Grassmann manifold of a JH-algebra

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Introduction

A JH-algebra is a real Jordan algebra as well as a real Hilbert space with a compatibility condition of two structures (associativity of the inner product — see Definition 1.1). This is a natural generalization of a finite-dimensional formally real Jordan algebra when one plans to investigate Riemannian geometric applications of infinite-dimensional Jordan algebras. In [**19**], we treated the Riemannian manifold of primitive idempotents in a JH-algebra. We proved the two-point homogeneity and obtained a sectional curvature formula. Continuing the way of this study, we deal with the idempotents of general rank — Grassmann manifold — in this paper.

Grassmann manifolds have been an object of a large variety of researches until now. For finite-dimensional Grassmann manifolds, we only cite here the papers whose approaches are directly related to ours : [**15**], [**27**], [**30**] for Riemannian geometry, and [**7**], [**8**] for the use of Jordan algebras. As regard infinite-dimensional Grassmann manifolds, we enumerate : [**21**, Chapter 7] in relation to loop groups, [**20**], [**24**] and [**29**] within the framework of C^*-algebras, and [**12**] from the viewpoint of symmetric Banach manifolds. Our standpoint, without assumption of finite dimensionality, is a Jordan theoretic one and thus motivated by the works [**7**], [**8**] quoted above. Furthermore, since the automorphism group $\mathrm{Aut}(V)$ of a JH-algebra V is merely a Banach-Lie group (not a Hilbert-Lie group in general),

its application to Riemannian geometry breaks down unlike the finite-dimensional case. Therefore, instead of working with Lie groups, we introduce a Riemannian manifold structure through a concrete atlas, which generalizes the one given in [**19**] for the case of primitive idempotents. The defining formulae (4.3) come from a Jordan theoretic interpretation on the works [**15**], [**27**], [**30**].

We now explain the contents of this paper section by section. In §1, we review basic facts concerning JH-algebras. Moreover, assuming the topological simplicity of the whole algebra, we prove the algebraic simplicity of the Peirce 1-spaces, which were stated only to be topologically simple in [**19**]. As a consequence of this, it turns out that the Peirce 1-spaces are of finite dimension for almost all idempotents (see Proposition 1.4).

In §2, we investigate subalgebras $V[a,b]$ generated by two idempotents a, b of the same rank in a topologically simple JH-algebra V. Here also, pairs of projections have repeatedly attracted mathematicians, see [**2**], [**17**], [**22**] as well as [**26**]. We refer to the Introduction of [**2**] for further details in this respect. In our case, we reveal first that $V[a,b]$ is finite-dimensional (Lemma 2.1). Hence it possesses the unit element e as a finite-dimensional formally real Jordan algebra. Then the spectral decomposition of $(a+b-e)^2$ leads us to the decomposition into a direct sum of ideals : $V[a,b] = \bigoplus V_j$ with $V_j \cong \mathrm{Sym}(2,\mathbb{R})$ except at most two of the V_j, where $\mathrm{Sym}(2,\mathbb{R})$ is the three dimensional JH-algebra of 2×2 real symmetric matrices. The excepted cases are $V_j \cong \mathbb{R}$ or $V_j \cong \mathbb{R}\bigoplus\mathbb{R}$ (cf. Theorem 2.6 and (2.9)). This is our JH-algebra version of the corresponding topics in C^* and von Neumann algebras described for example in [**22**], [**26**, V.1.41].

The purpose of §3 is two-fold. We first summarize fundamental properties of Hilbert Jordan triple system (Hilbert JTS for short). It is a real JTS and a real

Hilbert space satisfying a compatibility condition related to the inner product (see Definition 3.1). Secondly, we study the subalgebra $V[a,x]$ generated by an idempotent a and an element x in the Peirce 1/2-space $V_{1/2}(a)$. The observation that $V_{1/2}(a)$ forms a Hilbert JTS makes the structure of $V[a,x]$ clearer. In fact, through the spectral decomposition of x within the Hilbert JTS $V_{1/2}(a)$, which we will rename Schmidt expansion in §3 to make a distinction from the spectral decomposition in the JH-algebra V, we see that $V[a,x]$ is finite-dimensional (Proposition 3.5) and that it has the following decomposition into simple ideals : $V[a,x] = \oplus W_j$ with $W_j \cong \mathrm{Sym}(2,\mathbb{R})$, except for at most one of the W_j (Theorem 3.6).

Section 4 is devoted to constructing an atlas for the set $\mathfrak{I}_p$ of the idempotents of rank p. There, our structure theorems (Theorems 2.6 and 3.6) furnish the computation with visibility.

In §5, we introduce the canonical Levi-Civita connection relative to the embedding $\mathfrak{I}_p \hookrightarrow V$, and exhibit geodesics which are orbits of one-parameter subgroups of $\mathrm{Aut}(V)$. We will give these geodesics two types of expressions. One is operator-theoretic (Theorem 5.4), thus independent of the structure theorem, and another is based on Theorem 3.6 (Corollary 5.5). The diffeomorphism domain of the exponential mapping at $a \in \mathfrak{I}_p$ is the open ball of radius $\pi/2$ with respect to the spectral norm $|\cdot|_{1/2}$ of the Hilbert JTS $V_{1/2}(a)$ (see Theorem 5.7). Finally, we calculate the Riemannian distance 'dist' on $\mathfrak{I}_p$ and get $\mathrm{dist}(a,b) = \|\mathrm{Arcsin}(a-b)\|$ (Theorem 5.9), where Arcsin is defined through the Schmidt expansion on a domain in V regarded as a Hilbert JTS.

The last section, §6, concerns the sectional curvature $k_a(x,y)$ of $\mathfrak{I}_p$. In Theorem 6.1, we express $k_a(x,y)$ by using operators affiliated to the JTS structure of $V_{1/2}(a)$ and derive the estimate $0 \leqq k_a(x,y) \leqq 4\,\min(|x|_{1/2}^2, |y|_{1/2}^2) \leqq 2$. We also examine

the occurrence of $k_a(x,y) = 0$. Theorem 6.3 shows that this happens precisely when x, y are contained in a flat subspace, thus we obtain an infinite-dimensional generalization of [**31**, Theorem 4] with a Jordan theoretic interpretation.

§1. Preliminaries

A real *Jordan algebra* is an algebra over $\mathbb{R}$ in which one has the identities

$$xy = yx, \qquad x^2(xy) = x(x^2y). \tag{1.1}$$

The books [**1**], [**6**], [**10**] and [**28**] are our references for Jordan algebras.

Definition 1.1. Let V be a real Jordan algebra as well as a real Hilbert space with inner product $\langle \cdot \,|\, \cdot \rangle$. Then V is called a *JH-algebra* if the inner product is associative :

$$\langle\, xy \,|\, z \,\rangle = \langle\, y \,|\, xz \,\rangle \qquad \text{for all } x, y, z \in V. \tag{1.2}$$

Let V be a *JH*-algebra. For every $x \in V$, we denote by $L(x)$ the multiplication operator $y \mapsto xy$ on V. The associativity (1.2) of the inner product means that every $L(x)$ is a symmetric operator, and the closed graph theorem implies that each $L(x)$ is a bounded operator. Moreover, the Jordan product $V \times V \ni (x,y) \mapsto xy \in V$ is continuous by [**19**, Lemma 1.1]. Besides $L(x)$, we also use the operators

$$P(x) := 2L(x)^2 - L(x^2). \tag{1.3}$$

It is known that $P(x^n) = P(x)^n$ for $n = 1, 2, \ldots$.

