangle}{|W(1)|^2}.$$

This paper is organized as follows. In §2, we recall two integral representations of automorphic L-functions, which are the main ingredients in the proof of Theorem 1.1. We use the integral representation of the standard L-function $L(s, \pi, \mathrm{St})$ by Piatetski-Shapiro and Rallis [32] and that of the degree 16 L-function $L(s, \pi \times \pi^\vee) = \zeta(s) L(s, \pi, \mathrm{St}) L(s, \pi, \mathrm{Ad})$ by Jiang [18]. In §3, we review the results of Kudla and Rallis [25], Kudla [22], and Jiang [19] on the Siegel-Weil formula. This formula gives relations among the terms in the Laurent expansions of Eisenstein series and will be used to compare the two integral representations. In §4, we finish the proof of Theorem 1.1.

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2. Automorphic L-functions

2.1. **Preliminaries.** Let k be a totally real number field and $\mathbb{A} = \mathbb{A}_k$ the ring of adeles of k. If v is a finite place of k, let $\mathfrak{o}_v$, ϖ_v, q_v be the maximal compact subring of k_v, a generator of the maximal ideal of $\mathfrak{o}_v$, and the cardinality of $\mathfrak{o}_v/\varpi_v\mathfrak{o}_v$, respectively. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/k$. Let $\xi(s) = |D|^{\frac{s}{2}} \prod_{v \leq \infty} \zeta_v(s)$ denote the zeta function of k, where D is the discriminant of k. It satisfies the functional equation $\xi(1-s) = \xi(s)$ and has the Laurent expansion of the form

$$\xi(s) = \frac{\rho}{s-1} + \gamma + O(s-1).$$

Let $H = \mathrm{GSp}(8)$ and

$$\mathbf{G} = \{(g_1, g_2) \in \mathrm{GSp}(4) \times \mathrm{GSp}(4) \mid \nu(g_1) = \nu(g_2)\}.$$

Let Z_H be the center of H. We identify $\mathbf{G}$ with its image under the embedding

$$\mathbf{G} \longrightarrow H.$$

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & -b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{pmatrix}$$

Let $\pi = \otimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character. By [6], [3], π has a functorial lift Π to $\mathrm{GL}(4, \mathbb{A})$. By [11], Π is either cuspidal or $\Pi = \tau_1 \boxplus \tau_2$, where τ_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character such that $\tau_1 \neq \tau_2$. We say that π is stable (resp. endoscopic) if Π is cuspidal (resp. non-cuspidal). Let S be a finite set of places of k including all archimedean places such that, for $v \notin S$, π_v is unramified. Let $\mathrm{St} : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{GL}(5, \mathbb{C})$ and $\mathrm{Ad} : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{GL}(10, \mathbb{C})$ denote the standard representation and the adjoint representation, respectively. Then

$$L^S(s, \Pi, \wedge^2) = \zeta^S(s) L^S(s, \pi, \mathrm{St}),$$
$$L^S(s, \Pi, \mathrm{Sym}^2) = L^S(s, \pi, \mathrm{Ad}).$$

By [11], $L^S(s, \Pi, \wedge^2)$ has a simple (resp. double) pole at $s = 1$ if π is stable (resp. endoscopic). Hence $L^S(s, \pi, \mathrm{St})$ is holomorphic and non-zero (resp. has a simple pole) at $s = 1$ if π is stable (resp. endoscopic). Moreover, $L^S(s, \pi, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$.

Let v be a finite place of k such that π_v is unramified. Let

$$A_v = \mathrm{diag}(\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2) \in \mathrm{GSp}(4, \mathbb{C})$$

be the Satake parameter of π_v. Note that $\alpha_0^2\alpha_1\alpha_2 = 1$. Then Π_v is also unramified and its Satake parameter is $\iota(A_v)$, where $\iota : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{GL}(4, \mathbb{C})$ is the natural embedding. Since Π_v is generic and unitary, we have, by [34],

$$q_v^{-\frac{1}{2}} < |\alpha_0|, |\alpha_0\alpha_1|, |\alpha_0\alpha_2|, |\alpha_0\alpha_1\alpha_2| < q_v^{\frac{1}{2}}.$$

Hence there exists $0 \leq e_v < 1$ such that

$$q_v^{-e_v} \leq |\alpha_1|, |\alpha_2| \leq q_v^{e_v}. \tag{2.1}$$

2.2. **Standard L-functions.** In this section, we review the doubling method of Piatetski-Shapiro and Rallis [32], [13, §6.2].

Let

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \in H \;\middle|\; a \in \mathrm{GL}(4),\, \nu \in \mathbb{G}_m \right\}$$

be the Siegel parabolic subgroup of H. Then the modulus character δ_P of $P(\mathbb{A})$ is given by

$$\delta_P\left(\begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix}\right) = |\det a|^5 |\nu|^{-10}.$$

Let $I(s) = \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\delta_P^{\frac{s}{5}})$ denote the degenerate principal series representation of $H(\mathbb{A})$. For a holomorphic section F of $I(s)$, define a Siegel Eisenstein series $E(s, F)$ by

$$E(h; s, F) = \sum_{\gamma \in P(k)\backslash H(k)} F(\gamma h, s)$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. By Theorem 1.1 of [25], it has at most a simple pole at $s = \frac{1}{2}$.

For $f \in \pi$, let

$$Z(s, f, F) = \int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash \mathbf{G}(\mathbb{A})} E((g_1, g_2); s, F) f(g_1)\overline{f(g_2)}\, dg_1\, dg_2.$$

We assume that $f = \otimes_v f_v$ and $F = \otimes_v F_v$. Choose local pairings $\langle\ ,\ \rangle_v$ on π_v such that $\langle f, f\rangle = \prod_v \langle f_v, f_v\rangle_v$, and define a matrix coefficient ϕ_v of π_v by

$$\phi_v(g) = \langle f_v, f_v\rangle_v^{-1} \langle \pi_v(g) f_v, f_v\rangle_v.$$

Note that ϕ_v does not depend on the choice of $\langle\ ,\ \rangle_v$. Let S be a finite set of places of k including all archimedean places such that, for $v \notin S$,

- π_v is unramified,
- f_v is $\mathrm{GSp}(4, \mathfrak{o}_v)$-invariant,
- F_v is $H(\mathfrak{o}_v)$-invariant and $F_v(1, s) = 1$.

Put $d_P^S(s) = \zeta^S(s + \frac{5}{2})\zeta^S(2s+1)\zeta^S(2s+3)$. Define a local zeta integral $Z_v(s, \phi_v, F_v)$ by

$$Z_v(s, \phi_v, F_v) = \int_{\mathrm{Sp}(4,k_v)} F_v(\delta(g, 1), s)\phi_v(g)\, dg,$$

where

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\mathbf{1}_2 & \frac{1}{2}\mathbf{1}_2 \\ \frac{1}{2}\mathbf{1}_2 & \frac{1}{2}\mathbf{1}_2 & 0 & 0 \\ \mathbf{1}_2 & -\mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}.$$

Then, by [32], [13, §6.2],

$$Z(s, f, F) = \langle f, f\rangle d_P^S(s)^{-1} L^S(s + \tfrac{1}{2}, \pi, \mathrm{St}) \prod_{v \in S} Z_v(s, \phi_v, F_v)$$

for $\mathrm{Re}(s) \gg 0$.

