On height functions on Jacobian surfaces

By *Kentaro Yoshitomi*

Introduction

First we recall some fundamental facts on the canonical height function on an abelian variety. For an algebraic number field K, let A be an abelian variety defined over K. Take a very ample divisor D. Then we have the projective embedding $\phi_D \colon A \to \mathbf{P}^n$. Using this embedding, we define the logarithmic height h_D on A by $h \circ \phi_D$, where h is the logarithmic height on $\mathbf{P}^n$ (see [11],Chapter 3). The canonical height $\hat{h}_D$ on A attached to D is defined by $\hat{h}_D(z) = \lim_{n\to\infty} h_D(nz)/n^2$. The function $\hat{h}_D$ has a property that $\hat{h}_D(z) \geq 0$ for any $z \in A$ and $\hat{h}_D(z) = 0$ if and only if z is a torsion point. If D' is any ample divisor and mD' is very ample, then we define $\hat{h}_{D'}$ by $\dfrac{1}{m}\hat{h}_{mD'}$. The height pairing $\langle\, , \rangle_D$ is defined by $\langle x, y\rangle_D = \hat{h}_D(x+y) - \hat{h}_D(x) - \hat{h}_D(y)$. The regulator, which is an important factor of the Birch-Swinnerton-Dyer Conjecture, is defined via this height pairing. In general, it is very difficult to compute the canonical height directly by definition. Néron and Tate have shown that the canonical height decomposes into canonical local heights. In the case of elliptic curves, the archimedean local height is expressed using a theta function and the non-archimedean local height is expressed in a simple form (see [21],Chapter VI).

In this paper we compute canonical heights on Jacobian surfaces attached to the theta divisor and, as an example, we shall verify the Birch-Swinnerton-Dyer Conjecture numerically for some Jacobian surface. We use Néron's formula [9],p.332, which asserts that the computation of the height paring on Jacobian variety reduces to that of Néron's symbol. Néron's symbol is decomposed into Néron's local pairings first introduced by Néron [16],Chapter 2. Néron's local pairing is defined via Green's function at archimedean places and via intersection theory on an arithmetic surface at non-archimedean places; see [9].

The main result of this paper is the relation between the archimedean canonical local height and archimedean Néron's local pairing. In the case of elliptic curves, there exists the following relation between the canonical local height and Néron's local pairing at an archimedean place. Let $E : y^2 = 4x^3 - g_2 x - g_3$ be an elliptic curve defined over an algebraic number field K. Let v be an archimedean place of K. Let $E \simeq \mathbf{C}/\Lambda$ and $\wp(z)$ be the Weierstrass $\wp$-function relative to Λ. Let $k(z)$ be the Klein function, that is, $k(z) = \Delta(\Lambda)^{1/12} e^{-\frac{1}{2}z\eta(z)}\sigma(z)$, where $\Delta(\Lambda) = g_2^3 - 27g_3^2$, σ is the Weierstrass σ-function, and η is the quasi-period map associated to Λ (see [21],p.41 and p.465). For $z \in \mathbf{C}$, we denote by $\tilde{z}$ its image in $\mathbf{C}/\Lambda$. Then $\hat{\lambda}_v(\tilde{z}) = -\log|k(z)|$ is the archimedean canonical local height on $E \backslash \{O\}$, where O is the origin of E. Néron's local paring is defined via Green's function, which is, in this case, also expressed in terms of the Klein function. For any $P \in E(\bar{K})$, if we take, as an uniformizer, $\dfrac{y}{2x^2}$ at O and its translation at P, then we have

$$\langle (P)-(O),(P)-(O)\rangle_v = 2(\lambda_v(\tilde{z}_P) + \frac{1}{12}\log|\Delta(\Lambda)|),$$

where $P = (\wp(z_P), \wp'(z_P))$.

We shall generalize the relation above for the case of hyperelliptic curves of genus 2. That is : For an algebraic number field K, let C be a hyperelliptic curve of genus 2 defined by $y^2 = f(x) = x^5 + a_1x^4 + \cdots + a_5 \in K[x]$ and $\mathcal{B}$ be the set of finite Weierstrass points. Let J be the Jacobian variety of C and Θ be the theta divisor of J. For a divisor D of degree 0 on C, we denote its image in $J = \mathrm{Pic}^0(C)$ by $\overline{D}$. For $P \in C(\mathbf{C})$, we denote the hyperelliptic integral from ∞ to P by $u^P \in \mathbf{C}^2$, which is defined up to the period lattice Λ. For $z \in \mathbf{C}^2$, we denote its image in $J = \mathbf{C}^2/\Lambda$ by $\tilde{z}$. Let ϕ be the function as in Proposition 1.9. Let v be an archimedean place of K, $\hat{\lambda}_v$ be the canonical local height on $J \setminus \Theta$ which is normalized as in Definition 3.1 with the fixed function ϕ as above, and $\langle\ ,\ \rangle_v$ be Néron's local pairing explicitly defined as in (3.3). Then we have (Theorem 3.15):

Main Theorem *For $P_i(x_i, y_i) \in C(K)$, $(i = 1, 2)$, let $b = P_1 - P_2$ with $\bar{b} \notin \Theta$, and $z_b = u^{P_1} - u^{P_2} \in \mathbf{C}^2$. As the base of tangent space at P_i, we take $2\,y_i\,\dfrac{\partial}{\partial x} = f'(x_i)\,\dfrac{\partial}{\partial y}$. Then we can take $\dfrac{x - x_i}{2\,y_i}$ if $P_i \notin \mathcal{B}$ and $\dfrac{y - y_i}{f'(x_1)}$ if $P_i \in \mathcal{B}$ as an uniformizer at P_i. In both cases, for an archimedean place v, if we take the uniformizer as above, the relation*

$$\langle b, b\rangle_v = 2\,\hat{\lambda}_v(\tilde{z}_b)$$

between Néron's local pairing and the canonical local height holds.

We can compute the canonical local height at archimedean places numerically. In the case of elliptic curves, one can achieve this by evaluating a rapidly convergent series, which is called Tate's series [20]. In Call and Silverman [3], they generalized Tate's series for a class of varieties with a divisor and a morphism which satisfy certain conditions, including higher dimensional abelian varieties. Thus we can use their series to evaluate $\hat{\lambda}_v$ for archimedean places v. We shall give concrete expression of this series. In Grant [8], defining equations of Jacobian surfaces and the addition theorem are formulated by using analytic theory which goes back to an old book of Baker [1]. The formula of addition above is applicable only to the case where two points are distinct and both are not on the theta divisor. To construct the generalized Tate's series, we must first find appropriate domains in the Jacobian variety. We can take three domains which are obtained by partitioning the Jacobian surface by three translations of the theta divisor. We find the duplication formula(Theorem 2.5) and the additive formula in the limit case, in the sense that the addition of two points when one is not on the theta divisor and the other on the theta divisor(Theorem 2.3). By this formula, we can construct the generalized Tate's series explicitly and compute the archimedean canonical local height numerically. By virtue of Theorem 3.15, we can compute Néron's local pairing at archimedean places. At non-archimedean places, we compute Néron's local pairing using intersection theory on an arithmetic surface, and hence we can compute the canonical height.

In sections 1 and 2, we shall review some facts on Jacobian surfaces and find the formulae above by limiting procedures. In section 3, we shall find an explicit formula of Green's function attached to divisors of degree 0. Our formula uses naturally generalized Klein function (Theorem 3.11). The formula can be obtained from Bost's result [2], but he does not give a proof. We shall give a direct proof using the hyperelliptic $\mathfrak{p}$-functions. Using this formula, we shall prove Main Theorem. As a corollary of the proof of the theorem, for an archimedean place v and a point-wise rational divisor b of degree 0, we shall give a formula to compute $\langle b, b' \rangle_v$ where b' is a certain divisor of degree 0 which does not have common support with b and is linearly equivalent to b (Corollary 3.18). In section 4, we shall construct the generalized Tate's series in our case. Finally, in section 5, we shall give some examples. Especially, we shall check the Birch-Swinnerton-Dyer Conjecture numerically. Let A be an abelian variety over $\mathbf{Q}$. Then the conjecture is:

$$\lim_{s \to 1} (s-1)^{-r} L(s, A) = \frac{V_\infty V_S \det(\langle a_i, a'_j \rangle)_{1 \le i,j \le r} \#\text{Ш}}{\#A(\mathbf{Q})_{tors} \#A'(\mathbf{Q})_{tors}}.$$

See Example 5.4 for more details. We take A as follows. Let $X_0(23)$ be the modular curve of level 23. The canonical model of $X_0(23)$ is given by $y^2 = x^6 - 14x^5 + 57x^4 - 106x^3 + 90x^2 - 16x - 19 = f(x)$. Let χ be the character corresponding to $\mathbf{Q}(\sqrt{-7})$. We take $A = \mathrm{Jac}(X_0(23)_\chi)$, that is the Jacobian variety of the twisted curve $-7y^2 = f(x)$. We compute the real period of J using modular symbols [5]. Since the dimension of $X_0(23 \cdot 7^2)$ is 105, it is very hard to compute the eigenvectors of Hecke operators. Thus we use $7\times$ (the imaginary period of the Jacobian variety of $X_0(23)$), which equals the real period up to a rational number.

We take two points $\alpha_1 = \overline{D(P_1, P_2)}$ and $\alpha_2 = \overline{D(P_1, P_2')} \in J(\mathbf{Q})$ (see § 1.1 for the notation), where $P_1 = (1,1)$ and $P_2 = (3,5)$. If $R' = \det(\langle \alpha_i, \alpha_j \rangle) \neq 0$, then the points are independent and R' coincides with the regulator up to a multiple of an integer, and we shall show this is the case. Finally we have

$$\frac{196 L''(1,J)}{R' V'_\infty V_S} \fallingdotseq 1 \quad \text{(with accuracy up to about 50 digits)},$$

with $V'_\infty / V_\infty \in \mathbf{Q}$. This convince us that the conjecture holds.

The proofs of Proposition 3.13, addition and multiplication formulae, and some other assertions require manipulation of some very complicated polynomial expressions. These are computed using the mathematical computation system Maple V on Sun Sparc Station 20 workstations and Sun UltraSPARC workstations. The Tate's series was computed using GNU g++ Ver 2.7.2 and LiDIA library 1.2. [1]

I would like to express my gratitude to Prof. H. Yoshida and Prof. T. Ikeda for their many useful suggestions.

Notation and terminology

Throughout this paper, we use the following notation. By an algebraic number field, we understand a finite algebraic extension of $\mathbf{Q}$ in $\mathbf{C}$. For an algebraic number field K, let Σ_K^∞ denote the set of infinite places of K, Σ_K^0 the set of finite places of K, and $\Sigma_K = \Sigma_K^\infty \cup \Sigma_K^0$. For $v \in \Sigma_K^0$, let K_v be the completion of K at v, let π_v be an uniformizer at v, O_v the ring of v-adic integers, $k_v = O_v / \pi_v O_v$, q_v the number of elements of k_v, and p_v the residual characteristic. As usual, for $a \in K$ and $v \in \Sigma_K$, define:

$$|a|_v = \begin{cases} |a| & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a real place}, \\ |a|^2 & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a complex place}, \\ q_v^{-\operatorname{ord}_{\pi_v}(a)} & \text{if } v \in \Sigma_K^0. \end{cases}$$

As is well known, the product formula $\prod_{v \in \Sigma_K} |a|_v = 1$ holds. We also use additive notation $v(a) = -\log |a|_v$ for every $v \in \Sigma_K$.

For any finite set S, we denote by $\#S$ the cardinality of S. For any divisor a, we denote the support of a by $\operatorname{supp}(a)$. For divisors a and a', we write $a \sim a'$ if a is linearly equivalent to a'. For a complex number, a complex vector, or a complex matrix x, we denote by $\overline{x}$ its complex conjugate. For $z \in \mathbf{C}^2$, we denote by $\tilde{z}$ its image in $\mathbf{C}^2/\Lambda$(see § 1.2).

[1]This library may be available from anonymous ftp site:ftp://crypt1.cs.uni-sb.de/pub/LiDIA.

1. Review on Jacobian surfaces

We assume that the characteristic of a ground field K is not equal to 2; moreover, except for the section 1.1, we assume that K is a subfield of $\mathbf{C}$. Let C be a hyperelliptic curve of genus 2 over K, defined by the equation

$$y^2 = f(x) := x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = \prod_{i=1}^{5} (x - \beta_i)\,. \tag{1.1}$$

We consider C as a projective non-singular curve and we denote by ∞ the point at infinity. The double covering $C \to \mathbf{P}^1$, $P(x, y) \mapsto x$ is branched over 5 finite points and ∞. We denote the set of finite Weierstrass points by $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$, where $B_i = (\beta_i, 0)$, $(i = 1, \dots, 5)$. Let P^ι be the image of P under the hyperelliptic involution with respect to this covering, that is $P^\iota = (x, -y)$ when $P = (x, y)$.

1.1. Algebraic Theory.

We review the algebraic definition of the Jacobian variety $J = \mathrm{Jac}(C)$. Let $\mathrm{Div}_0(C)$ be the divisor group of degree zero and J be $\mathrm{Pic}_0(C)$ the Picard group of degree zero. For any $D \in \mathrm{Div}_0(C)$, denote by $\overline{D}$ its image in $\mathrm{Pic}_0(C)$. For m points $P_1, P_2, \dots, P_m \in C$, put $D(P_1, P_2, \dots, P_m) = P_1 + P_2 + \cdots + P_m - m\,\infty \in \mathrm{Div}_0(C)$.

Definition 1.1. For $m > 0$, we define

$$\begin{aligned}
\mathrm{Div}_0^m(C) &= \{D(P_1, P_2, \dots, P_m) \mid P_1, \dots, P_m \in C\} \subset \mathrm{Div}_0(C),\\
\mathrm{Div}_0^{+,m}(C) &= \left\{ D(P_1, P_2, \dots, P_m) \,\middle|\, \begin{array}{l} P_i \neq \infty \text{ for every } i, \\ P_i \neq P_j^\iota \text{ whenever } i \neq j \end{array} \right\} \subset \mathrm{Div}_0^m(C),
\end{aligned}$$

and denote their images in $\mathrm{Pic}_0(C)$ by $\overline{\mathrm{Div}_0^m(C)}$ and $\overline{\mathrm{Div}_0^{+,m}(C)}$.

Then $J = \overline{\mathrm{Div}_0^2(C)}$ (see [14],pp.3.28–3.31). If we put $\Theta = \overline{\mathrm{Div}_0^1(C)}$, which is the theta divisor of J, then we have $J - \Theta = \overline{\mathrm{Div}_0^{+,2}(C)}$ (loc.cit.). Hence any points of J can be written as $\overline{D(P_1, P_2)}$ with $P_1 \neq P_2^\iota$, $P_i \neq \infty$, or $\overline{D(P)}$, which belongs to the theta divisor. The zero element O_J of J is $\overline{D(\infty)} = \overline{D(P, P^\iota)}$. All 2-torsion points of J are given by $\overline{D(B_i)}$, $(i = 1, \dots, 5)$ and $\overline{D(B_i, B_j)}$, $(i, j = 1, \dots, 5,\ i \neq j)$. We abbreviate these to $\overline{B}_i$ and $\overline{B}_{ij}$.

Reduction of any divisor of degree 0 to the form $\overline{D(P_1, P_2)}$ is explicitly given as follows.

Let P_1, P_2, P_3 be three points of C and $P_i = (x_i, y_i)$, $(i = 1, 2, 3)$. For simplicity, we assume that the points P_i are distinct. Then we can find the polynomial $V(x)$

of degree 2 satisfying the equations $V(x_1)=y_1$, $V(x_2)=y_2$, $V(x_3)=y_3$. We write V_{P_1,P_2,P_3} for this V.

Then we define a rational function $\tilde{V}$ on C by

$$\tilde{V}(x,y)=\frac{y+V(x)}{(x-x_1)(x-x_2)(x-x_3)},$$

which has poles at P_i, $(i=1,2,3)$ and has a simple zero at ∞. Hence either $\tilde{V}$ has zeros of order 1 at two finite points P_4, P_5 or $\tilde{V}$ has a zero of order 2 at one point $P_4=P_5$. That is, $V(x_4)=-y_4$ and $V(x_5)=-y_5$, or $V(x_4)=-y_4$, $V'(x_4)=-\left.\dfrac{dy}{dx}\right|_{\substack{x=x_4\\ y=y_4}}$. Since x_i, $(i=1,\dots,5)$ is the solutions $f(x)-V^2(x)=0$, we can find x_4 and x_5 and by the equations $y_4=-V(x_4)$, $y_5=-V(x_5)$, we can find explicitly the coordinates of P_4, P_5 which satisfy $P_1+P_2+P_3\sim P_4+P_5+\infty$, that is $\overline{D(P_1,P_2,P_3)}=\overline{D(P_4,P_5)}$. For any divisor of degree 0, we can reduce it using the procedure above recursively.

