On the local Hamiltonian structure of vector fields

Dedicated to Ansgar Schnitzer who died tragically in a motorcycle accident on October 16th 1993

P. Crehan
Dept. of Mathematics, University of Kyoto
PAT@KUSM.KYOTO-U.AC.JP

Abstract

We derive a canonical form for smooth vector fields on $\Re^{n+1}$. We use this to demonstrate the local multi-Hamiltonian nature of the corresponding flows. Associated with the canonical form is an inhomogenious linear PDE whose solutions provide conserved measures. These can be used to construct the local Hamiltonians.

1 Introduction

The study of Poisson manifolds has recieved some attention recently as many interesting Poisson structures arise in the context of integrable systems, solvable models and quantum groups [1], [2], [3]. For this reason it is important to understand the scope of the Poisson manifold framework. Formally it is very similar to that of symplectic geometry, but extends it in a simple and non-trivial way.

Without the machinery of symplectic geometry and Hamiltonian mechanics, the concepts of solvability and full integrability, although still meaningful are slightly less clear [4]. Many dynamical systems of interest in mathematical physics are not usually treated by Hamiltonian methods. Examples include the damped harmonic oscillator, Lotka-Volterra systems and the Lorenz flow. Furthermore it has been claimed that if a system is not Hamiltonian (Lagrangian), then it cannot be consistently quantised [5]. Our results on the local structure of vector fields suggest however that the property of being Hamiltonian has broader scope than is generally afforded.

Wintner pointed out that every smooth vector field is locally fully integrable [6]. More sophisticated versions of this straigtening-out theorem arise as a consequence of the Lie-Frobenius theory of distributions of vector fields [7]. The straightening-out theorem on $\Re^{2n}$ has an important consequence for autonomous Hamiltonian vector fields. The $2n-1$ local integrals must close as an algebra under the Poisson bracket. If they did not they would generate a further independent integral and the flow lines would consist only of single points. In this way we know that autonomous Hamiltonian vector fields are locally fully integrable. The existence of integrals is intimately connected with the existence of symmetries. Doebner et al. clarified this connection at the local level when

they showed that the local symmetries of a Hamiltonian system depend only on the dimension of the space [8]. We extend these local analyses by proving some theorems, one of which asserts that every smooth vector field locally has a multiple Hamiltonian structure. A vector field is considered to be Hamiltonian with respect to a Poisson structure rather than a symplectic structure. This is true for odd as well as even dimensional manifolds. The odd dimensional case is not to be confused with a local contact structure.

In this article we deal with the Hamiltonian structure of generic vector fields only at the local level. Nevertheless we consider our results motivation for a study of general dynamical systems from a Poisson manifold point of view. Perhaps within a framework of the geometry of Poisson manifolds we can reach a more unified understanding of integrability. Perhaps the study of deformations of Poisson structures can provide a theoretical basis for considering the quantisation of a more general class of flows. Explicit examples which illustrate our results for simple but interesting flows on the plane will be published elsewhere.

2 The local structure of vector fields

We consider flows on $\Re^{n+1}$ generated by vector fields $v = v^i \partial_i$ according to

$$D_t \quad = \quad v^i \partial_i \tag{1}$$

where v^i, $i = 0, \ldots, n \in C^1(\Re^{n+1})$. From the theory of Lie and Frobenius, in the neighbourhood of a non-critical point of the flow there exist co-ordinate systems $\{x, \Phi^{(1)}, \ldots, \Phi^{(n)}\}$ in which $v = f\partial_x$ for some smooth function f. Furthermore the one dimensional distribution V generated by v, is tangent to the foliation of $\Re^{n+1}$ by the one dimensional submanifolds of the intersection of the level surfaces $\Phi^{(1)} = c_1$, ..., $\Phi^{(n)} = c_n$. In particular this means that $\partial_x \Phi^{(k)} = 0$ for $k = 1, \ldots, n$. In the original co- ordinate system we have

$$v^i \partial_i \Phi^{(1)} = 0, \ldots, v^i \partial_i \Phi^{(n)} = 0.$$

The components of v can therefore be represented as an antisymmetric product of gradient fields. We are led directly to the following lemma for smooth vector fields.

