THE QUASI-SURE EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS ON WIENER SPACE

By

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1. Introduction

A.B. Cruzeiro [3] proved the almost-sure existence of solutions for differential equations on the Wiener space. In this paper, we extend the almost-sure existence to the *quasi-sure existence*, that is, the existence of solutions for all initial values except in a set of (r,p)-capacity 0 for all $r \geq 0$ and $p > 1$. Here the capacity is associated with the Ornstein-Uhlenbeck operator. To show the quasi-sure existence we use the notion of Sobolev spaces of Banach valued functions (see Shigekawa [13] and Denis [4]). To be precise, let (X, H, μ) be an abstract Wiener space and A be a vector field on X which is a smooth mapping from X into H in the sense of Malliavin. Then under some conditions for A, which will be stated precisely in Theorem 5.5 below, the solutions $V_t(x)$ of the following differential equation exist for all $t \in \mathbb{R}$, quasi everywhere x (q.e.x).

$$\begin{cases} (dV_t/dt)(x) = A(V_t(x) + x), \\ V_0(x) = 0. \end{cases} \tag{1.1}$$

We first consider (1.1) in finite dimensional case. We show that for any $k \in \mathbb{N}$, $(L^k V_t)$ exists for all $t \in \mathbb{R}$, μ-a.e.x and thereby (V_t) admits a quasi-continuous modification as a $C([0,T] \to X)$-valued function for any $T > 0$. In finite dimensional case, one point has a positive (r,p)-capacity for sufficiently large r and p. Therefore we can show that a solution to (1.1) exists for every initial value $x \in X$. To deal with (LV_t), for example, we have to consider the following differential equation :

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$$\frac{d}{dt}V_t(x) = B(V_t(x)+x), \tag{1.2}$$

$$\frac{d}{dt}\nabla V_t(x) = \nabla B(V_t(x)+x)\cdot\nabla V_t(x) + \nabla B(V_t(x)+x), \tag{1.3}$$

$$\begin{aligned}\frac{d}{dt}LV_t(x) &= LB(V_t(x)+x) + \nabla B(V_t(x)+x)\cdot LV_t(x)\\ &\quad + \sum_i \nabla^2_{jk}B(V_t(x)+x)\cdot\partial_i V_t^j(x)\cdot\partial_i V_t^k(x)\\ &\quad + 2\sum_i \partial_i\nabla_j B(V_t(x)+x)\cdot\partial_i V_t^j(x),\end{aligned} \tag{1.4}$$

$$V_0(x) = \nabla V_0(x) = LV_0(x) = 0.$$

We need the differential equation for (LV_t), but ∇V_t appears in (1.4) and therefore we consider the equation (1.3). In general we have to solve the following system of differential equation successively.

$$\begin{aligned}\frac{d}{dt}L^m\nabla^n V_t &= \nabla B\cdot L^m\nabla^n V_t\\ &\quad + E^{m,n}(L^i\nabla^j B, L^l\nabla^r V_t : i=0,1,\cdots,m,\\ &\qquad j=0,1,\cdots,n,\quad l=0,1,\cdots,m-1,\\ &\qquad r=0,1,\cdots,n,\quad 2i+j\le 2m+n,\\ &\qquad 2l+r\le 2(m-1)+n),\end{aligned} \tag{1.5}$$

$$L^m\nabla^n V_0(x) = 0,\quad m=0,1,\cdots,k,\quad n=0,1,\cdots,2k,\quad 2m+n\le 2k,\quad k=2,3,\cdots,$$

where B is a vector field on $\mathbb{R}^n$, L is the Ornstein-Uhlenbeck operator and $E^{m,n}$ is some polynomial. Using these differential equations, we estimate (L^kV_t).

To proceed to infinite dimensional case, we adopt a finite dimensional approximation. To be precise, we take a sequence $\{A_n\}$ converging to A such that A_n depends only on finite number of coordinates and takes values in finite dimensional subspace of H. Denoting a solution for A_n by $(V_t^{(n)})$, we show that $(V_t^{(n)})$ converges quasi-everywhere and the limit satisfies (1.1). The organization of this paper is as follows. In the section 2, we review the capacity and Sobolev space of Banach valued functions. A sufficient condition of the existence of quasi-continuous modification given by Shigekawa [13] is crucial. In the section 3, we consider the finite dimensional case. A.B. Cruzeiro [2] proved the almost-sure existence of solutions for differential equation (1.2) on $\mathbb{R}^n$ which satisfy the flow property. We apply her idea to show the existence of solutions for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$ for the above system of differential equations (1.5). In the section 4, we prepare the approximations and prove the main result in the section 5.

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2. Sobolev spaces of Banach valued functions

Let (X, H, μ) be an abstract Wiener space introduced by L. Gross [6] where
(i) X is a real, separable Banach space with the norm $||\cdot||$,
(ii) H is a real, separable Hilbert space densely included in X with the inner product $\langle x, y\rangle_H$,
and
(iii) μ is the standard Gaussian measure, i.e., the Borel probability measure on X such that

$$\int_X \exp\{i(h,x)\}\mu(dx) = \exp(-\frac{1}{2}\langle h,h\rangle_H)$$

where $h \in X^* \subset H$ and $(\ ,\ )$ is a natural pairing of X^* and X. Note that $||x|| \leq k|x|_H = k\sqrt{\langle x,x\rangle_H}$, $x \in H$ for some constant $k > 0$ so that the inclusion map $i : H \to X$ is continuous. Hence we have $X^* \subset H^* = H$ and we regard X^* as a subset of H.

We denote the Borel σ-field by $\mathcal{B}(X)$ and define a transition probability $P_t(x, G)$, $t \geq 0, x \in X$, by

$$P_t(x,G) = \int_X 1_G(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(dy), \quad G \in \mathcal{B}(X).$$

Then the Ornstein-Uhlenbeck semigroup $\{T_t\}$ is defined by

$$T_t f(x) = \int_X f(y)P_t(x,dy), \quad f \in \mathcal{B}_b(X). \tag{2.1}$$

Let B be a real separable Banach space. We denote by $L^p(X,\mu;B)$ the space of all B-valued μ-measurable functions u satisfying

$$||u||_p := \left\{\int_X ||u(x)||_B^p \mu(dx)\right\}^{1/p} < \infty.$$

As usual, we identify two functions which are equal to each other μ-a.e. For $u \in L^p(X,\mu;B)$, we set

$$T_t u(x) = \int_X u(y)P_t(x,dy).$$

Here the integral in the right hand side is the Bochner integral. Then $T_t u(x)$ can be defined for μ-a.e.x and $\{T_t\}$ is a contraction semigroup on $L^p(X,\mu;B)$. Further $\{T_t\}$ is strongly continuous.

Let us denote the generator by L. Now for $r \geq 0$ we can define $(1-L)^{-r/2}$ by the following gamma transformation :

$$(1-L)^{-r/2} = \frac{1}{\Gamma(r/2)}\int_0^\infty t^{r/2-1}e^{-t}T_t dt.$$

$(1-L)^{-r/2}$ is a contraction operator since $\{T_t\}$ is a contraction semigroup. We set

$$W_r^p(B) := (1-L)^{-r/2}(L^p(X,\mu;B)).$$

Then $W_r^p(B)$ becomes a Banach space with a norm

$$||u||_{r,p} := ||f||_{L^p} \quad \text{for} \quad u = (1-L)^{-r/2}f, \quad f \in L^p.$$

We call $W_r^p(B)$ the Sobolev space with differentiability r and integrability index p. We denote the space $\cap_r W_r^p(B)$ by $W_\infty^p(B)$ for $p \in [1,\infty)$ and $W_\infty^\infty(B) = \cap_p W_\infty^p(B)$.

I. Shigekawa [13] gives a criterion in the following manner :

Proposition 2.1. Take any $v \in L^p(X,\mu;B)$. Then $v \in W_r^p(B)$ if and only if there exists a $u \in L^p(X,\mu;B)$ such that for any $\varphi \in B^*$,

$$< \varphi, v > \in W_r^p \quad \text{and} \quad (1-L)^{r/2} < \varphi, v > = < \varphi, u > .$$

Moreover, under the above condition, it holds that $(1-L)^{r/2}v = u$.

For a separable Banach space B and $d \in \mathbb{N}$, $C([0,1]^d \to B)$ (the set of all B-valued continuous function on $[0,1]^d$) is again a separable Banach space with a norm

$$||\xi||_C = \sup_{t\in[0,1]^d} ||\xi_t||_B.$$

$C([0,1]^d \to B)$-valued function ξ on X can be regarded as a family of B-valued functions with a parameter $t \in [0,1]^d$: $\xi = (\xi_t)_{t\in[0,1]^d}$.

Corollary 2.2. Take any $\xi \in L^p(X,\mu;C([0,1]^d \to B))$. Then $\xi \in W_r^p(C([0,1]^d \to B))$ if and only if there exists an $\eta \in L^p(X,\mu;C([0,1]^d \to B))$ such that for any $t \in [0,1]^d$,

$$\xi_t \in W_r^p(B) \quad \text{and} \quad (1-L)^{r/2}\xi_t = \eta_t.$$

Moreover, under the above condition, it holds that $(1-L)^{r/2}\xi = \eta$.

We use the above corollary as follows : if $((1-L)^{r/2}\xi_t)_{t\in[0,1]^d}$ admits a version (η_t) which is continuous in t μ-a.e. and

$$\int_X \sup_{t\in[0,1]^d} ||\eta_t||_B^p \mu(dx) < \infty,$$

then $\xi = (\xi_t)_{t\in[0,1]^d} \in W_r^p(C([0,1]^d \to B))$.

In the following, we discuss the connection to the (r,p)-capacity. First we review the (r,p)-capacity. For details, see [5] and [8].

For $[0,\infty]$-valued lower semicontinuous (l.s.c. in abbreviation) function h, define $C_{r,p}(h)$ by

$$C_{r,p}(h) := \inf\{||u||_{r,p}^p;\, u \in W_r^p(B), u \geq h, \mu\text{-a.e.}\}$$

and for an arbitrary $[-\infty,\infty]$-valued function f (not assumed to be μ-measurable),

$$C_{r,p}(f) := \inf\{C_{r,p}(h); h \text{ is l.s.c. and } h(x) \geq |f(x)|, \forall x \in X\}.$$

For a set G, we define

$$C_{r,p}(G) = C_{r,p}(1_G).$$

Here 1_G denotes the indicator function of G.

We say that a property holds quasi-everywhere (q.e. in abbreviation) if it holds except on a set of capacity 0 for all r, p. We note that the following property holds for Sobolev spaces on an abstract Wiener space.

$$W_r^p(B) \cap C_b(X \to B) \text{ is dense in } W_r^p(B) \text{ and } 1 \in W_r^p(B).$$

Then it has been proved by I. Shigekawa that any $v \in W_r^p(B)$ admits a quasi-continuous modification and denoting it by $\tilde{v}$, it holds that

$$C_{r,p}(||\tilde{v}||_B) \leq ||v||_{r,p}^p \tag{2.2}$$

and the Chebyshev type inequality holds

$$C_{r,p}(||\tilde{v}|| \geq \lambda) \leq \frac{1}{\lambda^p}||v||_{r,p}^p. \tag{2.3}$$

Using the above results, we prove the extension in section 5.