Lemma 2.1. *Let v be a finite place of k such that π_v is unramified. Assume that $\phi_v(k_1 g k_2) = \phi_v(g)$ for all $k_1, k_2 \in \mathrm{GSp}(4, \mathfrak{o}_v)$. Then $Z_v(s, \phi_v, F_v)$ is absolutely convergent for $\mathrm{Re}(s) > e_v - \frac{1}{2}$, where $0 \leq e_v < 1$ is as in (2.1).*

Proof. Let $s \in \mathbb{R}$. Let $K = \mathrm{Sp}(4, \mathfrak{o}_v)$ and

$$A^+ = \{\mathrm{diag}(\varpi_v^{n_1}, \varpi_v^{n_2}, \varpi_v^{-n_1}, \varpi_v^{-n_2}) \mid n_1 \geq n_2 \geq 0\}.$$

Then $\mathrm{Sp}(4, k_v) = KA^+K$. We may assume that F_v is $H(\mathfrak{o}_v)$-invariant and $F_v(1, s) = 1$. In particular,

$$F_v(\delta(k_1 g k_2, 1), s) = F_v(\delta(g, 1), s)$$

for all $k_1, k_2 \in K$. Let

$$a = \mathrm{diag}(\varpi_v^{n_1}, \varpi_v^{n_2}, \varpi_v^{-n_1}, \varpi_v^{-n_2}) \in A^+.$$

By [36, p. 241], there exists a constant $C > 0$ such that

$$\mathrm{vol}(KaK) \leq C q_v^{4n_1 + 2n_2}.$$

By Proposition 6.4 of [32],

$$F_v(\delta(a, 1), s) = q_v^{-(s+\frac{5}{2})(n_1+n_2)}.$$

By Macdonald's formula [27], [4], there exists a polynomial Ψ such that

$$|\phi_v(a)| \leq \Psi(n_1) q_v^{-2n_1 - n_2 + e_v(n_1+n_2)}.$$

Hence $|Z_v(s, \phi_v, F_v)|$ is majorized by

$$\begin{aligned}
&\sum_{a \in A^+} \mathrm{vol}(KaK) F_v(\delta(a, 1), s) |\phi_v(a)| \\
&\leq C \sum_{n_1 = n_2}^{\infty} \sum_{n_2 = 0}^{\infty} \Psi(n_1) q_v^{-(s+\frac{1}{2}-e_v)n_1} q_v^{-(s+\frac{3}{2}-e_v)n_2}.
\end{aligned}$$

This completes the proof. $\square$

Lemma 2.2. *Let v be a real place of k. Assume that the irreducible components of $\pi_v|_{\mathrm{Sp}(4,k_v)}$ are (limit of) discrete series representations of $\mathrm{Sp}(4, k_v)$. Then $Z_v(s, \phi_v, F_v)$ is absolutely convergent for $\mathrm{Re}(s) > -\frac{1}{2}$.*

Proof. Fix $s > -\frac{1}{2}$. By Hölder's inequality, it suffices to show that the integral

$$\int_{\mathrm{Sp}(4,k_v)} |F_v(\delta(g, 1), s)|^{2(1-\epsilon)} \, dg \tag{2.2}$$

is convergent for some $\epsilon > 0$. Let $K = \mathrm{Sp}(4, k_v) \cap \mathrm{O}(4)$ and

$$\mathfrak{a}^+ = \{\mathrm{diag}(X_1, X_2, -X_1, -X_2) \mid X_1 \geq X_2 \geq 0\}.$$

Then $\mathrm{Sp}(4,k_v) = K\exp(\mathfrak{a}^+)K$. We may assume that F_v is $(H(k_v)\cap \mathrm{O}(8))$-invariant and $F_v(1,s)=1$. In particular,

$$F_v(\delta(k_1 g k_2, 1), s) = F_v(\delta(g,1), s)$$

for all $k_1, k_2 \in K$. Let Δ^+ be the set of positive roots determined by the chamber $\mathfrak{a}^+$. Let

$$X = \mathrm{diag}(X_1, X_2, -X_1, -X_2) \in \mathfrak{a}^+$$

and $a_i = \exp(X_i)$. By Proposition 6.4 of [32],

$$F_v(\delta(\exp(X),1),s) = \left(\prod_{i=1}^{2}\sqrt{(1+a_i^2)(1+a_i^{-2})}\right)^{-s-\frac{5}{2}} \leq (a_1a_2)^{-s-\frac{5}{2}}.$$

Obviously,

$$\left|\prod_{\alpha\in\Delta^+}\sinh(\alpha(X))\right| \leq a_1^4 a_2^2.$$

Hence the integral (2.2) is majorized by

$$\int_{\mathfrak{a}^+}\int_{K\times K} F_v(\delta(k_1\exp(X)k_2,1),s)^{2(1-\epsilon)}\left|\prod_{\alpha\in\Delta^+}\sinh(\alpha(X))\right| dk_1\,dk_2\,dX$$
$$\leq \int_{a_2}^{\infty}\int_{1}^{\infty} a_1^{-2s-1+2\epsilon s+5\epsilon} a_2^{-2s-3+2\epsilon s+5\epsilon}\, d^\times a_2\, d^\times a_1.$$

This completes the proof. $\square$

2.3. ***L*-functions for** $\mathrm{GSp}(4)\times\mathrm{GSp}(4)$**.** In this section, we review Jiang's integral representation of the L-function for $\mathrm{GSp}(4)\times\mathrm{GSp}(4)$ [18].

Let

$$Q = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu\, {}^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in H \;\middle|\; \begin{array}{l} a \in \mathrm{GL}(3) \\ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}(2) \\ \nu = a'd' - b'c' \in \mathbb{G}_m \end{array} \right\}$$

be a maximal parabolic subgroup of H. Then the modulus character δ_Q of $Q(\mathbb{A})$ is given by

$$\delta_Q\left(\begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu\, {}^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix}\right) = |\det a|^6 |\nu|^{-9}.$$

Let $\mathcal{I}(s) = \mathrm{Ind}_{Q(\mathbb{A})}^{H(\mathbb{A})}(\delta_Q^{\frac{s}{5}})$ denote the degenerate principal series representation of $H(\mathbb{A})$. For a holomorphic section $\mathcal{F}$ of $\mathcal{I}(s)$, define an Eisenstein series $\mathcal{E}(s, \mathcal{F})$ by

$$\mathcal{E}(h; s, \mathcal{F}) = \sum_{\gamma \in Q(k)\backslash H(k)} \mathcal{F}(\gamma h, s)$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. By Theorem 4.0.1 of [18, Chapter 3], it has at most a double pole at $s = 1$.

For $f \in \pi$, let

$$\mathcal{Z}(s, f, \mathcal{F}) = \int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash \mathbf{G}(\mathbb{A})} \mathcal{E}((g_1, g_2); s, \mathcal{F}) f(g_1)\overline{f(g_2)}\, dg_1\, dg_2.$$

We assume that $f = \otimes_v f_v$ and $\mathcal{F} = \otimes_v \mathcal{F}_v$. Let $W = \otimes_v W_v$ be the Whittaker function of f. Let S be a finite set of places of k including all archimedean places such that, for $v \notin S$,

- π_v and ψ_v are unramified,
- f_v is $\mathrm{GSp}(4, \mathfrak{o}_v)$-invariant and $W_v(1) = 1$,
- $\mathcal{F}_v$ is $H(\mathfrak{o}_v)$-invariant and $\mathcal{F}_v(1, s) = 1$.

Put $d_Q^S(s) = \zeta^S(s+1)\zeta^S(s+2)\zeta^S(s+3)\zeta^S(2s+2)$. Define a local zeta integral $\mathcal{Z}_v(s, W_v, \mathcal{F}_v)$ by

$$\mathcal{Z}_v(s, W_v, \mathcal{F}_v) = \int_{Z_H(k_v)\tilde{U}(k_v)\backslash \mathbf{G}(k_v)} \mathcal{F}_v(\eta(g_1, g_2), s) W_v(g_1)\overline{W_v(g_2)}\, dg_1\, dg_2,$$

where

$$\tilde{U} = \left\{ \left(u \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, u \right) \,\middle|\, u \in U,\ x \in \mathbb{G}_a \right\}$$

and

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, by [18],

$$\mathcal{Z}(s, f, \mathcal{F}) = d_Q^S(s)^{-1} L^S(\tfrac{s+1}{2}, \pi \times \pi^\vee) \prod_{v \in S} \mathcal{Z}_v(s, W_v, \mathcal{F}_v)$$

for $\mathrm{Re}(s) \gg 0$.