We will assume throughout this paper that $\dim V > 0$ and that $L(x) = 0$ implies $x = 0$. Under this assumption, we know that V contains non-zero idempotents, cf. [**19**, Proposition 1.3]. Every idempotent $a \in V$ gives rise to the *Peirce decomposition* of V :

$$V = V_0(a) \oplus V_{1/2}(a) \oplus V_1(a) \qquad \text{(orthogonal direct sum)} \qquad \text{with}$$
$$V_k(a) := \{x \in V \,;\, ax = kx\} \qquad (k = 0,\, 1/2,\, 1). \tag{1.4}$$

The following multiplication rule holds:

$$\begin{cases} V_0V_0 \subset V_0, & V_0V_1 = \{0\}, & V_1V_1 \subset V_1, \\ V_kV_{1/2} \subset V_{1/2} \quad (k = 0, 1), & & V_{1/2}V_{1/2} \subset V_0 + V_1, \end{cases} \tag{1.5}$$

where V_k stands for $V_k(a)$. In particular, both $V_0(a)$ and $V_1(a)$ are subalgebras. Furthermore, setting $[\,S, T\,] := ST - TS$ for two operators S, T, we have

$$[\,L(u), L(v)\,] = 0 \qquad \text{if } u \in V_0(a) \text{ and } v \in V_1(a). \tag{1.6}$$

On the other hand, putting

$$\pi_a(x) = 2L(x)\Big|_{V_{1/2}(a)} \qquad \text{for } x \in V_1(a), \tag{1.7}$$

we know that π_a is a unital Jordan algebra representation of $V_1(a)$ by symmetric operators on $V_{1/2}(a)$: we have

$$2\pi_a(xy) = \pi_a(x)\pi_a(y) + \pi_a(y)\pi_a(x) \tag{1.8}$$

and $\pi_a(a) = I$. Let $E_k(a)$ $(k = 0,\, 1/2,\, 1)$ denote the orthogonal projections

$V \to V_k(a)$. They are expressed respectively as

$$
\begin{aligned}
E_1(a) &= P(a), \\
E_{1/2}(a) &= 4L(a) - 4L(a)^2 = 2L(a) - 2P(a), \\
E_0(a) &= I - 2L(a) + P(a).
\end{aligned} \tag{1.9}
$$

Now two idempotents a, b are said to be *orthogonal* if $ab = 0$. This is equivalent to $\langle\, a \,|\, b \,\rangle = 0$ by [**19**, Lemma 1.4]. Let c be a non-zero idempotent. The rank of the subalgebra $V_1(c)$, in which c is clearly the unit element, is called the *rank* of the idempotent c (the rank of 0 is defined to be 0). We recall that c is written as a finite sum $c = c_1 + \cdots + c_p$ of orthogonal primitive idempotents $\{c_j\}_{j=1}^{p}$ (cf. [**19**, §1]). Then p equals the rank of c. In particular, the primitive idempotents are those of rank one. We will denote by $\mathfrak{I}_p$ ($p = 0, 1, \dots$) the set of idempotents of rank p.

We assume from now on that V is *topologically simple*, that is, the only closed ideals of V are $\{0\}$ and V itself. Then all the primitive idempotents have the same norm [**19**, Proposition 2.2]. Hence we normalize the inner product of V in such a way that $\|c\| = 1$ for all $c \in \mathfrak{I}_1$. Thus, an idempotent c belongs to $\mathfrak{I}_p$ if and only if $\|c\| = p^{1/2}$. Let $c \in V$ be a non-zero idempotent. We know that $V_1(c)$ is topologically simple due to McCrimmon's theorem, cf. [**19**, Proposition 1.6]. However, we shall show in Proposition 1.2 a stronger fact that $V_1(c)$ is *algebraically simple*, the fact that the only ideals of $V_1(c)$ are $\{0\}$ and V itself. Although the proof given below is just an adaptation of a part of the proof of the first structure theorem in [**10**, p. 161] to the JH-algebra context, we write it down here for the readability of this paper.

Before proceeding we need some preparations. Let $c = c_1 + \cdots + c_p$ be a decomposition of c as a sum of orthogonal primitive idempotents. Putting $W = V_1(c)$ for brevity, we let

$$W = \bigoplus_{1 \leqq i \leqq j \leqq p} W_{ij} \qquad \text{with} \qquad \begin{cases} W_{ii} = W_1(c_i) = \mathbb{R}c_i, \\ W_{ij} = W_{1/2}(c_i) \cap W_{1/2}(c_j) \quad (i < j) \end{cases}$$

be the *Peirce decomposition* of W relative to the complete orthogonal system of primitive idempotents $\{c_j\}_{j=1}^p$. The orthogonal projections $E_{ij} : W \to W_{ij}$ are given by

$$E_{ii} = P(c_i), \qquad E_{ij} = 4L(c_i)L(c_j) \quad (i < j), \tag{1.10}$$

where it should be noted that the operators $L(c_i)$ commute with each other by virtue of (1.6). Since $W = V_1(c)$ is topologically simple, we know from [**19**, Lemma 2.1] that $W_{ij} \neq \{0\}$ for $i < j$.

Lemma 1.2. *If $i < j$, then the subalgebra $W^{ij} := W_{ii} + W_{ij} + W_{jj}$ is algebraically simple.*

Proof. Let I be an ideal of W^{ij} and put

$$I_{ii} := I \cap W_{ii}, \qquad I_{ij} := I \cap W_{ij}, \qquad I_{jj} := I \cap W_{jj}.$$

Since I is an ideal, explicit expressions (1.10) of orthogonal projections E_{ij} show that $I = I_{ii} + I_{ij} + I_{jj}$. Since $W_{ii} = \mathbb{R}c_i$, we have either $I_{ii} = \mathbb{R}c_i$ or $I_{ii} = \{0\}$. Similarly, either $I_{jj} = \mathbb{R}c_j$ or $I_{jj} = \{0\}$. Suppose first $I_{ii} = \mathbb{R}c_i$. Then $W_{ij} = L(c_i)W_{ij} \subset I$, so that $I_{ij} = W_{ij} \neq \{0\}$. Take $x \in I_{ij}$ such that $\|x\|^2 = 2$. Then

[**19**, Lemma 2.4] gives $c_i + c_j = \|x\|^2(c_i + c_j)/2 = x^2 \in I$. Since $c_i + c_j$ is the unit element of W^{ij}, we get $I = W^{ij}$. In the same way, the conclusion $I = W^{ij}$ follows from the assumption $I_{jj} = \mathbb{R}c_j$. Thus it only remains to treat the case $I_{ii} = \{0\} = I_{jj}$. Let $y \in I_{ij}$. Then, $y^2 \in I \cap (W_{ii} + W_{jj}) = I_{ii} + I_{jj} = \{0\}$. Hence $\|y\|^2 = 2\langle c_i y \,|\, y \rangle = 2\langle c_i \,|\, y^2 \rangle = 0$, so that we get $I = \{0\}$. ∎

Proposition 1.3. *Let $c \in V$ be a non-zero idempotent. Then $W := V_1(c)$ is algebraically simple.*

Proof. We retain the notation in Lemma 1.2. Suppose J is an ideal of W. Then the formulae in (1.10) give the decomposition $J = \bigoplus_{i \leqq j} J_{ij}$ with $J_{ij} := J \cap W_{ij}$. Since $J_{ii} + J_{ij} + J_{jj}$ is an ideal of W^{ij}, the algebraic simplicity of W^{ij} implies that for any $i < j$

$$J_{ii} + J_{ij} + J_{jj} = \{0\} \quad \text{or} \quad W^{ij}. \tag{1.11}$$

In particular, either $J_{ii} = \{0\}$ for all i, or $J_{ii} = \mathbb{R}c_i$ for all i. By (1.11), the former case gives $J_{ij} = \{0\}$ for all $i < j$, so that $J = \{0\}$, whereas the latter case $J_{ij} = W_{ij}$, so that $J = W$. ∎

To prove the next proposition, we need the *JH*-algebras $\mathcal{S}(e, \mathfrak{H})$ called *spin factors*, where $\mathfrak{H}$ is a real Hilbert space with inner product $\langle u \,|\, v \rangle_{\mathfrak{H}}$, e a symbol, and the underlying vector space is $\mathbb{R}e \oplus \mathfrak{H}$. The Jordan and the inner product are defined respectively by

$$(\alpha e + u)(\beta e + v) := \left[\alpha\beta + \langle u \,|\, v \rangle_{\mathfrak{H}}\right] e + \alpha v + \beta u, \tag{1.12}$$

$$\langle \alpha e + u \,|\, \beta e + v \rangle := 2\left(\alpha\beta + \langle u \,|\, v \rangle_{\mathfrak{H}}\right), \tag{1.13}$$

where $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathfrak{H}$. We note that the coefficient 2 appears on the right hand side of (1.13) to make the norm of the primitive idempotents equal to 1.