3. Results in the finite dimensional case

Definition 3.1. For a family σ_j, σ of Radon measures on $\mathbb{R}^n$, we call σ the *weak limit* of σ_j if

$$\int_{\mathbb{R}^n} \theta d\sigma_j \longrightarrow \int_{\mathbb{R}^n} \theta d\sigma, \quad \text{for all} \quad \theta \in C_c^1$$

where C_c^1 is the space of all C^1 functions with compact support. We denote it by w-$\lim_{j\to\infty} \sigma_j = \sigma$.

In this chapter we denote by μ the standard Gaussian measure on $\mathbb{R}^n$. By a vector field on $\mathbb{R}^n$, we mean a Borel measurable function $B : \mathbb{R}^n \ni x \mapsto B(x) \in \mathbb{R}^n$.

Definition 3.2. If B is a complete vector field on $\mathbb{R}^n$ and if $U_t^B(x)$ is a flow generated by B, we define $\mathrm{div}_\mu B = v$ for $v \in L^1(\mu)$ if the limit exists

$$\text{w-}\lim_{t\to 0} \frac{1}{t}[(U_t^B)_*\mu - \mu] = v\mu, \quad v \in L^1(\mu).$$

Let $W_1^2(\mu)$ be the space $\{u \in L^2(\mu) : ||\nabla u|| \in L^2(\mu)\}$ with the norm $||u||_{L^2} + ||\nabla u||_{L^2}$.

Definition 3.3. Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field in $L^2(\mu)$. The *divergence* of B with respect to μ (if it exists) is the element $\delta_\mu B \in L^2(\mu)$ satisfying

$$\int_{\mathbb{R}^n} \delta_\mu B \cdot u d\mu = \int_{\mathbb{R}^n} (B|\nabla u) d\mu$$

for all $u \in W_1^2(\mu)$. More generally, for $f \in L^1_{\text{loc}}(\mu)$, the *divergence* $\delta_{f\cdot\mu} B$ of B with respect to the Radon measure $f \cdot \mu$ is defined similarly.

Let B be a vector field of class C^2 on $\mathbb{R}^n$ such that $B \in L^2(\mu)$. Suppose further that $\text{div}_\mu B$ exists and in $L^2(\mu)$. Then $\delta_\mu B = \text{div}_\mu B$. Conversely, if $\delta_\mu B$ exists, $\text{div}_\mu B$ exists also and $\text{div}_\mu B = \delta_\mu B$, and further if f is a function of class C^1 such that $fB \in L^2(\mu)$ and $f \cdot \delta_\mu B - (B|\nabla f) \in L^2(\mu)$, then $\delta_\mu(f \cdot B)$ exists and $\delta_\mu(f \cdot B) = f \cdot \delta_\mu B - (B|\nabla f)$. Furthemore, if $f > 0$ and $\delta_{f\mu} B$ exists then $f \cdot \delta_{f\mu} B = \delta_\mu(fB)$. Thus if we denote by $\delta_L B$ the divergence of B with respect to the Lebesgue measure, which is given by $-\text{trace}\nabla B(x)$, then we have $\delta_\mu B(x) = (B(x)|x) - \text{trace}\nabla B(x)$.

For the existence of the flow $U_t^B(x)$, A.B. Cruzeiro proved the following fact.

Theorem 3.4. (Cruzeiro [2], Theorem 3.1) Let B be a vector field on $\mathbb{R}^n$ such that
(i) $B \in C^3$ and $\forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda ||B(x)||) d\mu(x) < +\infty$,
(ii) $\forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda |\delta_\mu B(x)|) d\mu(x) < +\infty$.

Then the flow $U_t^B(x)$ can be defined for all $t \in \mathbb{R}$, μ-a.e.x. Furthermore, if $k_t(x) = \exp(\int_0^t \delta_\mu B(U^B_{-\xi}(x)) d\xi)$, we have
(iii) $(d(U_t^B)_*\mu/d\mu)(x) = k_t(x)$, with $k_t \in L^p(\mu)$, $\forall p \geq 1$, and $||k_t||_{L^p} \leq C(M)$ if $|t| < M$, where $C(M)$ is a positive constant depending on M and p.

For the quasi-sure extension, we need to consider the differential equation to be satisfied by $LV_t(x)$ where $V_t(x) \equiv U_t^B(x) - x$ and L is the Ornstein-Uhlenbeck operator. But since

$$\begin{cases} (dLV_t/dt)(x) = L(B(V_t(x) + x)), \\ LV_0(x) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{aligned} L(B(V_t(x)+x)) &= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}(B(V_t(x)+x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(B(V_t(x)+x)) \cdot x_i \\ &= LB(V_t(x)+x) + \nabla B(V_t(x)+x) \cdot LV_t(x) \\ &\quad + \sum_{i=1}^n \nabla^2_{jk} B(V_t(x)+x) \cdot \partial_i V_t^j(x) \cdot \partial_i V_t^k(x) \\ &\quad + 2\sum_{i=1}^n \partial_i \nabla_j B(V_t(x)+x) \cdot \partial_i V_t^j(x), \end{aligned}$$

we should consider the following system of differential equations to be satisfied by $[V_t(x), \nabla V_t(x), LV_t(x)]$:

$$\frac{d}{dt}V_t(x) = B(V_t(x) + x), \tag{3.2}$$

$$\frac{d}{dt}\nabla V_t(x) = \nabla B(V_t(x) + x) \cdot \nabla V_t(x) + \nabla B(V_t(x) + x), \tag{3.3}$$

$$\begin{aligned}\frac{d}{dt}LV_t(x) &= LB(V_t(x) + x) + \nabla B(V_t(x) + x) \cdot LV_t(x) \\ &\quad + \sum_i \nabla^2_{jk} B(V_t(x) + x) \cdot \partial_i V_t^j(x) \cdot \partial_i V_t^k(x) \\ &\quad + 2\sum_i \partial_i \nabla_j B(V_t(x) + x) \cdot \partial_i V_t^j(x),\end{aligned} \tag{3.4}$$

$$V_0(x) = \nabla V_0(x) = LV_0(x) = 0.$$

More generally, in addition to (3.2)$\sim$ (3.4), we consider the following system of differential equations to be satisfied by

$$[L^m \nabla^n V_t(x) : m = 0, 1, \cdots, N, \quad n = 0, 1, \cdots, 2N, \quad 2m + n \le 2N] :$$

$$\begin{aligned}\frac{d}{dt}L^m \nabla^n V_t &= \nabla B \cdot L^m \nabla^n V_t \\ &\quad + E^{m,n}(L^i \nabla^j B, L^l \nabla^r V_t : i = 0, 1, \cdots, m, \\ &\qquad j = 0, 1, \cdots, n, \quad l = 0, 1, \cdots, m - 1, \\ &\qquad r = 0, 1, \cdots, n, \quad 2i + j \le 2m + n, \\ &\qquad 2l + r \le 2(m - 1) + n), \\ L^m \nabla^n V_0(x) = 0, &\quad m = 0, 1, \cdots, k, \quad n = 0, 1, \cdots, 2k, \\ &\quad 2m + n \le 2k, \quad k = 2, 3, \cdots, N,\end{aligned} \tag{3.5}$$

where $E^{m,n}$ is some polynomial which can be obtained successively (cf. (5.9) below).

Theorem 3.5. Suppose that $B \in C^\infty$ and
(i) $\forall m = 0, 1, 2, \cdots, \forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda \cdot ||\nabla^m B(x)||) d\mu(x) < +\infty$,
(ii) $\forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda |\delta_\mu B(x)|) d\mu(x) < +\infty$.
Then a solution of the above system of differential equations (3.2), (3.3), (3.4) and (3.5) exists also for all $t \in \mathbb{R}$ starting for all $x \in \mathbb{R}^n$.

Proof. We prove only the case of order 1. For the general cases of $L^2, L^3, \cdots$, we prove them in the section 5.

For the differential equation (3.2), by Theorem 3.4, $V_t(x)$ exists for all $t \in \mathbb{R}$, μ-a.e.x, and since $B \in C^\infty$, $V_t(x)$ is C^∞ with respect to t and x. For the equation (3.3), for fixed $T > 0$, we put $\sup_{0\le t\le T} ||\nabla B(V_t(x)+x)|| = C_1(T,x)$ which is finite since ∇B is C^∞. For $0 \le t \le T$, the integral form of (3.3) is

$$\nabla V_t(x) = \int_0^t \nabla B(V_s(x)+x)\cdot \nabla V_s(x) ds + \int_0^t \nabla B(V_s(x)+x) ds.$$

Then

$$||\nabla V_t(x)|| \le \int_0^t ||\nabla B(V_s(x)+x)|| \cdot (||\nabla V_s(x)|| + 1) ds,$$

and thus

$$||\nabla V_t(x)|| + 1 \le C_1(T,x) \int_0^t (||\nabla V_s(x)|| + 1) ds + 1.$$

By Gronwall's lemma,

$$||\nabla V_t(x)|| \le \exp(\int_0^t C_1(T,x) ds) - 1.$$

Thus

$$\sup_{0\le t\le T} ||\nabla V_t(x)|| \le \exp(T\cdot C_1(T,x)) - 1.$$

Since $T > 0$ is arbitrary, $\nabla V_t(x)$ exists also for all $t \in \mathbb{R}$, μ-a.e.x. Further $\nabla V_t(x)$ is C^∞ with respect to t and x.

We can estimate $\sup_{0\le t\le T} ||LV_t(x)||$ in the same way. Put $V_s(x) + x = V_s + x$. Since $\sum_i ||\partial_i V_t|| \cdot ||\partial_i V_t|| \le ||\nabla V_t||^2$ and $\sum_i ||\partial_i \nabla B|| \cdot ||\partial_i V_t|| \le ||\nabla^2 B|| \cdot ||\nabla V_t||$,

$$\begin{aligned} ||LV_t|| \le & \int_0^t ||\nabla B(V_s+x)|| \cdot ||LV_t|| ds + \int_0^t ||LB(V_s+x)|| ds \\ & + \int_0^t ||\nabla^2 B(V_s+x)|| \cdot ||\nabla V_s||^2 ds + 2\int_0^t ||\nabla^2 B(V_s+x)|| \cdot ||\nabla V_s|| ds. \end{aligned}$$

For arbitrary $T > 0$, put

$$\sup_{0\le t\le T} ||\nabla B(V_t(x)+x)|| = C_1(T,x),$$
$$\sup_{0\le t\le T} ||\nabla^2 B(V_t(x)+x)|| = C_2(T,x)$$

and

$$\sup_{0\le t\le T} ||LB(V_t(x)+x)|| = C_3(T,x).$$

By the previous estimation of $\nabla V_t(x)$, for $0 \le t \le T$,

$$\begin{aligned} & \int_0^t ||LB(V_s+x)|| ds \\ & + \int_0^t ||\nabla^2 B(V_s+x)|| \cdot ||\nabla V_s||^2 ds + 2\int_0^t ||\nabla^2 B(V_s+x)|| \cdot ||\nabla V_s|| ds \\ & \le T(C_3(T,x) + C_2(T,x)\cdot(\exp(2T\cdot C_1(T,x)) - 1 + 2\exp(T\cdot C_1(T,x)) - 2)) \\ & \equiv C(T, C_1, C_2, C_3) \equiv C \end{aligned}$$

for some positive constant $C < +\infty$. Then

$$\begin{aligned} ||LV_t(x)|| &\leq \int_0^t ||\nabla B(V_s + x)|| \cdot ||LV_s(x)||ds + C \\ &\leq C_1(T, x) \int_0^t ||LV_s(x)||ds + C. \end{aligned}$$