Let us consider a Jacobian surface over a finite field. Fix a prime $p\neq 2$. Take a hyperelliptic curve C of genus 2 over $\mathbf{Q}$ given by an equation of the form $y^2=f(x)$, where $f(x)$ is a polynomial of degree 5 or 6. Let J be the Jacobian variety of C. We assume that C has good reduction at p, and denote its reduction by $\tilde{C}$. Let $\tilde{J}$ be the reduction of J mod p. We count the number of $\mathbf{F}_p$-rational points of $\tilde{J}$.

Define integers m_1, m_2, n as follows

$m_1=$ the number of $\mathbf{F}_p$-rational points of $\tilde{C}$ which are Weierstrass points.

$2m_2=$ the number of $\mathbf{F}_p$-rational points of $\tilde{C}$ which are not Weierstrass points.

$2n=\#\left\{(x,y)\in\tilde{C}(\mathbf{F}_{p^2})\middle|x\notin\mathbf{F}_p\right\}$.

Then we have

Lemma 1.2. *Let $\tilde{J}$, m_1, m_2, n as above. Then*

$$\#\tilde{J}(\mathbf{F}_p)=1+\binom{m_1}{2}+2m_1m_2+m_2+\binom{2m_2}{2}+n.$$

Proof. Let $W=\{B_i|i=1,\dots,m_1\}$ be $\mathbf{F}_p$-rational Weierstrass points and $M=\{P_i,P_i^\iota|i=1,\dots,m_2\}$ be $\mathbf{F}_p$-rational non-Weierstrass points. Let σ be the generator of $\mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$, and $N=\{Q_i,Q_i^\sigma\neq Q_i^\iota|i=1,\dots,n\}$ be the $\mathbf{F}_{p^2}$-rational points where x-coordinates are not in $\mathbf{F}_p$. Then a point of $\tilde{J}(\mathbf{F}_p)$ is one of

(i) $\overline{D(P,P^\iota)}=O_J$, $P\in M\cup W$,
(ii) $\overline{D(P,P')}$, $\quad P,P'\in M, P'\neq P^\iota$,

(iii) $\overline{D(B,P)}$, $\quad B \in W, P \in M$,
(iv) $\overline{D(B,B')}$, $\quad B, B' \in W, B \neq B'$,
(v) $\overline{D(Q,Q^\sigma)}$, $\quad Q \in N$.

The number of each case are (i) 1, (ii) $\binom{2m_2}{2} + m_2$, (iii) $2m_1m_2$, (iv) $\binom{m_1}{2}$, (v) n. Thus we have the formula. $\square$

1.2. **Analytic Theory.** For convenience of the reader, we review analytic theory of hyperelliptic integrals following [1].

First we take a basis $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ of the first homology group $H_1(C, \mathbf{Z})$ with intersection numbers $\gamma_1 \cdot \gamma_2 = \gamma_1' \cdot \gamma_2' = 0$, $\gamma_i \cdot \gamma_j' = \delta_{ij}$ (Kronecker's δ). We take a basis of the differentials of the first kind, $\mu_1 = \dfrac{dx}{2\,y}$, $\mu_2 = \dfrac{x\,dx}{2\,y}$, and write $\mu = \begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}$. We denote their periods by

$$\omega_{ij} = \int_{\gamma_j} \mu_i, \quad \omega_{ij}' = \int_{\gamma_j'} \mu_i, \quad (i,j = 1,2).$$

It is well known that $\tau = \omega^{-1}\omega'$ belongs to the Siegel upper half space $\mathfrak{h}_2$. If we define the period lattice $\Lambda = \omega\mathbf{Z}^2 \oplus \omega'\mathbf{Z}^2$, then $J = \mathbf{C}^2/\Lambda$ is the Jacobian variety of C.

We define a hyperelliptic integral $u^{P,P_0} = \displaystyle\int_{P_0}^{P} \mu \in \mathbf{C}^2$, and $u^P = u^{P,\infty}$. Let u_i^{P,P_0} denote the i-th coordinate of u^{P,P_0}, $(i = 1,2)$. The integral u^{P,P_0} is determined up to Λ. For any divisor $b = \sum_P m_P P$, we denote the corresponding integral $\sum_P m_P u^P$ by z_b. For any vector $z \in \mathbf{C}^2$, we denote by $\tilde{z}$ its image in $J = \mathbf{C}^2/\Lambda$.

The map $(P_1, P_2) \mapsto u^{P_1} + u^{P_2}$ becomes a surjection from the symmetric 2-product of C $S^2(C)$ to J. The image of $\{(P, \infty) \in S^2(C)\}$ under this map is the theta divisor of J.

Next let ζ_1, ζ_2 be the differentials of the second kind on C defined by

$$\zeta_1 = \frac{(\,3\,x^3 + 2\,a_1\,x^2 + a_2\,x)\,dx}{2\,y}, \quad \zeta_2 = \frac{x^2\,dx}{2\,y},$$

and define their periods $\eta = (\eta_{ij})$, $\eta' = (\eta_{ij}')$ by

$$\eta_{ij} = \int_{\gamma_j} \zeta_i, \quad \eta_{ij}' = \int_{\gamma_j'} \zeta_i, \quad (i,j = 1,2).$$

We define an $\mathbf{R}$-linear map $\tilde{\eta}\colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\tilde{\eta}(u) = \eta\, r + \eta'\, r', \quad \text{where} \quad u = \omega\, r + \omega'\, r', r, r' \in \mathbf{R}^2 .$$

Between the periods, the following relation holds

$$\eta' = \eta\,\tau + 2\,\pi\, i\, {}^t\omega^{\,-1} . \tag{1.2}$$

Furthermore, we have

$$\eta\,\omega^{-1} \quad \text{is symmetric, which is equivalent to } \eta\, {}^t\eta' = \eta'\, {}^t\eta . \tag{1.3}$$

Also we define integrals of the second kind by

$$L_1^{P,P_0} = \int_{P_0}^{P} \zeta_1\,, \quad L_2^{P,P_0} = \int_{P_0}^{P} \zeta_2\,.$$

For P_1, $P_2 \in C$, we define the integral of the third kind by

$$M_{P_1,P_2}^{P,P_0} = \int_{P_0}^{P} \left(\frac{y + y_1}{2\,y\,(x - x_1)} - \frac{y + y_2}{2\,y\,(x - x_2)} \right) dx,$$

and define the normalized integral of the third kind by

$$\tilde{M}_{P_1,P_2}^{P,P_0} = M_{P_1,P_2}^{P,P_0} + C_1\, u_1^{P,P_0} + C_2\, u_2^{P,P_0}, \quad \text{for some} \quad C_1, C_2 \in \mathbf{C},$$

which has zero periods for γ_1, γ_2.

Define a polynomial of two variable F by

$$\begin{aligned} F(x_1, x_2) = {} & x_1^2\, x_2^2\,(x_1 + x_2) + 2\,a_1\, x_1^2\, x_2^2 + a_2\, x_1\, x_2\,(x_1 + x_2) \\ & + 2\,a_3\, x_1\, x_2 + a_4\,(x_1 + x_2) + 2\,a_5, \end{aligned} \tag{1.4}$$

and define a double integral

$$R_{Q,Q_0}^{P,P_0} = \int_{P_0}^{P} \int_{Q_0}^{Q} \frac{F(x,z) + 2\,y\,s}{4\,(x - z)^2}\, \frac{dx}{y}\, \frac{dz}{s},$$

with $s^2 = f(z)$, then we have a relation

$$R_{Q,Q_0}^{P,P_0} = \tilde{M}_{Q,Q_0}^{P,P_0} + {}^t u^{P,P_0}\ \eta\,\omega^{-1}\, u^{Q,Q_0} . \tag{1.5}$$

1.3. **σ-function.** As in [13],Chapter 2, for $\tau \in \mathfrak{h}_2$, we define a theta function on $\mathbf{C}^2$ by

$$\theta(z, \tau) = \sum_{n \in \mathbf{Z}^2} \exp\left[\pi\, i\, {}^t n\, \tau n + 2\pi i\, {}^t n\, z\right] ,$$

and for $a, b \in \mathbf{Q}^2$,

$$\begin{aligned}\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) &= \exp(\pi i^t a\,\tau a + 2\pi i^t a\,(z+b))\theta(z+\tau a+b)\\ &= \sum_{n\in\mathbf{Z}^2}\exp\left[\pi i^t(n+a)\,\tau(n+a) + 2\pi i^t(n+a)\,(z+b)\right].\end{aligned}$$

For $m,\ n \in \mathbf{Z}^2$, the factor of automorphy is given by

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z+\tau m+n) = \exp\left(2\pi i\,({}^t a\,n - {}^t b\,m - {}^t m\,z) - \pi i\,{}^t m\,\tau m\right)\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau).$$

For the theta characteristic $\delta = \begin{pmatrix}\delta'\\ \delta''\end{pmatrix}$, with $\delta' = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$ and $\delta'' = \begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix}$, the function $\theta\,[\delta]\,(z,\tau)$ has a simple zero only at the pullback of Θ in $\mathbf{C}^2$.

Now we define the hyperelliptic σ-function.

Definition 1.3. For $u \in \mathbf{C}^2$,

$$\sigma\,(u) = \exp\left(-\frac{1}{2}{}^t u\,\eta\,\omega^{-1}u\right)\theta\,[\delta]\left(\omega^{-1}u\right).$$

Here we shall list some fundamental properties of the σ-function.

Lemma 1.4. *For $p, p' \in \mathbf{Z}^2$ and $l = \omega'\,p' + \omega\,p$, let $\alpha_l(u)$ be the automorphy factor of $\sigma(u)$, that is $\sigma(u+l) = \alpha_l(u)\,\sigma(u)$, then we have $\alpha_l(u) = \exp(L_l\,u + C_l)$, where*

$$\begin{aligned}L_l &= -{}^t l\,\eta\,\omega^{-1} - 2\,\pi\,i\,{}^t p'\,\omega^{-1},\\ C_l &= -\frac{1}{2}\,{}^t l\,\eta\,\omega^{-1} l - \pi\,i\,{}^t p'\,\tau\,p' + 2\,\pi\,i\,{}^t\delta'\,p - 2\pi\,i\,{}^t\delta''\,p'.\end{aligned}$$

Let $c = \left.\dfrac{\partial}{\partial u_1}\sigma(u)\right|_{u=0}$. Then $\sigma(u)$ has, at $u=0$, the Taylor expansion

$$\sigma(u) = c\left(u_1 + \frac{a_3}{6}\,u_1^3 - \frac{1}{3}\,u_2^3 + (\text{terms of degree} \geq 5)\right).$$

Remark 1.5. Using the equations [8],(3.4) p.116, we have

$$\begin{aligned}c^{-1}\sigma(u) = u_1 &+ \frac{1}{6}a_3u_1^3 - \frac{1}{3}u_2^3 + \frac{1}{120}(a_3^2 - 8a_1a_5 + 2a_2a_4)u_1^5 - \frac{1}{3}a_5u_1^4u_2\\ &- \frac{1}{6}a_4u_1^3u_2^2 - \frac{1}{6}a_3u_1^2u_2^3 - \frac{1}{12}a_2u_1u_2^4 - \frac{1}{15}a_1u_2^5 + (\text{terms of degree} \geq 7).\end{aligned}$$

Proposition 1.6 ([1],p35). *We put* $u' = u^{P_1,A_1} + u^{P_2,A_2}$, $u'' = u^{Q_1,A_1} + u^{Q_2,A_2}$, *with* $A_i \in \mathcal{B}$. *For* $P, Q \in C$, $A \in \mathcal{B}$, *we have*

$$R^{P,Q}_{P_1,Q_1} + R^{P,Q}_{P_2,Q_2} = \log \frac{\sigma(u^{P,A} - u')}{\sigma(u^{P,A} - u'')} \Big/ \frac{\sigma(u^{Q,A} - u')}{\sigma(u^{Q,A} - u'')}.$$

Now we define $\mathfrak{p}$-functions.

Definition 1.7. For $i, j, \ldots, k = 1, 2$ and $u \in \mathbf{C}^2$, we define

$$\zeta_i(u) = \frac{\partial}{\partial u_i} \log \sigma(u), \quad \textit{and} \quad \mathfrak{p}_{ij..k}(u) = -\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} \cdots \frac{\partial}{\partial u_k} \sigma(u).$$

We define $\mathfrak{p}(u) = \mathfrak{p}_{11}(u)\,\mathfrak{p}_{22}(u) - \mathfrak{p}_{12}^2(u)$.

By Proposition 1.6, we can obtain the following [1],p.37: For $u = u^{P_1,A_1} + u^{P_2,A_2}$,

$$\begin{aligned} \zeta_1(u) &= -L_1^{P_1,A_1} - L_1^{P_2,A_2} + \frac{y_1 - y_2}{x_1 - x_2}, \\ \zeta_2(u) &= -L_2^{P_1,A_1} - L_2^{P_2,A_2}. \end{aligned} \tag{1.6}$$

Definition 1.8. For $u, v \in \mathbf{C}^2$,

$$q(u, v) = -c^2 \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}.$$

Then the following formula holds [1],p.100: For $u, v \in \mathbf{C}^2$,

$$q(u, v) = \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v) + \mathfrak{p}_{12}(u)\,\mathfrak{p}_{22}(v) - \mathfrak{p}_{12}(v)\,\mathfrak{p}_{22}(u). \tag{1.7}$$

The following proposition is the heart of the duplication theorem and the definition of the canonical local height.

Proposition 1.9. *For* $u \in \mathbf{C}^2$,

$$-c^3 \frac{\sigma(2u)}{\sigma^4(u)} = \mathfrak{p}_{111}(u) - \mathfrak{p}_{12}(u)\,\mathfrak{p}_{122}(u) + \mathfrak{p}_{22}(u)\,\mathfrak{p}_{112}(u).$$

We denote the right-hand side by $\phi(u)$.

Proof. By the definition of $q(u, v)$,

$$\frac{q(u,v)}{\sigma(u-v)} = -c^2 \frac{\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}. \tag{1.8}$$

Using the Taylor expansion of $\sigma(u)$ at $u = 0$,

$$\sigma(u - v) = c\left\{(u_1 - v_1) + \frac{a_3}{6}(u_1 - v_1)^3 - \frac{1}{3}(u_2 - v_2)^3 + \cdots\right\}.$$

Thus $\dfrac{\partial}{\partial v_1}\sigma(u-v)\Big|_{v=u} = -c$. By Rolle's Theorem,

$$\lim_{v\to u}\frac{q(u,v)}{\sigma(u-v)} = -\frac{1}{c}\frac{\partial}{\partial v_1}q(u,v)\Big|_{u=v}.$$

As for the right-hand side of (1.8),

$$\lim_{v\to u} -c^2\frac{\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)} = -c^2\,\frac{\sigma(2\,u)}{\sigma^4(u)}.$$

By differentiating the right-hand side of (1.7) with respect to v_1 and substituting $v=u$ into the result, the assertion follows. $\square$

Define a polynomial ψ by

$$\begin{aligned}(1.9)\quad \psi(x_1,x_2) = x_1^3\,x_2\,(3\,x_1+x_2) + 4\,a_1\,x_1^3\,x_2 + a_2\,x_1^2\,(x_1+3\,x_2)\\ + 2\,a_3\,x_1\,(x_1+x_2) + a_4\,(3\,x_1+x_2) + 4\,a_5,\end{aligned}$$

and let F be one as (1.4). Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be points on the curve C and we put $u = u^{P_1}+u^{P_2}$. Then we have

Proposition 1.10. (i) ([1],[8]) *When $P_1 \neq P_2$ and $P_1 \neq P_2^{\iota}$,*

$$\begin{aligned}\mathfrak{p}_{11}(u) &= 2\,\frac{F(x_1,x_2)-2\,y_1\,y_2}{(x_1-x_2)^2}, \quad \mathfrak{p}_{12}(u) = -x_1\,x_2, \;\; \mathfrak{p}_{22}(u) = x_1+x_2,\\ (1.10)\quad \mathfrak{p}_{111}(u) &= \frac{y_2\,\psi(x_1,x_2)-y_1\,\psi(x_2,x_1)}{(x_1-x_2)^3}, \quad \mathfrak{p}_{112}(u) = 2\,\frac{x_2^2\,y_1-x_1^2\,y_2}{x_1-x_2},\\ \mathfrak{p}_{122}(u) &= -2\,\frac{x_2\,y_1-x_1\,y_2}{x_1-x_2}, \quad \mathfrak{p}_{222}(u) = 2\,\frac{y_1-y_2}{x_1-x_2}\,.\end{aligned}$$