Proposition 1.4. *Let $c \in V$ be an idempotent such that $V_0(c) \neq \{0\}$. Then $\dim V_1(c)$ is finite.*

Proof. Suppose contrarily that $V_1(c)$ is of infinite dimension. Then $c \neq 0$. Take a non-zero idempotent $d \in V_0(c)$ and consider the idempotent $c + d$. By Proposition 1.3, the subalgebra $V_1(c+d)$ is algebraically simple. Moreover, we know that $V_1(c+d)$ is *reduced* in the sense of [**10**, p. 197]. Since $\dim V_1(c+d) \geqq \dim V_1(c) = \infty$, we see easily, with the aid of Corollary 1 in [**10**, p. 203], that the unital JH-algebra $V_1(c+d)$ is a spin factor $\mathcal{S}(e, \mathfrak{H})$ with $e = c + d$ and $\mathfrak{H}$ some real Hilbert space. Because the rank of $\mathcal{S}(e, \mathfrak{H})$ is two, the idempotent c must be of rank one. In other words, we get $c \in \mathfrak{I}_1$, so that $V_1(c) = \mathbb{R}c$, contradicting the assumption $\dim V_1(c) = \infty$. ∎

§2. Subalgebras generated by two idempotents

From now on, V will denote a topologically simple JH-algebra. For $u, v \in V$, we shall denote by $V[u, v]$ the subalgebra generated by u, v. Let $\mathfrak{I}_p$ $(p = 0, 1, \dots)$ be the set of idempotents of rank p in V. In this section, we study the subalgebra $V[a, b]$ generated by two idempotents a, b in $\mathfrak{I}_p$. We will assume without loss of generality that

$$p \text{ is finite with } 1 \leqq p \leqq r - p \qquad (r := \operatorname{rank}(V) \leqq \infty), \tag{2.1}$$

because if $p > r - p$, then $r < \infty$ and we consider $\mathfrak{I}_{r-p}$ instead.

Now by Shirshov-Cohn Theorem [**6**, 2.1.14] or [**10**, Theorem 10, p. 48], the subalgebra $V[a,b]$ is *special*. In other words, there is an associative algebra $\mathfrak{A}$ such that $V[a,b]$ is Jordan isomorphic to a subalgebra $\mathfrak{W}$ of $\mathfrak{A}^{\mathrm{J}}$, where $\mathfrak{A}^{\mathrm{J}}$ is the Jordan algebra obtained by defining a Jordan product $\circ$ in $\mathfrak{A}$ through $x \circ y = (xy + yx)/2$. In $\mathfrak{A}^{\mathrm{J}}$, the operators $P(x)$ in (1.3) are given by

$$P(x)y = 2x \circ (x \circ y) - x^2 \circ y = xyx. \tag{2.2}$$

In what follows, we fix the Jordan isomorphism $\varphi : V[a,b] \to \mathfrak{W} \subset \mathfrak{A}^{\mathrm{J}}$ mentioned above and we denote the image of an element of $V[a,b]$ under φ by the corresponding capital letter. So, we write $A = \varphi(a)$ for example.

Lemma 2.1. *Let $a, b \in \mathfrak{I}_p$ with p as in* (2.1). *Then* $\dim V[a,b] < \infty$.

Proof. Put $W = V[a,b]$ for simplicity. If a happens to be the unit element of W, then $W = \mathbb{R}a = \mathbb{R}b$, because b is of the same rank as a. Thus we may assume that neither a nor b is the unit element of W. Under this assumption, we have $W_0(a) \neq \{0\}$ and $W_0(b) \neq \{0\}$. In fact, suppose $W_0(a) = \{0\}$. If $y \in W_{1/2}(a)$, then (1.5) says $y^2 \in W_0(a) + W_1(a) = W_1(a)$, so that we have $y^3 = y(y^2a) = y^2(ya) = y^3/2$ by virtue of (1.1). Hence $y^3 = 0$, which in turn yields $\|y^2\|^2 = \langle\, y^3 \,|\, y \,\rangle = 0$. Thus, we get $\|y\|^2 = 2\,\langle\, ay \,|\, y \,\rangle = 2\,\langle\, a \,|\, y^2 \,\rangle = 0$, implying $W_{1/2}(a) = \{0\}$. Consequently, we arrive at a contradiction $W = W_1(a)$.

Let $\varphi : W \to \mathfrak{W} \subset \mathfrak{A}^{\mathrm{J}}$ be the Jordan isomorphism we are fixing. We shall prove $\dim \mathfrak{W} < \infty$. Put $A = \varphi(a)$, $B = \varphi(b)$. Then $\mathfrak{W}$ is generated by A and B. Since $A^2 = A$ and $B^2 = B$ in $\mathfrak{A}$, every element of $\mathfrak{W}$ is a linear combination of A, B, $(AB)^n$, $(BA)^n$, $(AB)^nA$ and $(BA)^nB$, where n runs through positive integers. Now, an easy induction shows $(AB)^nA = (ABA)^n$ and $(BA)^nB =$

$(BAB)^n$ $(n = 1, 2, \dots)$. Since (1.9) and (2.2) imply that ABA (resp. BAB) belongs to the Peirce 1-space $\mathfrak{W}_1(A)$ (resp. $\mathfrak{W}_1(B)$), we get $(AB)^n A \in \mathfrak{W}_1(A)$ (resp. $(BA)^n B \in \mathfrak{W}_1(B)$). Moreover, if $n \geqq 2$, we have

$$(AB)^n = (AB)^{n-1}AB = (ABA)^{n-1}B \in \mathfrak{W}_1(A)B,$$
$$(BA)^n = BA(BA)^{n-1} = B(ABA)^{n-1} \in B\mathfrak{W}_1(A).$$

Since $A \in \mathfrak{W}_1(A)$ and $B \in \mathfrak{W}_1(B)$, it is now clear that

$$\mathfrak{W} \subset \mathfrak{W}_1(A) + \mathfrak{W}_1(B) + \mathfrak{W}_1(A)B + B\mathfrak{W}_1(A).$$

Hence we arrive at $\dim \mathfrak{W} < \infty$ by virtue of Proposition 1.4. ∎

Let $a, b \in \mathfrak{I}_p$ with p satisfying (2.1). Then $\dim V[a, b] < \infty$ by Lemma 2.1. For every $x \in V[a, b]$, we denote by $L_0(x)$ the Jordan multiplication operator $y \mapsto xy$ on $V[a, b]$. Then the bilinear form $x, y \mapsto \operatorname{tr} L_0(xy)$ defines an associative inner product on $V[a, b]$, so that $V[a, b]$ has a unit element e as is well-known. Let

$$(a + b - e)^2 = \sum_{j=1}^{\sigma} \lambda_j e_j$$

be the spectral decomposition of $(a + b - e)^2 \in V[a, b]$, where λ_j are *distinct* real numbers and $\{e_j\}$ is a complete system of non-zero orthogonal idempotents. Let $V[a, b] = \bigoplus_{i \leqq j} W_{ij}$ be the Peirce decomposition of $V[a, b]$ relative to $\{e_j\}$. The corresponding decompositions of a, b are denoted as $a = \sum a_{ij}$, $b = \sum b_{ij}$ respectively.

Lemma 2.2. *If $i < j$, then $W_{ij} = \{0\}$.*

Proof. It suffices to show $a_{ij} = b_{ij} = 0$ for $i < j$. Let us put $f = (a + b - e)^2$ for

brevity. Since $f^2 = \sum \lambda_j^2 e_j$, we get

$$af^2 = \sum_j \lambda_j^2 a_{jj} + \sum_{i<j} (\lambda_i^2 + \lambda_j^2) a_{ij}/2.$$

Similarly, we have $af = \sum\limits_j \lambda_j a_{jj} + \sum\limits_{i<j} (\lambda_i + \lambda_j) a_{ij}/2$, so that

$$f(af) = \sum_j \lambda_j^2 a_{jj} + \sum_{i<j} (\lambda_i + \lambda_j)^2 a_{ij}/4.$$

We assume for a while that $af^2 = f(af)$. Then,

$$\sum_{i<j} (\lambda_i - \lambda_j)^2 a_{ij} = 4 \left[af^2 - f(af) \right] = 0.$$

Hence $a_{ij} = 0$ if $i < j$. In the same way, we obtain $b_{ij} = 0$ for $i < j$.