By Gronwall's lemma,

$$||LV_t(x)|| \leq C \exp(\int_0^t C_1(T, x)ds),$$

we have $\sup_{0\leq t\leq T} ||LV_t(x)|| \leq C \exp(T \cdot C_1(T, x))$. Thus there exists a quasi-continuous modification $\tilde{V}_t$ of V_t and

$$\tilde{V}_t(x) = \int_0^t A(V_s(x) + x)ds, \quad \forall t \in \mathbb{R}, \quad \forall x \in X_0,$$

where $\mu(X_0^c) = 0$. Since the capacity of one point is positive, $\tilde{V}_t$ is a continuous modification. For $y \in \mathbb{R}^n$, there exists a sequence $\{x_n\} \subset X_0$ converging to y. Since $\tilde{V}_t$ is continuous, $\tilde{V}_t(x_n) \to \tilde{V}_t(y)$ as $n \to \infty$. By the hypothesis of A and the fact $V_s(x) = \tilde{V}_s(x)$ if $x \in X_0$,

$$\begin{aligned} \int_0^t A(V_s(x_n) + x_n)ds &= \int_0^t A(\tilde{V}_s(x_n) + x_n)ds \\ &\to \int_0^t A(\tilde{V}_s(y) + y)ds \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence V_t exists for all $x \in \mathbb{R}^n$ and satisfies

$$V_t(x) = \int_0^t A(V_s(x) + x)ds, \quad \forall t \in \mathbb{R}. \qquad \square$$

4. Finite dimensional approximations and a formula of pertubation of flows

Let (X, H, μ) be an abstract Wiener space such that $\dim H = \infty$. We consider an orthonomal basis $e_1, \cdots, e_k, \cdots$ on H such that e_k belongs to X^*, so that, for every $k = 1, 2, \cdots$, the map $x \mapsto \langle x, e_k \rangle$ on H has a continuous extension to the space X which we denote by $\langle x, e_k \rangle$ also.

We can define on X the Hermite polynomial

$$\mathcal{H}_s(x) = \prod_i \mathcal{H}_{s_i} \langle x, e_i \rangle,$$

where $s = (s_1, \cdots, s_n, \cdots)$, $s_i \in \mathbb{N}$, $|s| = \sum s_i < +\infty$ and $\mathcal{H}_{s_i}$ is the usual Hermite polynomial on $\mathbb{R}$

$$\mathcal{H}_{s_i}(\xi) = \frac{(-1)^{s_i}}{(s_i!)^{1/2}} e^{\xi^2/2} \frac{d^{s_i}}{d\xi^{s_i}} e^{-\xi^2/2}, \quad \xi \in \mathbb{R}.$$

Then we know that these functions form an orthonomal basis of $L^2(X;\mathbb{R})$ and if G is an abstract separable Hilbert space having $\{g_q\}$ for base, then $g_q \otimes \mathcal{H}_s(x)$ is a base of $L^2(X;G)$.

If ϕ is a μ-measurable function having some regularity defined on X with values in the Hilbert space G, and $x \in X$, then $\nabla\phi(x)$ is the linear map defined by

$$\nabla\phi(x)(h) = D_h\phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\phi(x + \varepsilon h) - \phi(x)], \quad h \in H.$$

Then $\nabla\phi(x) \in \mathfrak{L}(H;G)$, the space of linear operators from H into G. Let $\mathfrak{L}_{(2)}(H;G)$ be the class of Hilbert-Schmidt operators, i.e., the class of linear operators satisfying

$$||\psi||^2_{\mathfrak{L}_{(2)}(H;G)} = \sum_{k=1}^{\infty} ||\psi(e_k)||^2_G < +\infty.$$

$\mathfrak{L}_{(2)}(H;G)$ is a Hilbert space with inner product

$$\langle \psi, \theta \rangle = \sum_{k=1}^{\infty} \langle \psi(e_k) | \theta(e_k) \rangle_G.$$

We can also define $\nabla^2\phi(x), \cdots$, and $\nabla^i\phi(x)$ can be considered as an i multi-linear operator on H with values in G. We consider the correspond space $\mathfrak{L}^i_{(2)}(H;G)$ with the Hilbert-Schmidt norm

$$||\psi||^2_{H\text{-}S} = ||\psi||^2_{\mathfrak{L}^i_{(2)}(H;G)} = \sum_{k_1,\cdots,k_i=1}^{\infty} ||\psi(e_{k_1}, \cdots, e_{k_i})||^2_G.$$

Let f be a vector field on X that is a mapping from X into H. If $f \in L^2(X;H)$, we denote by δf the divergence, i.e., δ is the adjoint operator of ∇ for the Wiener measure, more precisely, δf is the element of $L^2(X)$ (if it exists) satisfying

$$\int_X \delta f \theta d\mu = \int_X \langle f | \nabla\theta \rangle_H d\mu$$

for all function $\theta \in W_1^2(X)$.

If G is a Hilbert space, for $f \in L^2(X;G)$, we have the orthogonal decomposition $f(x) = \sum_s c_s \mathcal{H}_s(x)$, where $s = (s_1, \cdots, s_i, \cdots)$, $s_i \in \mathbb{N}$, $|s| = \sum_i s_i < +\infty$, and $\sum_s ||c_s||^2_G < +\infty$.

Let $\mathcal{F}_n = \{s = (s_1, \cdots, s_m, \cdots) | s_m = 0, \forall m > n\}$ and let H_n be the subspace of H generated by $e_1, \cdots, e_n$. We say $f \in L^2(H_n;G)$ if $f(x) = \sum_{s \in \mathcal{F}_n} c_s \mathcal{H}_s(x)$, in other words, if $f(x) = F(x_1, \cdots, x_n)$, where $x_i = \langle x, e_i \rangle$, for some $F \in L^2(\mathbb{R}^n \to$

$G, \mu_n)$ where μ_n is the standard Gaussian measure on $\mathbb{R}^n$. We can regard $L^2(H_n; G)$ as a subspace of $L^2(X; G)$, and we denote by E^{H_n} the operator of orthogonal projection of $L^2(X; G)$ on $L^2(H_n; G)$. If $f \in L^2(X; G)$ and $f(x) = \sum_s c_s \mathcal{H}_s(x)$, then $E^{H_n} f(x) = \sum_{s \in \mathcal{F}_n} c_s \mathcal{H}_s(x)$.

If we put $\mathcal{F}_n = \sigma(\langle x, e_1 \rangle, \cdots, \langle x, e_n \rangle)$ and $E_{H_n} f = E[f|\mathcal{F}_n]$, then it is easy to show that $E_{H_n} f = E^{H_n} f$. Thus $E^{H_n} f$ is a martingale and by the convergence theorem of martingales, $E[f|\mathcal{F}_n] \longrightarrow f$ as $n \to \infty$ in L^p if $f \in L^p$, $p \geq 1$. And we can easily prove the following lemmas (Cruzeiro [3], Lemma 2.1.1, 2.1.2).

Lemma 4.1. Let $f \in W_1^2(X; G)$. Then

$$D_h(E^{H_n} f) = E^{H_n}(D_h f), \quad \text{for all } h \in H_n.$$

Lemma 4.2. For all $1 \leq p < +\infty$ and $0 \leq r < +\infty$,

$$||E^{H_n} f||_{r,p} \leq ||f||_{r,p}, \quad f \in W_r^p,$$

and

$$||E^{H_n} f - f||_{r,p} \longrightarrow 0 \quad \text{as } n \to \infty, \quad f \in W_r^p.$$

Let $A : X \to H$ be a vector field on X such that $A \in L^2(X; H)$ and consider the vector fields $E^{H_n} A$, $n = 1, 2, \cdots$. These fields depend only on n-coordinates and thus define vector fields on $\mathbb{R}^n$ with values in H. If $A \in W_\infty^2(X; H)$, then $E^{H_n} A \in W_\infty^2(\mathbb{R}^n; H)$ by Lemmas 4.1 and 4.2.

We project $E^{H_n} A$ on $\mathbb{R}^n$, in other words, P_{H_n} denotes the orthogonal projection of H on H_n, we set

$$A_n(x) = P_{H_n}[E^{H_n} A(x)].$$

Then we have $||A_n(x)|| \leq ||E^{H_n} A(x)||$, $D_h A_n(x) = P_{H_n}[D_h E^{H_n} A(x)]$ for all $h \in H$ and $||A_n||_{r,p} \leq ||E^{H_n} A||_{r,p}$. Consequently $A_n \in W_\infty^2(\mathbb{R}^n; \mathbb{R}^n)$. By the Sobolev's immersion theorem, this is a C^∞-function.

Lemma 4.3. For all $r > 0$ and all $n \in \mathbb{N}$, we have

$$\int_X \exp(\lambda ||A_n(x)||) d\mu(x) \leq \int_X \exp(\lambda ||A(x)||) d\mu(x)$$

and

$$\int_X \exp(\lambda ||\nabla A_n(x)||) d\mu(x) \leq \int_X \exp(\lambda ||\nabla A(x)||) d\mu(x).$$

We can easily prove the following general facts for flows on $\mathbb{R}^n$ (Cruzeiro [3], Lemma 3.1.1, 3.1.2).

Lemma 4.4. Let A and B be C^3-vector fields on $\mathbb{R}^n$ generating the global flows U_t^A and U_t^B. Let $h = A - B$. Then

$$U_t^A = U_t^B \circ U_0^{t,Z},$$

where $U_s^{t,Z}$ is defined by

$$(\frac{\partial U_s^{t,Z}}{\partial t})(x) = Z^t(U_s^{t,Z}(x)), \quad U_s^s(x) = x,$$

and Z is given by

$$Z^t(y) = (U_{-t}^B)'(y)h(U_t^B(y)).$$

Corollary 4.5. With the same notation as Lemma 4.4, we have

(i) $$U_t^A = (U_{-t}^{0,Z}) \circ U_t^B,$$

(ii) $$U_t^A(x) - U_t^B(x) = -\int_0^{-t} (U_{t+\xi}^B)'(U_\xi^{0,Z}(y)) \cdot h(U_{-t-\xi}^B(U_\xi^{0,Z}(y)))d\xi,$$

where $y = U_t^B(x)$,

(iii) $$\frac{d}{d\lambda}U_\lambda^{0,Z}(x) = -(U_0^B)'(U_\lambda^{0,Z}(x)) \cdot (A-B)(U_\lambda^{0,Z}(x)) - \int_0^\lambda [\frac{d}{d\lambda}(U_{-\lambda+\xi}^B)'(y) \cdot (A-B)(U_{\lambda-\xi}^B)(y) + (U_{-\lambda+\xi}^B)'(y)\frac{d}{d\lambda}((A-B)(U_{\lambda-\xi}^B(y)))]d\xi,$$

where $y = U_\xi^{0,Z}(x)$.

By Theorem 3.5, the field A_n has the flow $U_t^{A_n}$ which can be defined for all $t \in \mathbb{R}$ starting for all $x \in \mathbb{R}^n$. Further we have, by the formula of change of variables (Theorem 3.4),

(4.1) $$\frac{d(U_t^{A_n})_*\mu}{d\mu}(x) = \exp(\int_0^t \delta A_n(U_{-\xi}^{A_n}(x))d\xi) = k_t^n(x), \quad x \in \mathbb{R}^n.$$

This flow is a transformation of $\mathbb{R}^n$. We must modify the transformation on X and at the same time it does not change the formula (4.1).