(ii) *When $P_1 = P_2 = (x,y) \notin \mathcal{B}$ (equivalently $y \neq 0$),*

$$\begin{aligned}&\mathfrak{p}_{11}(u) = 4\,x^3+2\,a_1\,x^2+a_2\,x+\frac{f'^2(x)-2\,f(x)\,f''(x)}{4\,y^2}\\ &\mathfrak{p}_{12}(u) = -x^2\,, \quad \mathfrak{p}_{22}(u) = 2\,x,\\ (1.11)\quad &\mathfrak{p}_{111}(u) = -(14\,x^2+8\,a_1\,x+2\,a_2)\,y - \frac{(2\,x^3-f''(x))\,f'(x)}{2\,y} - \frac{f'^3(x)}{4\,y^3},\\ &\mathfrak{p}_{112}(u) = \frac{x^2\,f'(x)-4\,x\,f(x)}{y}, \quad \mathfrak{p}_{122}(u) = -\frac{x\,f'(x)-2\,f(x)}{y},\\ &\mathfrak{p}_{222}(u) = \frac{f'(x)}{y}.\end{aligned}$$

Proof. By (1.6), differentiating with respect to u_i, we have the formulae in (i). As for (ii), the formulae can be deduced from (1.10) by Rolle's Theorem. $\square$

Immediately, by the lemma above, we have

Corollary 1.11. *If $u = u^{P_1} + u^{P_2}$, then*

$$
\begin{aligned}
& x_1 + x_2 = \mathfrak{p}_{22}(u)\,, \quad x_1\, x_2 = -\mathfrak{p}_{12}(u), \\
& y_1 + y_2 = \mathfrak{p}_{122}(u) + \frac{1}{2}\, \mathfrak{p}_{22}(u)\, \mathfrak{p}_{222}(u), \\
& y_1\, y_2 = \frac{1}{4}\left(\mathfrak{p}_{122}^2(u) - \mathfrak{p}_{112}(u)\, \mathfrak{p}_{222}(u)\right).
\end{aligned}
\tag{1.12}
$$

1.4. **Defining Equations.** We review defining equations for the affine model using $\mathfrak{p}$-functions([8]). We define coordinate functions X_{ij}, X_{ijk}, and X as follows.

$$
X_{ij} = \mathfrak{p}_{ij}, \quad X_{ijk} = \frac{1}{2}\, \mathfrak{p}_{ijk}, \quad \text{and} \quad X = \frac{1}{2}\,(\mathfrak{p} + a_2\, \mathfrak{p}_{12} - a_4).
$$

Then we have

Theorem 1.12 ([8])**.** *Let E_i, $(i = 1, \ldots, 14)$, be the equations f_i in [8],pp.103–107. The ideal $I(J - \Theta)$ is generated by $E_2, \ldots, E_7$.*

Remark 1.13. There is a misprint in [8],p.106. The equation f_{10} should be read

$$
f_{10} = X_{112}^2 - X_{111}\, X_{122} + \cdots .
$$

1.5. **The projective model.** We write X_0^h, X_{ij}^h, X_{ijk}^h, X^h for homogeneous coordinates of a points of $\mathbf{P}^8$ with $X_{ij} = X_{ij}^h / X_0^h$, $X_{ijk} = X_{ijk}^h / X_0^h$, and $X = X^h / X_0^h$. We write $[X_0^h : X_{11}^h : X_{12}^h : X_{22}^h : X_{111}^h : X_{112}^h : X_{122}^h : X_{222}^h : X^h]$ for a set of coordinates. Then we have

Theorem 1.14 ([8])**.** *Let E_i^h be the homogenization of E_i, $(i = 1, \ldots, 14)$. Then their Jacobian matrix is of full rank at every point of J and the equations give a non-singular model in $\mathbf{P}^8$.*

The theta divisor on $J \subset \mathbf{P}^8$ is given by $X_0^h = 0$. Substituting $X_0^h = 0$ into the equations E_i^h, we have

$$
\Theta \subset \left\{ [0 : 0 : 0 : 0 : X_{111}^h : X_{112}^h : X_{122}^h : X_{222}^h : X^h] \right\}.
$$

Proposition 1.15. *For $P(x, y) \in C$, the coordinates of $\overline{D(P)} \in \Theta \subset \mathbf{P}^8$ is given by*

$$
[0 : 0 : 0 : 0 : -x^3 : x^2 : -x : 1 : -y]\,.
$$

Especially, the zero element O_J of J has the coordinates

$$
[0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0].
$$

Proof. Take another point $Q(x_1, y_1) \in C$. By substituting $X^h_0 = X^h_{ij} = 0$ into the equations E^h_{10}, E^h_{11}, and E^h_{12}, we get

$$X^h_{112}{}^2 = X^h_{111}\,X^h_{122}, \quad X^h_{111}\,X^h_{222} = X^h_{112}\,X^h_{122}, \quad X^h_{122}{}^2 = X^h_{112}\,X^h_{222}\,. \tag{1.13}$$

It is easy to show that $\lim_{P_1\to\infty} X^h_{122}/X^h_{222} = -x$, and using (1.13), we get the limit of X^h_{ijk}/X^h_{222}. On the other hand, $\overline{D(P,Q)} \in \mathbf{P}^8$ has coordinates $[(x-x_1)^3 : \cdots : X^h_{222} : X^h]$, where

$$\begin{aligned} X^h = \big(&2\,x^3\,x_1{}^3 + a_1\,x^3\,x_1{}^2 + a_1\,x^2\,x_1{}^3 + 2\,a_2\,x^2\,x_1{}^2 + a_3\,x^2\,x_1 + a_3\,x\,x_1{}^2 \\ &+ 2\,a_4\,x\,x_1 + a_5\,x + a_5\,x_1 - x\,y\,y_1 - x_1\,y\,y_1\big)\,(x-x_1), \end{aligned}$$

and

$$X^h_{222} = (x-x_1)^2\,(y-y_1).$$

At ∞, x_1 has a pole of order 2 and y_1 has a pole of order 5. Take an uniformizer s at ∞ such that $x_1 = 1/s^2$. Using the equation $y_1^2 = f(x_1)$, we have

$$\frac{X^h}{X^h_{222}} = -y + (a_1\,x_1^2 + 2\,x_1^3)\,s + (\text{terms of degree} \geq 2).$$

Hence the assertion follows. □

2. Arithmetic on Jacobian surfaces

In this section, we omit the superscript 'h' of the homogeneous coordinates and for a letter '$*$', denote the set of coordinates $[*_0 : *_{11} : *_{12} : *_{22} : \cdots : *]$ by D_*. We shall omit the formulae for the X-coordinate, because it is easily computable using the defining equation E_2.

2.1. **The addition theorem.** In Grant [8], pp.114–116, the addition formula in the case where both points are not in the theta divisor is described, except for the 3 coordinates X_{112}, X_{122}, X_{222}. For convenience, we describe the formulae for all coordinates except for X in homogeneous coordinates.

Theorem 2.1. *Let $D_Z = D_X + D_Y$. Then we have*

$$\begin{aligned} Z_0 &= 2\,X_0\,Y_0\,Q^3, \\ Z_{11} &= -2\,(Y_0\,X_{11} + X_0\,Y_{11})\,Q^3 + 2\,X_0\,Y_0\,Q\,Q_1^2 - 2\,X_0\,Y_0\,Q^2\,Q_{11}, \\ Z_{12} &= -2\,(Y_0\,X_{12} + X_0\,Y_{12})\,Q^3 + 2\,X_0\,Y_0\,Q\,Q_1\,Q_2 - 2\,X_0\,Y_0\,Q^2\,Q_{12}, \\ Z_{22} &= -2\,(Y_0\,X_{22} + X_0\,Y_{22})\,Q^3 + 2\,X_0\,Y_0\,Q\,Q_2^2 - 2\,X_0\,Y_0\,Q^2\,Q_{22}, \\ Z_{111} &= -(Y_0\,X_{111} + X_0\,Y_{111})\,Q^3 + 3\,X_0\,Y_0\,Q\,Q_1\,Q_{11} - Q^2\,Q_{111} \end{aligned}$$

$$
\begin{aligned}
&\quad - 2\, X_0\, Y_0\, Q_1^3 + 3\,(Y_0\, X_{11} + X_0\, Y_{11})\, Q^2\, Q_1,\\
Z_{112} &= -(Y_0\, X_{112} + X_0\, Y_{112})\, Q^3 + 2\, X_0\, Y_0\, Q\, Q_1\, Q_{12} + X_0\, Y_0\, Q\, Q_2\, Q_{11} - Q^2\, Q_{112}\\
&\quad - 2\, X_0\, Y_0\, Q_1^2\, Q_2 + 3\,(Y_0\, X_{12} + X_0\, Y_{12})\, Q^2\, Q_1,\\
Z_{122} &= -(Y_0\, X_{122} + X_0\, Y_{122})\, Q^3 + 2\, X_0\, Y_0\, Q\, Q_2\, Q_{12} + X_0\, Y_0\, Q\, Q_1\, Q_{22} - Q^2\, Q_{122}\\
&\quad - 2\, X_0\, Y_0\, Q_2^2\, Q_1 + 3\,(Y_0\, X_{12} + X_0\, Y_{12})\, Q^2\, Q_2,\\
Z_{222} &= -(Y_0\, X_{222} + X_0\, Y_{222})\, Q^3 + 3\, X_0\, Y_0\, Q\, Q_2\, Q_{22} - Q^2\, Q_{222}\\
&\quad - 2\, X_0\, Y_0\, Q_2^3 + 3\,(Y_0\, X_{22} + X_0\, Y_{22})\, Q^2\, Q_2,
\end{aligned}
$$

where

$$
\begin{aligned}
Q\quad &= Y_0\, X_{11} - X_0\, Y_{11} + X_{12}\, Y_{22} - Y_{12}\, X_{22},\\
Q_1\quad &= Y_0\, X_{111} - X_0\, Y_{111} + X_{112}\, Y_{22} - Y_{112}\, X_{22} + Y_{122}\, X_{12} - X_{122}\, Y_{12},\\
Q_2\quad &= Y_0\, X_{112} - X_0\, Y_{112} + X_{122}\, Y_{22} - Y_{122}\, X_{22} + Y_{222}\, X_{12} - X_{222}\, Y_{12},\\
Q_{11}\quad &= a_3 Q - 2(Y X_{12} - X Y_{12}) - 2a_5(Y_0 X_{22} - X_0 Y_{22}) + 2(X_{112}\, Y_{122} - Y_{112}\, X_{122}),\\
Q_{12}\quad &= a_3\,(Y_0\, X_{12} - X_0\, Y_{12}) - (X_{11}\, Y_{12} - Y_{11}\, X_{12}) + (X\, Y_{22} - Y\, X_{22})\\
&\quad - a_4\,(Y_0\, X_{22} - X_0\, Y_{22}) + (X_{112}\, Y_{222} - Y_{112}\, X_{222}),\\
Q_{22}\quad &= 2\, a_2\,(Y_0\, X_{12} - X_0\, Y_{12}) - 2\,(Y_0\, X - X_0\, Y) - 2\,(X_{11}\, Y_{22} - Y_{11}\, X_{22})\\
&\quad + 2\, a_1\,(X_{12}\, Y_{22} - Y_{12}\, X_{22}) + 2\,(X_{122}\, Y_{222} - Y_{122}\, X_{222}),\\
Q_{111} &= a_3\, X_0\, Y_0\, Q_1 + (Y_0\, X_{111}\, X_{22}\, Y_{12} - X_0\, Y_{111}\, Y_{22}\, X_{12})\\
&\quad + Y_{122}\,(X_{12}\,(3\, Y_0\, X_{11} - X_0\, Y_{11} + 2\, a_3\, X_0\, Y_0) - a_4\, X_0\, Y_0\, X_{22})\\
&\quad - X_{122}\,(Y_{12}\,(3\, X_0\, Y_{11} - Y_0\, X_{11} + 2\, a_3\, Y_0\, X_0) - a_4\, Y_0\, X_0\, Y_{22})\\
&\quad + X_{112}\,(Y_{12}\,(3\, X_0\, Y_{12} - 2\, Y_0\, X_{12} + a_2\, X_0\, Y_0) + a_4\, X_0\, Y_0^2)\\
&\quad - Y_{112}\,(X_{12}\,(3\, Y_0\, X_{12} - 2\, X_0\, Y_{12} + a_2\, Y_0\, X_0) + a_4\, Y_0\, X_0^2),\\
Q_{112} &= a_3\, X_0\, Y_0\, Q_2 + Y_0\, X_{222}\,(X_{11}\, Y_{12} - 2\, a_5\, X_0\, Y_0)\\
&\quad - X_0\, Y_{222}\,(Y_{11}\, X_{12} - 2\, a_5\, Y_0\, X_0)\\
&\quad + Y_{122}\,(X_{12}\,(2\, X_0\, Y_{12} + 2\, Y_0\, X_{12}) + 2\, Y_0\, X_0\, X + Y_0\, X_{11}\, X_{22})\\
&\quad - X_{122}\,(Y_{12}\,(2\, Y_0\, X_{12} + 2\, X_0\, Y_{12}) + 2\, X_0\, Y_0\, Y + X_0\, Y_{11}\, Y_{22})\\
&\quad + X_{112}\,(Y_{12}\,(Y_0\, X_{22} + 3\, X_0\, Y_{22} + 2\, a_1\, X_0\, Y_0) - X_0\, Y_0\, Y_{11})\\
&\quad - Y_{112}\,(X_{12}\,(X_0\, Y_{22} + 3\, Y_0\, X_{22} + 2\, a_1\, Y_0\, X_0) - Y_0\, X_0\, X_{11}),\\
Q_{122} &= X_0\, Y_{111}\,(X_0\, Y_{22} + 2\, Y_0\, X_{22}) - Y_0\, X_{111}\,(Y_0\, X_{22} + 2\, X_0\, Y_{22})\\
&\quad + X_{112}\,(2\, Y_0^2\, X_{12} + 2\, a_1\, X_0\, Y_0\, Y_{22} + a_2\, X_0\, Y_0^2)\\
&\quad - Y_{112}\,(2\, X_0^2\, Y_{12} + 2\, a_1\, Y_0\, X_0\, X_{22} + a_2\, Y_0\, X_0^2)\\
&\quad + X_{122}\,(3\, X_0\, Y_{12}\, Y_{22} + X_0\, Y_0\, Y_{11} - Y_0^2\, X_{11})
\end{aligned}
$$

$$
\begin{aligned}
&\quad - Y_{122}\,(3\,Y_0\,X_{12}\,X_{22} + Y_0\,X_0\,X_{11} - X_0^2\,Y_{11})\\
&\quad + Y_{222}\,(3\,Y_0\,X_{12}^2 + 2\,X_0\,Y_0\,X + a_4\,X_0^2\,Y_0)\\
&\quad - X_{222}\,(3\,X_0\,Y_{12}^2 + 2\,Y_0\,X_0\,Y + a_4\,Y_0^2\,X_0),\\
Q_{222} &= X_0\,Y_{112}\,(X_0\,Y_{22} + 2\,Y_0\,X_{22}) - Y_0\,X_{112}\,(Y_0\,X_{22} + 2\,X_0\,Y_{22})\\
&\quad + X_{122}\,(3\,X_0\,Y_{22}^2 + 4\,a_1\,X_0\,Y_0\,Y_{22} + 2\,Y_0^2\,X_{12} + 2\,X_0\,Y_0\,Y_{12} + 2\,a_2\,X_0\,Y_0^2)\\
&\quad - Y_{122}\,(3\,Y_0\,X_{22}^2 + 4\,a_1\,Y_0\,X_0\,X_{22} + 2\,X_0^2\,Y_{12} + 2\,Y_0\,X_0\,X_{12} + 2\,a_2\,Y_0\,X_0^2)\\
&\quad + Y_{222}\,(3\,Y_0\,X_{12}\,X_{22} + 4\,a_1\,X_0\,Y_0\,X_{12} - 3\,X_0\,Y_0\,X_{11} + X_0^2\,Y_{11})\\
&\quad - X_{222}\,(3\,X_0\,Y_{12}\,Y_{22} + 4\,a_1\,Y_0\,X_0\,Y_{12} - 3\,Y_0\,X_0\,Y_{11} + Y_0^2\,X_{11}).
\end{aligned}
$$

Proof. Put $D_i = \dfrac{\partial}{\partial u_i} + \dfrac{\partial}{\partial v_i}$. In the formula (1.7) take log and operate $D_i\,D_j$ or $D_i\,D_j\,D_k$ on both sides, we get the formulae above. $\square$

Remark 2.2. As in [8], we define q_{ij} and q_{ijk} by

$$
\begin{aligned}
q_i(u,v) &= D_i\,q(u,v)\\
q_{ij}(u,v) &= D_i\,D_j\,q(u,v) - 6\,(\mathfrak{p}_{ij}(u) + \mathfrak{p}_{ij}(v))\,q(u,v),
\end{aligned}
$$

As for q_{ijk}, we set

$$
\begin{aligned}
q_{111}(u,v) &= D_1\,q_{11}(u,v), \quad & q_{112}(u,v) &= D_2\,q_{11}(u,v),\\
q_{122}(u,v) &= D_1\,q_{22}(u,v), \quad & q_{222}(u,v) &= D_2\,q_{22}(u,v)\,.
\end{aligned}
$$

Remark that $q_{112}(u,v) \neq D_1\,q_{12}(u,v)$ and $q_{122}(u,v) \neq D_2 q_{12}(u,v)$. The functions $Q_{ij\ldots k}$ in the theorem is 2^n-multiplied $q_{ij\ldots k}$ where n is the length of the index $ij\ldots k$.