Let v be a real place of k. By Theorem 3.2.1 of [18, Chapter 5], $\mathcal{Z}_v(s, W_v, \mathcal{F}_v)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, but the meromorphic continuation of $\mathcal{Z}_v(s, W_v, \mathcal{F}_v)$ has not been proved.

3. The Siegel-Weil formula

3.1. **Eisenstein series.** Let n, r be positive integers such that $n \geq r$. Let $G = \mathrm{Sp}(2n)$ denote the symplectic group of rank n defined by

$$\mathrm{Sp}(2n) = \left\{ g \in \mathrm{GL}(2n) \;\middle|\; g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \right\}.$$

Let

$$P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in G \;\middle|\; \begin{array}{l} a \in \mathrm{GL}(r) \\ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{Sp}(2n-2r) \end{array} \right\}$$

be a maximal parabolic subgroup of G. We define a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ by

$$K_v = \begin{cases} G(\mathfrak{o}_v) & \text{if } v \text{ is finite,} \\ G(k_v) \cap \mathrm{O}(2n) & \text{if } v \text{ is real.} \end{cases}$$

Let $\mathfrak{g}_\infty$ be the Lie algebra of $\prod_{v|\infty} G(k_v)$ and $K_\infty = \prod_{v|\infty} K_v$ a maximal compact subgroup of $\prod_{v|\infty} G(k_v)$.

Let $I_{n,r}(s) = \mathrm{Ind}_{P_{n,r}(\mathbb{A})}^{G(\mathbb{A})}(|\det|^s \otimes \mathbf{1})$ denote the degenerate principal series representation of $G(\mathbb{A})$. For a holomorphic section F of $I_{n,r}(s)$, define an Eisenstein series $E^{(n,r)}(s, F)$ by

$$E^{(n,r)}(g; s, F) = \sum_{\gamma \in P_{n,r}(k) \backslash G(k)} F(\gamma g, s)$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. Let

$$E^{(n,r)}(s, F) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(n,r)}(s_0, F)$$

be the Laurent expansion of $E^{(n,r)}(s, F)$ at $s = s_0$. When F is K-invariant and $F(1, s) = 1$, we write $E^{(n,r)}(s) = E^{(n,r)}(s, F)$ and $E_d^{(n,r)}(s_0) = E_d^{(n,r)}(s_0, F)$.

3.2. **Theta integrals.** In this section, we review the result of Kudla and Rallis [25] on the regularization of theta integrals.

Assume that $2 \leq r \leq n$. Let $V_{r,r} = k^{2r}$ be the space of column vectors equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$ given by

$$(x, y) = {}^t x \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} y$$

for $x, y \in V_{r,r}$. Let $G' = \mathrm{O}(r, r)$ denote the orthogonal group of $V_{r,r}$ defined by

$$\mathrm{O}(r, r) = \left\{ g' \in \mathrm{GL}(2r) \,\middle|\, {}^t g' \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} g' = \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} \right\}.$$

Let

$$P' = \left\{ \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \in G' \,\middle|\, a \in \mathrm{GL}(r) \right\}$$

be the Siegel parabolic subgroup of G' and N' its unipotent radical. We define a maximal compact subgroup $K' = \prod_v K'_v$ of $G'(\mathbb{A})$ by

$$K'_v = \begin{cases} G'(\mathfrak{o}_v) & \text{if } v \text{ is finite,} \\ G'(k_v) \cap \mathrm{O}(2r) & \text{if } v \text{ is real.} \end{cases}$$

Let dg' be the Haar measure on $G'(\mathbb{A})$ such that $\mathrm{vol}(G'(k)\backslash G'(\mathbb{A})) = 1$, da (resp. dn') the Tamagawa measure on $\mathrm{GL}(r, \mathbb{A})$ (resp. $N'(\mathbb{A})$), and dk' the Haar measure on K' such that $\mathrm{vol}(K') = 1$. By [17, §9],

$$\int_{G'(\mathbb{A})} \phi(g')\, dg' = C_r \int_{\mathrm{GL}(r,\mathbb{A}) \times N'(\mathbb{A}) \times K'} \phi\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} n'k' \right) da\, dn'\, dk'$$

for $\phi \in L^1(G'(\mathbb{A}))$, where

$$C_r = \frac{\rho}{\xi(r)} \prod_{i=1}^{\left[\frac{r-1}{2}\right]} \frac{\xi(2i+1)}{\xi(2r-2i)}.$$

Let $\omega = \omega_{n,V_{r,r}}$ denote the Weil representation of $G(\mathbb{A}) \times G'(\mathbb{A})$ on $\mathcal{S}(V_{r,r}(\mathbb{A})^n) = \mathcal{S}(\mathrm{Mat}_{2r\times n}(\mathbb{A}))$ with respect to ψ. Let $S_{n,r}$ be the subspace of $\mathcal{S}(V_{r,r}(\mathbb{A})^n)$ consisting of functions which correspond to polynomials in the Fock model at every archimedean place. For $g \in G(\mathbb{A})$, $g' \in G'(\mathbb{A})$, and $\Phi \in S_{n,r}$, let

$$\Theta(g, g'; \Phi) = \sum_{x \in V_{r,r}(k)^n} \omega(g, g')\Phi(x).$$

Fix a real place v of k and let $z = z_{r-1,n} \in \mathfrak{z}(\mathfrak{g}_v)$ be the regularizing differential operator as in Corollary 5.1.2 of [25]. By Proposition 5.3.1

of [25], the function $g' \mapsto \Theta(g, g'; z \cdot \Phi)$ on $G'(k)\backslash G'(\mathbb{A})$ is rapidly decreasing. Here $\mathfrak{z}(\mathfrak{g}_v)$ acts on $S_{n,r}$ via the differential of ω.

Put $s_0' = \frac{r-1}{2}$. Let F be the K'-invariant holomorphic section of $\mathrm{Ind}_{P'(\mathbb{A})}^{G'(\mathbb{A})}(|\det|^s)$ such that $F(1, s) = 1$. Define an auxiliary Eisenstein series $E(s)$ by

$$E(g'; s) = \sum_{\gamma' \in P'(k)\backslash G'(k)} F(\gamma' g', s)$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. Note that $E(s)$ has a simple pole at $s = s_0'$ with constant residue. Following [25, §5.5], we consider the integral

$$I^{(n,r)}(g; s, \Phi) = \rho_{n,r}(s)^{-1} \int_{G'(k)\backslash G'(\mathbb{A})} \Theta(g, g'; z \cdot \Phi) E(g'; s)\, dg',$$

where

$$\rho_{n,r}(s) = \prod_{i=0}^{r-1} ((s - s_0' + i)^2 - (n + 1 - r)^2).$$

Let

$$I^{(n,r)}(s, \Phi) = \sum_{d \gg -\infty} (s - s_0')^d I_d^{(n,r)}(\Phi)$$

be the Laurent expansion of $I^{(n,r)}(s, \Phi)$ at $s = s_0'$. Note that $I^{(n,r)}(s, \Phi)$ has at most a simple (resp. double) pole at $s = s_0'$ if $r \le \frac{n+1}{2}$ (resp. $\frac{n+1}{2} < r \le n$).