LEMMA 1 *In a sufficiently small but finite neighbourhood U of a non-critical point of a vector field $v = v^i \ \partial_i$ on $\Re^{n+1}$, there exist functions $\rho \in C^1(U)$ and $\Phi^{(1)}, \ldots, \Phi^{(n)} \in C^2(U)$ such that*

$$v^i \quad = \quad \rho^{-1} \ \epsilon^{ij_1 \ldots j_n} \ \partial_{j_1} \Phi^{(1)} \ldots \partial_{j_n} \Phi^{(n)} \tag{2}$$

where $\epsilon^{ij_1 \ldots j_n}$ is the totally antisymmetric tensor.

The inverse of ρ has been used instead of ρ for convenience later on. Before we state the main theorem we need to know a little bit about Poisson structures of rank 2. Poisson structures on a manifold $\mathcal{M}$ are tensor fields Λ^{ij} which satisfy the following tensorial relations

$$\begin{aligned} \Lambda^{ij} + \Lambda^{ji} &= 0, & (3) \\ \Lambda^{it}\partial_t\Lambda^{jk} + \Lambda^{jt}\partial_t\Lambda^{ki} + \Lambda^{kt}\partial_t\Lambda^{ij} &= 0. & (4) \end{aligned}$$

Tensors which satisfy (3) and (4) can be used to construct Poisson brackets. The Poisson bracket of two functions $A, B \in C^\infty(\mathcal{M})$ being given by $\{A, B\} = \Lambda^{ij}\ \partial_i A\ \partial_j B$. It turns out that all brackets which satisfy the axioms for a Poisson bracket are of this form. A review of the theory of Poisson manifolds with references can be found in [9].

In $\Re^2$ the most general Poisson structure is of the form

$$\Lambda^{ij} = \Theta\ \epsilon^{ij} \quad (5)$$

where $\Theta \in C^1(\Re^2)$ and ϵ^{ij} is the totally antisymmetric tensor. In $\Re^3$ a Poisson tensor must have rank 2. In general it has the form

$$\Lambda^{ij} = \Theta\ \epsilon^{ijk}\ \partial_k\Phi \quad (6)$$

where $\Theta \in C^1(\Re^3)$, $\Phi \in C^2\Re^3$ and ϵ^{ijk} is the totally antisymmetric tensor on $\Re^3$. This was shown in [10] but has more recenly appeared in a slightly different form in [2]. Due to the antisymmetry condition eqn(3) Λ must have even rank. In $\Re^4$ Poisson tensors must therefore have rank 2 or 4. Those of rank 2 are of the general form [10]

$$\Lambda^{ij} = \Theta\ \epsilon^{ijkl}\ \partial_k\Phi^{(1)}\ \partial_l\Phi^{(2)}. \quad (7)$$

A glance at equations (5), (6) and (7) suggests the following lemma.

LEMMA 2 *Tensors on $\Re^{m+2}$ of the form*

$$\Lambda^{ij} = \Theta\ \epsilon^{ijs_1\ldots s_m}\ \partial_{s_1}\Phi^{(1)}\ldots\partial_{s_n}\Phi^{(n)}$$

where $\Theta \in C^1(\Re^{m+2})$, $\Phi^{(k)} \in C^2(\Re^{m+2})$ $k = 1, \ldots, m$ and $\epsilon^{ijs_1\ldots s_n}$ is the totally antisymmetric tensor, provide Poisson structures of rank 2.

Eqn (3) is automatically satisfied. To verify eqn (4) it is best to move to the co-ordinate system $\{X, Y, \Phi^{(1)}, \ldots, \Phi^{(m)}\}$. In these co-ordinates the only non-zero terms of Λ are $\Lambda^{XY} = -\Lambda^{YX}$. Λ is therefore of rank 2 and eqn (4) can be

verified by a trivial calculation. The converse is also true and can be proven by a similar argument.

The expression for v given by eqn (2) can be rewritten as

$$v^i = \Lambda^{ij}_{(k)}\ \partial_j \Phi^{(k)}$$

where

$$\Lambda^{ij}_{(k)} = \rho^{-1}\ \epsilon^{is_1\ldots s_{k-1}js_{k+1}\ldots s_n}\ \partial_{s_1}\Phi^{(1)}\ldots\partial_{s_{k-1}}\Phi^{(k-1)}\ \partial_{s_{k+1}}\Phi^{(k+1)}\ldots\partial_{s_n}\Phi^{(n)}.$$