2.2. **Addition formulae in the limit case.** In Theorem 2.1, taking limit of Y to $\overline{D(P)}$, we obtain explicit formulae of $D_Z = \overline{D(P)} + D_X$.

Theorem 2.3. *Let $D_Z = \overline{D(P)} + D_X$, where $P = (x,y)$. Then we can take the coordinate Z_0 as*

$$Z_0 = 2\,X_0\,\left(x^2\,X_0 - X_{12} - x\,X_{22}\right)^3,$$

and then we have

$$
\begin{aligned}
Z_{11} &= 2\,\left(X_0\,x^2 - X_{22}\,x - X_{12}\right)\left({X_0}^2\,X_{11}\,x^4 + \cdots\right),\\
Z_{12} &= \left(X_0\,x^2 - X_{22}\,x - X_{12}\right)\left(2\,{X_0}^2\,X_{12}\,x^4 + \cdots\right),\\
Z_{22} &= 2\,\left(X_0\,x^2 - X_{22}\,x - X_{12}\right)\left({X_0}^2\,X_{22}\,x^4 + \cdots\right),\\
Z_{111} &= 2\,{X_0}^3\,X_{111}\,x^6 + 2\,a_2\,{X_0}^3\,X_{112}\,x^5 + \cdots\,,\\
Z_{112} &= 2\,{X_0}^3\,X_{112}\,x^6 - {X_0}^3\,X_{111}\,x^5 + \cdots\,,
\end{aligned}
$$

$$\begin{aligned} Z_{122} &= 2\,{X_0}^3\,X_{122}\,x^6 + {X_0}^3\,X_{112}\,x^5 + \cdots\,, \\ Z_{222} &= 2\,{X_0}^3\,X_{222}\,x^6 + 4\,{X_0}^3\,X_{122}\,x^5 + \cdots\,. \end{aligned}$$

Proof. First we find local parameters of $\mathcal{O}_P = \mathcal{O}_{P,J}$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_P$, and take parameters d_{11}, d_{12}, d_{22}, d_{111}, d_{112}, d_{122}, and d_3 in $\mathcal{M}$ as

$$\begin{aligned} &\frac{X_{ij}}{X_{222}} = d_{ij}, \\ &\frac{X_{111}}{X_{222}} + x^3 = d_{111}, \qquad \frac{X_{112}}{X_{222}} - x^2 = d_{112}, \\ &\frac{X_{122}}{X_{222}} + x = d_{122}, \qquad \frac{X}{X_{222}} + y = d_3. \end{aligned} \tag{2.1}$$

We substitute (2.1) into the defining equations E_i^h and write $\tilde{E}_i$ for them. By $\tilde{E}_2$, $d_0 \in \mathcal{M}^2$, and with $\tilde{E}_1$, $d_0 \in \mathcal{M}^3$. By $\tilde{E}_3$, $d_{12} + x\,d_{22} \in \mathcal{M}^2$, and by $\tilde{E}_4$, $d_{11} - x\,d_{12} - 2\,x^2\,d_{22} \in \mathcal{M}^2$, hence $d_{11} - x^2\,d_{22} \in \mathcal{M}^2$. By $\tilde{E}_5$ modulo $\mathcal{M}^4$, $d_{11}\,d_{22}^2 \in x^2\,d_0 + \mathcal{M}^4$, by $\tilde{E}_6$, $d_{22}^3 \in d_0 + \mathcal{M}^4$. $\tilde{E}_7$ reads $d_{12}\,d_{22}^2 \in -x\,d_0 + \mathcal{M}^4$, and $\tilde{E}_8$ reads $d_{11}^3 \in x^6\,d_0 + \mathcal{M}^4$. Similarly by $\tilde{E}_{10}$, $\tilde{E}_{11}$, $\tilde{E}_{12}$, $\tilde{E}_{13}$, and $\tilde{E}_{14}$, we have

$$\begin{gathered} x\,d_{111} + 2\,x^2\,d_{112} + x^3\,d_{122} + y\,d_{11} \in \mathcal{M}^2, \\ d_{111} + x\,d_{112} - x^2\,d_{122} - 2\,y\,d_{12} \in \mathcal{M}^2, \\ -d_{112} - 2\,x\,d_{122} + y\,d_{22} \in \mathcal{M}^2, \\ d_{11} + x\,d_{12} \in \mathcal{M}^2, \\ -2\,x\,d_{11} - x^2\,d_{12} + x^3\,d_{22} \in \mathcal{M}^2\,. \end{gathered}$$

From these we have $\mathcal{M} = (d_{22}, d_{122})$, that is d_{22} and d_{122} are regular local parameters. Substituting (2.1) into D_Z in Theorem 2.1, we write $D_{\tilde{Z}}$ for the results. We use the next substitutions.

$$\begin{gathered} d_0 = d_0 - \tilde{E}_6,\;\; d_{11} = d_{11} - \tilde{E}_4,\;\; d_{12} = d_{12} + \tilde{E}_3, \\ y\,d_3 = y\,d_3 - \tilde{E}_9/2,\;\; d_{111} = d_{111} - \tilde{E}_{11},\;\; d_{112} = d_{112} - \tilde{E}_{12}. \end{gathered} \tag{2.2}$$

For example, substituting (2.2) into the formula of $\tilde{Z}_0$ in the order d_0, d_{11}, d_{12} and reducing modulo $\mathcal{M}^7$, we get $\tilde{Z}_0 = \hat{Z}_0\,d_{22}^6$, where $\hat{Z}_0$ is free of parameters d_{ij} and d_{ijk}. As for other coordinates, we list the order of substitutions of local parameters.

$$\begin{aligned} Z_{11} &: d_0, d_{11}, d_{12}, d_{111}, d_{12}, d_{112}, d_{11}, d_{12}, d_0, \\ Z_{12} &: d_0, d_{11}, d_{12}, d_{111}, d_{12}, d_{112}, d_{11}, d_{12}, d_0, \\ Z_{22} &: d_0, d_{11}, d_{12}, d_{112}, d_{11}, d_{12}, d_0, \\ Z_{111} &: d_0, d_{11}, d_{12}, d_{111}, d_{112}, d_{12}, d_0, d_{11}, d_3, d_{11}, d_{12}, d_{111}, d_{112}, d_{12}, \\ Z_{112} &: d_0, d_{11}, d_{12}, d_{111}, d_{112}, d_{12}, d_0, d_{11}, d_{12}, d_{112}, \\ Z_{122} &: d_0, d_{11}, d_{12}, d_{111}, d_{112}, d_{11}, d_{12}, d_0, d_{112}, \\ Z_{222} &: d_0, d_0, d_{11}, d_{12}, d_{111}, d_{112}, d_{11}, d_{12}, d_0, d_{112}\,. \end{aligned}$$

The coefficient of $d_{22}^i d_{122}^j$, $i+j=6$, $j>0$ are eliminated by using the relations $y^2=f(x)$ and the defining equations for D_X. Cancelling d_{22}^6, we have the formulae desired. □

Remark 2.4. By the formulae above, for $D_X=\overline{D(P,P)}$, $D_Z=D_X+\overline{D(P_1)}$ is computable if $P_1\neq P$. But if $P_1=P$, the coordinates are all 0. If we want to compute $\overline{D(P,P,P)}$, we must differentiate the coordinate of $D_Z=D_X+\overline{D(P_1)}$ with respect to x_1 six times and substitute x for x_1. The formula for $\overline{D(P,P,P)}$ is too long to describe here. Also if we want to compute $\overline{D(P,P_1)}+P$, we cannot use the above formulae directly. In spite of this, we use $\overline{D(P,P)}+P_1$.

2.3. **The duplication theorem.** In the formula of Proposition 1.9, taking log and operating $\dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}$ or $\dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}\dfrac{\partial}{\partial u_k}$ on both sides, we get the duplication theorem. We have

Theorem 2.5. *Put* $\phi_{ij}(u)=\dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}\phi(u)$ *and* $\phi_{ijk}(u)=\dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}\dfrac{\partial}{\partial u_k}\phi(u)$*. Then*

$$\begin{aligned}\mathfrak{p}_{ij}(2\,u)&=\mathfrak{p}_{ij}(u)-\frac{\phi_{ij}(u)}{4\,\phi(u)},\\ \mathfrak{p}_{ijk}(2\,u)&=\frac{1}{2}\,\mathfrak{p}_{ijk}(u)-\frac{\phi_{ijk}(u)}{8\,\phi(u)}+\\ &\quad\frac{\phi_{ij}(u)\,\phi_k(u)+\phi_{jk}(u)\,\phi_i(u)+\phi_{ki}(u)\,\phi_j(u)}{8\,\phi^2(u)}-\frac{\phi_i(u)\,\phi_j(u)\,\phi_k(u)}{4\,\phi^3(u)}.\end{aligned}$$

Since it is easy to formulate above in homogeneous coordinates, we do not describe the formulae here.

3. Archimedean local heights

3.1. **Canonical local heights.** In this section we review on the Néron-Tate local heights (the canonical local heights) on a Jacobian surface J. See [3],Section 2 for more details.

Let $\Psi_2\colon J\to J$ be the multiplication by 2 map. The theta divisor Θ satisfies

$$\Psi_2^*\Theta\sim 4\,\Theta\,.$$

Now we define the canonical local height $\hat{\lambda}_v : J\setminus\Theta\to\mathbf{R}$ for $v\in\Sigma_K$ as follows:

Definition 3.1. (1) $\hat{\lambda}_v$ is a Weil local height funtion corresponding the divisor Θ.

(2) Let ϕ be a function such that $\Psi_2^* \Theta = 4\,\Theta + \mathrm{div}(\phi)$, then

$$\hat{\lambda}_v(2\,z) = 4\,\hat{\lambda}_v(z) + v(\phi(z))\,.$$

Remark 3.2. As in [3], p.171, $\hat{\lambda}_v$ is uniquely determined if we fix ϕ, since we assume the equation of the second condition holds with $v(\phi(z))$ in spite of $v(a\phi(z))$. After this, we shall fix ϕ as in Proposition 3.3 below.

Proposition 3.3. *Let ϕ be the function defined in Proposition 1.9. Then* $[2]^* \Theta = 4\,\Theta + \mathrm{div}(\phi)$ *.*

Proof. Since $\sigma(u)$ has zero at Θ of order 1, we have

$$[2]^* \Theta = 4\,\Theta + \mathrm{div}\left(\frac{\sigma \circ \Psi_2}{\sigma^4}\right).$$

By Proposition 1.9 we have the assertion. ☐

3.2. **Néron's local paring.** We review Néron's local paring following [9]. For any $v \in \Sigma_K$, let $\mathrm{Div}_0(C)_{/K_v}$ be the K_v-rational subgroup of $\mathrm{Div}_0(C)$ and let $Z_0(C)_{/K_v}$ be the point-wise K_v-rational subgroup. Then

Proposition 3.4 ([9],p.328)**.** *There is a unique pairing $\langle a, b\rangle_v$ on relatively prime divisors $a \in Z_0(C)_{/K_v}$, $b \in \mathrm{Div}_0(C)_{/K_v}$ with values in $\mathbf{R}$ which satisfies the following properties:*

(i) $\langle a, b\rangle_v + \langle a, c\rangle_v = \langle a, b + c\rangle_v$.
(ii) $\langle a, b\rangle_v = \langle b, a\rangle_v$ *for any* $b \in Z_0(C)_{/K_v}$.
(iii) $\langle a, \mathrm{div}(g)\rangle_v = \log|g(a)|_v$.
(iv) *For fixed b and $x_0 \in C(K_v) \setminus \mathrm{supp}(b)$, $C(K_v) \setminus \mathrm{supp}(b) \ni x \to \langle x - x_0, b\rangle_v \in \mathbf{R}$ is continuous.*

This pairing is called Néron's local pairing. This pairing satisfies functoriality. That is, let C' be an another curve and $\Phi \in C \times C'$ be a correspondence rational over K_v, then we have $\langle a, \Phi^* b\rangle_C = \langle \Phi_* a, b\rangle_{C'}$ for $a \in \mathrm{Div}_0(C)$, $b \in \mathrm{Div}_0(C')$ when the both sides are defined. If L_w be an extension of K_v,

$$\langle a, b\rangle_w = [L_w : K_v]\,\langle a, b\rangle_v\,. \tag{3.1}$$

If $\mathrm{supp}(a) \cap \mathrm{supp}(b) \neq \emptyset$, we modify the pairing by

$$\langle a, b\rangle_v = \log|g[a]|_v + \langle a, b'\rangle_v\,, \tag{3.2}$$

where $b = b' + \mathrm{div}(g)$ such that $\mathrm{supp}(a) \cap \mathrm{supp}(b') = \emptyset$. We define a modified value of f at x as follows: Fix a tangent vector $\frac{\partial}{\partial t}$, and take an uniformizer z around x with $\frac{\partial z}{\partial t} = 1$. Then we define the modified value $f[x]$ of f at x by

$$f[x] = \frac{f}{z^m}\bigg|_{z=0},$$

where m is the order of f at x. For any divisor $a = \sum a_x\, x$, we define $f[a] = \prod f[x]^{a_x}$.

For the archimedean place v, the pairing is explicitly given by

$$\langle a, b\rangle_v = g_a(b), \tag{3.3}$$

where g_a is Green's function attached to a. (See § 3.3 for the definition of Green's function).

For the non-archimedean place v, the pairing is explicitly given by

$$\langle a, b\rangle_v = -(A \cdot B) \log q_v,$$

where A, B are extensions of divisors a, b in a regular model $\mathcal{C}$ of C over O_v that satisfy $(A \cdot \mathcal{F}) = (B \cdot \mathcal{F}) = 0$ for any fibral irreducible divisor $\mathcal{F}$ of $\mathcal{C}/O_v$. The following lemma is useful to compute the "correction term" (cf. [4]).

Lemma 3.5. *Let $\mathcal{C}/O_v$ be a regular arithmetic surface. Let $\sum_{i=0}^{n} m_i C_i$ be the special fibre, where C_i is an irreducible divisor, and σ be a horizontal divisor of degree 0. We assume that $m_0 = 1$. Let M be a matrix given by $M_{ij} = (C_i \cdot C_j)$, for $1 \le i, j \le n$. Define rational numbers a_i $(i = 1, \ldots, n)$ by*

$$(a_1, \ldots, a_n) = -((\sigma \cdot C_1), \ldots, (\sigma \cdot C_n))M^{-1}.$$

Then we have $(\sigma + \sum_{i=1}^{n} a_i C_i \cdot C_j) = 0$ for any $j = 0, \ldots, n$.

Finally Néron's formula is $\langle \overline{a}, \overline{b}\rangle = \sum_{v \in \Sigma_K} \langle a, b\rangle_v$; see Néron [16],pp.295–296. Here $\langle \overline{a}, \overline{b}\rangle$ is the height paring on $J \times J$ satisfying

$$\langle \overline{a}, \overline{b}\rangle = \hat{h}(\overline{a} + \overline{b}) - \hat{h}(\overline{a}) - \hat{h}(\overline{b}),$$

where $\hat{h}$ is the Néron-Tate height $\hat{h}_\Theta$ attached to Θ(see Introduction). (We identify J and $\hat{J}$ by $J \to \hat{J}$, $a \mapsto$ (class of $(\Theta - a) - \Theta$)). If L is a finite extension of K, we have

$$\langle \alpha, \beta\rangle_L = [L : K]\ \langle \alpha, \beta\rangle_K\ . \tag{3.4}$$

3.3. **Green's Function.** A meromorphic differential ϱ on the Riemann surface $C(\mathbf{C})$ is said to be of the third kind if $\mathrm{ord}_x(\varrho) \geq -1$ for all $x \in C(\mathbf{C})$. For such ϱ, we define the divisor $\mathrm{Res}(\varrho)$ by $\sum\limits_{x \in C(\mathbf{C})} \mathrm{res}_x(\varrho)\, x$, which belongs to $\mathrm{Div}_0(C(\mathbf{C}))$. Conversely, by the Riemann-Roch theorem, for any $D \in \mathrm{Div}_0(C(\mathbf{C}))$, there exists a differential of the third kind ϱ such that $\mathrm{Res}(\varrho) = D$, and it is determined up to an addition of a differential of the first kind.