Let $\hat{\omega}$ be the Weil representation of $G(\mathbb{A}) \times G'(\mathbb{A})$ on $\mathcal{S}(\mathrm{Mat}_{r\times 2n}(\mathbb{A}))$ defined via the partial Fourier transform

$$\begin{aligned} \mathcal{S}(\mathrm{Mat}_{2r\times n}(\mathbb{A})) &\longrightarrow \mathcal{S}(\mathrm{Mat}_{r\times 2n}(\mathbb{A})), \\ \Phi &\longmapsto \hat{\Phi} \end{aligned}$$

where

$$\hat{\Phi}(u, v) = \int_{\mathrm{Mat}_{r\times n}(\mathbb{A})} \Phi \begin{pmatrix} x \\ u \end{pmatrix} \psi(\mathrm{tr}(v\,{}^t x))\, dx$$

for $u, v \in \mathrm{Mat}_{r\times n}(\mathbb{A})$. For $\mathrm{Re}(s) > -n + \frac{3r-1}{2}$, define a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-intertwining map

$$F : S_{n,r} \longrightarrow I_{n,r}(s)$$

by

$$F(\Phi)(g, s) = C_r \int_{\mathrm{GL}(r,\mathbb{A})} \hat{\omega}(g, 1)\widehat{\Phi_{K'}}(0_{r\times n}, {}^t a, 0_{r\times(n-r)})|\det a|^{s+n-\frac{r-1}{2}}\, da,$$

where

$$\Phi_{K'}(x) = \int_{K'} \omega(1, k')\Phi(x)\, dk'.$$

Then, by [25, §5.5],

$$I^{(n,r)}(s,\Phi) = E^{(n,r)}(s, F(\Phi)).$$

3.3. **First and second term identities.** In this section, we give certain identities relating the terms in the Laurent expansions of regularized theta integrals to that of Siegel Eisenstein series, which have been studied by Kudla and Rallis [25], Kudla [22], and Jiang [19].

Let $n = 4$. Assume that $k = \mathbb{Q}$. Then $\rho = \mathrm{Res}_{s=1}\,\xi(s) = 1$. Let ψ be the standard additive character of $\mathbb{A}/\mathbb{Q}$. Let $\mathcal{A}$ denote the space of automorphic forms on $G(\mathbb{A}) = \mathrm{Sp}(8,\mathbb{A})$ and $\mathcal{R}$ the subspace of $\mathcal{A}$ spanned by residues $E^{(4,4)}_{-1}(\frac{1}{2}, F)$, where F is a holomorphic section of $I_{4,4}(s)$. Define a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-intertwining map

$$\begin{aligned} S_{4,3} &\longrightarrow I_{4,4}(\tfrac{1}{2}) \\ \Phi &\longmapsto F_\Phi \end{aligned}$$

by

$$F_\Phi(g, \tfrac{1}{2}) = \omega(g,1)\Phi(0).$$

We extend F_Φ to the holomorphic section F_Φ of $I_{4,4}(s)$ such that its restriction to K is independent of s. We define $\Phi^0 = \otimes_v \Phi^0_v \in S_{4,3}$ as follows:

- Φ^0_p is the characteristic function of $V_{3,3}(\mathbb{Z}_p)^4$ for all primes p,
- $\Phi^0_\infty(x) = \exp(-\pi\,\mathrm{tr}(x{}^t\!x))$.

Then Φ^0 is $(K \times K')$-invariant. By a direct calculation,

$$I^{(4,3)}(s,\Phi^0) = \frac{\xi(s+1)\xi(s+2)\xi(s+3)}{\xi(2)\xi(3)\xi(4)} E^{(4,3)}(s).$$

Hence, by Corollary 6.3 of [25] and Lemma A.8, for all $\Phi \in S_{4,3}$,

$$I^{(4,3)}_{-2}(\Phi) = \frac{1}{\xi(4)} E^{(4,4)}_{-1}(\tfrac{1}{2}, F_\Phi). \tag{3.1}$$

Let $S^0_{4,3}$ be the $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-submodule of $S_{4,3}$ generated by Φ^0. As in [17, §5], define a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-intertwining map

$$\mathrm{pr} : S_{4,3} \longrightarrow S_{4,2}$$

by

$$\mathrm{pr}(\Phi)\begin{pmatrix} x \\ y \end{pmatrix} = \int_{\mathbb{A}^4} \Phi_{K'}\begin{pmatrix} u \\ x \\ 0 \\ y \end{pmatrix} du$$

for $x, y \in \mathrm{Mat}_{2\times 4}(\mathbb{A})$. By a direct calculation,

$$I^{(4,2)}(s, \mathrm{pr}(\Phi^0)) = \frac{\xi(s+\frac{5}{2})\xi(s+\frac{7}{2})}{\xi(2)^2} E^{(4,2)}(s).$$

Proposition 3.1. *For all* $\Phi \in S^0_{4,3}$,

$$I^{(4,3)}_{-1}(\Phi) - \frac{\xi(2)^2}{\xi(3)\xi(4)^2} I^{(4,2)}_0(\mathrm{pr}(\Phi)) - \frac{1}{\xi(4)} E^{(4,4)}_0(\tfrac{1}{2}, F_\Phi) \in \mathcal{R}.$$

Proof. Define $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-intertwining maps

$$\begin{aligned} A_1 &: S_{4,3} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{R}, \\ A_2 &: S_{4,3} \longrightarrow S_{4,2} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{R}, \\ A_3 &: S_{4,3} \longrightarrow I_{4,4}(\tfrac{1}{2}) \longrightarrow \mathcal{A}/\mathcal{R}, \end{aligned}$$

by

$$A_1(\Phi) = I^{(4,3)}_{-1}(\Phi), \quad A_2(\Phi) = I^{(4,2)}_0(\mathrm{pr}(\Phi)), \quad A_3(\Phi) = E^{(4,4)}_0(\tfrac{1}{2}, F_\Phi).$$

Put

$$A = A_1 - \frac{\xi(2)^2}{\xi(3)\xi(4)^2} A_2 - \frac{1}{\xi(4)} A_3.$$

Since the map

$$A : S_{4,3} \longrightarrow \mathcal{A}/\mathcal{R}$$

is $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-intertwining, it suffices to show that

$$A(\Phi^0) \equiv 0 \mod \mathcal{R}.$$

By Lemma A.8,

$$\begin{aligned} I^{(4,3)}_{-1}(\Phi^0) &\equiv E^{(4,3)}_{-1}(1) \mod \mathcal{R}, \\ I^{(4,2)}_0(\mathrm{pr}(\Phi^0)) &\equiv \frac{\xi(3)\xi(4)}{\xi(2)^2} E^{(4,2)}_0(\tfrac{1}{2}) \mod \mathcal{R}. \end{aligned}$$

Hence, by Lemma A.9,

$$A(\Phi^0) \equiv E^{(4,3)}_{-1}(1) - \frac{1}{\xi(4)} E^{(4,2)}_0(\tfrac{1}{2}) - \frac{1}{\xi(4)} E^{(4,4)}_0(\tfrac{1}{2}) \equiv 0 \mod \mathcal{R}.$$

This completes the proof. □

4. Proof of Theorem 1.1

Let $f \in \pi$ be as in §1. We normalize W_v so that $W_v(1) = 1$ for all places v. For an automorphic form φ on $\mathrm{Sp}(8, \mathbb{A})$, let

$$\langle \varphi, \bar{f} \otimes f \rangle = \int_{(\mathrm{Sp}(4,\mathbb{Q})\backslash \mathrm{Sp}(4,\mathbb{A}))^2} \varphi((g_1, g_2)) f(g_1) \overline{f(g_2)} \, dg_1 \, dg_2.$$

Lemma 4.1. *Assume that π is stable. For all $\varphi \in \mathcal{R}$,*

$$\langle \varphi, \bar{f} \otimes f \rangle = 0.$$

Proof. Since $L^S(s, \pi, \mathrm{St})$ is holomorphic at $s = 1$, the assertion follows. □