We can set, for $x \in X$,

$$x = y + z, \quad z \in H_n.$$

If $U_t^{A_n}$ is the flow for A_n defined on $\mathbb{R}^n$, set

$$V_t^{A_n}(x) = U_t^{A_n}(z) + y - x = U_t^{A_n}(z) - z.$$

Then

$$\frac{V_t^{A_n}(x)}{dt} = A_n(V_t^{A_n}(x) + x) \quad \text{and} \quad V_0^{A_n}(x) = 0. \tag{4.2}$$

If ϕ is a function defined on H_n, then we have $\phi(V_t^{A_n}(x) + x) = \phi(U_t^{A_n}(x))$.

On the other hand, the measure μ can be decomposed as $\mu = \mu_n \otimes \nu_n$ with ν_n defined on $H_n^{\perp}$. Then we have

$$\begin{aligned}\int_X \psi(x) d(V_t^{A_n}(\cdot) + \cdot)_* \mu(x) &= \int_{H_n^{\perp}} (\int_{H_n} \psi(V_t^{A_n}(y+z) + y + z) d\mu_n(z)) d\nu_n(y) \\ &= \int_X \psi(x) \exp(\int_0^t \delta A_n(U_{-\xi}^{A_n}(x)) d\xi) d\mu(x).\end{aligned}$$

Thus we have the following

$$\frac{d(V_t^{A_n}(\cdot) + \cdot)_* \mu}{d\mu}(x) = \exp(\int_0^t \delta A_n(V_{-\xi}^{A_n}(x) + x) d\xi) = k_t^n(x). \tag{4.3}$$

We consider $V_t^{A_n}(x) = \tilde{U}_t^{A_n}(x) - x \in H$ and have to show the convergence of these functions. For this, we prepare the estimation of the norm of $(U_t^{A_n})'(x)$ which can be proved easily (Cruzeiro [3], Lemma 3.2.1).

Lemma 4.6. Suppose that $A \in W_\infty^2(X; H)$ and $\int_X \exp(\lambda ||\nabla A(x)||) d\mu(x) < +\infty$, $\forall \lambda > 0$. For all $t \in \mathbb{R}$, the matrix $(U_t^{A_n})'(x)$ is in the space $L^p(X; \mathrm{End}\ H)$ for all $1 \leq p < +\infty$ and if $|t| < M$, we have

$$\int_X ||(U_t^{A_n})'(x)||^p d\mu(x) \leq C(M),$$

where $C(M)$ does not depend on n.

Lemma 4.7. For all fixed t, $V_t^{A_n}(x) + x$ is a Cauchy sequence in the space $L^p(X; X)$ for all $1 \leq p < +\infty$, and uniformly in $t \in [-M, M]$. Denote the limit by $V_t^A(x) + x$.

Proof. By the Corollary 4.5, if $x = y + z$, $z \in H_n$, we have

$$\begin{aligned}V_t^{A_{n+m}}(x) - V_t^{A_n}(x) &= U_t^{A_{n+m}}(z) - U_t^{A_n}(z) \\ &= -\int_0^{-t} (U_{t+\xi}^{A_n})'(U_\xi^{0,Z}(U_t^{A_n}(z))) \\ &\qquad \cdot (A_{n+m} - A_n)(U_{-t-\xi}^{A_n})(U_\xi^{0,Z}(U_t^{A_n}(z))) d\xi.\end{aligned}$$

Since $U_\xi^{0,Z} = (U_0^{\xi,Z})^{-1} = U_{-\xi}^{A_{n+m}} \circ U_\xi^{A_n}$,

$$\begin{aligned} V_t^{A_{n+m}}(x) - V_t^{A_n}(x) \\ = -\int_0^{-t} (U_{t+\xi}^{A_n})'(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(V_t^{A_n}(x)+x)+x)+x) \\ \cdot (A_{n+m}-A_n)(V_{-t-\xi}^{A_n}(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(V_t^{A_n}(x)+x)+x)+x)+x)d\xi. \end{aligned}$$

By the formula (4.3) and Hölder's inequality,

$$\begin{aligned} \int_X ||V_t^{A_{n+m}}(x) - V_t^{A_n}(x)||_H^p d\mu(x) \\ \le \int_0^{-t} d\xi \Big(\int_X ||(U_{t+\xi}^{A_n})'||^{2p}(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(x)+x)+x)\cdot k_t^n(x)d\mu(x)\Big)^{1/2} \\ \cdot \Big(\int_X ||A_{n+m}-A_n||^{2p}(V_{-t-\xi}^{A_n}(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(x)+x)+x)+x) \\ \cdot k_t^n(x)d\mu(x)\Big)^{1/2}. \end{aligned}$$

By the application of the formula of change of variables successively and Hölder's inequality,

$$\begin{aligned} \int_X ||(U_{t+\xi}^{A_n})'||^{2p}(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(x)+x)+x)\cdot k_t^n(x)d\mu(x) \\ \le ||k_t^n||_{L^2}\cdot\Big(\int_X ||(U_{t+\xi}^{A_n})'||^{4p}(V_{-\xi}^{A_{n+m}}(x)+x)\cdot k_\xi^n(x)d\mu(x)\Big)^{1/2} \\ \le ||k_t^n||_{L^2}\cdot||k_\xi^n||_{L^2}^{1/2}\cdot||k_{-\xi}^{n+m}||_{L^2}^{1/4}\cdot\Big(\int_X ||(U_{t+\xi}^{A_n})'||^{16p}d\mu(x)\Big)^{1/8}. \end{aligned}$$

By Lemma 4.6, $||(U_{t+\xi}^{A_n})'||_{L^{16p}} \le C(t+\xi)$, and by Theorem 3.4, if $|t| < M$, $||k_t^n||_{L^p} \le C(M)$, $\forall p$. Similarly, we can estimate the second integral

$$\begin{aligned} \int_X ||A_{n+m}-A_n||^{2p}(V_{-t-\xi}^{A_n}(V_{-\xi}^{A_{n+m}}(V_\xi^{A_n}(x)+x)+x)+x)\cdot k_t^n(x)d\mu(x) \\ \le ||k_t^n||_{L^2}\cdot||k_\xi^n||_{L^2}^{1/2}\cdot||k_{-\xi}^{n+m}||_{L^2}^{1/4}\cdot||k_{-t-\xi}^n||_{L^2}^{1/8}\cdot\Big(\int_X ||A_{n+m}-A_n||^{32p}d\mu(x)\Big)^{1/16}. \end{aligned}$$

Since $||A_{n+m}-A_n||_{L^p} \to 0$ as $n,m \to \infty$, $\forall p$, the above integral tends to 0 as $n,m \to \infty$. Hence for all fixed t ($|t| < M$),

$$\int_X ||V_t^{A_{n+m}}(x) - V_t^{A_n}(x)||_H^p d\mu(x) \le C(M)\cdot||A_{n+m}-A_n||_{L^{p'}}. \tag{4.4}$$

□

Corollary 4.8. $(V_t^{A_n}(x)+x)$ forms a Cauchy sequence in $L^p([-M,M]\times X;X)$ for the norm of H (in particular, for the norm in X) and $V_t(x)\in H$.

Furthermore, we have

$$\begin{aligned}
&\int_X \sup_{|t|\le M}||V_t^{A_{n+m}}(x)-V_t^{A_n}(x)||d\mu(x)\\
&\quad=\int_X \sup_{|t|\le M}||\int_0^t (A_{n+m}(V_\xi^{A_{n+m}}(x)+x)-A_n(V_\xi^{A_n}(x)+x))d\xi||d\mu(x)\\
&\quad\le\int_X \sup_{|t|\le M}\int_0^t||A_{n+m}(V_\xi^{A_{n+m}}(x)+x)-A_n(V_\xi^{A_n}(x)+x)||d\xi d\mu(x)\\
&\quad\le\int_0^M\int_X||A_{n+m}(V_\xi^{A_{n+m}}(x)+x)-A_n(V_\xi^{A_n}(x)+x)||d\mu(x)d\xi\\
&\quad\le\int_0^M\int_X||A_{n+m}(V_\xi^{A_{n+m}}(x)+x)-A(V_\xi(x)+x)||d\mu(x)d\xi\\
&\qquad+\int_0^M\int_X||A_n(V_\xi^{A_n}(x)+x)-A(V_\xi(x)+x)||d\mu(x)d\xi\\
&\quad\longrightarrow 0\quad\text{as } n,m\to\infty\quad\text{(by Lemma 5.2).}
\end{aligned}$$

Corollary 4.9. The map $t\mapsto V_t(\cdot)+\cdot$ is continuous on $\mathbb{R}$ with values in $L^p(X;X)$.

Proof. For every n, $t\mapsto V_t(\cdot)+\cdot$ is continuous with values in $L^p(X;X)$ and $V_t^{A_n}(x)$ satisfies the Cauchy's criterion for uniform convergence by Corollary 4.8. □

Remark 4.10. For all fixed t, we can choose a sub-sequence of $(V_t^{A_n}(x)+x)$ which converges to $V_t(x)+x$ for a.e.x.

5. Differential equation (1.1) on the Wiener space

We assume the following in Lemmas 5.1, 5.2 and Theorem 5.3.

Assumption 1. Suppose that a vector field A on X satisfies the following conditions :

(i) $A\in W_\infty^2(X;H)$ and $\forall\lambda>0$, $\int_X\exp(\lambda||A(x)||)d\mu(x)<+\infty$,
(ii) $\forall\lambda>0$, $\int_X\exp(\lambda||\nabla A(x)||)d\mu(x)<+\infty$,
(iii) $\forall\lambda>0$, $\int_X\exp(\lambda|\delta A(x)|)d\mu(x)<+\infty$.

Lemma 5.1. For all $M>0$, we have

$$\lim_{n,l\to\infty}[\sup_{|t|<M}\int_X||A_n(\tilde U_t^{A_n}(x))-A_n(\tilde U_t^{A_l}(x))||d\mu(x)]=0. \tag{5.1}$$

Proof. By the formula (i) of Corollary 4.5,

$$U_t^{A_n} = U_{-t}^{0,Z} \circ U_t^{A_l},$$

where $Z^t(y) = (U_{-t}^{A_l})'(y) \cdot (A_n - A_l)(U_t^{A_l}(y))$. Then if $x = y + z$, $z \in H_n$,

$$||A_n(\tilde{U}_t^{A_n}(x)) - A_n(\tilde{U}_t^{A_l}(x))|| = ||A_n(\tilde{U}_{-t}^{0,Z}(\tilde{U}_t^{A_l}(x))) - A_n(\tilde{U}_t^{A_l}(x))||.$$

By the formula (4.3), if $\tilde{U}_t^{0,Z} = \tilde{U}_{-t}^{A_n} \circ \tilde{U}_t^{A_l}$,

$$\begin{aligned}\int_X ||A_n(\tilde{U}_t^{A_n}(x)) - A_n(\tilde{U}_t^{A_l}(x))||d\mu(x) \\ &= \int_X ||k_t^l(x)|| \cdot ||A_n(\tilde{U}_{-t}^{0,Z}(x)) - A_n(x)||d\mu(x) \\ &= ||k_t^l(x)||_{L^2} \cdot (\int_X ||A_n(\tilde{U}_{-t}^{0,Z}(x)) - A_n(x)||^2 d\mu(x))^{1/2}.\end{aligned}$$

We estimate the last intergal

$$\begin{aligned}\int_X ||A_n(\tilde{U}_{-t}^{0,Z}(x)) - A_n(x)||^2 d\mu(x) \\ &\le \int_X \int_0^{-t} ||\nabla A_n(\tilde{U}_\lambda^{0,Z}(x))||^2 \cdot ||\frac{d}{d\lambda}\tilde{U}_\lambda^{0,Z}(x)||^2 d\lambda d\mu(x) \\ &\le 2(I(t) + II(t) + III(t)),\end{aligned}$$

where by (iii) of Corollary 4.5,

$$\begin{aligned}I(t) &= \int_0^{-t} \int_X ||\nabla A_n(\tilde{U}_\lambda^{0,Z}(x))||^2 \\ &\quad \cdot ||(U_0^{A_l})'(\tilde{U}_\lambda^{0,Z}(x))||^2 \cdot ||(A_n - A_l)(\tilde{U}_\lambda^{0,Z}(x))||^2 d\mu(x) d\lambda, \\ II(t) &= 2\int_0^{-t} \lambda \int_X ||\nabla A_n(\tilde{U}_\lambda^{0,Z}(x))||^2 \\ &\quad \cdot [\int_0^\lambda ||\frac{d}{d\lambda}(U_{-\lambda+\xi}^{A_l})'(y) \cdot (A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y))||^2 d\xi] d\mu(x) d\lambda, \\ III(t) &= 2\int_0^{-t} \lambda \int_X ||\nabla A_n(\tilde{U}_\lambda^{0,Z}(x))||^2 \\ &\quad \cdot [\int_0^\lambda ||(U_{-\lambda+\xi}^{A_l})'(y) \cdot \frac{d}{d\lambda}((A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y)))||^2 d\xi] d\mu(x) d\lambda,\end{aligned}$$

where $y = \tilde{U}_\xi^{0,Z}(x)$.