Lemma 3.6. *We can choose ϱ uniquely with pure imaginary periods.*

Proof. For any differential of the third kind $\tilde{\varrho}$, we define

$$\mathbf{r} = -2\left(\mathrm{Re}\left(\int_{\gamma_1} \tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_2} \tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_1'} \tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_2'} \tilde{\varrho}\right)\right).$$

Then we can define complex numbers c_1, c_2 by

$$(c_1, c_2, \overline{c_1}, \overline{c_2}) = \mathbf{r}\tilde{\Omega}^{-1},$$

where $\tilde{\Omega} = \begin{pmatrix} \omega & \omega' \\ \overline{\omega} & \overline{\omega'} \end{pmatrix}$. Then $\varrho = \tilde{\varrho} + c_1\, \mu_1 + c_2\, \mu_2$ has pure imaginary periods. $\square$

Lemma 3.7. *If $a = D(P_1)$ with $P_1(x_1, y_1) \in C$, then ω_a is explicitly given by*

$$\omega_a = \frac{y + y_1}{2\, y\, (x - x_1)} dx + \omega_h,$$

where ω_h is any differential of the first kind.

Proof. Noting that $\mathrm{div}(dx) = \sum B_i - 3\,\infty$ and $\mathrm{div}(y) = \sum B_i - 5\,\infty$, it is obvious that ω_a has simple poles at only P_1 and ∞. The function $x - x_1$ is an uniformizer at P_1 and $\mathrm{res}_{P_1}(\omega_a) = 1$. By the residue theorem, $\mathrm{res}_\infty(\omega_a) = -1$ and $\mathrm{Res}(\omega_a) = P_1 - \infty = a$. $\square$

Definition 3.8. For any $v \in \Sigma_K^\infty$ and for each $a \in \mathrm{Div}_0(C)$, Green's function on $C(\mathbf{C}) - |a|$ attached to a is a real valued harmonic function g_a such that

(1) $g_a - m_x \log |z|_v$ is harmonic near x, where z is a local parameter at x and m_x is the order of x in a.
(2) g_a is a solution of a differential equation $\partial\bar{\partial} g_a = -2\,\pi\, i \delta_a$, where δ_a is (1,1)-current which represents the evaluation of (0,0)-forms at a.

For $a \in \mathrm{Div}_0(C)$, choose ω_a so that it has pure imaginary periods, then the differential equation $\omega_a + \bar{\omega}_a = d\, g$ has a solution g and we can take g as g_a [9].

Now we define a modified σ-function $k(u)$, which is a natural generalization of the Klein function, by

$$k(u) = c^{-1} \exp\left(\frac{1}{2} u \tilde{\eta}(u)\right) \sigma(u). \tag{3.5}$$

Proposition 3.9. *The function $|k(u)|$ on $\mathbf{C}^2$ is periodic for Λ.*

Proof. First note that ${}^t z\ \tilde{\eta}(w) = {}^t\tilde{\eta}(w)\ z$, since this is a scalar. Let $\kappa_l(u) = k(u+l)/k(u)$ for $l = \omega\, p + \omega'\, p'$, p, $p' \in \mathbf{Z}^2$, then by Lemma 1.4,

$$\kappa_l(u) = \begin{pmatrix} {}^t p & {}^t p' \end{pmatrix}\ M_r \begin{pmatrix} r \\ r' \end{pmatrix} + \begin{pmatrix} {}^t p & {}^t p' \end{pmatrix}\ M_p \begin{pmatrix} p \\ p' \end{pmatrix}.$$

where

$$M_r = \begin{pmatrix} \frac{1}{2}\,({}^t\eta\ \omega - {}^t\omega\ \eta) & \frac{1}{2}\,({}^t\eta\ \omega' + {}^t\omega\ \eta' - 2\,{}^t\omega\ \eta\,\tau) \\ \frac{1}{2}\,({}^t\eta'\ \omega - {}^t\omega'\ \eta) & \frac{1}{2}\,({}^t\eta'\ \omega' + {}^t\omega'\ \eta' - 2\,{}^t\omega'\,\eta\,\tau) \end{pmatrix} - 2\,\pi\, i \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix},$$

and

$$M_p = \begin{pmatrix} 0 & \frac{1}{2}\,({}^t\omega\ \eta' - {}^t\omega\ \eta\,\tau) \\ 0 & \frac{1}{2}\,({}^t\omega'\ \eta' - {}^t\omega'\ \eta\,\tau - 2\,\pi\, i\,\tau) \end{pmatrix}.$$

By (1.3), $\eta\,\omega^{-1} = {}^t(\eta\,\omega^{-1}) = {}^t\omega^{-1}\,{}^t\eta$, thus, ${}^t\eta\ \omega = {}^t\omega\ \eta$. That is $(M_r)_{11} = 0$. By (1.2), ${}^t\eta'\ \omega = {}^t(\eta\,\tau + 2\,\pi\, i\,{}^t\omega^{-1})\ \omega = \tau\,{}^t\eta\ \omega + 2\,\pi\, i\,\mathbf{1}_2$ and ${}^t\omega'\ \eta = {}^t({}^t\eta\ \omega') = {}^t({}^t\eta\ \omega\,\tau) = \tau\,{}^t({}^t\eta\,\omega) = \tau\,{}^t\omega\ \eta$. As above, ${}^t\eta\ \omega = {}^t\omega\ \eta$, so $(M_r)_{21} = -\pi\, i\,\mathbf{1}_2$. Similarly, $(M_r)_{12} = \pi\, i\,\mathbf{1}_2$. Finally, since $\tau\,{}^t\eta\ \omega' = \tau\,{}^t(\eta\,\omega^{-1}\,\omega)\ \omega' = {}^t\omega'\ \eta\,\tau$, using (1.2), we get $(M_r)_{22} = 0$. Also by (1.2), we conclude that

$$M_p = \begin{pmatrix} 0 & \pi\, i\,\mathbf{1}_2 \\ 0 & 0 \end{pmatrix}.$$

Since every p, p', r, r' belongs to $\mathbf{R}^2$, $\kappa_l(u)$ is of the form $\exp(i \cdot \text{'real number'})$ and $|\kappa_l(u)| = 1$. □

Corollary 3.10. *For $v \in \Sigma_K^\infty$ and $u \in \mathbf{C}^2$,*

$$\hat{\lambda}_v(\tilde{u}) = -\log |k(u)|_v\,. \tag{3.6}$$

Proof. By the proposition above, the right-hand side of (3.6) is well defined and clearly it is a Weil function for Θ. Furthermore

$$\frac{k(2\,u)}{k^4(u)} = c^3 \frac{\sigma(2\,u)}{\sigma^4(u)} = -\phi(u),$$

thus the right-hand side of (3.6) satisfies the property (2) in the definition of $\hat{\lambda}_v$. By the uniqueness, the assertion follows. □

Now we have an explicit formula of Green's function.

Theorem 3.11. (1) *When* $a = P_1 - \infty$,

$$g_a \equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})}{k(2\,u^P)}\right|_v .$$

(2) *When* $a = P_1 + P_2 - 2\,\infty$,

$$\begin{aligned} g_a &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})}{k(2\,u^P + u^{P_2})}(x - x_2)\right|_v \\ &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_2})}{k(2\,u^P + u^{P_1})}(x - x_1)\right|_v \\ &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})\,k(2\,u^P - u^{P_2})}{k(2\,u^P)^2}\right|_v . \end{aligned}$$

In the both cases, the symbol $\equiv$ *means equality up to a constant.*

Proof. (1) We assume $K_v = \mathbf{C}$. From the result of Bost [2], we can prove the theorem. In fact if we view $\overline{D} \in \mathrm{Pic}^1(C)$ as $\overline{D - \infty} \in \mathrm{Pic}^0(C)$, then the function $||\theta||$ coincide the generalized Klein function. But Bost does not give a proof in [2], therefore we give a direct proof by checking that the g_a really satisfies $d\,g_a = \omega_a + \overline{\omega_a}$.

Let $u = 2\,u^P - u^{P_1}$, $v = 2\,u^P$, and $d\,g_a = G\,dx + G'\,d\bar{x}$. Then

$$G = \frac{1}{2}\,\frac{\partial}{\partial x}\left(\log\,k(u) + \log\,\overline{k(u)} - \log\,k(v) - \log\,\overline{k(v)}\right) .$$

In general, for $z = 2\,u^P + v_0$, $v_0 \in \mathbf{C}^2$,

$$\frac{\partial}{\partial\,x}\,\log k(z) = \frac{\partial}{\partial\,x}\left(\frac{1}{2}\,{}^t z\,\tilde{\eta}(z)\right) + \frac{\partial}{\partial\,x}\,\log\sigma(z) .$$

If we put $(Y\ Y') = (\eta\ \eta')\,\tilde{\Omega}^{-1}$, where $\tilde{\Omega} = \begin{pmatrix}\omega & \omega' \\ \overline{\omega} & \overline{\omega'}\end{pmatrix}$, then $Y, Y' \in M_2(\mathbf{C})$, $Y = {}^tY$, ${}^tY' = \overline{Y'}$, and

$$\begin{aligned} {}^t z\,\tilde{\eta}(z) =& Y_{11}\,z_1^2 + 2\,Y_{12} z_1\,z_2 + Y_{22}\,z_2^2 \\ &+ Y'_{11}\,z_1\,\overline{z_1} + Y'_{12}\,z_1\,\overline{z_2} + \overline{Y'_{12}}\,\overline{z_1}\,z_2 + Y'_{22}\,z_2\,\overline{z_2} . \end{aligned}$$

Since $\dfrac{dz_1}{dx} = \dfrac{1}{y}$, $\dfrac{dz_2}{dx} = \dfrac{x}{y}$,

$$\begin{aligned} \frac{\partial}{\partial\,x}\left(\frac{1}{2}\,{}^t z\,\tilde{\eta}(z)\right) &= \frac{1}{2}\left(\frac{\partial}{\partial z_1}\,{}^t z\,\tilde{\eta}(z)\frac{dz_1}{dx} + \frac{\partial}{\partial z_2}\,{}^t z\,\tilde{\eta}(z)\frac{dz_2}{dx}\right) \\ &= \left(Y_{11}\,z_1 + Y_{12}\,z_2 + \frac{1}{2}\,Y'_{11}\overline{z_1} + \frac{1}{2}\,Y'_{12}\overline{z_2}\right)\frac{1}{y} \\ &\quad + \left(Y_{12}\,z_1 + Y_{22}\,z_2 + \frac{1}{2}\,\overline{Y'_{12}}\overline{z_1} + \frac{1}{2}\,Y'_{22}\overline{z_2}\right)\frac{x}{y} . \end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial x} \log \overline{k(z)} = \frac{\partial}{\partial x} \left(\frac{1}{2} \overline{{}^t z\, \tilde{\eta}(z)} \right) + \frac{\partial}{\partial x} \log \overline{\sigma(z)} .$$

Since $\dfrac{\partial}{\partial x} \log \overline{\sigma(z)} = 0$,

$$\frac{\partial}{\partial x} \log \overline{k(z)} = \frac{1}{2} \left(\overline{Y'_{11}}\, \overline{z_1} + Y'_{12}\, \overline{z_2} \right) \frac{1}{y} + \frac{1}{2} \left(\overline{Y'_{12}}\, \overline{z_1} + \overline{Y'_{22}}\, \overline{z_2} \right) \frac{x}{y} .$$

Thus, using the fact that $Y'_{11} = \overline{Y'_{11}}$, $Y'_{22} = \overline{Y'_{22}}$, $Y'_{12} = \overline{Y'_{21}}$,

$$G = \frac{1}{2} \frac{\partial}{\partial x} \log \frac{\sigma(u)}{\sigma(v)} + L(u - v),$$

where

$$\begin{aligned} L(z) = {} & \frac{1}{y} \left(Y_{11}\, z_1 + Y_{12}\, z_2 + Y'_{11}\, \overline{z_1} + Y'_{12}\, \overline{z_2} \right) \\ & + \frac{x}{y} \left(Y_{12}\, z_1 + Y_{22}\, z_2 + Y'_{21}\, \overline{z_1} + Y'_{22}\, \overline{z_2} \right) . \end{aligned}$$

Now we want to prove

$$\frac{d^2}{dx^2}\, 2\, y\, G = \frac{d^2}{dx^2} \frac{2\, y\, \omega_a}{dx}. \tag{3.7}$$

If this is proved, then for some $a, b \in \mathbf{C}$, we have

$$G\, dx = \omega_a + \frac{a + b\, x}{2\, y} dx,$$

That is $G\, dx$ and ω_a is equal up to addition of differential of the first kind. Since we can prove $\partial \overline{\partial} g_a = 0$ or the harmonicity of g_a in the similar way as above, we shall verify (3.7).

Observing that

$$\frac{d^2}{dx^2} \left(2\, y\, L(z) \right) = 2 \frac{d}{dx} \left(\frac{Y_{11} + 2 Y_{12} x + Y_{22} x^2}{y} \right) + 2\, Y_{12} + 2\, Y_{22}\, x,$$

is independent of v_0, we have

$$\frac{d^2}{dx^2} \left(2\, y\, L(u - v) \right) = 0 .$$

On the other hand,

$$\frac{d^2}{dx^2}\left(y\,\frac{d}{dx}\,\log\,\sigma(z)\right) = \frac{d^2}{dx^2}\,(\zeta_1(z) + \zeta_2(z)\,x)$$

$$= \frac{d}{dx}\left(-\mathfrak{p}_{11}(z)\,\frac{1}{y} - 2\,\mathfrak{p}_{12}(z)\,\frac{x}{y} - \mathfrak{p}_{22}(z)\,\frac{x^2}{y} + \zeta_2(z)\right)$$

$$= \frac{f'(x)}{2\,y^3}\,\mathfrak{p}_{11}(z) + 2\,\frac{x\,f'(x) - 2\,f(x)}{2\,y^3}\,\mathfrak{p}_{12}(z) + \frac{x^2\,f'(x) - 4\,x\,f(x)}{2\,y^3}\,\mathfrak{p}_{22}(z)$$

$$-\frac{1}{y^2}\,\mathfrak{p}_{111}(z) - \frac{3\,x}{y^2}\,\mathfrak{p}_{112}(z) - \frac{3\,x^2}{y^2}\,\mathfrak{p}_{122}(z) - \frac{x^3}{y^2}\,\mathfrak{p}_{222}(z) - \mathfrak{p}_{12}(z)\,\frac{1}{y} - \mathfrak{p}_{22}(z)\,\frac{x}{y}.$$

As for the right-hand side of (3.7),

$$\frac{d^2}{dx^2}\frac{2y\omega_a}{dx} = \frac{y''\,(x-x_1)^2 - 2\,(x-x_1)\,y' + 2\,(y+y_1)}{(x-x_1)^3},$$

where $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2\,y}{dx^2}$. Using (1.11) and Theorem 2.3, computer-assisted computations for the right-hand side leads the desired equation.

(2) The third form is obvious by (1) . The first and second ones may be deduced from Bost's result [2]. Or we can deduce them by directly computing in the same manner as (1). Alternatively, noticing that the correspondence

$$P \to \frac{\sigma(2\,u^P - u^{P_1})\,\sigma(2\,u^P + u^{P_1})}{\sigma^2(2\,u^P)}$$

is independent of a choice of the integral and becomes a rational function on C, hence we have the equation

$$\frac{\sigma(2\,u^P - u^{P_1})\,\sigma(2\,u^P + u^{P_1})}{\sigma^2(2\,u^P)} = \mathrm{Const}\,\cdot\,(x-x_1)^2. \tag{3.8}$$

This formula also can be checked by a direct computation similarly as above. Since if $a = b + \mathrm{div}(\alpha)$, then $g_a(P) = g_b(P) + \log|\alpha(P)|_v$, the assertion follows. □

Remark 3.12. The constant appeared in the right-hand side of (3.8) is given by M/c^2 where $M = \lim\limits_{u\to u^{P_1}} \sigma^2(u)\mathfrak{p}_{22}(u)$.