Lemma 4.2. *Assume that π is stable. For all $d \in \mathbb{Z}$ and all $\Phi \in S_{4,2}$,*

$$\langle I_d^{(4,2)}(\Phi), \bar{f} \otimes f \rangle = 0.$$

Proof. Let π_0 be an irreducible component of the restriction of π to $\mathrm{Sp}(4, \mathbb{A})$. Let $\theta_{r,r}(\pi_0)$ be the theta lift of π_0 to $\mathrm{O}(r, r)(\mathbb{A})$. By (2.1), $\theta_{1,1}(\pi_0) = 0$. If $\theta_{2,2}(\pi_0) \neq 0$, then

$$L^S(s, \pi, \mathrm{St}) = \zeta^S(s) L^S(s, \tau_1 \times \tau_2),$$

where S is a sufficiently large finite set of places of $\mathbb{Q}$ and τ_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$. This contradicts the holomorphy of $L^S(s, \pi, \mathrm{St})$ at $s = 1$. Hence

$$\theta_{2,2}(\pi_0) = 0. \tag{4.1}$$

Let $z \cdot \Phi = \sum_j \Phi_{1j} \otimes \overline{\Phi_{2j}} \in S_{4,2}$ with $\Phi_{ij} \in S_{2,2}$. Then

$$\Theta((g_1, g_2), g'; z \cdot \Phi) = \sum_j \Theta(g_1, g'; \Phi_{1j}) \overline{\Theta(g_2, g'; \Phi_{2j})}.$$

By (4.1), the theta lift $\theta(f, \Phi_{ij})$ is identically zero, where

$$\theta(g'; f, \Phi_{ij}) = \int_{\mathrm{Sp}(4,\mathbb{Q})\backslash \mathrm{Sp}(4,\mathbb{A})} \Theta(g, g'; \Phi_{ij}) f(g) \, dg$$

for $g' \in \mathrm{O}(2, 2)(\mathbb{A})$. Hence $\langle I^{(4,2)}(s, \Phi), \bar{f} \otimes f \rangle$ is equal to

$$\begin{aligned}
&\rho_{4,2}(s)^{-1} \int_{(\mathrm{Sp}(4,\mathbb{Q})\backslash \mathrm{Sp}(4,\mathbb{A}))^2} \int_{\mathrm{O}(2,2)(\mathbb{Q})\backslash \mathrm{O}(2,2)(\mathbb{A})} \\
&\times \Theta((g_1, g_2), g'; z \cdot \Phi) E(g'; s) f(g_1) \overline{f(g_2)} \, dg' \, dg_1 \, dg_2 \\
&= \rho_{4,2}(s)^{-1} \int_{\mathrm{O}(2,2)(\mathbb{Q})\backslash \mathrm{O}(2,2)(\mathbb{A})} \sum_j \theta(g'; f, \Phi_{1j}) \overline{\theta(g'; f, \Phi_{2j})} E(g'; s) \, dg' \\
&= 0.
\end{aligned}$$

This completes the proof. □

Let $\Phi = \otimes_v \Phi_v \in S_{4,3}$ such that $\Phi_p = \Phi_p^0$ for all primes p. Then there exists a holomorphic section $F = \otimes_v F_v$ of $I(s)$ satisfying the following conditions:

- F_p is $H(\mathbb{Z}_p)$-invariant and $F_p(1,s) = 1$ for all primes p,
- $F_\infty = F_{\Phi_\infty}$ depends only on Φ_∞,
- $E(s,F) = E^{(4,4)}(s, F_\Phi)$ on $\mathrm{Sp}(8,\mathbb{A})$.

Also, there exists a holomorphic section $\mathcal{F} = \otimes_v \mathcal{F}_v$ of $\mathcal{I}(s)$ for $\mathrm{Re}(s) > 0$ satisfying the following conditions:

- $\mathcal{F}_p$ is $H(\mathbb{Z}_p)$-invariant and $\mathcal{F}_p(1,s) = 1$ for all primes p,
- $\mathcal{F}_\infty = \mathcal{F}_{\Phi_\infty}$ depends only on Φ_∞,
- $\mathcal{E}(s,\mathcal{F}) = I^{(4,3)}(s,\Phi)$ on $\mathrm{Sp}(8,\mathbb{A})$.

Since f is $\mathrm{GSp}(4,\hat{\mathbb{Z}})$-invariant, we have

$$\langle E^{(4,4)}(s,F_\Phi), \bar{f} \otimes f\rangle = \langle f, f\rangle d_{P,\mathrm{fin}}(s)^{-1} L_{\mathrm{fin}}(s+\tfrac{1}{2}, \pi, \mathrm{St}) Z_\infty(s, \phi_\infty, F_{\Phi_\infty}),$$
$$\langle I^{(4,3)}(s,\Phi), \bar{f} \otimes f\rangle = d_{Q,\mathrm{fin}}(s)^{-1} L_{\mathrm{fin}}(\tfrac{s+1}{2}, \pi \times \pi^\vee) \mathcal{Z}_\infty(s, W_\infty, \mathcal{F}_{\Phi_\infty}).$$

Let $S_{4,3,\infty}$ (resp. $S^0_{4,3,\infty}$) be the archimedean component of $S_{4,3}$ (resp. $S^0_{4,3}$).

Lemma 4.3. *For all $\Phi_\infty \in S_{4,3,\infty}$, $\mathcal{Z}_\infty(s, W_\infty, \mathcal{F}_{\Phi_\infty})$ has a meromorphic continuation and is holomorphic at $s = 1$.*

Proof. By (3.1) and Lemma 4.1, $\langle I^{(4,3)}(s,\Phi), \bar{f} \otimes f\rangle$ has at most a simple (resp. double) pole at $s = 1$ if π is stable (resp. endoscopic). Since $L_{\mathrm{fin}}(s, \pi \times \pi^\vee) = L_{\mathrm{fin}}(s, \Pi \times \Pi^\vee)$ has a simple (resp. double) pole at $s = 1$ if π is stable (resp. endoscopic), the assertion follows. $\square$

Lemma 4.4. *There exists $\Phi_\infty \in S^0_{4,3,\infty}$ such that*

$$Z_\infty(\tfrac{1}{2}, \phi_\infty, F_{\Phi_\infty}) \neq 0.$$

Proof. For convenience, we suppress the subscript ∞ from the notation. Let p, q be odd positive integers such that $p + q = 6$. Let $V_{p,q}$ denote the quadratic space over $\mathbb{R}$ of signature (p,q) and $\omega_{4,V_{p,q}}$ the Weil representation of $\mathrm{Sp}(8,\mathbb{R}) \times \mathrm{O}(p,q)$ on $S(V^4_{p,q})$. Let $R(p,q)$ be the image of the $(\mathfrak{sp}(8,\mathbb{R}), \mathrm{U}(4))$-intertwining map

$$\begin{aligned} S(V^4_{p,q}) &\longrightarrow I_{4,4}(\tfrac{1}{2}), \\ \Phi &\longmapsto F_\Phi \end{aligned}$$

where $F_\Phi(g, \frac{1}{2}) = \omega_{4,V_{p,q}}(g,1)\Phi(0)$. By [23], [26],

$$I_{4,4}(\tfrac{1}{2}) = R(5,1) + R(3,3) + R(1,5).$$

Let $S(V^4_{3,3})^0$ be the $(\mathfrak{sp}(8,\mathbb{R}), \mathrm{U}(4))$-submodule of $S(V^4_{3,3})$ generated by the Gaussian Φ^0. Since $R(3,3)$ is generated by F_{Φ^0}, the natural map

$$S(V^4_{3,3})^0 \longrightarrow R(3,3)$$

is surjective.