Now it only remains to prove $af^2 = f(af)$. This is done by transferring both sides to $\mathfrak{W}$ under the Jordan isomorphism φ. In $\mathfrak{W} \subset \mathfrak{A}$, we have $F = -A - B + E + AB + BA$. Hence

$$AF = ABA = FA, \tag{2.3}$$

so that $A \circ F = AF = FA$. Thus, $A \circ F^2 = AF^2 = F \circ (A \circ F)$. ∎

For simplicity, we set $V_j := W_{jj}$, $a_j := a_{jj}$ and $b_j := b_{jj}$ respectively. Note that a_j, b_j are also idempotents. We have

$$V[a, b] = \bigoplus_{j=1}^{\sigma} V_j \qquad \text{with } V_j = V[a_j, b_j], \tag{2.4}$$

$$f_j := (a_j + b_j - e_j)^2 = \lambda_j e_j \qquad (j = 1, 2, \ldots, \sigma). \tag{2.5}$$

Now, transferring the decomposition (2.4) to $\mathfrak{W}$ under φ, we get a decomposition $\mathfrak{W} = \overset{\sigma}{\underset{j=1}{\oplus}} \mathfrak{V}_j$. Then the equality (2.5) is transferred to $F_j = \lambda_j E_j$ in $\mathfrak{V}_j$. Since (2.3) gives $A_j F_j = A_j B_j A_j$, we get $A_j B_j A_j = \lambda_j A_j$. Similarly, $B_j A_j B_j = \lambda_j B_j$. Returning to V_j, we obtain

$$P(a_j)b_j = \lambda_j a_j, \qquad P(b_j)a_j = \lambda_j b_j. \tag{2.6}$$

Lemma 2.3. *If $\lambda_j = 0$, then $a_j b_j = 0$.*

Proof. If $\lambda_j = 0$, then $\langle\, a_j \,|\, b_j \,\rangle = \langle\, a_j \,|\, P(a_j)b_j \,\rangle = 0$ by (2.6). This implies $a_j b_j = 0$ owing to [**19**, Lemma 1.4]. ∎

Proposition 2.4. *Suppose $\lambda_j \neq 0$. Then,*

(1) $\|a_j\| = \|b_j\|$, *so that the idempotents a_j and b_j are of the same rank.*

(2) $\lambda_j = \dfrac{\langle\, a_j \,|\, b_j \,\rangle}{\|a_j\|\|b_j\|}$.

(3) $\|a_j b_j\|^2 = \lambda_j(\lambda_j + 1)\|a_j\|\|b_j\|/2$.

(4) (i) $0 < \lambda_j \leqq 1$, (ii) $\lambda_j = 1$ *if and only if* $a_j = b_j$.

Proof. (1) By (2.6), we have

$$\begin{aligned} \lambda_j \|a_j\|^2 &= \langle\, \lambda_j a_j \,|\, a_j \,\rangle = \langle\, P(a_j)b_j \,|\, a_j \,\rangle = \langle\, b_j \,|\, a_j \,\rangle \\ &= \langle\, b_j \,|\, P(b_j)a_j \,\rangle = \lambda_j \|b_j\|^2. \end{aligned} \tag{2.7}$$

Since $\lambda_j \neq 0$, we get $\|a_j\| = \|b_j\|$. The assertion (2) also follows from (2.7).

(3) We have

$$\begin{aligned} 2\,\|a_j b_j\|^2 &= 2\,\langle\, L(a_j)^2 b_j \,|\, b_j \,\rangle = \langle\, P(a_j)b_j \,|\, b_j \,\rangle + \langle\, L(a_j)b_j \,|\, b_j \,\rangle \\ &= \lambda_j \langle\, a_j \,|\, b_j \,\rangle + \langle\, a_j \,|\, b_j \,\rangle = \lambda_j(\lambda_j + 1)\|a_j\|\|b_j\|. \end{aligned}$$

(4) (i) This is a consequence of (2) and (3).

(ii) If $\lambda_j = 1$, then a_j and b_j are linearly dependent by (2). Since both a_j and b_j are non-zero idempotents, we get $a_j = b_j$. ∎

Let $\mathrm{Sym}(2,\mathbb{R})$ denote the three dimensional JH-algebra of 2×2 real symmetric matrices with inner product $\langle Y \mid Z \rangle := \mathrm{tr}(YZ)$. The following matrices A, $B(\theta)$ $(\theta \in \mathbb{R})$ and X in $\mathrm{Sym}(2,\mathbb{R})$ will be frequently used in the sequel :

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B(\theta) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

Proposition 2.5. *Suppose $\lambda_j \neq 0$ and $a_j \neq b_j$. Then there is a Jordan isomorphism $\Theta_j : V_j \to \mathrm{Sym}(2,\mathbb{R})$ such that $\Theta_j(a_j) = A$, $\Theta_j(b_j) = B(\theta_j)$, where θ_j $(0 < \theta_j < \pi/2)$ is determined by $\cos^2\theta_j = \lambda_j$. Moreover, if p_j denotes the common rank of the idempotents a_j, b_j, then $p_j^{1/2}\Theta_j$ is an isometry.*

We omit the proof, because it is completely parallel to that of [**19**, Proposition 2.6]. We only note that $\|a_j\| = \|b_j\| = p_j^{1/2}$ in the present case. Now we summarize the discussion above in Theorem 2.6. Notice that in view of Proposition 2.4, the rank of a_j is equal to that of b_j even if $\lambda_j = 0$, because a, b are of the same rank at the start.

Theorem 2.6. *Put $\mu_j := \langle a_j \mid b_j \rangle$. Then in the decomposition $V[a,b] = \overset{\sigma}{\underset{j=1}{\bigoplus}} V_j$ of (2.4), one has the Jordan isomorphism $V_j \cong \mathrm{Sym}(2,\mathbb{R})$, $\mathbb{R}$, or $\mathbb{R} \bigoplus \mathbb{R}$ according to the cases $0 < \mu_j < \|a_j\|\|b_j\|$, $\mu_j = \|a_j\|\|b_j\| \neq 0$, or $\mu_j = 0$.*

Based on Theorem 2.6, we shall rewrite the decomposition (2.4) as

$$V[a,b] = V_* \bigoplus \Big[\overset{\tau}{\underset{j=0}{\bigoplus}} V_j \Big], \tag{2.9}$$

where we have the Jordan isomorphisms $V_* \cong \mathbb{R}$, $V_0 \cong \mathbb{R} \bigoplus \mathbb{R}$ and $V_j \cong \mathrm{Sym}(2, \mathbb{R})$ for $1 \leqq j \leqq \tau$ respectively. Of course, we do not exclude the possibility such that V_0 or V_* is missing. Accordingly, we write

$$a = a_* + \sum_{j=0}^{\tau} a_j, \qquad b = b_* + \sum_{j=0}^{\tau} b_j, \tag{2.10}$$

where we have $a_* = b_*$ and $a_0 b_0 = 0$. Note that a_0 and b_0 are also of the same rank. We denote by $V_1(a)^\times$ the set of invertible elements of the unital Jordan algebra $V_1(a)$.