3.4. **The canonical local height and Néron's local pairing.** Let P_1, P_2 be K-rational points on C. Take P_3, P_4, P_5, P_6 which satisfy $P_1 + P_3 + P_4 \sim P_2 + P_5 + P_6$. Define polynomials $G_1 = V_{P_1,P_3,P_4}$ and $G_2(x) = V_{P_2,P_5,P_6}$ (see § 1.1). For simplicity, we write u^i for u^{P_i}.

Proposition 3.13. *Let P_3, P_4, P_5, P_6 and G_1, G_2 as above. Then we have*

$$q(u^1-u^2,u^3+u^4-u^2)(=q(u^1-u^2,u^5+u^6-u^1))$$
$$=\frac{(x_1-x_3)(x_1-x_4)(x_2-x_5)(x_2-x_6)(y_2+G_1(x_2))(y_1+G_2(x_1))}{(x_2-x_3)(x_2-x_4)(x_1-x_5)(x_1-x_6)(x_1-x_2)^2}.$$

Proof. Let $\overline{D(P_1,P_3,P_4)}=\overline{D(P_2,P_5,P_6)}=\overline{D(P_{11},P_{12})}$, $P_{1j}=(x_{1j},y_{1j})$. Then G_1 is characterized by $G_1(x_1)=y_1$, $G_1(x_{11})=-y_{11}$, $G_1(x_{12})=-y_{12}$, G_2 is characterized in the similar way. By the periodicity of the left-hand side we can assume

$$u^1+u^3+u^4=u^2+u^5+u^6=u^{11}+u^{12} \tag{3.9}$$

Using (2.3), the terms $(x_1-x_3)(x_1-x_4)$, $(x_1-x_5)(x_1-x_6)$, $(x_2-x_3)(x_2-x_4)$, $(x_2-x_5)(x_2-x_6)$, $(y_2+G_1(x_2))(y_1+G_2(x_1))$ are explicitly described via x_1, y_1, x_2, y_2 and $\mathfrak{p}$-functions of $u^{11}+u^{12}$. Since $u^3+u^4-u^2=u^{11}+u^{12}-u^1-u^2$, the left-hand side is also expressed via x_1, y_1, x_2, y_2 and $\mathfrak{p}$-functions of $u^{11}+u^{12}$. Take the quotient of the right-hand side by the left-hand side, and expand the difference of the numerator and the denominator of the quotient uniquely using $y_i^2=f(x_i)$. All the coefficients of $x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2}$, $(j_k=0,1)$, are expressed via $\mathfrak{p}$-functions of $u^{11}+u^{12}$. Finally substitute (1.10) into the each coefficient, and using relations $y_{1j}^2=f(x_{1j})$, we can show the difference is equal to 0. □

Remark 3.14. The above proof is computational and cost quite long computational time. Even if we consider the equation as meromorphic function of P_1, we cannot resolve the zeros and poles since $P_i\,(i=3,4,5,6)$ are not independent, depend on P_1. The author does not have any idea to prove the proposition theoretically.

Theorem 3.15. *For $P_i(x_i,y_i)\in C(K)$, $i=1,2$, let $b=P_1-P_2$ with $\bar{b}\notin\Theta$, and $z_b=u^1-u^2\in\mathbf{C}^2$. As the base of tangent space at P_i, we take $2\,y_i\,\dfrac{\partial}{\partial x}=f'(x_i)\,\dfrac{\partial}{\partial y}$. Then we can take $\dfrac{x-x_i}{2\,y_i}$ if $P_i\notin\mathcal{B}$ and $\dfrac{y-y_i}{f'(x_1)}$ if $P_i\in\mathcal{B}$ as an uniformizer at P_i. In both cases, for an archimedean place v, if we take the uniformizer as above, the relation*

$$\langle b,b\rangle_v=2\,\hat{\lambda}_v(\tilde{z}_b) \tag{3.10}$$

between Néron's local pairing and the canonical local height holds.

Proof. For any divisor $b'\in\mathrm{Div}_0(C)$ which is linearly equivalent to b, let $G=G_{b,b'}$ be a rational function such that $b=b'+\mathrm{div}(G_{b,b'})$. Then we have

$$\langle b,b\rangle_v=\langle b,b'\rangle_v+\log|G[b]|_v\,,\quad \langle b,b'\rangle_v=g_b(b'),$$

where $G[b]$ means the modified value of G at b (see (3.2)). As in the proof of Proposition 3.13, we can take b' in the form $b'=P_5+P_6-P_3-P_4$ and $P_1+P_3+P_4\sim P_{11}+P_{12}+\infty$.

If we define

$$\tilde{G_1}(P) = \frac{y + G_1(x)}{(x-x_1)\,(x-x_3)\,(x-x_4)},$$
$$\tilde{G_2}(P) = \frac{y + G_2(x)}{(x-x_2)\,(x-x_5)\,(x-x_6)},$$

then,

$$\operatorname{div}(\tilde{G_1}) = P_{11} + P_{12} + \infty - P_1 - P_3 - P_4,$$
$$\operatorname{div}(\tilde{G_2}) = P_{11} + P_{12} + \infty - P_2 - P_5 - P_6\,.$$

Thus if we define $G = \tilde{G_2}/\tilde{G_1}$, then $\operatorname{div}(G) = b - b'$, that is we can take G as $G_{b,b'}$.

When both P_i, $(i = 1, 2)$ are not Weierstrass points, we may take the local parameter at P_i as mentioned in the theorem, and we have

$$G[b] = \frac{(x_1-x_3)(x_1-x_4)(x_2-x_5)(x_2-x_6)(y_2+G_1(x_2))(y_1+G_2(x_1))}{-(x_2-x_3)\,(x_2-x_4)\,(x_1-x_5)\,(x_1-x_6)\,(x_1-x_2)^2}\,.$$

By Theorem 3.11, we have

$$\begin{aligned}\langle b, b'\rangle_v &= g_b(P_5) + g_b(P_6) - g_b(P_3) - g_b(P_4)\\ &= \frac{1}{2}\,\log\left|\frac{k(2\,u^5-u^1)\,k(2\,u^6-u^1)\,k(2\,u^3-u^2)\,k(2\,u^4-u^2)}{k(2\,u^5-u^2)\,k(2\,u^6-u^2)\,k(2\,u^3-u^1)\,k(2\,u^4-u^1)}\right|_v\,.\end{aligned}$$

Put $V_{i,j} = {}^t u^{\,i}\tilde{\eta}(u^j)$. Then the formula in the $\log|\quad|$ is of the form

$$\exp(E)\,\Sigma,$$

where

$$\begin{aligned}E = &-(V_{1,5} - V_{2,5} + V_{1,6} - V_{2,6}) + (V_{1,3} + V_{1,4} - V_{2,3} - V_{2,3})\\ &- (V_{5,1} + V_{6,1} - V_{5,2} - V_{6,2}) + (V_{3,1} + V_{4,1} - V_{3,2} - V_{4,2}),\end{aligned}$$

and

$$\Sigma = \frac{\sigma(2\,u^5-u^1)\,\sigma(2\,u^6-u^1)\,\sigma(2\,u^3-u^2)\,\sigma(2\,u^4-u^2)}{\sigma(2\,u^5-u^2)\,\sigma(2\,u^6-u^2)\,\sigma(2\,u^3-u^1)\,\sigma(2\,u^4-u^1)}.$$

By the assumption (3.9), we have

$$E = -2\,z_b\,\tilde{\eta}(z_b).$$

On the other hand, by (1.7), we have

$$\Sigma = \Sigma_1\,\times\,\frac{\sigma^2(u^5+u^6-u^1)\,\sigma^2(u^3+u^4-u^2)}{\sigma^2(u^5+u^6-u^2)\,\sigma^2(u^3+u^4-u^1)},$$

where

$$\Sigma_1 = \frac{q(u^5+u^6-u^1,u^5-u^6)\,q(u^3+u^4-u^2,u^3-u^4)}{q(u^5+u^6-u^2,u^5-u^6)\,q(u^3+u^4-u^1,u^3-u^4)}\,.$$

Using (1.7) again, we have

$$\begin{aligned}\sigma^2(u^5+u^6-u^2)&\sigma^2(u^3+u^4-u^1)\\ &=\frac{1}{c^4}q^2(u^3+u^4-u^2,u^1-u^2)\sigma^4(u^3+u^4-u^2)\sigma^4(z_b),\end{aligned}$$

and by the assumption (3.9),

$$\Sigma = \Sigma_1\,\Sigma_2\,(c^{-1}\sigma(z_b))^{-4},$$

where

$$\Sigma_2 = \frac{1}{q^2(u^3+u^4-u^2,u^1-u^2)}\,.$$

Here we prove the following lemma.

Lemma 3.16. *Let Σ_1 be as above, then $\Sigma_1 = 1$.*

Proof. Let i, j, $k \in \{1,\dots,6\}$ be distinct indices. We put $v^+ = u^i + u^j$, $v^- = u^i - u^j$, and $u = v^+ - u^k$. By (1.7), $q(u,v^+) = 0$ and, by (1.10), $\mathfrak{p}_{12}(v^-) = \mathfrak{p}_{12}(v^+)$ and $\mathfrak{p}_{22}(v^-) = \mathfrak{p}_{22}(v^+)$. Hence we have

$$\begin{aligned}q(u,v^-) &= \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v^-) + \mathfrak{p}_{12}(u)\,\mathfrak{p}_{22}(v^-) - \mathfrak{p}_{12}(v^-)\,\mathfrak{p}_{22}(u) - q(u,v^+)\\ &= \mathfrak{p}_{11}(v^+) - \mathfrak{p}_{11}(v^-) = \frac{-4\,y_i\,y_j}{(x_i-x_j)^2},\end{aligned}$$

and this does not depend on the index k, thereby completing the proof. □

Finally, by Proposition 3.13, we have

$$\langle b,b\rangle_v = \frac{1}{2}\,\log\left|\Sigma_2\,k^{-4}(z_b)\right|_v + \frac{1}{2}\,\log|G[b]|_v^2 = -2\,\log|k(z_b)|_v\,.$$

As for the case where $P_i \in \mathcal{B}$, we can prove the equation similarly, noticing that

$$\lim_{P\to P_i}\frac{x-x_i}{y-y_i} = \frac{2\,y_i}{f'(x_i)}.$$

In any case, by Corollary 3.10, we have $\langle b,b\rangle_v = 2\,\hat{\lambda}_v(\tilde{z}_b)$. This completes the proof. □

Remark 3.17. In Proposition 1.6, substituting P_1, P_2, P_3, P_3, P_5, P_5 for P, Q, P_1, P_2, Q_1, Q_2 respectively, we have

$$2\,R^{P_1,P_2}_{P_3,P_5} = \log\frac{\sigma(u^1-2\,u^3)}{\sigma(u^1-2\,u^5)}\Big/\frac{\sigma(u^2-2\,u^3)}{\sigma(u^2-2\,u^5)}\,.$$

Since $\dfrac{d}{dx}\,\dfrac{d}{dz}\log\sigma(u^P-u^Q) \;=\mathfrak{p}_{11}(u^P-u^Q)/\,4\,y\,s$, where $P=(x,y), Q=(z,s)$, so

$$R^{P_1,P_2}_{P_3,P_5}=\int_{P_2}^{P_1}\int_{P_5}^{P_3}\frac{d}{dx}\,\frac{d}{dz}\log\sigma(u^P-u^Q)dxdz\,.$$

Thus we have

$$\log\frac{\sigma(u^1-2\,u^3)}{\sigma(u^1-2\,u^5)}\Big/\frac{\sigma(u^2-2\,u^3)}{\sigma(u^2-2\,u^5)}=2\,\log\frac{\sigma(u^1-u^3)}{\sigma(u^1-u^5)}\Big/\frac{\sigma(u^2-u^3)}{\sigma(u^2-u^5)},$$

and we can verify the symmetry of the pairing.

By Proposition 3.13 and the proof of Theorem 3.15, we have

Corollary 3.18. *Let $b=P_1-P_2\in Z_0(C)_{K_v}$, $b'=P_5+P_6-P_3-P_4\in \mathrm{Div}_0(C)_{K_v}$ and u^i be as above. Then we have*

$$\langle b,b\rangle_v=\langle b,b'\rangle_v+\log|q(u^1-u^2,u^3+u^4-u^2)|_v.$$

4. Tate's series

In this section, we shall give concrete expression of Tate's series for the canonical local height.

4.1. **Generalities.**

We review Tate's series [3]. In general let V be a non-singular projective variety, Ψ be a morphism $V\to V$, and Θ be a divisor $\Theta\in\mathrm{Div}(V)\otimes\mathbf{R}$, with $\Psi^*\,\Theta=\alpha\,\Theta+\mathrm{div}(\phi)$, for some real number $\alpha>1$ and a function ϕ. Let $t_1,\dots,t_r$, $t_i\in K(V)\otimes\mathbf{R}$ be functions with $\mathrm{div}(t_i)=\Theta-D_i$ satisfying $\bigcap_i\mathrm{supp}(D_i)=\emptyset$. For each $i=1,\dots,r$, we define functions $w_i=\phi\cdot t_i^\alpha$, $z_i=\dfrac{\phi\,\cdot\,t_i^\alpha}{t_i\circ\Psi}$, and for $i,j=1,\dots,r$, we define $s_{ij}=\dfrac{z_j\,w_i}{w_j}$. For any ample divisor D, we define a distance function λ_D as in [3],pp.191–192. Then we have

Theorem 4.1 ([3])**.** *Let $P\in V(\overline{K_v})\setminus\mathrm{supp}(\Theta)$ be given. Define a sequence of indices $i_0,\,i_1,\,\dots,\,i_n,\,\dots,$ by*

$$\lambda_{\mathrm{supp}(D_{i_n})}(\Psi^n\,P)=\min_{1\le i\le r}\lambda_{\mathrm{supp}(D_i)}(\Psi^n\,P)\,.$$

Define a sequence of real numbers c_n as

$$c_n=-v(s_{i_n\,i_{n+1}}(\Psi^n\,P)),\quad n=0,1,2,\dots,$$

which is bounded independently of n and P. Then

$$\hat{\lambda}_\Theta(P) = v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + O(\alpha^{-N}),$$

where the constant of $O(\alpha^{-N})$ is independent of both P and N.

4.2. **The case of Jacobian surfaces.** Now we apply the above to the case of Jacobian surfaces. That is $V = J$, Θ is the theta divisor, $\Psi = \Psi_2$, that is the multiplication by 2 map, and $\alpha = 4$.

For $P \in J$, we denote by T_P the translation map $J \to J$, $D \mapsto D + P$.

Lemma 4.2. *Let $D_1 = T^*_{\overline{B_1}}\Theta$, $D_2 = T^*_{\overline{B_2}}\Theta$, and $D_3 = T^*_{\overline{B_{13}}}\Theta$. Then D_i is irreducible and $\bigcap D_i = \emptyset$.*

Proof. The first assertion is obvious since Θ is irreducible. Any point in D_1 can be written $\overline{D(P, B_1)}$. If this point belongs to D_2, then, for some $Q \in C$, $P + B_1 \sim Q + B_2$. If $P \neq B_1 (= B_1^\iota)$ and $P \neq \infty$, by the uniqueness, $P = B_2$ and $Q = B_1$, hence the point is $\overline{D(P, B_1)} = \overline{B_{12}}$. If $P = B_1$, then $Q = B_2$ and $\overline{D(P, B_1)} = O_J$. The case $P = \infty$ does not occur. Thus $D_1 \cap D_2 = \{O_J, \overline{B_{12}}\}$, hence we have to prove that both O_J and $\overline{B_{12}}$ do not belong to D_3. If $O_J \in D_3$, that is for some $P \in C$, $B_1 + B_3 \sim P + \infty$. Since $\overline{B_{13}} \notin \Theta$, this case does not occur. If $\overline{B_{12}} \in D_3$, then $B_2 \sim B_3$, which leads to contradiction. $\square$

Lemma 4.3. *Let t_i be the functions corresponding to the divisors D_i of Lemma 4.2 (see § 4.1). We can take t_i as follows:*

$$t_1 = \left(\frac{X_0}{X_{12} + \beta_1 X_{22} - \beta_1^2 X_0}\right)^{1/2},$$
$$t_2 = \left(\frac{X_0}{X_{12} + \beta_2 X_{22} - \beta_2^2 X_0}\right)^{1/2}, \tag{4.1}$$
$$t_3 = \left(\frac{X_0}{X_{11} + (\beta_1 + \beta_3) X_{12} + \beta_1 \beta_3 X_{22} + A_{13} X_0}\right)^{1/2},$$

where

$$A_{13} = (\beta_1 + \beta_3)(\beta_1^2 + \beta_1\beta_3 + \beta_3^2) + a_1(\beta_1 + \beta_3)^2 + a_2(\beta_1 + \beta_3) + a_3.$$

Proof. First, as for t_1, t_2, we can show that the function $\mathfrak{p}_{12}(u) + \beta_i \mathfrak{p}_{22}(u) - \beta_i^2$ vanishes only when $u = u^{B_i} + u^P$, since if we put $u = u^{P_1} + u^{P_2}$, then this function is equal to

$(x_1-\beta_i)\,(x_2-\beta_i)$. The function has poles at Θ of order 2, thus $\operatorname{div}(t_i^2) = 2\,T^*_{B_i}\,\Theta - 2\,\Theta$, that is, $\operatorname{div}(t_i) = D_i - \Theta$.