By the formula (4.3) and Hölder's inequality,

$$I(t) \leq \int_0^{-t} ||k_\lambda^l||_{L^2} \cdot ||k_{-\lambda}^n||_{L^2}^{1/2} \cdot (\int_X ||\nabla A_n||^8 \cdot ||(U_0^{A_l})'||^8 \cdot ||A_n - A_l||^8 d\mu(x))^{1/4} d\lambda.$$

Since $||\nabla A_n||_{L^p} \leq ||\nabla A||_{L^p}$, $||(U_0^{A_l})'||_{L^p} \leq C(M)$ by Lemma 4.6 and $||A_n - A_l||_{L^p} \to 0$ as $n, l \to \infty$, we have $I(t) \to 0$ as $n, l \to \infty$ uniformly in $t \in [-M, M]$.

By Lemma 4.6,

$$\frac{d}{dt}(U_t^{A_l})'(x) = Y_l(U_t^{A_l}(x)) \cdot (U_t^{A_l})'(x),$$

where $Y_l = (\partial_i A_l^j)$. Then if $x = y + z$, $z \in H_n$,

$$||\frac{d}{dt}(U_t^{A_l})'(z)||_{\mathrm{End}(H)} \leq ||Y_l(U_t^{A_l}(z))||_{\mathrm{End}(H)} \cdot ||(U_t^{A_l})'(z)||_{\mathrm{End}(H)}. \tag{5.2}$$

We can estimate $II(t)$ similarly as $I(t)$:

$$\begin{aligned} II(t) &\leq 2 \int_0^{-t} \lambda \int_0^{\lambda} [\int_X ||\nabla A_n(\tilde{U}_\lambda^{0,Z}(x))||^4 d\mu(x)]^{1/2} \\ &\quad \cdot [\int_X ||\frac{d}{d\lambda}(U_{-\lambda+\xi}^{A_l})'(y) \cdot (A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y))||^4 d\mu(x)]^{1/2} d\xi d\lambda \\ &\leq 2 \int_0^{-t} \lambda \int_0^{\lambda} ||k_\lambda^l||_{L^2}^{1/2} \cdot ||k_{-\lambda}^n||_{L^2}^{1/4} \\ &\qquad \cdot (\int_X ||\nabla A_n||^{16} d\mu(x))^{1/8} \cdot ||k_\xi^l||_{L^2}^{1/2} \cdot ||k_{-\xi}^n||_{L^2}^{1/4} \\ &\qquad \cdot (\int_X ||\frac{d}{d\lambda}(U_{-\lambda+\xi}^{A_l})'(x)||^{16} \cdot ||A_n - A_l||^{16} d\mu(x))^{1/8} d\xi d\lambda. \end{aligned}$$

By the hypothesis, the formula (5.2) and Lemma 4.6, we have $II(t) \to 0$ as $n, l \to \infty$ uniformly in $t \in [-M, M]$.

Since

$$\frac{d}{d\lambda}((A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y))) = \sum_i D_{e_i}(A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y)) \cdot (A_l(\tilde{U}_{\lambda-\xi}^{A_l}(y))|e_i),$$

$$\begin{aligned} \int_X ||\frac{d}{d\lambda}((A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y)))||^p d\mu(x) &\leq (\int_X ||\nabla(A_n - A_l)(\tilde{U}_{\lambda-\xi}^{A_l}(y))||^{2p} d\mu(x))^{1/2} \\ &\quad \cdot (\int_X ||A_l(\tilde{U}_{\lambda-\xi}^{A_l}(y))||^{2p} d\mu(x))^{1/2}, \quad \forall p. \end{aligned}$$

Thus by the same way, $III(t)$ tends to 0 as $n, l \to \infty$ uniformly in $t \in [-M, M]$. □

By the same technique used in the above lemma, we can show the following lemma more easily.

Lemma 5.2. For all $M > 0$, we have

$$\lim_n \int_{-M}^{M} \int_X ||A_n(V_t^{A_n}(x) + x) - A_n(V_t(x) + x)|| d\mu(x) dt = 0$$

and for all $M > 0$, $A_n(V_t^{A_n}(x) + x) - A(V_t(x) + x) \to 0$ in the space $L^1([-M, M] \times X; H)$.

Using the above lemmas and results in the section 4, we can prove the following fact.

Theorem 5.3. (Cruzeiro [3], Theorem 1.6.1) If A is a vector field on X satisfying the Assumption 1, then $V_t(x)$ exists for all $t \in \mathbb{R}$, $d\mu$-a.e.x and satisfies the following differential equation.

$$\begin{cases} (dV_t/dt)(x) = A(V_t(x) + x), & d\mu\text{-a.e.}x, \\ V_0(x) = 0. \end{cases} \tag{5.3}$$

Furthermore, for all $t \in \mathbb{R}$, the measure $(V_t(\cdot) + \cdot)_*\mu$ is absolutely continuous with respect to μ and if we set $(d(V_t(\cdot) + \cdot)_*\mu / d\mu)(x) = k_t(x)$, then $k_t \in L^p(\mu)$, $\forall p$.

From now on, we wish to prove the quasi-sure existence of the solution. For that, we put the following assumption.

Assumption 2. Suppose that a vector field A on X satisfies the following conditions :

(i) $A \in W_\infty^\infty(X; H)$ and $\forall \lambda > 0$, $\int_X \exp(\lambda ||A(x)||) d\mu(x) < +\infty$,
(ii) $\forall \lambda > 0$, $\forall n = 1, 2, \cdots$, $\int_X \exp(\lambda ||\nabla^n A(x)||) d\mu(x) < +\infty$,
(iii) $\forall \lambda > 0$, $\int_X \exp(\lambda |\delta A(x)|) d\mu(x) < +\infty$.

We have proved the existence for the finite dimensional case in Theorem 3.5. Namely, if B is a finite dimensional vector field in Theorem 3.5, then there exists $V_t(x) = U_t^B(x) - x$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ satisfying (3.2), (3.3), (3.4) and (3.5). Put $V_t(x) \equiv V_t^{(n)}(x)$, $A \equiv A^{(n)}$, $\nabla V_t \equiv \nabla V_t^{(n)}, \cdots$, for n-dimensional case.

For the purpose we have to consider the limits of $\nabla V_t(x)$, $LV_t(x)$, $L^2 V_t(x), \cdots$, as $n \to \infty$.

Lemma 5.4. For all $M > 0$, $p > 0$ and $m = 1, 2, \cdots$,

$$\lim_{n,l \to \infty} [\sup_{|t| < M} \int_X ||\nabla^m A^{(n)}(V_t^{(n)}(x) + x) - \nabla^m A^{(n)}(V_t^{(l)}(x) + x)||^p d\mu(x)] = 0. \tag{5.4}$$

Proof. By Lemma 4.4, $U_t^{A^{(n)}} = U_{-t}^{0,Z} \circ U_t^{A^{(l)}}$, where

$$Z^t(y) = (U_{-t}^{A^{(l)}})'(y) \cdot (A^{(n)} - A^{(l)})(U_t^{A^{(l)}}(y)).$$

If $x = y + z$, $z \in H_n$,

$$
\begin{aligned}
&||\nabla A^{(n)}(V_t^{(n)}(x)+x)-\nabla A^{(n)}(V_t^{(l)}(x)+x)|| \\
&\qquad = ||\nabla A^{(n)}(U_t^{(n)}(x))-\nabla A^{(n)}(U_t^{(l)}(x))|| \\
&\qquad = ||\nabla A^{(n)}(\tilde{U}_{-t}^{0,Z}(\tilde{U}_t^{(l)}(x)))-\nabla A^{(n)}(\tilde{U}_t^{(l)}(x))||,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\int_X ||\nabla A^{(n)}(V_t^{(n)}(x)+x)-\nabla A^{(n)}(V_t^{(l)}(x)+x)||^p d\mu(x) \\
&\qquad = \int_X k_t^l \cdot ||\nabla A^{(n)}(\tilde{U}_{-t}^{0,Z}(x))-\nabla A^{(n)}(x)||^p d\mu(x) \\
&\qquad \leq ||k_t^l||_{L^2} \cdot (\int_X ||\nabla A^{(n)}(\tilde{U}_{-t}^{0,Z}(x))-\nabla A^{(n)}(x)||^{2p} d\mu(x))^{\frac{1}{2}}.
\end{aligned}
$$

By the mean value theorem,

$$
\nabla A^{(n)}(\tilde{U}_{-t}^{0,Z})(x)-\nabla A^{(n)}(x) = \int_0^{-t} \nabla^2 A^{(n)}(\tilde{U}_\lambda^{0,Z}(x)) \cdot \frac{d}{d\lambda}(\tilde{U}_\lambda^{0,Z}(x)) d\lambda.
$$

Thus

$$
\begin{aligned}
&||\nabla A^{(n)}(\tilde{U}_{-t}^{0,Z})-\nabla A^{(n)}||^{2p} \\
&\qquad = ||\int_0^{-t} \nabla^2 A^{(n)}(\tilde{U}_\lambda^{0,Z}(x)) \cdot \frac{d}{d\lambda}(\tilde{U}_\lambda^{0,Z}(x)) d\lambda||^{2p} \\
&\qquad \leq (-t)^{2p-1} \cdot \int_0^{-t} ||\nabla^2 A^{(n)}(\tilde{U}_\lambda^{0,Z}(x))||^{2p} \cdot ||\frac{d}{d\lambda}(\tilde{U}_\lambda^{0,Z}(x))||^{2p} d\lambda.
\end{aligned}
$$

The proof for higher order cases is the same as that of Lemma 5.1 if we replace $\nabla A^{(n)}$ by $\nabla^2 A^{(n)}$. In the proof of the case $\nabla^m A^{(n)}$, the sufficient condition is $||\nabla^{m+1} A^{(n)}||_{L^p} < +\infty$. Thus under the Assumption 2, the formula (5.4) can be asserted. □

Now we state our main result :

Theorem 5.5. If A is a vector field on X satisfying the Assumption 2, then, choosing a suitable quasi continuous modification $\tilde{A}$ of A, $V_t(x)$ exists for all $t \in \mathbb{R}$, q.e.x satisfying the following differential equation.

$$
\left\{
\begin{array}{l}
(dV_t/dt)(x) = A(V_t(x)+x), \\
V_0(x) = 0.
\end{array}
\right. \tag{1.1}
$$

The rest of this paper is devoted to the proof of this theorem.