According to lemma (2) each of these $\Lambda_{(k)}$ is a Poisson tensor of rank 2. They are independent in the sense that they determine different foliations of $\Re^{n+1}$ by 2 dimensional symplectic leaves. In this sense eqn(1) can be thought of as generating a flow which is Hamiltonian with Hamiltonian $\Phi^{(k)}$ with respect to the Poisson structure $\Lambda_{(k)}$. We can even drop the need for labels on the $\Phi^{(k)}$ by using instead of a $\Phi^{(k)}$, a single $\Phi = \Phi^{(1)} + \Phi^{(2)} + \ldots + \Phi^{(n)}$. Now we can state the main theorem

THEOREM 1 *In a finite but sufficiently small neighbourhood U of a non-critical point of a smooth vector field $v = v^i\partial_i$ on $\Re^{n+1}$, there exists a function $\Phi \in C^2(U)$ and a family of n smooth independent rank 2 Poisson structures $\Lambda_{(k)}$ $k = 1,\ldots,n$ so that*

$$v^i = \Lambda^{ij}_{(k)}\ \partial_j\Phi \qquad k = 1,\ldots,n.$$

In this sense all smooth vector fields locally have a multiple Hamiltonian structure. The trajectories of the flow (1) are determined by the $\Phi^{(k)}$ in eqn(2). They are not affected by ρ, although the spreading of classical wavepackets along the trajectories is. The $\Phi^{(k)}$ are readily interpreted as local Hamiltonians but to understand the geometrical and physical significance of ρ we have to do a little more work.

3 An equation for ρ, conserved measures and the local Hamiltonians

The functional form of the $\Phi^{(k)}$ is not unique and nor is that of ρ. As long as F is a well behaved non-trivial function of $\Re^n$, eqn(2) is invariant under the replacements

$$\begin{aligned} \Phi^{(k)} &\rightarrow F(\Phi^{(1)},\ldots,\Phi^{(n)}) \\ \rho &\rightarrow \rho\ \partial_k F(\Phi^{(1)},\ldots,\Phi^{(n)}). \end{aligned} \tag{8}$$

One can show by direct substsitution of eqn(2) that ρ satisfies the following first order linear PDE

$$\rho(\nabla \cdot v) + v \cdot (\nabla \rho) = 0. \tag{9}$$

It is customary to study such equations as initial value problems. One takes a set of initial data Γ on a hypersurface Σ in $\Re^{n+1}$. If Σ is nowhere tangent to v, then the initial data can be extended from Σ to $\Re^{n+1}$ so that eqn(9) is satisfied. In the theory of partial differential equations, the surfaces of intersection of the level surfaces of the local Hamiltonians $\Phi^{(k)}$ are known as the characteristics. In other words Σ must be tranverse to each of the level surfaces of the local Hamiltonians. It is easy to show that if ρ is a solution of eqn(9), then so is $\rho F(\Phi^{(1)}, \ldots, \Phi^{(n)})$ where $F \in C^1(\Re^n)$.

One can also show that the ratio of two independant solutions of eqn(9) provides a conservation law. To see this compute $D_t(\rho_1 \rho_2^{-1})$ using $D_t = v \cdot \nabla$ and eqn(9). More precisely we can state the following lemma

LEMMA 3 *Given Σ a hypersurface in $\Re^{n+1}$ nowhere tangent to the vector field v. If ρ_A and ρ_B are solutions of $\rho(\nabla \cdot v) + v \cdot (\nabla \rho) = 0$ with inital data Γ_A and Γ_B on Σ. Then $\rho_A \rho_B^{-1}$ is an integral of the flow $\dot{x}^i = v^i$.*

A theorem due to Liouville tells us that the volume element is conserved by flows which are Hamiltonian with respect to the standard symplectic structure. We know that this is not true for general dynamical systems. To see how we might characterize measures conserved by (1), we consider Δ - a smooth function on $\Re^{n+1}$ with compact support. Δ_t is defined by transporting Δ with the flow for a time t. A measure on $\Re^{n+1}$ is denoted $d\mu = \mu \; dx^0 \wedge \ldots \wedge dx^n = \mu \; dV$. The measure of Δ is denoted by $\mu(\Delta)$ and given by $\mu(\Delta) = \int \Delta \; d\mu$. To determine for what ρ is $D_t \mu(\Delta_t) = 0$ we compute

$$D_t \mu(\Delta_t) = \int (D_t \Delta_t) d\mu = - \int \rho \Delta_t \; D_t(\rho^{-1} \; \mu) \; dV.$$

On the support of ρ the conserved measure must therefore satisfy $D_t(\rho^{-1}\mu) = 0$. One solution is given by $\mu = \rho$. By inspection any $\mu = \rho F(\Phi^{(1)}, \ldots, \Phi^{(n)})$ also provides a solution. From lemma(3) if ρ satisfies eqn(9) then so must $\mu = \rho F(\Phi^{(1)}, \ldots, \Phi^{(n)})$. Furthermore if ρ' is any other solution of eqn(9), lemma(3) implies that $D_t(\rho/\rho') = 0$. We therefore conclude with the following theorem.