Theorem 2.7. *With the notation in* (2.10), *there are* $d_j \in V_{1/2}(a_j)$ *satisfying* $d_j^3 = d_j$ $(1 \leqq j \leqq \tau)$ *such that*

$$b = a_* + b_0 + \sum_{j=1}^{\tau} \left[(\cos 2\theta_j) a_j + (\sin 2\theta_j) d_j/2 + (1 - \cos 2\theta_j) d_j^2/2 \right], \tag{2.11}$$

where θ_j $(0 < \theta_j < \pi/2)$ *are all distinct and determined by* $\cos^2 \theta_j = \dfrac{\langle\, a_j \,|\, b_j \,\rangle}{\|a_j\| \, \|b_j\|}$. *Moreover, if* $P(a)b \in V_1(a)^\times$, *then* a_0 *and* b_0 *are absent.*

Proof. For $1 \leqq j \leqq \tau$, let Θ_j be the Jordan isomorphism described in Proposition 2.5. Let $A, B(\theta), X$ be as in (2.8) and put $d_j = \Theta_j^{-1}(X) \in V_{1/2}(a_j)$. Since

$$B(\theta_j) = (\cos 2\theta_j) A + (\sin 2\theta_j) X/2 + (1 - \cos 2\theta_j) X^2/2,$$

we have $b_j = (\cos 2\theta_j) a_j + (\sin 2\theta_j) d_j/2 + (1 - \cos 2\theta_j) d_j^2/2$ for $1 \leqq j \leqq \tau$. The property $d_j^3 = d_j$ comes from $X^3 = X$. The final assertion is clear. ∎

Proposition 2.8. *Let* $a, b \in \mathfrak{I}_p$ *with* p *as in* (2.1). *Suppose* $ab = 0$, *then there is* $d \in V_{1/2}(a) \cap V_{1/2}(b)$ *with* $\|d\|^2 = 2p$ *such that* $\mathbb{R}a \bigoplus \mathbb{R}d \bigoplus \mathbb{R}b \cong \mathrm{Sym}(2, \mathbb{R})$.

Proof. Let $a = \sum_{j=1}^{p} a_j$, $b = \sum_{j=1}^{p} b_j$ be decompositions of a, b into sums of orthogonal primitive idempotents in $V_1(a)$, $V_1(b)$ respectively. Taking $y_j \in V_{1/2}(a_j) \cap V_{1/2}(b_j)$ with $\|y_j\|^2 = 2$, we have $y_j^2 = a_j + b_j$ by [**19**, Lemma 2.4]. Put $d = \sum_{j=1}^{p} y_j$. Since $y_j \in V_1(a_j+b_j)$ and since $y_i \in V_0(a_j+b_j)$ for $i \neq j$, we get $d^2 = \sum\ y_j^2 = a+b$. The proposition follows easily from this. ∎

§3. Certain subalgebras generated by two elements

Keeping the notation of §2, we let $a \in \mathfrak{I}_p$ $(p > 0)$ and $x \in V_{1/2}(a)$. In this section, we investigate the subalgebra $V[a, x]$ generated by a, x. To do so, we need the notion of Hilbert Jordan triple system (Hilbert JTS for short) and some of its fundamentals. Our references for JTS are the books [**4**, Chapter 9], [**16**], [**18**], [**23**], [**25**], [**28**] and the papers [**13**].

Definition 3.1. Let Z be a real Hilbert space with inner product $\langle \cdot \,|\, \cdot \rangle$. Let $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ be a trilinear mapping which makes Z a Jordan triple system : one has for all a, b, x, y, $z \in Z$,

$$\{x, y, z\} = \{z, y, x\}, \tag{1}$$

$$\{a, b, \{x, y, z\}\} - \{x, y, \{a, b, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\}. \tag{2}$$

Then Z is called a *Hilbert JTS* if the inner product satisfies

$$\langle\, \{x, y, z\} \,|\, w \,\rangle = \langle\, z \,|\, \{y, x, w\} \,\rangle \qquad \text{for all } x, y, z, w \in Z. \tag{3.1}$$

Let Z be a Hilbert JTS. We shall denote by $x \,\square\, y$ $(x, y \in Z)$ the linear operator $z \mapsto \{x, y, z\}$ on Z. Then (3.1) implies that the linear operator $x \,\square\, y$ has the adjoint $y \,\square\, x$. In particular, $x \,\square\, y$ is a closed operator, being the adjoint of $y \,\square\, x$. Hence

$x \,\square\, y$ is a bounded operator by the closed graph theorem. Moreover, we have the following

Lemma 3.2. *The Jordan triple product* $Z \times Z \times Z \ni (x, y, z) \mapsto \{x, y, z\} \in Z$ *is continuous.*

Proof. Let $\boldsymbol{B}(Z)$ denote the set of all bounded linear operators on Z. For every $y \in Z$, we consider the linear operator $M_y : Z \ni x \mapsto x \,\square\, y \in \boldsymbol{B}(Z)$. Using $(x \,\square\, y)z = (z \,\square\, y)x$, we see easily that the operator M_y is closable. Hence it is bounded by the closed graph theorem. Now if $\|y\| \leqq 1$, we have

$$\|M_y(x)\| = \|x \,\square\, y\| = \|y \,\square\, x\| = \|M_x(y)\| \leqq \|M_x\|,$$

for any $x \in Z$. This implies that the family of operators $\mathcal{M} := \{M_y \ ; \ \|y\| \leqq 1\}$ is pointwise bounded. The Banach-Steinhaus theorem then says that $\mathcal{M}$ is uniformly bounded. Since $y \mapsto M_y$ is clearly linear, there is $C > 0$ such that $\|M_y\| \leqq C\|y\|$ for all $y \in Z$. Therefore we get

$$\|\{x, y, z\}\| \leqq \|M_y(x)\|\|z\| \leqq \|M_y\|\|x\|\|z\| \leqq C\|x\|\|y\|\|z\|. \quad (3.2) \qquad \blacksquare$$

An element $d \in Z$ is called a *tripotent* if $\{d, d, d\} = d$. Two tripotents $d_1, d_2 \in Z$ are said to be *orthogonal* if $d_1 \,\square\, d_2 = 0$. If $d_1, \ d_2 \in Z$ are orthogonal tripotents, then we have $\langle\, d_1 \,|\, d_2 \,\rangle = \langle\, (d_2 \,\square\, d_2)d_1 \,|\, d_2 \,\rangle = \langle\, (d_1 \,\square\, d_2)d_2 \,|\, d_2 \,\rangle = 0$. But it should be remarked that the opposite implication $\langle\, d_1 \,|\, d_2 \,\rangle = 0 \implies d_1 \,\square\, d_2 = 0$ is not true in general unlike idempotents in a JH-algebra.

We now assume that Z is non-trivial : $Z \,\square\, Z \neq \{0\}$. Under this assumption, one knows that there are non-zero tripotents in Z, cf. [**13**, II, 3.3], [**18**, IV.2.4] or

[**3**, 1.8]. Moreover, every element $x \in Z$ has the expansion

$$x = \sum \xi_j d_j, \tag{3.3}$$

where ξ_j are distinct *positive* numbers and $\{d_j\}$ is a system of non-zero orthogonal tripotents. Here if the series is infinite, it converges relative to the norm of Z. In the literature, the expansion (3.3) is called the spectral decomposition of x, cf. [**16**, 3.12], [**18**, IV.2.9] and [**4**, p. 161]. In this paper, however, the term 'spectral decomposition' is reserved for JH-algebras. So we shall call (3.3) the *Schmidt expansion* of x after [**5**, p. 28], where this term is utilized for an expansion of compact operators from a Hilbert space to another. Though other terms such as singular value decomposition are also used in the literature, see [**9**, p. 415] for example, we prefer the former one here.