Proof. In the first place, we consider $\nabla V_t^{(n)}(x)$. Put $\mathfrak{a}_t^{(n)} = \nabla A^{(n)}(V_t^{(n)}(x) + x)$, then since $\nabla V_t^{(n)}(x)$ satisfies the equation (3.3),

$$\begin{aligned}
&\nabla V_t^{(n)}(x) - \nabla V_t^{(m)}(x) \\
&\qquad = \int_0^t (\mathfrak{a}_s^{(n)} \nabla V_s^{(n)}(x) - \mathfrak{a}_s^{(m)} \nabla V_s^{(m)}(x)) ds + \int_0^t (\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}) ds.
\end{aligned}$$

Since $\mathfrak{a}_t^{(m)}$ and $\nabla V_t^{(n)}(x)$ are continuous in t, for fixed T, $0 \le t \le T$, by Gronwall's lemma,

$$\begin{aligned}
&|\nabla V_t^{(n)}(x) - \nabla V_t^{(m)}(x)| \\
&\qquad \le \int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}| \cdot (1 + |\nabla V_s^{(n)}|) ds + \int_0^t |\mathfrak{a}_s^{(m)}| \cdot |\nabla V_s^{(n)} - \nabla V_s^{(m)}| ds \\
&\qquad \le (\int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}| \cdot (1 + |\nabla V_s^{(n)}|) ds) \cdot \exp(\int_0^t |\mathfrak{a}_s^{(m)}| ds).
\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}
&\sup_{0 \le s \le t} |\nabla V_s^{(n)} - \nabla V_s^{(m)}|^p \\
&\qquad \le (T \int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}| \cdot (1 + |\nabla V_s^{(n)}|) \frac{ds}{T})^p \cdot \exp(p \int_0^t |\mathfrak{a}_s^{(m)}| ds) \\
&\qquad \le T^{p-1} \cdot \int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^p \cdot (1 + |\nabla V_s^{(n)}|)^p ds \cdot \exp(p \int_0^t |\mathfrak{a}_s^{(m)}| ds).
\end{aligned}$$

Thus

$$\begin{aligned}
&E[\sup_{0 \le s \le t} |\nabla V_s^{(n)} - \nabla V_s^{(m)}|^p] \\
&\le T^{p-1} E[(\int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^p \cdot (1 + |\nabla V_s^{(n)}|)^p ds)^2]^{1/2} \cdot E[\exp(2p \int_0^t |\mathfrak{a}_s^{(m)}| ds)]^{1/2} \\
&\le T^{p-\frac{1}{2}} E[\int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^{2p} \cdot (1 + |\nabla V_s^{(n)}|)^{2p} ds]^{1/2} \\
&\quad \cdot (\int_0^t E[\exp(2p \cdot t \cdot |\mathfrak{a}_s^{(m)}|)] \frac{ds}{t})^{1/2} \\
&\le T^{p-\frac{1}{2}} (\int_0^T E[|\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^{2p} \cdot (1 + |\nabla V_s^{(n)}|)^{2p}] ds)^{1/2} \\
&\quad \cdot (\int_0^t E[\exp(2p \cdot t \cdot |\mathfrak{a}_s^{(m)}|)] \frac{ds}{t})^{1/2}.
\end{aligned}$$

Note that

$$E[|\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^{2p} \cdot (1 + |\nabla V_s^{(n)}|)^{2p}] \le E[|\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^{4p}]^{1/2} \cdot E[(1 + |\nabla V_s^{(n)}|)^{4p}]^{1/2}$$

and

$$\begin{aligned} E[|\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}|^{2p}] \le 2^{4p}(&E[|\nabla A^{(n)}(V_s^{(n)}(x) + x) - \nabla A^{(n)}(V_s^{(m)}(x) + x)|^{4p}] \\ &+ E[|\nabla A^{(n)}(V_s^{(m)}(x) + x) - \nabla A^{(m)}(V_s^{(m)}(x) + x)|^{4p}]). \end{aligned}$$

Since

$$E[|\nabla A^{(n)}(V_s^{(n)}(x) + x) - \nabla A^{(n)}(V_s^{(m)}(x) + x)|^{4p}] \longrightarrow 0 \quad \text{as } n, m \to \infty$$

(uniformly in s by Lemma 5.4),

$$\begin{aligned} &E[|\nabla A^{(n)}(V_s^{(m)}(x) + x) - \nabla A^{(m)}(V_s^{(m)}(x) + x)|^{4p}] \\ &\quad = E[|\nabla A^{(n)} - \nabla A^{(m)}|^{4p} \cdot k_s^m] \\ &\quad \le E[|\nabla A^{(n)} - \nabla A^{(m)}|^{8p}]^{1/2} \cdot ||k_s^m||_{L_2} \longrightarrow 0 \quad \text{as } n, m \to \infty \end{aligned}$$

(by the fact $||k_s^m||_{L^p} < +\infty, \forall p$),

$$E[(1 + |\nabla V_s^{(n)}|)^{4p}] \le 2^{4p}(E[1] + E[|\nabla V_s^{(n)}|^{4p}]) < +\infty,$$

and

$$\begin{aligned} E[\exp(2t \cdot p \cdot |\mathfrak{a}_s^m|)] &= E[\exp(2t \cdot p \cdot |\nabla A^{(m)}|) \cdot k_s^{(m)}] \\ &\le E[\exp(4t \cdot p \cdot |\nabla A^{(m)}|)] \cdot ||k_s^m||_{L^2} \\ &< +\infty \quad \text{by hypothesis,} \end{aligned}$$

we have

$$E[\sup_{0 \le s \le t} |\nabla V_s^{(n)} - \nabla V_s^{(m)}|^p] \longrightarrow 0 \quad \text{as } n, m \to \infty \text{ uniformly in } t.$$

For the case $LV_t^{(n)}$, by the equation (3.4),

$$\begin{aligned} &|LV_t^{(n)} - LV_t^{(m)}| \\ &\quad \le \int_0^t |\mathfrak{d}_s^{(n)} - \mathfrak{d}_s^{(m)}| ds \\ &\qquad + \sum_i \int_0^t |\mathfrak{b}_s^{(n)} \cdot \partial_i V_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{b}_s^{(m)} \cdot \partial_i V_s^{(m)} \cdot \partial_i V_s^{(m)}| ds \\ &\qquad + \int_0^t |\mathfrak{a}_s^{(n)} \cdot LV_s^{(n)} - \mathfrak{a}_s^{(m)} \cdot LV_s^{(m)}| ds \\ &\qquad + 2 \sum_i \int_0^t |\mathfrak{c}_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{c}_s^{(m)} \cdot \partial_i V_s^{(m)}| ds, \end{aligned}$$

where $\mathfrak{a}_s^{(n)} = \nabla A^{(n)}(V_s^{(n)}(x)+x)$, $\mathfrak{b}_s^{(n)} = \nabla^2 A^{(n)}(V_s^{(n)}(x)+x)$, $\mathfrak{c}_s^{(n)} = \partial_i \nabla A^{(n)}(V_s^{(n)}(x)+x)$ and $\mathfrak{d}_s^{(n)} = LA^{(n)}(V_s^{(n)}(x)+x)$.
Let $0 \le t \le T$. Then

$$\begin{aligned}
&|LV_t^{(n)} - LV_t^{(m)}| \\
&\quad \le \int_0^T |\mathfrak{d}_s^{(n)} - \mathfrak{d}_s^{(m)}| ds \\
&\qquad + \sum_i \int_0^T |\mathfrak{b}_s^{(n)} \cdot \partial_i V_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{b}_s^{(m)} \cdot \partial_i V_s^{(m)} \cdot \partial_i V_s^{(m)}| ds \\
&\qquad + \int_0^T |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}| \cdot |LV_s^{(n)}| ds \\
&\qquad + \int_0^t |\mathfrak{a}_s^{(m)}| \cdot |LV_s^{(n)} - LV_s^{(m)}| ds \\
&\qquad + 2 \sum_i \int_0^T |\mathfrak{c}_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{c}_s^{(m)} \cdot \partial_i V_s^{(m)}| ds.
\end{aligned}$$

Put

$$\begin{aligned}
F(T) = \int_0^T (&|\mathfrak{d}_s^{(n)} - \mathfrak{d}_s^{(m)}| \\
&+ \sum_i |\mathfrak{b}_s^{(n)} \cdot \partial_i V_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{b}_s^{(m)} \cdot \partial_i V_s^{(m)} \cdot \partial_i V_s^{(m)}| \\
&+ |\mathfrak{a}_s^{(n)} - \mathfrak{a}_s^{(m)}| \cdot |LV_s^{(n)}| \\
&+ 2 \sum_i |\mathfrak{c}_s^{(n)} \cdot \partial_i V_s^{(n)} - \mathfrak{c}_s^{(m)} \cdot \partial_i V_s^{(m)}|) ds.
\end{aligned}$$

Since $\mathfrak{a}_s^{(m)} = \nabla A^{(m)}(V_s^{(m)}(x)+x)$ and $LV_s^{(n)} - LV_s^{(m)}$ are continuous in s,

$$|LV_s^{(n)} - LV_s^{(m)}| \le F(T) \cdot \exp\Big(\int_0^s |\mathfrak{a}_u^{(m)}| du\Big).$$

Then

$$E[\sup_{0 \le s \le t} |LV_s^{(n)} - LV_s^{(m)}|^p] \le E[F(T)^{2p}]^{1/2} \cdot E[\exp(2p \cdot \int_0^t |\mathfrak{a}_u^{(m)}| du)]^{1/2}.$$

Since

$$E[\exp(2p\cdot\int_0^t|\mathfrak{a}_u^{(m)}|du)] = E[\exp(2pt\cdot\int_0^t|\nabla A^{(m)}(V_u^{(m)}(x)+x)|\frac{du}{t})]$$
$$\leq \int_0^t E[\exp(2pt|\nabla A^{(m)}(x)|)\cdot k_u^m]\frac{du}{t}$$
$$\leq \int_0^t E[\exp(4pt|\nabla A^{(m)}(x)|)]^{1/2}\cdot||k_u^m||_{L^2}\frac{du}{t}$$
$$< +\infty \quad \text{by hypothesis,}$$

there is nothing to prove if we only prove the following fact

$$E[F(T)^{2p}]$$
$$\leq T^{2p-1}C(p)\int_0^T(E[|\mathfrak{d}_s^{(n)}-\mathfrak{d}_s^{(m)}|^{2p}]$$
$$+E[(\sum_i|\mathfrak{b}_s^{(n)}\cdot\partial_iV_s^{(n)}\cdot\partial_iV_s^{(n)}$$
$$-\mathfrak{b}_s^{(m)}\cdot\partial_iV_s^{(m)}\cdot\partial_iV_s^{(m)}|)^{2p}]$$
$$+\cdots)ds$$
$$\equiv T^{2p-1}C(p)\int_0^T(E_1+E_2+E_3+E_4)ds\to 0 \quad \text{as } n,m\to\infty.$$

$$E_1 = E[|\mathfrak{d}_s^{(n)}-\mathfrak{d}_s^{(m)}|^{2p}]$$
$$= E[|LA^{(n)}(V_s^{(n)}(x)+x)-LA^{(m)}(V_s^{(m)}(x)+x)|^{2p}]$$
$$\leq 2^{2p}(E[|LA^{(n)}(V_s^{(n)}(x)+x)-LA^{(n)}(V_s^{(m)}(x)+x)|^{2p}]$$
$$+E[|LA^{(n)}(V_s^{(m)}(x)+x)-LA^{(m)}(V_s^{(m)}(x)+x)|^{2p}]).$$