THEOREM 2 *Measures of the form $d\mu = \rho \; dx^0 \wedge \ldots \wedge dx^n$ are preserved by the flow $\dot{x}^i = v^i$ if and only if $v \cdot (\nabla \rho) + \rho(\nabla \cdot v) = 0$. Ratios of independent local measures provide local integrals.*

The ρ which appeared in the canonical form eqn(2) is therefore a local conserved measure.

4 Discussion

Eqn(2) provides an expression for flows in terms of local Hamiltonians $\Phi^{(k)}$ and locally conserved measures ρ. Theorem(2) links these through a linear PDE (9) derived from the flow. In general the $\Phi^{(k)}$ are single valued only locally. They cannot be extended to globally defined single valued functions on $\Re^{n+1}$. Even in the case of fully integrable systems on $\Re^{2m}$, which are Hamiltonian with respect to the standard symplectic structure, it is rare for all of the integrals to be single valued. A Hamiltonian system on $\Re^{2m}$ is said to be fully integrable if there exist m globally defined single valued functions including the Hamiltonian, which are in involution under the Poisson bracket. If we call these $\Phi^{(k)}$ for $k = 1, \ldots, m$, the method of solution is to find a further set of functions $\phi_{(k)}$, which are conjugate to the $\Phi^{(k)}$. In the co-ordinate system $(\Phi^{(1)}, \ldots, \Phi^{(n)}, \phi^{(1)}, \ldots, \phi^{(n)})$ the equations of motion are $D_t\phi^{(k)} = -\partial_{\Phi^{(k)}} H(\Phi)$, and $D_t\Phi^{(k)} = 0$, for $k = 1, \ldots, m$. In these coordinates the equations are linear and the system can be solved. Although only m integrals have been used in the solution of the problem, it is possible to construct a further $m-1$ integrals of the motion $\Phi^{(i\ i+1)}$ as follows

$$\Phi^{(i\ i+1)} \quad = \quad \phi^{(i)} D_t \Phi^{(i+1)} - \phi^{(i+1)} D_t \Phi^{(i)}.$$

The functions $\Phi^{(i)}$ for $i = 1, \ldots, m$ and $\Phi^{(i\ i+1)}$ for $i = 1, \ldots, m-1$ constitute the full set of $2m-1$ local integrals. The simplest illustration of this is of course the system of two uncoupled harmonic oscillators. The Hamiltonian of this system with respect to the standard symplectic structure on $\Re^4$ is $(p_x^2 + w_x^2 x^2)/2 + (p_y^2 + w_y^2 y^2)/2$. In accordance with lemma(1) we can write the equations of motion in the form

$$\dot{x}^i \quad = \quad \rho^{-1}\ \epsilon^{ijkl}\ \partial_j \Phi^{(1)}\ \partial_k \Phi^{(2)}\ \partial_l \Phi^{(12)},$$

where $\rho = 1$, and the three local Hamiltonians are

$$\begin{aligned}
\Phi^{(1)} &= \frac{1}{2}(p_x^2 + w_x^2 x^2), \\
\Phi^{(2)} &= \frac{1}{2}(p_y^2 + w_y^2 y^2), \\
\Phi^{(12)} &= \frac{1}{w_x} \arctan(w_x \frac{x}{p_x}) - \frac{1}{w_y} \arctan(w_y \frac{y}{p_y}).
\end{aligned}$$

In general $\Phi^{(12)}$ is multivalued and unless the winding frequencies are commensureable, it takes infinitely many values. This gives rise to a flow which is ergodic on the surface defined by the intersection of the level surfaces of $\Phi^{(1)}$

and $\Phi^{(2)}$. The case where all of the integrals are single valued is considered to be degenerate. In the case of a generic vector field, there are no single valued integrals. A generic vector field can therefore be thought of as a flow, none of whose Hamiltonians are isolating. A careful study of the relationship between constants of motion and degeneration for systems which are Hamiltonian with respect to standard Hamiltonian structures is provided by Onofri and Pauri [11]. It seems reasonable to extend this classification to arbitrary vector fields on the basis of lemma(1) and theorem(1).