To prove the formula for t_3, we use the following lemma.

Lemma 4.4. *Let $u \in \mathbf{C}^2$ and $v_0 = u^{P_1} + u^{P_2}$ for $P_1, P_2 \in C$. Define a rational function ϕ_{P_1,P_2} on $\mathbf{C}^2$ by*

$$\phi_{P_1,P_2}(u) = \mathfrak{p}_{11}(u) + \mathfrak{p}_{22}(v_0)\,\mathfrak{p}_{12}(u) - \mathfrak{p}_{12}(v_0)\,\mathfrak{p}_{22}(u) - \mathfrak{p}_{11}(v_0)\,.$$

Then

$$\operatorname{div}(\phi_{P_1,P_2}) = T^*_{P_1,P_2}\,\Theta + T^*_{P_1',P_2'}\,\Theta - 2\,\Theta.$$

Proof. By (1.7), the function ϕ_{P_1,P_2} vanishes at $T^*_{P_1,P_2}\,\Theta$ and $T^*_{P_1',P_2'}\,\Theta$, has poles at Θ of order 2 and has no poles at elsewhere. Thus the lemma follows. ☐

Proof of Lemma 4.3. By Lemma 4.4, we can take $\phi^{1/2}_{B_1,B_3}$ as t_3. Using the fact that $f(\beta_i) = 0$, we have $F(\beta_1,\beta_3) + A_{13}\,(\beta_1-\beta_3)^2 = 0$, and from (1.10), the assertion follows. ☐

Finally, for $u \in \mathbf{C}^2$ and $\tilde{u} \in J$, as a function measuring the distance of $\tilde{u}$ and Θ, we take

$$\lambda_\Theta(\tilde{u}) = \max\left(\log|\mathfrak{p}_{ij}(u)|\,,\log|\mathfrak{p}_{ijk}(u)|\,,\log|\mathfrak{p}(u)|\right)\,.$$

If $\mathfrak{p}_I(u) = 0$ for some index I, we regard the value $\log|\mathfrak{p}_I(u)|$ as $-\infty$ and may ignore it.

5. Examples

In this section, we give some examples. Throughout this section, we denote by N_T the number of terms of the summation of Tate's series of Theorem 4.1. For the archimedean place $v = v_\infty$ of $\mathbf{Q}$, we write $\hat{\lambda}_\infty$ for $\hat{\lambda}_v$. For the symbols I_{a-b-c}, I^*_{a-b-c} etc., see [12], also [15].

Example 5.1. Let $C\colon y^2 = f(x) = x^5 - x + \dfrac{1}{4}$. Take points $P_1(0,\frac{1}{2})$, $P_2(1,-\frac{1}{2})$, $P_3(2,\frac{11}{2})$ on C, and let $D = \overline{D(P_1,P_2,P_3)}$. By Theorem 2.3 we have

$$D = [1 : 7/4 : -7/4 : 37/4 : -3/2 : -13/4 : 45/8 : -223/8 : 113/16],$$

and by Theorem 2.5 we have

$$[2]\,D = [671404750 : 2388195140 : -665639030 : -1169137340 : 4898463477 \\ : -1641940454 : -964796127 : -38418056 : -2073577148]\,.$$

On the other hand, we have ($N_T \geq 50$):

$$\begin{aligned} \hat{\lambda}_\infty(D) &= 1.1525641924492670933394535885 7\ldots \\ \hat{\lambda}_\infty(2\,D) &= 0.8389336464720574792371985880 2\ldots \\ v_\infty(\phi(D)) &= -3.7713231233250108941206157662 6\ldots \end{aligned}$$

and we can check the equation $\hat{\lambda}_v(2\,D) = 4\,\hat{\lambda}_v(D) + v(\phi(D))$ holds approximately. In fact, the difference of the left-hand and the right-hand side is less than 10^{-29} if $N_T \geq 50$ and less than 10^{-179} if $N_T \geq 300$.

Example 5.2. Let $f(x)$ be the same as in the previous example. The curve C has the model over $\mathbf{Z}$, $\mathcal{C}$: $y^2 + y = x^5 - x$. This arithmetic surface has singular fibre at $p = 139 (= p_1), 449 (= p_2)$, but it is regular at any point on the surface. In fact, we can prove the singular fibers $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$ are both of genus 1 with one normal singularity and they are of type I_{1-0-0}.

Let $\mathcal{C}_\eta$ be the generic fibre of $\mathcal{C}$ and $\alpha : C \to \mathcal{C}_\eta$ be an isomorphism $(x, y) \mapsto (x, y - \frac{1}{2})$. Take points $P_1(1, \frac{1}{2})$ and $P_2(-1, \frac{1}{2})$ on C and put $b = P_1 - P_2$. Then for $N_T \geq 50$,

$$\hat{\lambda}_\infty(z_b) = 0.3479557596566240490280900180 47\ldots.$$

Take points $P_3(0, \frac{1}{2})$ and $P_4(-1, -\frac{1}{2})$ on C. Let P_5, P_6, with $\alpha(P_5) = (x_5, y_5)$ and $\alpha(P_6) = (x_6, y_6)$ be the points which satisfy $b \sim b'$ for $b' = P_5 + P_6 - P_3 - P_4$. Then, by the addition theorem,

$$x_5 + x_6 = 93/11^2, \quad x_5\,x_6 = -68/11^2,$$
$$y_5 + y_6 = 3 \cdot 61 \cdot 1031/11^5, \quad y_5\,y_6 = 2^2 \cdot 3^2 \cdot 17 \cdot 73/11^5.$$

We write $\tilde{P}$ for the section corresponding to the point $\alpha(P) \in \mathcal{C}_\eta$. Then $\tilde{P}_i$ and $\tilde{P}_j$ do not intersect for $i = 1, 2$ and $j = 3, 4$. Also $\tilde{P}_1$ intersects neither $\tilde{P}_5$ nor $\tilde{P}_6$. One of $\tilde{P}_5$ and $\tilde{P}_6$ intersects $\tilde{P}_2$ with multiplicity 1 on the fibre $\mathcal{C}_{73}$. Thus we have $\langle b, b' \rangle_v = \log 73$ if $p_v = 73$ and $\langle b, b' \rangle_v = 0$ for other finite places v. Since $q(u^1 - u^2, u^3 + u^4 - u^2) = -73/4$, by Corollary 3.18, we have

$$\begin{aligned} \langle \overline{b}, \overline{b} \rangle &= 2\,\hat{\lambda}_\infty(z_b) + 2\,\log 2 \\ &= 2.0822058804331387168906442790105599508865\ldots. \end{aligned}$$

Remark 5.1. In the example above, if we take $P_3 = (2, \dfrac{11}{2})$, $P_4 = (-1, -\dfrac{1}{2})$, then $\tilde{G}[b] = -31/4$. For the place v with $p_v = 31$, $\langle b, b' \rangle_v = \log 31$, and for the other places v, $\langle b, b' \rangle_v = 0$. Hence we obtain the same result for $\langle \overline{b}, \overline{b} \rangle$ and the global height is surely independent of P_3, P_4. In this way, we can check the computation of Néron's symbol.

Example 5.3. Let

$$f(x) = x^5 - 2\,x^4 + 2\,x^3 - x^2 + \frac{1}{4},$$

and C be the hyperelliptic curve $y^2 = f(x)$. Take points $P_1(0, \frac{1}{2})$, $P_2(1, -\frac{1}{2})$, $P_3(2, -\frac{7}{2})$, and $P_4(1, \frac{1}{2})(= P_2^{\iota})$ on C and put $b = P_1 - P_2$. Then we can check $3\,\overline{b} = O$, that is $\overline{b}$ is a 3-torsion point on J using Theorem 2.5. We want to compute $\langle \overline{b}, \overline{b} \rangle$, which is, by the property of the Néron-Tate height, should be 0.

First we compute the archimedean local height of $\overline{b}$. We have $\hat{\lambda}_\infty(\tilde{z}_b) < 10^{-30}$ with $N_T \geq 50$ and $\hat{\lambda}_\infty(\tilde{z}_b) < 10^{-181}$ with $N_T \geq 300$. Thus we have approximately $\hat{\lambda}_v(\tilde{z}_b) = 0$ for the archimedean place v.

Let the arithmetic surface $\mathcal{C}/\mathbf{Z}$ be given by the model

$$\mathcal{C}\colon\, y^2 + y = x^5 - 2\,x^4 + 2\,x^3 - x^2\,.$$

Let $\mathcal{C}_\eta$ be the generic fibre and $\mathcal{C}_p$ be the special fibre over p. The curve C has bad reduction at only $p = 997$, and the surface $\mathcal{C}$ has no singular point. In fact, the Igusa invariants [10](see also [12]) are

$$J_2 = -1, \quad J_4 = 1/2^5, \quad J_6 = -13/2^8,$$
$$J_8 = 3 \cdot 17/2^{12}, \quad J_{10} = 997/2^{20}.$$

The curves C and $\mathcal{C}_\eta$ are isomorphic via the morphism $\alpha\colon (x, y) \mapsto (x, y - \frac{1}{2})$. We have

$$\alpha(P_1) = (0, 0), \quad \alpha(P_2) = (1, -1),$$
$$\alpha(P_3) = (2, -4), \quad \alpha(P_4) = (1, 0)\,.$$

Let P_5 and P_6 be points which satisfy that the divisor $b' = P_5 + P_6 - P_3 - P_4$ is linearly equivalent to b. Then the points P_5 and P_6 are given by

$$\alpha(P_5) = \left(\frac{7 - \sqrt{-23}}{18}, \frac{-19 + 4\sqrt{-23}}{243} \right),$$
$$\alpha(P_6) = \left(\frac{7 + \sqrt{-23}}{18}, \frac{-19 - 4\sqrt{-23}}{243} \right).$$

Now we put $K = \mathbf{Q}(\sqrt{-23})$, and $\omega_1 = \dfrac{1+\sqrt{-23}}{2}$, $\omega_2 = \dfrac{1-\sqrt{-23}}{2}$. Let O_K be the ring of integers of K, then $O_K = \mathbf{Z} \bigoplus \mathbf{Z}\,\omega_1 = \mathbf{Z} \bigoplus \mathbf{Z}\,\omega_2$, and ω_i $(i = 1, 2)$ satisfy the equation $\omega_i^2 - \omega_i + 6 = 0$. We regard naturally $\mathcal{C}$ as an arithmetic surface over O_K. First we prove the following lemma.

Lemma 5.2. *Let b, b' as above. Then, for $v \in \Sigma^0_{\mathbf{Q}}$, we have*

$$\langle b, b' \rangle_v = \begin{cases} 3 \log 2 & \textit{if} \quad p_v = 2, \\ 0 & \textit{if} \quad p_v \neq 2. \end{cases}$$

Proof. Let $\tilde{P}_i$ be the section of the arithmetic surface $\mathcal{C}/O_K$ corresponding to the points $\alpha(P_i) \in \mathcal{C}_\eta$, $(i = 1, \ldots, 6)$. The section $\tilde{P}_1$ intersect $\tilde{P}_3$ over on the fibre $\mathcal{C}_2$ with multiplicity 1 and do not intersect elsewhere. The section $\tilde{P}_1$ do not intersect $\tilde{P}_4$ and $\tilde{P}_2$ do not intersect $\tilde{P}_3$, $\tilde{P}_4$ anywhere. Put $\alpha(P_i) = (x_i, w_i)$ and $x' = \dfrac{1}{x-1}$, $x'' = \dfrac{1}{x}$. The coordinate x', x'' give affine charts containing the point at infinity of C. Then $N_{K/\mathbf{Q}}(x_5 - x_1) = \dfrac{2}{9}$ and $N_{K/\mathbf{Q}}(x'_5 - x'_1) = \dfrac{1}{2}$. Thus the sections $\tilde{P}_5$ and $\tilde{P}_6$ intersect to the section $\tilde{P}_1$ at most over $\mathfrak{P}$ dividing 2. Similarly, $N_{K/\mathbf{Q}}(x_5 - x_2) = \dfrac{4}{9}$ and $N_{K/\mathbf{Q}}(x''_5 - x''_2) = 2$. Thus it is enough to consider over $\mathfrak{P}$ dividing 2.

The ideal $(2) = 2O_K$ decomposes in K as

$$(2) = \mathfrak{P}_{2,1}\,\mathfrak{P}_{2,2}, \quad \mathfrak{P}_{2,i} = [2, \omega_i]\,, i = 1, 2,$$

where, for simplicity, we write $[\alpha_1, \alpha_2]$ for $\alpha_1 \mathbf{Z} \oplus \alpha_2 \mathbf{Z}$, $\alpha_1, \alpha_2 \in K$. We have $\mathfrak{P}^2_{2,i} = [4, \omega_i - 6]$, $\mathfrak{P}^3_{2,i} = [8, \omega_i + 30]$. Let us consider over $\mathfrak{P} = \mathfrak{P}_{2,1}$. Since $w_5 - w_1 = -\dfrac{8\omega_1 - 23}{243}$ and $w_6 - w_1 = -\dfrac{-8\omega_1 - 15}{243}$, the sections $\tilde{P}_j$ and $\tilde{P}_1$ do not intersect over $\mathfrak{P}$ for $j = 5, 6$. Similarly we have $x_5 - x_2 \notin \mathfrak{P}$ and $x_6 - x_2 \in \mathfrak{P}^2 \setminus \mathfrak{P}^3$. Since $\alpha(P_2)$ and $\alpha(P_2^\iota)$ are not congruent mod 2, the intersection multiplicity of the sections $\tilde{P}_2$ and $\tilde{P}_6$ is 2. By (3.1), we obtain the lemma. $\square$

By the lemma above, we have $\sum_{v \in \Sigma^0_{\mathbf{Q}}} \langle b, b' \rangle_v = 3 \log 2$. Since we have $\tilde{G}[b] = -8$, by Corollay 3.18, we have $\sum_{v \in \Sigma^0_{\mathbf{Q}}} \langle b, b \rangle_v = 0$. Summing up over all places, we obtain

$$\langle \bar{b}, \bar{b} \rangle = 0,$$

as desired.

Example 5.4. Let $N = 23$ and $X_0(N)$ be the modular curve. It has the canonical model [7],p.416:

$$y^2 = f(x) = x^6 - 14\,x^5 + 57\,x^4 - 106\,x^3 + 90\,x^2 - 16\,x - 19.$$

Let χ be the quadratic character corresponding to the quadratic field $\mathbf{Q}(\sqrt{-7})$, let $X_0(N)_\chi$ be the twisted modular curve which is given by

$$-7\,y^2 = f(x) = x^6 - 14\,x^5 + 57\,x^4 - 106\,x^3 + 90\,x^2 - 16\,x - 19, \tag{5.1}$$

and denote this by C. Let J be the Jacobian variety of C. We want to verify the Birch-Swinnerton-Dyer Conjecture for J.