Let π_0 be a (limit of) discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$. Fix a pairing $\langle\ ,\ \rangle$ on π_0. For $f_1, f_2 \in \pi_0$, define a matrix coefficient $\phi_{f_1\otimes f_2}$ of π_0 by

$$\phi_{f_1\otimes f_2}(g) = \langle \pi_0(g) f_1, f_2\rangle.$$

Then the local zeta integral $Z(\frac{1}{2}, \phi_{f_1\otimes f_2}, F)$ is absolutely convergent and defines a $(\mathfrak{sp}(4,\mathbb{R}), \mathrm{U}(2)) \times (\mathfrak{sp}(4,\mathbb{R}), \mathrm{U}(2))$-intertwining map

$$\begin{aligned} I_{4,4}(\tfrac{1}{2}) &\longrightarrow \pi_0^\vee \otimes \pi_0. \\ F &\longmapsto [f_1\otimes f_2 \mapsto Z(\tfrac{1}{2}, \phi_{f_1\otimes f_2}, F)] \end{aligned}$$

By Corollary 3.2.3 of [24], for fixed non-zero elements $f_1, f_2 \in \pi_0$, there exists F such that $Z(\frac{1}{2}, \phi_{f_1\otimes f_2}, F) \neq 0$.

Assume that π_0 is large. Let $\theta_{p,q}(\pi_0^\vee)$ be the theta lift of $\pi_0^\vee$ to $\mathrm{O}(p,q)$. By Theorem 15 of [31], $\theta_{5,1}(\pi_0^\vee) = \theta_{1,5}(\pi_0^\vee) = 0$. Hence, as in the proof of Proposition 3.1 of [15],

$$\begin{aligned} \mathrm{Hom}_{(\mathfrak{sp}(4,\mathbb{R}),\mathrm{U}(2))\times(\mathfrak{sp}(4,\mathbb{R}),\mathrm{U}(2))}(R(5,1), \pi_0^\vee\otimes\pi_0) = 0, \\ \mathrm{Hom}_{(\mathfrak{sp}(4,\mathbb{R}),\mathrm{U}(2))\times(\mathfrak{sp}(4,\mathbb{R}),\mathrm{U}(2))}(R(1,5), \pi_0^\vee\otimes\pi_0) = 0. \end{aligned}$$

This completes the proof. □

Let $\Phi = \otimes_v \Phi_v \in S^0_{4,3}$ such that $\Phi_p = \Phi_p^0$ for all primes p. By Proposition 3.1 and Lemmas 4.1, 4.2,

$$\langle I_{-1}^{(4,3)}(\Phi), \bar{f}\otimes f\rangle = \xi(4)^{-1}\langle E_0^{(4,4)}(\tfrac{1}{2}, F_\Phi), \bar{f}\otimes f\rangle$$

if π is stable. By (3.1),

$$\langle I_{-2}^{(4,3)}(\Phi), \bar{f}\otimes f\rangle = \xi(4)^{-1}\langle E_{-1}^{(4,4)}(\tfrac{1}{2}, F_\Phi), \bar{f}\otimes f\rangle$$

if π is endoscopic. Hence

$$\begin{aligned} &2 d_{Q,\mathrm{fin}}(1)^{-1} L_{\mathrm{fin}}(1,\pi,\mathrm{St}) L_{\mathrm{fin}}(1,\pi,\mathrm{Ad}) \mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_{\Phi_\infty}) \\ &\quad = \xi(4)^{-1}\langle f, f\rangle d_{P,\mathrm{fin}}(\tfrac{1}{2})^{-1} L_{\mathrm{fin}}(1,\pi,\mathrm{St}) Z_\infty(\tfrac{1}{2}, \phi_\infty, F_{\Phi_\infty}) \end{aligned}$$

if π is stable and

$$\begin{aligned} &4 d_{Q,\mathrm{fin}}(1)^{-1} \mathrm{Res}_{s=1} L_{\mathrm{fin}}(s,\pi,\mathrm{St}) L_{\mathrm{fin}}(1,\pi,\mathrm{Ad}) \mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_{\Phi_\infty}) \\ &\quad = \xi(4)^{-1}\langle f, f\rangle d_{P,\mathrm{fin}}(\tfrac{1}{2})^{-1} \mathrm{Res}_{s=1} L_{\mathrm{fin}}(s,\pi,\mathrm{St}) Z_\infty(\tfrac{1}{2}, \phi_\infty, F_{\Phi_\infty}) \end{aligned}$$

if π is endoscopic. Since $L_{\mathrm{fin}}(1,\pi,\mathrm{St}) \neq 0$ (resp. $\mathrm{Res}_{s=1} L_{\mathrm{fin}}(s,\pi,\mathrm{St}) \neq 0$) if π is stable (resp. endoscopic) and $L_\infty(s,\pi_\infty,\mathrm{Ad})$ is holomorphic and non-zero at $s=1$, we have

$$2^c L(1,\pi,\mathrm{Ad}) \mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_{\Phi_\infty}) = C'_\infty \langle f, f\rangle Z_\infty(\tfrac{1}{2}, \phi_\infty, F_{\Phi_\infty}).$$

Here

$$c = \begin{cases} 1 & \text{if } \pi \text{ is stable,} \\ 2 & \text{if } \pi \text{ is endoscopic,} \end{cases}$$

and

$$C'_\infty = \zeta_\infty(4)^{-1} L_\infty(1, \pi_\infty, \mathrm{Ad}) \in \mathbb{C}^\times. \tag{4.2}$$

By Lemma 4.4, we may assume that $Z_\infty(\frac{1}{2}, \phi_\infty, F_{\Phi_\infty}) \neq 0$. Then $\mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_{\Phi_\infty}) \neq 0$ and

$$2^c \frac{L(1, \pi, \mathrm{Ad})}{\langle f, f \rangle} = C'_\infty \frac{Z_\infty(\frac{1}{2}, \phi_\infty, F_{\Phi_\infty})}{\mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_{\Phi_\infty})}. \tag{4.3}$$

Since ϕ_∞ and W_∞ depend only on π_∞, the right-hand side depends only on π_∞ and Φ_∞. However, the left-hand side is independent of Φ_∞. This completes the proof of Theorem 1.1.

Appendix. Spherical Eisenstein series

In this appendix, we recall some relations among the terms in the Laurent expansions of spherical Eisenstein series, which have been studied by Kudla [22] and Jiang [19], and give a proof for the sake of completeness.

We retain the notation of §3.1. For an automorphic form φ on $G(\mathbb{A})$, let $\varphi_{P_{n,1}}$ be the constant term of φ along $P_{n,1}$. We regard $\varphi_{P_{n,1}}$ as an automorphic form on $\mathbb{A}^\times \times \mathrm{Sp}(2n-2, \mathbb{A})$. Let $\mathbf{1}$ denote the constant function.