Now by the continuity of the bilinear mapping $x, y \mapsto x \,\square\, y$ (see (3.2)), we have $x \,\square\, x = \sum \xi_j^2 d_j \,\square\, d_j$. Since the spectrum of the operator $d_j \,\square\, d_j$ is a subset of $\{\, 0,\ 1/2,\ 1 \,\}$ for every j, the operator $x \,\square\, x$ is positive semi-definite. In fact, setting for $0 \leqq i \leqq j$

$$Z_{ij} = \{ z \in V_{1/2}(a)\,;\, (d_k \,\square\, d_k)z = (\delta_{ik} + \delta_{jk})z/2 \quad \text{for } 1 \leqq k \leqq n \}, \tag{3.4}$$

we have $Z = \bigoplus_{0 \leqq i \leqq j} Z_{ij}$ (orthogonal direct sum) and

$$(x \,\square\, x)z = (\xi_i^2 + \xi_j^2)z/2 \qquad \text{if } z \in Z_{ij}, \tag{3.5}$$

where we put $\xi_0 = 0$. Let $\| x \|$ denote the operator norm of the positive semi-definite

square root $(x \,\square\, x)^{1/2}$ of the operator $x \,\square\, x$:

$$\|x\| := \|(x \,\square\, x)^{1/2}\| = \|x \,\square\, x\|^{1/2}. \tag{3.6}$$

Then it is clear from (3.5) that $|x\| = \sup \xi_j$, hence $\|x\| = 0$ implies $x = 0$. By [**28**, 20.17], we know that $|\cdot\|$ defines a norm on Z, called the *spectral norm*. Note (3.2) implies that the spectral norm $|\cdot\|$ is a continuous function on Z and that $\|d\| \geqq C^{-1/2}$ for all non-zero tripotents d. Therefore in the Schmidt expansion $x = \sum \xi_j d_j$, we have

$$\sum \xi_j^2 \leqq C \sum \xi_j^2 \|d_j\|^2 = C\|x\|^2 < \infty.$$

Thus we can renumber $\{\xi_j\}$ in such a way that $\xi_1 > \xi_2 > \cdots > \xi_j \to 0$, in case $\{\xi_j\}$ is infinite.

We let

$$x = \sum \xi_j d_j \qquad (|x\| = \xi_1 > \xi_2 > \cdots > \xi_j \to 0)$$

be the Schmidt expansion of $x \in Z$ anew. With every *odd* real analytic function f defined on an open interval $(-\alpha, \alpha)$ $(\alpha > 0)$ in $\mathbb{R}$, we associate a Z-valued function on the domain $D_\alpha := \{x \in Z\,;\, \|x\| < \alpha\}$ by

$$f(x) = \sum f(\xi_j) d_j. \tag{3.7}$$

It is easy to see that the right hand side of (3.7) converges and defines an element of Z. The Z-valued function $x \mapsto f(x)$ is real analytic on D_α, cf. [**16**, 3.19].

Let us return to our JH-algebra V. Let $a \in \mathfrak{I}_p$ with p as in (2.1). Then each of the Peirce spaces $V_k(a)$ $(k = 0,\ 1/2,\ 1)$ as well as V itself becomes a Hilbert JTS

by defining a triple product through

$$\{x, y, z\} := (xy)z + x(yz) - y(xz), \tag{3.8}$$

where the multiplication rule (1.5) should be recalled. We have

$$x \,\square\, y = L(xy) + [\, L(x), L(y)\,] \tag{3.9}$$

in these cases. Therefore the requirement $(x \,\square\, y)^* = y \,\square\, x$ is obviously satisfied by (1.1) and (1.2).

We now focus on $V_{1/2}(a)$. First of all, it is clear that $d \in V_{1/2}(a)$ is a tripotent if and only if $d^3 = d$ by the definition (3.8) of the triple product.

Lemma 3.3. (1) *Let $d \in V_{1/2}(a)$ be a non-zero tripotent. Then one has $d^2 \in V_0(a) + V_1(a)$, $ad^2 \in V_1(a)$, and they are non-zero idempotents such that $\|d^2\|^2 = 2\,\|ad^2\|^2 = \|d\|^2$.*

(2) *Let d_1, $d_2 \in V_{1/2}(a)$ be non-zero orthogonal tripotents. Then,*

(i) $\langle\, d_1^2 \,|\, d_2^2 \,\rangle = \langle\, d_1^2 \,|\, ad_2^2 \,\rangle = \langle\, ad_1^2 \,|\, ad_2^2 \,\rangle = 0$, (ii) $d_1 d_2 = 0$.

Proof. (1) Since $d^3 = d$, we have $(d^2)^2 = d^4 = d^2$. Hence d^2 is an idempotent. Moreover, $d^2 \in V_0(a) + V_1(a)$, so that $ad^2 \in V_1(a)$. Note that ad^2 is the orthogonal projection of d^2 onto $V_1(a)$. It is evident that ad^2 is also an idempotent. We now compute the norms of d^2 and ad^2. First, we have $\|d^2\|^2 = \langle\, d \,|\, d^3 \,\rangle = \|d\|^2$. Next, since (1.1) yields

$$2(ad^2)d = 2(ad)d^2 = d^3 = d, \tag{3.10}$$

we get $2\,\|ad^2\|^2 = 2\,\langle\, ad^2 \,|\, d^2 \,\rangle = 2\,\langle\, (ad^2)d \,|\, d \,\rangle = \langle\, d \,|\, d \,\rangle$.

(2) (i) We have

$$\langle\, d_1^2 \,|\, d_2^2 \,\rangle = \langle\, d_2 d_1^2 \,|\, d_2 \,\rangle = \langle\, (d_2 \,\square\, d_1) d_1 \,|\, d_2 \,\rangle = 0, \tag{3.11}$$

$$\langle\, ad_1^2 \,|\, ad_2^2 \,\rangle = \langle\, d_1^2 \,|\, ad_2^2 \,\rangle = \langle\, d_1^2 d_2^2 \,|\, a \,\rangle = 0 \quad \text{(by (1) and (3.11))}. \tag{3.12}$$

(ii) Since $d_2 d_1^2 = (d_2 \,\square\, d_1) d_1 = 0$, we obtain

$$2\,\|d_1 d_2\|^2 = 2\,\langle\, d_1(d_1 d_2) \,|\, d_2 \,\rangle = \langle\, (d_1 \,\square\, d_2) d_1 \,|\, d_2 \,\rangle = 0. \quad \blacksquare$$

Let $a = \sum_{j=1}^{p} a_j$ be the decomposition of a as a sum of orthogonal primitive idempotents $\{a_j\}_{j=1}^{p}$. By our assumption (2.1), we can add $a_{p+1}, \ldots, a_{2p} \in \mathfrak{I}_1$ to $\{a_j\}_{j=1}^{p}$, so that $\{a_j\}_{j=1}^{2p}$ is a system of orthogonal primitive idempotents in V. Put $U := V_1(a_1 + \cdots + a_{2p})$ and let $U = \bigoplus_{i \leqq j} U_{ij}$ be the Peirce decomposition of U corresponding to $\{a_j\}_{j=1}^{2p}$. By [**19**, Lemma 2.4], there is $d_j \in U_{j,j+p} = U_{1/2}(a_j) \cap U_{1/2}(a_{j+p})$ $(1 \leqq j \leqq p)$ satisfying $d_j^2 = a_j + a_{j+p}$. We have $d_j \in V_{1/2}(a)$ and $d_j^3 = d_j$. Furthermore, the observation

$$d_i \in U_1(a_i + a_{i+p}), \qquad d_j \in U_0(a_i + a_{i+p}) \quad (j \neq i)$$

gives $d_i d_j = 0$ and $[\, L(d_i), L(d_j) \,] = 0$ for $i \neq j$ by virtue of (1.5), (1.6). Hence $\{d_j\}_{j=1}^{p}$ is a system of non-zero orthogonal tripotents in $V_{1/2}(a)$ in view of (3.9). This implies that the rank ρ of Hilbert JTS $V_{1/2}(a)$ is at least p. But Lemma 3.3 shows that ρ does not exceed the rank of the *JH*-algebra $V_1(a)$. We have thus shown

Proposition 3.4. *If $a \in \mathfrak{I}_p$, then the Hilbert JTS $V_{1/2}(a)$ is of rank p.*

Proposition 3.5. *Let $a \in \mathfrak{J}_p$, $x \in V_{1/2}(a)$. Then the subalgebra $V[a,x]$ is of finite dimension.*