By Lemma 5.4,

$$E[|LA^{(n)}(V_s^{(n)}(x)+x)-LA^{(n)}(V_s^{(m)}(x)+x)|^{2p}]\to 0 \quad \text{as } n,m\to\infty$$

and

$$E[|LA^{(n)}(V_s^{(m)}(x)+x)-LA^{(m)}(V_s^{(m)}(x)+x)|^{2p}]$$
$$\leq E[|LA^{(n)}-LA^{(m)}|^{4p}]^{1/2}\cdot||k_s^m||_{L^2}$$
$$\leq C(p)\cdot||A^{(n)}-A^{(m)}||_{2,4p}^{1/2}\cdot||k_s^m||_{L^2}$$
$$\text{(by the Meyer's equivalence)}$$
$$\longrightarrow 0 \quad \text{as } n,m\to\infty \text{ by hypothesis.}$$

$$
\begin{aligned}
E_2 = E[(\sum_i |\nabla^2 A^{(n)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)} \\
- \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}\cdot \partial_i V_s^{(m)}|)^{2p}] \\
\le C(p)\cdot (E[(\sum_i |\nabla^2 A^{(n)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)} \\
- \nabla^2 A^{(n)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)}|)^{2p}] \\
+ E[(\sum_i |\nabla^2 A^{(n)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)} \\
- \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)}|)^{2p}] \\
+ E[(\sum_i |\nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)}\cdot \partial_i V_s^{(n)} \\
- \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}\cdot \partial_i V_s^{(n)}|)^{2p}]) \\
+ E[(\sum_i |\nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}\cdot \partial_i V_s^{(n)} \\
- \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}\cdot \partial_i V_s^{(m)}|)^{2p}]) \\
\equiv C(p)\cdot (E_{2,1}+E_{2,2}+E_{2,3}+E_{2,4}).
\end{aligned}
$$

$$
\begin{aligned}
E_{2,1} &\le E[\sum_i |\nabla^2 A^{(n)}(V_s^{(n)}(x)+x) - \nabla^2 A^{(n)}(V_s^{(m)}(x)+x)|^{2p}\cdot |\partial_i V_s^{(n)}|^{4p}] \\
&\le E[|\nabla^2 A^{(n)}(V_s^{(n)}(x)+x) - \nabla^2 A^{(n)}(V_s^{(m)}(x)+x)|^{4p}]^{1/2}\cdot E[|\nabla V_s^{(n)}|^{4p}]^{1/2}.
\end{aligned}
$$

Since the first term of right hand side tends to zero as $n, m \to \infty$ by Lemma 5.4, and the second is finite by Theorem 3.5, we have $E_{2,1} \to 0$ as $n, m \to \infty$.

$$
\begin{aligned}
E_{2,2} &\le E[|\nabla^2 A^{(n)}(V_s^{(m)}(x)+x) - \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)|^{2p}\cdot \nabla V_s^{(n)}|^{2p}] \\
&\le E[|\nabla^2 A^{(n)} - \nabla^2 A^{(m)}|^{8p}]^{1/2}\cdot ||k_s^m||_{L^2}\cdot E[|\nabla V_s^{(n)}|^{4p}]^{1/2} \\
&\longrightarrow 0 \quad \text{as } n, m \to \infty.
\end{aligned}
$$

By the same way,

$$
E_{2,3} \le 2^{2p} E[|\nabla^2 A^{(m)}|^{8p}]^{1/4}\cdot ||k_s^m||_{L^2}^{1/2}\cdot E[|\nabla V_s^{(n)}|^{8p}]^{1/4}\cdot E[|\nabla V_s^{(n)} - \nabla V_s^{(m)}|^{8p}]^{1/4}.
$$

$E_{2,4}$ can be estimated by the same way as $E_{2,3}$. In the first part of infinite case, we have proved $E[|\nabla V_s^{(n)} - \nabla V_s^{(m)}|^{8p}]^{1/4} \to 0$ as $n, m \to \infty$ and the others are all finite.

$$
\begin{aligned}
E_3 &= E[|\nabla A^{(n)}(V_s^{(n)}(x)+x) - \nabla A^{(m)}(V_s^{(m)}(x)+x)|^{2p} \cdot |LV_s^{(n)}|^{2p}] \\
&\le E[|\nabla A^{(n)}(V_s^{(n)}(x)+x) - \nabla A^{(n)}(V_s^{(n)}(x)+x)|^{4p}]^{1/2} \cdot E[|LV_s^{(n)}|^{4p}]^{1/2} \\
&\longrightarrow 0 \quad \text{as } n, m \to \infty
\end{aligned}
$$

by the same method as for the previous case E_2 and Meyer's equivalence.

Similarly, E_4 can be calculated as follows :

$$
\begin{aligned}
&E[(2\sum_i |\partial_i \nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)} - \partial_i \nabla A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}|)^{2p}] \\
&\le C(p)(E[\sum_i |\partial_i \nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)} - \partial_i \nabla A^{(m)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)}|^{2p}] \\
&\quad + E[\sum_i |\partial_i \nabla A^{(m)}(V_s^{(n)}(x)+x)\cdot \partial_i V_s^{(n)} - \partial_i \nabla A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)}|^{2p}] \\
&\quad + E[\sum_i |\partial_i \nabla A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(n)} - \partial_i \nabla A^{(m)}(V_s^{(m)}(x)+x)\cdot \partial_i V_s^{(m)}|^{2p}]) \\
&\equiv C(p)(E_{4,1} + E_{4,2} + E_{4,3}).
\end{aligned}
$$

Since

$$
\begin{aligned}
E_{4,1} &\le E[|\nabla^2 A^{(n)}(V_s^{(n)}(x)+x)\cdot \nabla V_s^{(n)} - \nabla^2 A^{(m)}(V_s^{(n)}(x)+x)\cdot \nabla V_s^{(n)}|^{2p}] \\
&\le E[|\nabla^2 A^{(n)} - \nabla^2 A^{(m)}|^{8p}]^{1/4} \cdot ||k_s^n||_{L^2}^{1/2} \cdot E[|\nabla V_s^{(n)}|^{4p}]^{1/2} \\
&\longrightarrow 0 \quad \text{as } n, m \to \infty, \\
E_{4,2} &\le E[|\nabla^2 A^{(m)}(V_s^{(n)}(x)+x) - \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)|^{2p} \cdot |\nabla V_s^{(n)}|^{2p}] \\
&\le E[|\nabla^2 A^{(m)}(V_s^{(n)}(x)+x) - \nabla^2 A^{(m)}(V_s^{(m)}(x)+x)|^{4p}]^{1/2} \cdot E[|\nabla V_s^{(n)}|^{4p}]^{1/2} \\
&\longrightarrow 0 \quad \text{as } n, m \to \infty \text{ by Lemma 5.4,}
\end{aligned}
$$

and

$$
\begin{aligned}
E_{4,3} &\le E[|\nabla^2 A^{(m)}(V_s^{(m)}(x)+x)|^{2p} \cdot |\nabla V_s^{(n)} - \nabla V_s^{(m)}|^{2p}] \\
&\le E[|\nabla^2 A^{(m)}|^{8p}]^{1/4} \cdot ||k_s^m||_{L^2}^{1/2} \cdot E[|\nabla V_s^{(n)} - \nabla V_s^{(m)}|^{4p}]^{1/2} \\
&\longrightarrow 0 \quad \text{as } n, m \to \infty
\end{aligned}
$$

by the same reason as in the previous cases and hypothesis, E_4 also tends to 0 as $n, m \to \infty$. Thus we have proved that

$$
E[\sup_{0\le s\le t} |LV_s^{(m)} - LV_s^{(n)}|^p] \to 0 \quad \text{as} \quad m, n \to \infty.
$$

Using the inequality (2.3) and Borel-Cantelli's Lemma, there exists a sub-sequence $\{V_s^{(n_j)}\}$ and quasi-continuous limit V_s such that

$$\sup_{0\le s\le t}|V_s^{(n_j)}-V_s|\to 0 \quad \text{as} \quad j\to\infty, \quad (2,p)\text{-quasi-surely.}$$

On the other hand, the right hand side $\int_0^t A^{(n)}(V_s^{(n)}(x)+x)ds$ also converges to $\int_0^t A(V_s(x)+x)ds$, $(2,p)$-quasi-surely. In fact, since

$$\begin{aligned}L\{A^{(n)}(V_s^{(n)}(x)+x)\} &= LA^{(n)}(V_s^{(n)}(x)+x)+\nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot LV_s^{(n)}(x)\\ &\quad+\sum\nabla^2A^{(n)}(V_s^{(n)}(x)+x)\cdot\partial_iV_s^{(n)}(x)\cdot\partial_iV_s^{(n)}(x)\\ &\quad+2\sum\partial_i\nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot\partial_iV_s^{(n)}(x)\end{aligned}$$

and

$$\begin{aligned}L\{A^{(n)}(V_s^{(m)}(x)+x)\} &= LA^{(n)}(V_s^{(m)}(x)+x)+\nabla A^{(n)}(V_s^{(m)}(x)+x)\cdot LV_s^{(m)}(x)\\ &\quad+\sum\nabla^2A^{(n)}(V_s^{(m)}(x)+x)\cdot\partial_iV_s^{(m)}(x)\cdot\partial_iV_s^{(m)}(x)\\ &\quad+2\sum\partial_i\nabla A^{(n)}(V_s^{(m)}(x)+x)\cdot\partial_iV_s^{(m)}(x),\end{aligned}$$

we can prove that

$$\lim_{n,m\to\infty}\sup_{0\le s\le t}E[|L\{A^{(n)}(V_s^{(n)}(x)+x)\}-L\{A^{(n)}(V_s^{(m)}(x)+x)\}|^p]=0$$

and thus

$$\lim_{n\to\infty}\sup_{0\le s\le t}E[|L\{A^{(n)}(V_s^{(n)}(x)+x)\}-L\{A^{(n)}(V_s(x)+x)\}|^p]=0$$

by the same technique used in this proof. Let $X^{(n)}(\cdot)=A^{(n)}(V_\cdot^{(n)}(x)+x)$ and $Y^{(n)}(\cdot)=A^{(n)}(V_\cdot(x)+x)$. Then $X^{(n)}$, $Y^{(n)}\in L^p([0,t])$ and $X^{(n)}$, $Y^{(n)}$ are quasi-continuous and

$$\begin{aligned}&E[||LX^{(n)}-LY^{(n)}||_p^p]\\ &\quad=E[\int_0^t|LX^{(n)}(s)-LY^{(n)}(s)|^pds]\\ &\quad\le\sup_{0\le s\le t}t\cdot E[|LX^{(n)}(s)-LY^{(n)}(s)|^p]\to 0 \quad\text{as}\quad n\to\infty,\end{aligned}$$

where $||\cdot||_p$ is the L^p-norm in $L^p([0,t])$. Thus, by (2.2)

$$C_{2,p}(||X^{(n)}-Y^{(n)}||_p)\le E[||LX^{(n)}-LY^{(n)}||_p^p]\to 0 \quad\text{as}\quad n\to\infty. \tag{5.5}$$