Lemma A.1. *Let φ be a K-invariant automorphic form on $G(\mathbb{A})$. Assume that φ is concentrated on the Borel subgroup. If $\varphi_{P_{n,1}} = 0$, then*

$$\varphi = 0.$$

Proof. See Corollary 3.1 of [19]. $\square$

Proposition A.2. *If $1 < r < n$, then $E^{(n,r)}(s)_{P_{n,1}}$ is equal to*

$$\begin{aligned} &|\ |^{s+n-\frac{r-1}{2}} \otimes E^{(n-1,r-1)}(s+\tfrac{1}{2}) \\ &+ \frac{\xi(s+n-\frac{3r-1}{2})}{\xi(s+n-\frac{r-1}{2})} \cdot |\ |^r \otimes E^{(n-1,r)}(s) \\ &+ \frac{\xi(s-n+\frac{3r-1}{2})\xi(2s)}{\xi(s+n-\frac{r-1}{2})\xi(2s+r-1)} \cdot |\ |^{-s+n-\frac{r-1}{2}} \otimes E^{(n-1,r-1)}(s-\tfrac{1}{2}). \end{aligned}$$

If $r = 1$, then $E^{(n,1)}(s)_{P_{n,1}}$ is equal to

$$|\ |^{s+n} \otimes \mathbf{1} + \frac{\xi(s+n-1)}{\xi(s+n)} \cdot |\ |^1 \otimes E^{(n-1,1)}(s) + \frac{\xi(s-n+1)}{\xi(s+n)} \cdot |\ |^{-s+n} \otimes \mathbf{1}.$$

If $r=n$, then $E^{(n,n)}(s)_{P_{n,1}}$ is equal to

$$\begin{aligned} &|\ |^{s+\frac{n+1}{2}} \otimes E^{(n-1,n-1)}(s+\tfrac{1}{2}) \\ &+ \frac{\xi(s+\frac{n-1}{2})\xi(2s)}{\xi(s+\frac{n+1}{2})\xi(2s+n-1)} \cdot |\ |^{-s+\frac{n+1}{2}} \otimes E^{(n-1,n-1)}(s-\tfrac{1}{2}). \end{aligned}$$

Proof. See Proposition 2.6 of [19]. □

Obviously,

$$E_{-1}^{(1,1)}(1) = \frac{\rho}{\xi(2)}\mathbf{1}, \tag{A.1}$$

$$E_{-1}^{(2,1)}(2) = E_{-1}^{(2,2)}(\tfrac{3}{2}) = \frac{\rho}{\xi(4)}\mathbf{1}, \tag{A.2}$$

and

$$\xi(s+1)\xi(2s)E^{(2,2)}(s-\tfrac{1}{2}) = \xi(s-1)\xi(2s-1)E^{(2,2)}(-s+\tfrac{1}{2}). \tag{A.3}$$

Lemma A.3.

$$E_{-1}^{(2,2)}(\tfrac{1}{2}) = \frac{\rho}{2\xi(2)}E_0^{(2,1)}(0).$$

Proof. By Proposition A.2 and (A.1),

$$\begin{aligned} E_{-1}^{(2,2)}(\tfrac{1}{2})_{P_{2,1}} &= \frac{\rho}{\xi(2)} \cdot |\ |^2 \otimes \mathbf{1} + \frac{\rho^2}{2\xi(2)^2} \cdot |\ |^1 \otimes E_1^{(1,1)}(0), \\ E_0^{(2,1)}(0)_{P_{2,1}} &= 2 \cdot |\ |^2 \otimes \mathbf{1} + \frac{\rho}{\xi(2)} \cdot |\ |^1 \otimes E_1^{(1,1)}(0). \end{aligned}$$

Hence

$$E_{-1}^{(2,2)}(\tfrac{1}{2})_{P_{2,1}} = \frac{\rho}{2\xi(2)}E_0^{(2,1)}(0)_{P_{2,1}}.$$

This completes the proof. □

Lemma A.4.

$$E_1^{(2,1)}(0) = -\frac{\xi'(2)}{\xi(2)}E_0^{(2,1)}(0).$$

Proof. Put $\tilde{E}^{(2,1)}(s) = \xi(s+2)E^{(2,1)}(s)$. Since $\tilde{E}^{(2,1)}(-s) = \tilde{E}^{(2,1)}(s)$, we have

$$\frac{\partial}{\partial s}\tilde{E}^{(2,1)}(s)\bigg|_{s=0} = \xi(2)E_1^{(2,1)}(0) + \xi'(2)E_0^{(2,1)}(0) = 0.$$

□

Lemma A.5.

$$E_{-1}^{(3,2)}(\tfrac{3}{2}) = \frac{\rho}{\xi(4)} E_0^{(3,1)}(1),$$
$$E_{-1}^{(3,3)}(1) = \frac{\rho}{2\xi(4)} E_0^{(3,1)}(0).$$

Proof. By Proposition A.2 and (A.2),

$$E_{-1}^{(3,2)}(\tfrac{3}{2})_{P_{3,1}} = \frac{\rho}{\xi(4)} \cdot |\ |^4 \otimes \mathbf{1} + \frac{\rho\xi(2)}{\xi(4)^2} \cdot |\ |^2 \otimes \mathbf{1} + \frac{\rho\xi(3)}{\xi(4)^2} \cdot |\ |^1 \otimes E_0^{(2,1)}(1),$$
$$E_0^{(3,1)}(1)_{P_{3,1}} = |\ |^4 \otimes \mathbf{1} + \frac{\xi(3)}{\xi(4)} \cdot |\ |^1 \otimes E_0^{(2,1)}(1) + \frac{\xi(2)}{\xi(4)} \cdot |\ |^2 \otimes \mathbf{1},$$
$$E_{-1}^{(3,3)}(1)_{P_{3,1}} = \frac{\rho}{\xi(4)} \cdot |\ |^3 \otimes \mathbf{1} + \frac{\xi(2)^2}{\xi(3)\xi(4)} \cdot |\ |^1 \otimes E_{-1}^{(2,2)}(\tfrac{1}{2}),$$
$$E_0^{(3,1)}(0)_{P_{3,1}} = 2 \cdot |\ |^3 \otimes \mathbf{1} + \frac{\xi(2)}{\xi(3)} \cdot |\ |^1 \otimes E_0^{(2,1)}(0).$$

Hence, by Lemma A.3,

$$E_{-1}^{(3,2)}(\tfrac{3}{2})_{P_{3,1}} = \frac{\rho}{\xi(4)} E_0^{(3,1)}(1)_{P_{3,1}},$$
$$E_{-1}^{(3,3)}(1)_{P_{3,1}} = \frac{\rho}{2\xi(4)} E_0^{(3,1)}(0)_{P_{3,1}}.$$

This completes the proof. $\square$

Lemma A.6.

$$E_{-1}^{(3,2)}(\tfrac{1}{2}) = \frac{\rho}{2\xi(3)} E_0^{(3,3)}(0).$$

Proof. By Proposition A.2 and (A.3), $E_{-1}^{(3,2)}(\tfrac{1}{2})_{P_{3,1}}$ is equal to

$$\frac{\rho}{\xi(3)} \cdot |\ |^2 \otimes E_0^{(2,2)}(\tfrac{1}{2}) + \frac{\rho}{\xi(3)} \left(\frac{\gamma}{\rho} - \frac{\xi'(3)}{\xi(3)} \right) \cdot |\ |^2 \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})$$
$$- \frac{\rho^2}{2\xi(2)\xi(3)} \cdot |\ |^2 \otimes E_1^{(2,1)}(0) + \frac{\rho^2}{2\xi(2)\xi(3)} \cdot |\ |^2 \log |\ | \otimes E_0^{(2,1)}(0)$$
$$+ \frac{\rho^2}{2\xi(2)\xi(3)} \left(-\frac{\gamma}{\rho} + \frac{2\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) \cdot |\ |^2 \otimes E_0^{(2,1)}(0),$$

and $E_0^{(3,3)}(0)_{P_{3,1}}$ is equal to

$$2 \cdot |\ |^2 \otimes E_0^{(2,2)}(\tfrac{1}{2}) + 2 \cdot |\ |^2 \log |\ | \otimes E_{-1}^{(2,2)}(\tfrac{1}{2}) + \frac{6\xi'(2)}{\xi(2)} \cdot |\ |^2 \otimes E_{-1}^{(2,2)}(\tfrac{1}{2}).$$

Hence, by Lemmas A.3, A.4,

$$E_{-1}^{(3,2)}(\tfrac{1}{2})_{P_{3,1}} = \frac{\rho}{2\xi(3)} E_0^{(3,3)}(0)_{P_{3,1}}.$$