Proof. Let $x = \sum_{j=1}^{v} \xi_j d_j$ $(\xi_1 > \cdots > \xi_v > 0)$ be the Schmidt expansion of x in the Hilbert JTS $V_{1/2}(a)$ of rank $p < \infty$. Then it holds that

$$x^{2k-1} = \sum_{j=1}^{v} \xi_j^{2k-1} d_j, \qquad x^{2k} = \sum_{j=1}^{v} \xi_j^{2k} d_j^2 \quad \text{(by Lemma 3.3 (2) (ii))} \tag{3.13}$$

for $k = 1, 2, \ldots$. We have

$$\begin{aligned} ad_j &= d_j/2, \qquad d_i d_j^2 = (d_i \,\square\, d_j) d_j = \delta_{ij} d_j, \\ a(ad_j^2) &= ad_j^2, \qquad d_i^2 (ad_j^2) = (ad_i^2)(ad_j^2) = \delta_{ij} ad_j^2. \end{aligned} \tag{3.14}$$

Recalling the unital Jordan algebra representation π_a in (1.7), we obtain

$$d_i(ad_j^2) = \pi_a(ad_j^2)\pi_a(ad_i^2)d_i = \delta_{ij} d_i/2 \tag{3.15}$$

by (3.10) and (i) of Lemma 3.3 (2). Therefore setting

$$a_\natural := a - \sum_{j=1}^{v} ad_j^2, \qquad W_j := \mathbb{R}(ad_j^2) + \mathbb{R}d_j + \mathbb{R}d_j^2,$$

we get $V[a,x] \subset \mathbb{R}a_\natural \oplus \Big[\bigoplus_{j=1}^{v} W_j \Big]$, where $a_\natural = 0$ possibly. Moreover, (3.13) together with a Vandermonde argument shows that each d_j is a finite linear combination of $x, x^3, \ldots$. Hence, $d_j \in V[a,x]$, so that

$$V[a,x] = \mathbb{R}a_\natural \bigoplus \Big[\bigoplus_{j=1}^{v} W_j \Big]. \tag{3.16}$$

In particular, $\dim V[a,x] < \infty$. ∎

In (3.16), it is easy to see that the JH-algebra W_j is Jordan isomorphic to $\mathrm{Sym}(2,\mathbb{R})$ through the linear mapping Ξ_j defined by

$$\Xi_j(ad_j^2) = A, \qquad \Xi_j(d_j) = X, \qquad \Xi_j(d_j^2) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right). \tag{3.17}$$

We note that if p_j denotes the rank of the idempotent ad_j^2, then $p_j^{1/2}\Xi_j$ is an isometry. Summarizing all the above, we get

Theorem 3.6. *Let $a \in \mathfrak{I}_p$ with p as in (2.1) and $x \in V_{1/2}(a)$. Then one has $V[a,x] = W_\natural \oplus \Big[\bigoplus_{j=1}^{\upsilon} W_j\Big]$, where $W_j \cong \mathrm{Sym}(2,\mathbb{R})$ for $1 \leqq j \leqq \upsilon$ and $W_\natural$ is at most one dimensional.*

§4. $\mathfrak{I}_p$ as a Riemannian manifold

Let V be a topologically simple JH-algebra. In this section, we will introduce a Riemannian manifold structure on $\mathfrak{I}_p$ by giving a concrete atlas, so that the inclusion $\mathfrak{I}_p \hookrightarrow V$ is an embedding. Our structure theorems, Theorems 2.6 and 3.6, make the way of defining the relevant mappings almost the same as the case $\mathfrak{I}_1$ treated in [**19**], though the formulae (4.3) below are a little different in appearance.

First we give $\mathfrak{I}_p$ the norm topology of V. Then $\mathfrak{I}_p$ is a complete metric space. Next we note that $\mathfrak{I}_p$ is arcwise connected. In fact, if $a,\ b \in \mathfrak{I}_p$ satisfy $P(a)b \in V_1(a)^\times$, then we have (2.11) with b_0 missing. Putting

$$\gamma(t) := a_* + \sum_{j=1}^{\tau} \Big[(\cos 2t\theta_j)a_j + (\sin 2t\theta_j)d_j/2 + (1-\cos 2t\theta_j)d_j^2/2\Big] \tag{4.1}$$

for $0 \leqq t \leqq 1$, we obtain a continuous curve $t \mapsto \gamma(t) \in \mathfrak{I}_p$ such that $\gamma(0) = a$, $\gamma(1) = b$. In the general case where a_0 and b_0 in (2.10) are present, take $d_0 \in$

$V_{1/2}(a_0) \cap V_{1/2}(b_0)$ with $\|d_0\|^2 = 2\|a_0\|^2$ such that $\mathbb{R}a_0 \bigoplus \mathbb{R}d_0 \bigoplus \mathbb{R}b_0 \cong \mathrm{Sym}(2,\mathbb{R})$ according to Proposition 2.8. Then we only have to add the term

$$(\cos \pi t)a_0 + (\sin \pi t)d_0/2 + (1 - \cos \pi t)d_0^2/2$$

to the $\gamma(t)$ in (4.1) in order to obtain a continuous curve joining a, b.

Before giving an atlas, we note here that for every $u \in V$, the operator $L(u^2)$ on V is positive semi-definite. In fact, if $u = \sum_{j=1}^{\infty} \lambda_j c_j$ is the spectral decomposition [**19**, (1.5)] of u, then $u^2 = \sum \lambda_j^2 c_j$, so that $L(u^2) = \sum \lambda_j^2 L(c_j)$ is positive semi-definite because so is each of the $L(c_j)$ as says the Peirce decomposition (1.4).

Now for every $a \in \mathfrak{J}_p$ with p as in (2.1), we put

$$N_a := \{b \in \mathfrak{J}_p;\ P(a)b \in V_1(a)^\times\}. \tag{4.2}$$

Then it is clear that N_a is an open subset of $\mathfrak{J}_p$ containing a, and so the family $\{N_a\,;\, a \in \mathfrak{J}_p\}$ is an open covering of $\mathfrak{J}_p$. Recalling the unital Jordan algebra representation π_a introduced in (1.7), we define

$$\begin{aligned} \xi_a(b) &:= 2\pi_a(P(a)b)^{-1}(ab - P(a)b) \qquad (b \in N_a), \\ \eta_a(x) &:= (I + L(x^2))^{-1}\left[(I - L(x^2))a + x + x^2\right] \qquad (x \in V_{1/2}(a)), \end{aligned} \tag{4.3}$$

where notice that the operators $I + L(x^2)$ are positive definite.

Remark 4.1. If $p = 1$, then $V_1(a) = \mathbb{R}a$, so that we have $P(a) = E_1(a) = \langle\, \cdot \,|\, a \,\rangle a$. Therefore we see that

$$\xi_a(b) = 2\,\langle\, b \,|\, a \,\rangle^{-1}(ab - \langle\, b \,|\, a \,\rangle a),$$

agreeing with the expression of ξ_a in [**19**, (3.3)]. Moreover, [**19**, Lemma 2.3] implies that if $x \in V_{1/2}(a)$, then the operator $L(x^2)$ acts as a scalar $\|x\|^2/2$ on $\mathbb{R}a+\mathbb{R}x+\mathbb{R}x^2$. Hence we get

$$\eta_a(x) = (2 + \|x\|^2)^{-1} \left[(2 - \|x\|^2)a + 2x + 2x^2 \right],$$

which is the expression of η_a in [**19**, (3.3)].

Proposition 4.2. *One has*

(1) $\mathrm{Image}(\xi_a) = V_{1/2}(a), \qquad \mathrm{Image}(\eta_a) = N_a,$

(2) *ξ_a and η_a are inverses to each other.*

Proof. Step 1. Let $b \in N_a$ and let $b = b_0 + b_{1/2} + b_1$ ($b_j \in V_j(a)$) be the Peirce decomposition of b relative to a. Then we have $\xi_a(b) = \pi_a(b_1)^{-1}b_{1/2} \in V_{1/2}(a)$.