Now we recall the Birch-Swinnerton-Dyer Conjecture. Let A be an abelian variety defined over $\mathbf{Q}$, let A' be the dual abelian variety of A, let V_∞ be the volume of real periods $\mathrm{Vol}(A(\mathbf{R}))$, let S be the finte set of bad primes, let V_S be $\mathrm{Vol}(\prod\limits_{p\in S} A(\mathbf{Q}_p))$, let Ш be the Tate-Shafarevich group of A, let $A(\mathbf{Q})_{tors}$ be the torsion part of the Mordell-Weil group of A, and let r be the Mordell-Weil rank of A, which conjecturally equals the order of the Hasse-Weil zeta function $L(s, A)$ at $s = 1$. Let α_i, $1 \leq i \leq r$ be a system of generators of $A(\mathbf{Q}) \otimes \mathbf{Q}$ and $R = \det(\langle \alpha_i, \alpha_j\rangle)_{1\leq i,j\leq r}$ be the regulator of A. Then the conjecture is as follows [22],p.51,Conjecture 2.8.2:

$$\lim_{s\to 1}(s-1)^{-r}\,L(s, A) = \frac{R\,V_\infty\,V_S\,\#\text{Ш}}{\#A(\mathbf{Q})_{tors}\#A'(\mathbf{Q})_{tors}}. \tag{5.2}$$

Since we do not have methods to compute the order of Ш, we want to check

$$\frac{\lim\limits_{s\to 1}(s-1)^{-r}\,L(s, A)\#A(\mathbf{Q})_{tors}\#A'(\mathbf{Q})_{tors}}{R\,V_\infty\,V_S} \in \mathbf{Q}. \tag{5.3}$$

Let $S_2(N)$ be the space of cusp forms of weight 2 with respect to $\Gamma_0(N)$. The space $S_2(23)$ is 2-dimensional. Let $g \in S_2(23)$ be the one of the eigen cusp forms which has the Fourier expansion $g(q) = a_1 + a_2 q + \cdots$, with $a_1 = 1$, $a_2 = \dfrac{-1+\sqrt{5}}{2}$ (cf. [6]). It is well known that the coefficients a_n belong to $K = \mathbf{Q}(\sqrt{5})$ and g, g^σ are basis of $S_2(23)$ where σ is the generator of $\mathrm{Gal}(K/\mathbf{Q})$. Let g_χ be a cusp form given by $\sum\limits_{n\geq 1} \chi(n) a_n\, q^n$, which belongs to $S_2(23\cdot 7^2)$. Then the Hasse-Weil ζ-function $L(s, J)$ equals $L(s, g_\chi)L(s, g_\chi^\sigma)$. Since the signs of the functional equations are -1, both of $L(s, g_\chi)$ and $L(s, g_\chi^\sigma)$ have odd analytic rank(analytic rank means the order at $s = 1$). In fact, they are of analytic rank 1, that is the first derivatives of them do not vanish at $s = 1$. We check this by computing the special value of the derivatives of the L-functions using the following proposition.

Proposition 5.3 ([5],p.31, Prop.2.13.1). *For* $g = \sum_{n=1}^{\infty} a_n q^n \in S_2(N)$,

$$L'(g,1) = 2\sum_{n=1}^{\infty} \frac{a_n}{n} G_1\left(\frac{2\pi n}{\sqrt{N}}\right) \quad \textit{where} \quad G_1(x) = \int_1^{\infty} e^{-xy}\frac{dy}{y}.$$

By this method, we have

$$\begin{aligned} L'(g_\chi,1) &= 3.32367015912761142112490902457175944194175 \\ &\qquad 4825617012739979983630403310 8\cdots, \\ L'(g_\chi^\sigma,1) &= 1.22357337805505770149941672608138385308754 \\ &\qquad 691091007879090110751843133 38\cdots. \end{aligned}$$

Thus, by the conjecture, the Mordell-Weil rank of J should be 2. On the other hand, we have four rational points of C; $P_1(1,1)$, $P_2(3,5)$, and their images of the hyperelliptic involution: P_1^ι, P_2^ι. We define $b_1 = P_1 - P_2$, $b_2 = P_1 - P_2^\iota$, and $b_3 = P_1 - P_2^\iota$, $b_i \in Z_0(C)_{\mathbf{Q}}$. Put $\alpha_i = \overline{b_i} \in J(\mathbf{Q})$, $(i = 1,2,3)$, then we have $\alpha_3 = \alpha_1 + \alpha_2$. If $R' := \det\left(\langle \alpha_i, \alpha_j \rangle_{1 \le i,j \le 2}\right)$ is not 0, then α_1, α_2 are independent and R' is the regulator up to a multiple of an integer.

Next we compute the archimedean local height. We have the factorization

$$f(x) = (x^3 - 3x^2 + 2x + 1)(x^3 - 11x^2 + 22x - 19).$$

Take the real root x_0 of $x^3 - 3x^2 + 2x + 1$, then we have an isomorphism over $\mathbf{Q}(x_0)$

$$(x,y) \mapsto \left(\frac{f'(x_0)}{x - x_0}, \frac{f'(x_0)^2\, y}{(x - x_0)^3}\right),$$

the image of which is the curve given by $y^2 = x^5 + \cdots \in \mathbf{Q}(x_0)[x]$. Using this equation, we compute the Tate's series, and the results are as follows($N_T \geq 150$):

$$\begin{aligned} \hat{\lambda}_\infty(\alpha_1) &= 8.74171083024832967671545571797907091200 77 \\ &\qquad 08804445670485790231576423903384449099 42\ldots, \\ \hat{\lambda}_\infty(\alpha_2) &= 8.58243933600655667351212350930368395258 37 \\ &\qquad 43969971366096662429024626388022973606 34\ldots, \\ \hat{\lambda}_\infty(\alpha_3) &= 8.75615433645837162589297698399512703307 61 \\ &\qquad 34101740369342389009193082624291398760 24\ldots. \end{aligned}$$

Take points $P_3(5 - \sqrt{13}, 10 - 3\sqrt{13})$, $P_4(5 + \sqrt{13}, 10 + 3\sqrt{13})$ on C. Then we obtain points P_j, $(j = 5, \ldots, 10)$ on C such that b_i is linearly equivalent to $b_i' =$

$P_{2i+3} + P_{2i+4} - P_3 - P_4$. Let $G_i = G_{b_i, b_i'}$ be functions satisfying $\mathrm{div}(G_i) = b_i - b_i'$. We have, by the addition theorem,

$$\begin{aligned}
P_5 &= \left(3\sqrt{-1}, 2 + 21\sqrt{-1}\right), \\
P_6 &= \left(-3\sqrt{-1}, 2 - 21\sqrt{-1}\right), \\
P_7 &= \left(\frac{1015 + 3\sqrt{32009}}{256}, \frac{-70569557 - 419001\sqrt{32009}}{4194304}\right), \\
P_8 &= \left(\frac{1015 - 3\sqrt{32009}}{256}, \frac{-70569557 + 419001\sqrt{32009}}{4194304}\right), \\
P_9 &= \left(\frac{437 + \sqrt{147206}}{107}, \frac{27721445 + 67635\sqrt{147206}}{1225043}\right), \\
P_{10} &= \left(\frac{437 - \sqrt{147206}}{107}, \frac{27721445 - 67635\sqrt{147206}}{1225043}\right).
\end{aligned}$$

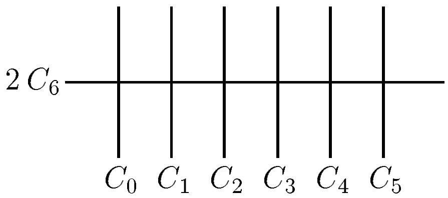
All of these points are not same as P_1 and P_2. Thus we compute the real archimedean Néron's local paring $r_i := \langle b_i, b_i' \rangle_\infty = 2\hat{\lambda}_\infty(\alpha_i) - \log|G_i[b]|$. We have

$$\begin{aligned}
r_1 &= -0.377968203847479323956651621406492851 1906 \\
&\qquad 518770297380695556416800300082746053 9180\ldots, \\
r_2 &= -0.1995352620720013629776611485766647308863 \\
&\qquad 6166002662262541504391361101778520199515\ldots, \\
r_3 &= -2.3135175421748408234366903964991804162103 \\
&\qquad 7008163231687118098438209046679595805722\ldots.
\end{aligned}$$

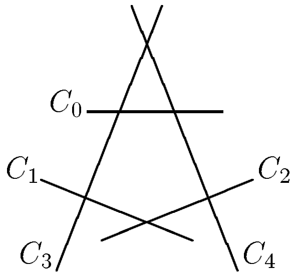
Next we consider Néron's local parings at non-archimedean places. At $p = 2$, the model (5.1) over $\mathbf{Z}$ is not normal, thus we must blow up by $y = (x^3 + x^2 + 1) + 2Y$. Then we have

$$\langle b_1, b_1' \rangle_2 = -\log 2, \quad \langle b_2, b_2' \rangle_2 = 0, \quad \langle b_3, b_3' \rangle_2 = -\log 2.$$

Let $k = k_i$ be a biquadratic field which is generated by the coordinates of the points of the support of b_i'. Following [12], we have: At $p = 7$, the fibre of the minimal regular model is of type I^*_{0-0-0}, and if $p = 7$ is ramified in k, $C \times O_k$ has good fibre over the primes lying over 7; at $p = 23$, if p is unramified in the field k, then the fibre of the minimal regular model is of type I_{1-2-3}, and if ramified, ramification index is 2 and the type is I_{2-4-6}. In our case, both of primes 7 and 23 are unramified in k. We figure the fibres at each prime:



$p = 7$



$p = 23$

The fibres C_i are (-2)-curves, except for C_6 at $p = 7$ and C_3, C_4 at $p = 23$, all of which are (-3)-curves. At $p = 7$, we can decide which fibre the section hits by looking up the x coordinates. At $p = 23$, the sections corresponding to the P_i do not hit C_0, C_1, C_2, and we can decide which fibre they hit, by checking whether $(x+2)(x+5)(x+9)$ equals $4y \mod 23$ or $-4y \mod 23$.

Remark 5.4. At $\mathfrak{P}|23$, if the reduction of x-coordinate is -2, then the section hits C_0. If the reduction is -5, putting $\xi = \dfrac{x+5}{y} \mod \mathfrak{P}$, and $\eta = \dfrac{23}{y} \mod \mathfrak{P}$, then; if $11\xi - 5\eta = 4$, hit C_1, else hit C_2, when C_3 is the fibre $(x+2)(x+5)(x+9) = 4y$. This can be proved by chasing the procedures of blowing-ups.

Thus, using Lemma 3.5, we have

$$\langle b_1, b_1' \rangle_7 = \langle b_2, b_2' \rangle_7 = \langle b_3, b_3' \rangle_7 = 0$$
$$\langle b_1, b_1' \rangle_{23} = -\frac{6}{11}\log 23, \quad \langle b_2, b_2' \rangle_{23} = 0, \quad \langle b_3, b_3' \rangle_{23} = -\frac{6}{11}\log 23.$$

It is easy to compute the intersection numbers at other primes. Summing up over all finite places, we have

$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_1, b_1' \rangle_v = -\log 2 + 3\log 3 - \frac{6}{11}\log 23,$$
$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_2, b_2' \rangle_v = -3\log 3 + \log 887,$$
$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_3, b_3' \rangle_v = -\log 2 + \log 3 + \log 179 - \frac{6}{11}\log 23.$$

Put $h_i = \langle \alpha_i, \alpha_i \rangle$ for $i = 1, 2, 3$. Summing up the above with the archimedean part, we have

$$\begin{aligned} h_1 &= 0.5144519092719137003718049687337098118895 \\ &\quad\; 6676307438099043759656003472103072539585 \ldots, \\ h_2 &= 3.2924728542332490532396517934684869119793 \\ &\quad\; 9026185015676151351622678083750406914366 \ldots, \\ h_3 &= 1.5690637994490878142790801498173730131778 \\ &\quad\; 0086100745083786934319002486632428241500 \ldots . \end{aligned}$$

Since $\langle \alpha_1, \alpha_2 \rangle = (h_3 - h_1 - h_2)/2$, we have the regulator up to a multiple of an integer

$$\begin{aligned} R' &= 0.4418135224747459009837796585512486028911 \\ &\quad\; 9027232784016701161394069098323593670992 \ldots . \end{aligned}$$

Next we compute the real periods using the method in Cremona [5]. We compute imaginary periods of $\Gamma_0(23)$, and multiply it by $\sqrt{-7}$ to get the real period of J up to a multiple of a rational number.

Let $(c : d)$ be the M-symbol [5],p.8. Let $\gamma_1^- = \frac{1}{2}(3 : 1) - \frac{1}{2}(-3 : 1)$, and $\gamma_2^- = \frac{1}{2}(2 : 1) - \frac{1}{2}(3 : 1) + (4 : 1) - \frac{1}{2}(-3 : 1)$, then these are basis of the anti-holomorphic homology space $H^-(N)$. Define V'_∞ be $\sqrt{7}^2 \left\langle \gamma_1^-, f_\chi \right\rangle \left\langle \gamma_1^-, f_\chi^\sigma \right\rangle$. Then this is the real period of J up to a multiple of a rational number. Computing by direct method(see [5],p.25,Prop.2.10.1),

$$\begin{aligned} V'_\infty &= 10.2506719848116009699526519174413057653616 \\ &\quad\; 5631926561357938968846741216424637142414 \ldots . \end{aligned}$$

The number of connected components of Néron model of $J/\mathbf{Z}_{11}$ is 16 and that of $J/\mathbf{Z}_{23}$ is 11. Hence we have $V_S = 11 \cdot 16$. Let $\tilde{J}_p$ be the reduction of J at a prime p. By Lemma 1.2, we have $\#\tilde{J}_3(\mathbf{F}_3) = 21$ and $\#\tilde{J}_{11}(\mathbf{F}_{11}) = 221$. Since C and J has good reduction at $p = 3$ and 11, and $J(\mathbf{Q})_{tors}$ is injectively mapped to $\tilde{J}_3(\mathbf{F}_3)$ and $\tilde{J}_{11}(\mathbf{F}_{11})$. Thus we have $\#J(\mathbf{Q})_{tors} = 1$. If we put

$$T := \frac{\frac{1}{2} L'(g_\chi, 1) L'(g_\chi^\sigma, 1) \# J(\mathbf{Q})_{tors}^2}{R' V'_\infty V_S},$$

then

$$392 \cdot T = 0.9999999999 \cdots (\text{'9' repeats at least 50 times}).$$

Thus we can guess $T = \dfrac{1}{392}$ and is a rational number as desired.

References

[1] *H. F. Baker*, Multiply periodic functions, Cambridge University Press, 1907.
[2] *J.-B. Bost*, Fonctions de Green-Arakelov, fonctions thêta et courbes de genre 2, C.R. Acad. Sci. Paris **305** (1987), 643–646.
[3] *G. S. Call and J. H. Silverman*, Canonical heights on varieties with morphisms, Composito Math. **89** (1993), 163–205.
[4] *D. A. Cox and S. Zucker*, Intersection numbers of sections of elliptic surfaces, Inv. math. **53** (1979), 1–44.
[5] *J. E. Cremona*, Algorithms for modular elliptic curves, Cambridge University Press, 1992.
[6] *K. Doi*, On the jacobian varieties of the fields of elliptic modular functions, Osaka Math. J. **15** (1963), 249–256.
[7] *R. Fricke*, Die Elliptischen Funktionén und ihre Anwendungen, Verlag von B.G.Teubner, 1921.
[8] *D. Grant*, Formal groups in genus two, J. reine angew. Math. **411** (1990), 96–121.
[9] *B. H. Gross*, Local Heights on Curves, Arithmetic Geometry (*G. Cornell and J. H. Silverman*, eds.), Springer-Verlag, 1985, pp. 327–339.
[10] *J. Igusa*, Arithmetic variety of moduli for genus two, Ann. of Math. **72** (1960), 612–649.
[11] *S. Lang*, Fundamentals of Diophantine Geometry, Springer-Verlag, 1983.
[12] *Q. Liu*, Modèles minimaux des courbes de genre deux, J. reine angew. Math. **453** (1994), 137–164.
[13] *D. Mumford*, Tata Lectures on Theta I, Birkhäuser, 1983.
[14] ______, Tata Lectures on Theta II, Birkhäuser, 1984.
[15] *Y. Namikawa and K. Ueno*, The complete classification of fibres in pencils of curves of genus two, Manuscripta Math. **9** (1973), 143–186.
[16] *A. Néron*, Quasi-fonctions et hauteurs sur les variétés abéliennes, Ann. of Math. **82** (1965), 249–331.
[17] *G. Shimura*, On the factors of the jacobian variety of a modular function field, J. Math. Soc. of Japan **25** (1973), 523–544.
[18] ______, On the periods of modular forms, Math. Ann. **229** (1977), 211–221.
[19] *J. H. Silverman*, The Arithmetic of Elliptic Curves, GTM106, Springer-Verlag, 1986.
[20] ______, Computing heights on elliptic curves, Mathematics of Computation **51** (1988), 339–358.
[21] ______, Advanced Topics in the Arithmetic of Elliptic Curves, GTM151, Springer-Verlag, 1994.
[22] *W. W.J.Hulsbergen*, Conjectures in Arithmetic Algebraic Geometry, Aspects of Mathematics, Vieweg, 1992.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-01, Japan

Fax: +81-75-753-3711

E-mail address: yositomi@kusm.kyoto-u.ac.jp