This completes the proof. $\square$

Lemma A.7.

$$E_1^{(3,1)}(0) = -\frac{\xi'(3)}{\xi(3)} E_0^{(3,1)}(0),$$
$$E_1^{(3,3)}(0) = -\frac{3\xi'(2)}{\xi(2)} E_0^{(3,3)}(0).$$

Proof. Put $\tilde{E}^{(3,1)}(s) = \xi(s+3)E^{(3,1)}(s)$ and $\tilde{E}^{(3,3)}(s) = \xi(s+2)\xi(2s+2)E^{(3,3)}(s)$. Since $\tilde{E}^{(3,1)}(-s) = \tilde{E}^{(3,1)}(s)$ and $\tilde{E}^{(3,3)}(-s) = \tilde{E}^{(3,3)}(s)$, we have

$$\left.\frac{\partial}{\partial s}\tilde{E}^{(3,1)}(s)\right|_{s=0} = \xi(3)E_1^{(3,1)}(0) + \xi'(3)E_0^{(3,1)}(0) = 0,$$
$$\left.\frac{\partial}{\partial s}\tilde{E}^{(3,3)}(s)\right|_{s=0} = \xi(2)^2 E_1^{(3,3)}(0) + 3\xi(2)\xi'(2)E_0^{(3,3)}(0) = 0.$$

$\square$

Lemma A.8.

$$E_{-2}^{(4,3)}(1) = \frac{\rho}{\xi(4)} E_{-1}^{(4,2)}(\tfrac{1}{2}) = \frac{\rho}{\xi(4)} E_{-1}^{(4,4)}(\tfrac{1}{2}).$$

Proof. By Proposition A.2,

$$E_{-2}^{(4,3)}(1)_{P_{4,1}} = \frac{\rho}{\xi(4)} \cdot |\ |^3 \otimes E_{-1}^{(3,3)}(1) + \frac{\rho\xi(2)}{\xi(4)^2} \cdot |\ |^2 \otimes E_{-1}^{(3,2)}(\tfrac{1}{2}),$$
$$E_{-1}^{(4,2)}(\tfrac{1}{2})_{P_{4,1}} = \frac{\xi(2)}{\xi(4)} \cdot |\ |^2 \otimes E_{-1}^{(3,2)}(\tfrac{1}{2}) + \frac{\rho}{2\xi(4)} \cdot |\ |^3 \otimes E_0^{(3,1)}(0),$$
$$E_{-1}^{(4,4)}(\tfrac{1}{2})_{P_{4,1}} = |\ |^3 \otimes E_{-1}^{(3,3)}(1) + \frac{\rho\xi(2)}{2\xi(3)\xi(4)} \cdot |\ |^2 \otimes E_0^{(3,3)}(0).$$

Hence, by Lemmas A.5, A.6,

$$E_{-2}^{(4,3)}(1)_{P_{4,1}} = \frac{\rho}{\xi(4)} E_{-1}^{(4,2)}(\tfrac{1}{2})_{P_{4,1}} = \frac{\rho}{\xi(4)} E_{-1}^{(4,4)}(\tfrac{1}{2})_{P_{4,1}}.$$

This completes the proof. $\square$

Lemma A.9.

$$E_{-1}^{(4,3)}(1) - \frac{\rho}{\xi(4)} E_0^{(4,2)}(\tfrac{1}{2}) - \frac{\rho}{\xi(4)} E_0^{(4,4)}(\tfrac{1}{2})$$
$$= \frac{\rho}{\xi(4)} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) E_{-1}^{(4,4)}(\tfrac{1}{2}).$$

Proof. By Proposition A.2, $E^{(4,3)}_{-1}(1)_{P_{4,1}}$ is equal to

$$\begin{aligned}
&|\ |^4 \otimes E^{(3,2)}_{-1}(\tfrac{3}{2})\\
&+ \frac{\rho}{\xi(4)} \cdot |\ |^3 \otimes E^{(3,3)}_{0}(1) + \frac{\rho}{\xi(4)} \left(\frac{\gamma}{\rho} - \frac{\xi'(4)}{\xi(4)} \right) \cdot |\ |^3 \otimes E^{(3,3)}_{-1}(1)\\
&+ \frac{\rho\xi(2)}{\xi(4)^2} \cdot |\ |^2 \otimes E^{(3,2)}_{0}(\tfrac{1}{2}) - \frac{\rho\xi(2)}{\xi(4)^2} \cdot |\ |^2 \log|\ | \otimes E^{(3,2)}_{-1}(\tfrac{1}{2})\\
&+ \frac{\rho\xi(2)}{\xi(4)^2} \left(\frac{\gamma}{\rho} + \frac{2\xi'(2)}{\xi(2)} - \frac{3\xi'(4)}{\xi(4)} \right) \cdot |\ |^2 \otimes E^{(3,2)}_{-1}(\tfrac{1}{2}),
\end{aligned}$$

$E^{(4,2)}_{0}(\frac{1}{2})_{P_{4,1}}$ is equal to

$$\begin{aligned}
&|\ |^4 \otimes E^{(3,1)}_{0}(1)\\
&+ \frac{\xi(2)}{\xi(4)} \cdot |\ |^2 \otimes E^{(3,2)}_{0}(\tfrac{1}{2}) + \frac{\xi(2)}{\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot |\ |^2 \otimes E^{(3,2)}_{-1}(\tfrac{1}{2})\\
&+ \frac{\rho}{2\xi(4)} \cdot |\ |^3 \otimes E^{(3,1)}_{1}(0) - \frac{\rho}{2\xi(4)} \cdot |\ |^3 \log|\ | \otimes E^{(3,1)}_{0}(0)\\
&+ \frac{\rho}{2\xi(4)} \left(\frac{2\gamma}{\rho} - \frac{3\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot |\ |^3 \otimes E^{(3,1)}_{0}(0),
\end{aligned}$$

and $E^{(4,4)}_{0}(\frac{1}{2})_{P_{4,1}}$ is equal to

$$\begin{aligned}
&|\ |^3 \otimes E^{(3,3)}_{0}(1) + |\ |^3 \log|\ | \otimes E^{(3,3)}_{-1}(1)\\
&+ \frac{\rho\xi(2)}{2\xi(3)\xi(4)} \cdot |\ |^2 \otimes E^{(3,3)}_{1}(0) - \frac{\rho\xi(2)}{2\xi(3)\xi(4)} \cdot |\ |^2 \log|\ | \otimes E^{(3,3)}_{0}(0)\\
&+ \frac{\rho\xi(2)}{2\xi(3)\xi(4)} \left(\frac{2\gamma}{\rho} + \frac{\xi'(2)}{\xi(2)} - \frac{\xi'(3)}{\xi(3)} - \frac{2\xi'(4)}{\xi(4)} \right) \cdot |\ |^2 \otimes E^{(3,3)}_{0}(0).
\end{aligned}$$

Hence, by Lemmas A.5–A.7,

$$\begin{aligned}
&E^{(4,3)}_{-1}(1)_{P_{4,1}} - \frac{\rho}{\xi(4)} E^{(4,2)}_{0}(\tfrac{1}{2})_{P_{4,1}} - \frac{\rho}{\xi(4)} E^{(4,4)}_{0}(\tfrac{1}{2})_{P_{4,1}}\\
&= \frac{\rho}{\xi(4)} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) \cdot |\ |^3 \otimes E^{(3,3)}_{-1}(1)\\
&+ \frac{\rho^2\xi(2)}{2\xi(3)\xi(4)^2} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) \cdot |\ |^2 \otimes E^{(3,3)}_{0}(0)\\
&= \frac{\rho}{\xi(4)} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) E^{(4,4)}_{-1}(\tfrac{1}{2})_{P_{4,1}}.
\end{aligned}$$

This completes the proof. $\square$

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