A lot of work has been done on the subject of integrability. Much of which centers around applications of Painlevé analysis. According to one reviewer [4], the conceptual basis for this is less clear in the non-Hamiltonian case than in the Hamiltonian case. Painlevé analysis proceeds by complexifying the dynamical system. It then deals with the integrability of the complex system by investigating the branching of solutions of the system in the complex time plane. Rigorous results such as those of Ziglin [12] provide criteria for integrability in terms of the monodromony group of the system in the complex time plane. If a system is fully integrable when it is complexified, then it must be integrable. On the other hand Kozlov gives an example of a system which although non-integrable in the complex domain is completely integrable in the real domain [13]. The concept of integrability of a real dynamical system is therefore slightly different from that which is directly addressed by Painlevé analysis.

Theorem(1) suggests an alternative framework for investigating arbitrary smooth vector fields. A framework which incorporates essential geometrical ideas of Hamiltonian mechanics. It might provide an interesting standpoint from which to understand concepts of integrability. In particular it might be instructive to consider the singularities of solutions of the associated PDE (9) and the relationship they bear to those encountered in Painlevé analysis.

It seems natural to pursue a deformation approach to the quantisation of vector fields [14], based on the geometry of Poisson manifolds rather than that of symplectic manifolds. Barriers to the quantisation of systems indicated in [5] could be removed at least locally in this way. An interesting question is whether the symmetry between local Hamiltonians which is present at the classical level, will also be present at the quantum level.

5 Acknowledgement

I would like to thank Michio Jimbo for his hospitality at Kyoto University Dept. of Mathematics, as well as for his interest in and discussion of this work. I would also like to thank Hiroshi Kokubu for his interest and discussion.

References

[1] Luen-Chau Li & Serge Parmentier: Commun. Math. Phys. 125 (1989) p.545-563. Non-linear Poisson Structures and r-Matrices.

[2] J. Grabowski, G. Marmo & A. M. Perelomov: Modern Phys Letts A8(18) 1993 p.1719-1733. Poisson Structures - towards a classification.

[3] F. LIzzi, G. Marmo, G. Sparano & P. Vitale: Modern Phys. Letters A8(31) 1993 p.2973-2987. Dynamical Aspects of Lie-Poisson Structures

[4] Research Reports in Physics: Symmetries and Singularities. Springer Verlag 1990. Eds. Lakshmanan and M. Daniels. S. Puri: Painleve Property in Hamiltonian and Non- Hamiltonian Systems. pp 93-103:

[5] Sergio A. Hojman & L.C. Shepley: J.Math Phys. 32(1) January 1991 p142 - 146. No Lagrangian? no Quantisation!

[6] The Analytic Foundations of Celestial Mechanics. Aurel Wintner. Princeton University Press 1947.

[7] Westenholz. Differential Forms in Mathematical Physics. Studies in Mathematics and its Applications Vol.3. Amsterdam North Holland 1978.

[8] E. Aguirre, H. D. Doebner, & J. D. Hennig: Lett. Math. Phys. 7 (1983) p.85. All local classical symmetries in Hamiltonian mechanics.

[9] Alan Weinstein: J. Diff. Geom 18 1983 p.523-557. The local structure of Poisson manifolds.

[10] P. Crehan: Proceedings of **New Aspects of Field Theory** held at Tokyo University Nuclear Physics Research Institute Feb 17th 1992: INS-T-153 p.45. The Geometry of Poisson Manifolds - as an approach to constrained dynamical systems.

[11] E. Onofri, & M. Pauri: J. Math. Phys.14 1973 p.1106. Constants of motion and degeneration in Hamiltonian systems.

[12] S. L. Ziglin: Funct. Anal. Appl. Vol.16 p.181-189 & Vol.17 p.6-17 1983. Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I, II.

[13] V. Kozlov: Russ. Math. Sur. Vol38(1) 1983 p.3-67. Integrability and non-integrability in Hamiltonian mechanics.

[14] F Bayen, M Flato, C Fronsdal, A Lichnerowicz, & D Sternheimer: Ann. Phys. (NY) Vol.111 1978 p.61 p.111.