Since $L\{A(V_s(x)+x)\}$ is the same type as $L\{A^{(n)}(V_s(x)+x)\}$ calculated above, we have

$$\lim_{n\to\infty}\sup_{0\le s\le t} E[|L\{A^{(n)}(V_s(x)+x)\}-L\{A(V_s(x)+x)\}|^p]=0.$$

Taking the quasi-continuous modification $\tilde{A}$ of A by

$$\tilde{A}(x)=\begin{cases}\lim_n A^{(n)}(x), & \text{if it converges,}\\ 0, & \text{otherwise,}\end{cases}$$

there exists a sub-sequence $A^{(n_i)}(V_s(x)+x)$ which converges to $A(V_s(x)+x)$, $(2,p)$-quasi-surely and the limit is quasi-continuous. To be more precise, the sub-sequence $\{n_i\}$ of $A^{(n_i)}(V_s(x)+x)$ is taken from the sub-sequence $\{n_j\}$ of $\{V_s^{(n_j)}(x)\}$ which converges to $V_s(x)$. Then $A^{(n_i)}(V_s(x)+x)$ converges to $A(V_s(x)+x)$, $(2,p)$-quasi-surely. Since

$$\begin{aligned}&\lim_{n\to\infty} E[\int_0^t |L\{A^{(n)}(V_s(x)+x)\}-L\{A(V_s(x)+x)\}|^p ds]\\ &\quad\le \lim_{n\to\infty}\sup_{0\le s\le t} t\cdot E[|L\{A^{(n)}(V_s(x)+x)\}-L\{A(V_s(x)+x)\}|^p]=0,\end{aligned}$$

we have

$$C_{2,p}(||Y^{(n)}-Y^{(m)}||_p)\to 0 \quad \text{as} \quad n,m\to\infty.$$

Thus, taking a sub-sequence if necessary, we have

$$\sum_n C_{2,p}(||Y^{(n)}-Y^{(n+1)}||_p\ge 2^{-(n+1)})<\infty.$$

By Borel-Cantelli's Lemma,

$$||Y^{(n)}-Y^{(n+1)}||_p<2^{-(n+1)}, \quad (2,p)-\text{q.e.}x.$$

Therefore for $(2,p)$-q.e.x, $Y^{(n)}(t)=A^{(n)}(V_t(x)+x)$ converges to $X(t):=A(V_t(x)+x)$, a.e.t and thus

$$C_{2,p}(||Y^{(n)}-X||_p)\to 0 \quad \text{as} \quad n\to\infty. \tag{5.6}$$

By (5.5) and (5.6),

$$C_{2,p}(||X^{(n)}-X||_p)\to 0 \quad \text{as} \quad n\to\infty.$$

So $X^{(n)}$ converges to X, $(2,p)$-quasi-surely in L^p taking a sub-sequence if necessary. Thus we have proved $\int_0^t A^{(n)}(V_s^{(n)}(x)+x)ds$ converges to $\int_0^t A(V_s(x)+x)ds$, $(2,p)$-quasi-surely. Hence $V_t(x)$ exists for all $t\in\mathbb{R}$, $(2,p)$-quasi-surely. Moreover, taking $n\to\infty$ in (4.2), we have

$$\begin{cases} (dV_t/dt)(x) = A(V_t(x)+x), \\ V_0(x) = 0. \end{cases}$$

We can rewrite the formula (3.4) as follows :

$$\frac{d}{dt}LV_t^{(n)} = \nabla A^{(n)} \cdot LV_t^{(n)} + F^1(LA^{(n)}, \nabla^2 A^{(n)}, \nabla V_t^{(n)})$$

for some polynomial F^1. Since

$$\begin{aligned} LF &= \sum_{j=1}^{m} \partial_j F \cdot Lu_j + \sum_{j,k}^{m} \partial_j\partial_k F \cdot (\sum_{i=1}^{n} \partial_i u_j \cdot \partial_i u_k) \\ &= \sum_{j=1}^{m} \partial_j F \cdot Lu_j + \sum_{j,k}^{m} \partial_j\partial_k F \cdot (\nabla u_j, \nabla u_k), \end{aligned}$$

for $F = F(u_1, \cdots, u_m)$, a polynomial and $u_j = u_j(x_1, \cdots, x_n), j = 1, \cdots, m$, we have

$$\begin{aligned} \frac{d}{dt}L^2V_t^{(n)} &= \nabla A^{(n)} \cdot L^2V_t^{(n)} \\ &\quad + F^2(L^2A^{(n)}, L(\nabla^2 A^{(n)}), L(\nabla A^{(n)}), LA^{(n)}, \\ &\qquad \nabla^4 A^{(n)}, \nabla^3 A^{(n)}, \nabla^2 A^{(n)}, \nabla A^{(n)}, \\ &\qquad L(\nabla V_t^{(n)}), LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}), \end{aligned} \tag{5.7}$$

for some polynomial F^2. In fact, since

$$\begin{aligned} \frac{d}{dt}L^2V_t^{(n)} &= L(\nabla A^{(n)} \cdot LV_t^{(n)}) + LF^1(LA^{(n)}, \nabla^2 A^{(n)}, \nabla V_t^{(n)}) \\ &= L(\nabla A^{(n)}) \cdot LV_t^{(n)} + \nabla A^{(n)} \cdot L^2V_t^{(n)} \\ &\quad + \sum_{j=1}^{3} \partial_j F^1 \cdot (Lu_j) + \sum_{j,k=1}^{3} \partial_j\partial_k F^1 \cdot (\sum_{i=1}^{n} \partial_i u_j \cdot \partial_i u_k) \\ &\quad (\text{where} \quad u_1 = LA^{(n)}, u_2 = \nabla^2 A^{(n)}, u_3 = \nabla V_t^{(n)}), \\ L(\nabla A^{(n)}) \cdot LV_t^{(n)} &= [\sum_i (\nabla^3 A^{(n)}(V_t^{(n)}(x)+x) \cdot (\partial_i V_t^{(n)}(x) + I_i)^2 \\ &\quad + \nabla^2 A^{(n)}(V_t^{(n)}(x)+x) \cdot (\partial_i\partial_i V_t^{(n)}(x))) \\ &\quad + \sum_i (\nabla^2 A^{(n)}(V_t^{(n)}(x)+x) \cdot (\partial_i V_t^{(n)}(x) + I_i)) \\ &\quad \times x_i] \cdot LV_t^{(n)}(x) \\ &\quad (\text{where} \quad I_i = (0, \cdots, 0, \overset{i}{\overset{\vee}{1}}, 0, \cdots, 0)) \end{aligned}$$

$$
\begin{aligned}
&= [\sum_i \nabla^3 A^{(n)}(V_t^{(n)}(x)+x)\cdot(\partial_i V_t^{(n)}(x)+I_i)^2 \\
&\quad + \nabla^2 A^{(n)}(V_t^{(n)}(x)+x)\cdot\nabla^2 V_t^{(n)}(x) \\
&\quad + \sum_i \nabla^2 A^{(n)}(V_t^{(n)}(x)+x)\cdot I_i\cdot x_i]\cdot LV_t^{(n)}(x) \\
&= F_1^2(L(\nabla A^{(n)}), \nabla^3 A^{(n)}, \nabla^2 A^{(n)}, LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}), \\
\partial_1 F^1\cdot(L(LA^{(n)})) &= F_2^2(L^2 A^{(n)}, \nabla^2(LA^{(n)}), \nabla(LA^{(n)}), LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}), \\
\partial_2 F^1\cdot(L(\nabla^2 A^{(n)})) &= F_3^2(L(\nabla^2 A^{(n)}), \nabla^4 A^{(n)}, \nabla^3 A^{(n)}, LV_t^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}), \\
\partial_3 F^1\cdot(L(\nabla V_t^{(n)})) &= F_4^2(L(\nabla V_t^{(n)})),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j,k=1}^{3}\partial_j\partial_k F^1\cdot(\nabla u_j,\nabla u_k) = F_5^2(&L(\nabla A^{(n)}), LA^{(n)}, \nabla^3 A^{(n)}, \\
&\nabla^2 A^{(n)}, \nabla^2 V_t^{(n)}, \nabla V_t^{(n)}),
\end{aligned}
$$

for some polynomials $F_1^2,\cdots,F_5^2$, the formula (5.7) holds by setting

$$
F^2 = \sum_{i=1}^{5} F_i^2.
$$

Since F^2 includes $L(\nabla V_t^{(n)})$ in its components, we should consider the following differential equation for $L(\nabla V_t^{(n)})$.

$$
\begin{aligned}
\frac{d}{dt}L(\nabla V_t^{(n)}(x)) &= L(\nabla A^{(n)}(V_t^{(n)}(x)+x))\cdot\nabla V_t^{(n)}(x) \\
&\quad + \nabla A^{(n)}(V_t^{(n)}(x)+x)\cdot L(\nabla V_t^{(n)}(x)) \\
&\quad + L(\nabla A^{(n)}(V_t^{(n)}(x)+x)) \\
&= \nabla A^{(n)}(V_t^{(n)}(x)+x)\cdot L(\nabla V_t^{(n)}(x)) \\
&\quad + E^{1,1}(L(\nabla A^{(n)}), \nabla^3 A^{(n)}, \nabla^2 A^{(n)}, \nabla V_t^{(n)}),
\end{aligned}
\tag{5.8}
$$

where $E^{1,1}$ is some polynomial which can be calculated by the same method as F^2. Under Assumption 2, since it is the same type as in the previous case $LV_t^{(n)}$, we can prove the almost sure existence of $L(\nabla V_t^{(n)}(x))$ satisfying the equation (5.8). And then for the formula (5.7), it can be also proved by the same reason. For the general case $L^m\nabla^n V_t^{(n)}(x)$ $(m=0,1,\cdots,k,\quad n=0,1,\cdots,2k,\quad 2m+n\le 2k,\quad k=2,3,\cdots)$, the differential equation can be calculated by

$$(5.9)\quad \frac{d}{dt}L^m\nabla^n V_t^{(n)} = \nabla A^{(n)}\cdot L^m\nabla^n V_t^{(n)} + E^{m,n}(L^l A^{(n)}, LA^{(n)}, L^i(\nabla^j A^{(n)}), \nabla^{2l}A^{(n)}, \nabla^2 A^{(n)}, \nabla A^{(n)}, L^{l-1}(V_t^{(n)}), \nabla^{2l-1}A^{(n)}, L^{i-1}(\nabla^{j-1}V_t^{(n)}), \nabla^{2l-2}V_t^{(n)}, \nabla^{2l-3}V_t^{(n)}) :$$

$$i = 1, \cdots, m-1, \quad j = 1, \cdots, 2n-2,$$

$$l = 2, \cdots, m, \quad 2i + j \le 2(m-1) + n),$$

for some polynomial $E^{m,n}$ which can be calculated successively. Since it is the same type as the case $LV_t^{(n)}$, we can prove the almost-sure existence of solutions satisfying the formula (5.9) and

$$E[\sup_{0\le s\le t}|L^k V_s^{(m)} - L^k V_s^{(n)}|^p] \longrightarrow 0 \quad \text{as} \quad m, n \to \infty.$$

Using the same technique to the case $LV_s^{(n)}$, we can prove the $(2k, p)$-quasi-sure existence of V_t. Since the above fact can be asserted for all $k = 1, 2, \cdots$ and $p \ge 1$, the proof is complete. □

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