We next prove that $\mathrm{Image}(\eta_a) \subset N_a$. Let $x \in V_{1/2}(a)$ and we decompose $V[a,x]$ as in (3.16) : $V[a,x] = \mathbb{R}a_\natural \oplus \Big[\bigoplus_{j=1}^{v} W_j \Big]$. Let $a = a_\natural + \sum a_j$ ($a_j := ad_j^2$) and $x = \sum \xi_j d_j$ be the corresponding decompositions of a, x respectively, where $\{d_j\}$ is a system of non-zero orthogonal tripotents and $\xi_j > 0$. Moreover, we have the Jordan isomorphisms $\Xi_j : W_j \to \mathrm{Sym}(2,\mathbb{R})$ given by (3.17). Setting $x_j := \xi_j d_j$ for simplicity, we put $\theta_j := \mathrm{Arctan}\, \xi_j$ ($0 < \theta_j < \pi/2$) and

$$b_\natural := a_\natural, \qquad b_j := (I + L(x_j^2))^{-1} \left[(I - L(x_j^2))a_j + x_j + x_j^2 \right] \quad (1 \leqq j \leqq v).$$

By a simple computation we obtain $\Xi_j(b_j) = B(\theta_j)$ for $1 \leqq j \leqq v$, where $B(\theta)$ stands for the matrix in (2.8). Hence, $b := \eta_a(x) = b_\natural + \sum_{j=1}^{v} b_j \in \mathfrak{I}_p$. Moreover, transferring $P(a)b$ by Ξ_j, we see easily that

$$P(a)b = a_\natural + \sum_{j=1}^{v} (\cos^2 \theta_j)a_j \in V_1(a)^\times, \tag{4.4}$$

because $0 < \theta_j < \pi/2$ for $1 \leqq j \leqq \upsilon$. Hence $b \in N_a$.

Step 2. Let $b \in N_a$ and we shall show $\eta_a(\xi_a(b)) = b$. We decompose $V[a,b]$ as in (2.9) : $V[a,b] = \mathbb{R}a_* \oplus \Big[\bigoplus_{j=1}^{\tau} V_j\Big]$, where V_0 is missing because $b \in N_a$. The corresponding decompositions of a, b are $a = a_* + \sum_{j=1}^{\tau} a_j$, $b = a_* + \sum_{j=1}^{\tau} b_j$ respectively. Moreover, b has the expression (2.11) with b_0 absent. For $1 \leqq j \leqq \tau$, let $\Theta_j : V_j \to \mathrm{Sym}(2,\mathbb{R})$ be the Jordan isomorphism given by Proposition 2.5. Then putting

$$x_j = 2\pi_{a_j}(P(a_j)b_j)^{-1}(a_jb_j - P(a_j)b_j) \in V_j \qquad (1 \leqq j \leqq \tau),$$

we have $\xi_a(b) = \sum x_j$ and $\Theta_j(x_j) = (\tan\theta_j)X$ with X as in (2.8). Let us set $d_j := \Theta_j^{-1}(X)$. Then d_j is a non-zero tripotent and $x_j = (\tan\theta_j)d_j$. Because θ_j are distinct positive numbers, $\xi_a(b) = \sum_{j=1}^{\tau} (\tan\theta_j)d_j$ is the Schmidt expansion. Now it is clear that the subalgebra V_j coincides with $W_j := V[a_j, x_j]$. Hence we have $V[a,b] = V[a,x] = \mathbb{R}a_* \oplus \Big[\bigoplus_{j=1}^{\tau} W_j\Big]$. The isomorphism Θ_j also agrees with the Ξ_j used in Step 1, so that the computation in Step 1 gives

$$(I + L(x_j^2))^{-1}\left[(I - L(x_j^2))a_j + x_j + x_j^2\right] = b_j.$$

This says that $\eta_a(\xi_a(b)) = a_* + \sum b_j = b$.

The equality $\xi_a(\eta_a(x)) = x$ $(x \in V_{1/2}(a))$ can be proved similarly. ∎

Now we can introduce a Riemannian manifold structure on $\mathfrak{I}_p$. Since the way is parallel to that of $\mathfrak{I}_1$ treated in [**19**], we only outline it here. First we note that the Peirce 1/2-spaces $V_{1/2}(a)$ $(a \in \mathfrak{I}_p)$ are mutually unitarily isomorphic, because $\mathfrak{I}_p$ is arcwise connected. So, take and fix an element $c \in \mathfrak{I}_p$ and let

$\Upsilon_a : V_{1/2}(a) \to \mathfrak{H} := V_{1/2}(c)$ be the unitary isomorphism. Then the collection $\{(N_a, \Upsilon_a \circ \xi_a, \mathfrak{H})\,;\, a \in \mathfrak{I}_p\}$ gives an atlas to $\mathfrak{I}_p$, so that $\mathfrak{I}_p$ becomes a Hilbert manifold modeled on $\mathfrak{H}$. Moreover, the inclusion $\mathfrak{I}_p \hookrightarrow V$ is an embedding, hence $\mathfrak{I}_p$ inherits a Riemannian manifold structure from the ambient Hilbert space V. The tangent space $T_a(\mathfrak{I}_p)$ at a will be identified with $V_{1/2}(a)$.

Let $GL(V)$ denote the multiplicative group of the invertible elements $\boldsymbol{B}(V)^\times$ in the Banach algebra $\boldsymbol{B}(V)$ of all bounded linear operators on V. The group $GL(V)$ is a Banach-Lie group in the operator-norm topology. Let $\mathrm{Aut}(V)$ be the automorphism group of the JH-algebra V. We know from Harris-Kaup's theorem [**28**, 7.14] that $\mathrm{Aut}(V)$ is also a Banach-Lie group in the operator-norm topology as an algebraic subgroup of degree $\leqq 2$ of $GL(V)$. Moreover, [**19**, Proposition 2.8] says $\mathrm{Aut}(V) \subset O(V)$, the orthogonal group of the underlying Hilbert space V. The Peirce reflections $T(a) := I - E_{1/2}(a)$ $(a \in \mathfrak{I}_p)$ belong to $\mathrm{Aut}(V)$. It is easy to see that every $a \in \mathfrak{I}_p$ is an isolated fixed point of the action of $T(a)$ on $\mathfrak{I}_p$, whence we have the first assertion of the following proposition.

Proposition 4.3. *The Riemannian manifold $\mathfrak{I}_p$ is symmetric. Furthermore, if a, $b \in \mathfrak{I}_p$, then there is an idempotent $c \in \mathfrak{I}_p$ such that $T(c)a = b$. In particular,* $\mathrm{Aut}(V)$ *acts transitively on $\mathfrak{I}_p$.*

In view of Theorem 2.6 and Proposition 2.8, proof of the second assertion of Proposition 4.3 can be given in a similar fashion as that of Proposition 2.9 in [**19**], and so omitted.

§5. Geodesics and Riemannian distance

Let $\mathfrak{X}(\mathfrak{I}_p)$ denote the set of tangent vector fields on $\mathfrak{I}_p$. Since $\mathfrak{I}_p$ is a Rieman-

nian manifold embedded in V, and since the tangent space $T_a(\mathfrak{I}_p)$ at a is identified with the Peirce 1/2-space $V_{1/2}(a)$, every element $X \in \mathfrak{X}(\mathfrak{I}_p)$ is regarded as a V-valued smooth function on $\mathfrak{I}_p$ such that $X(a) \in V_{1/2}(a)$ for all $a \in \mathfrak{I}_p$. Recalling the Peirce projection $E_{1/2}(a)$ in (1.9), we now define the *canonical connection* ∇ of $\mathfrak{I}_p$ by

$$\nabla_X Y(a) := E_{1/2}(a)\big(Y'(a)(X(a))\big) \qquad (X, Y \in \mathfrak{X}(\mathfrak{I}_p) \text{ and } a \in \mathfrak{I}_p), \tag{5.1}$$

where Y' stands for the Fréchet derivative of the V-valued function Y. The connection ∇ is Levi-Civita and $\mathrm{Aut}(V)$-invariant.

We first exhibit geodesics in $\mathfrak{I}_p$ which are orbits of one-parameter subgroups in $\mathrm{Aut}(V)$. Let $\mathrm{Der}(V)$ be the Banach-Lie algebra of all continuous derivations of the *JH*-algebra V. Then $\mathrm{Der}(V)$ is the Lie algebra of the Banach-Lie group $\mathrm{Aut}(V)$. Moreover, we know that the operators of the form $[\,L(x), L(y)\,]$ $(x, y \in V)$ are derivations. We put

$$D(x, y) := 4\,[\,L(x), L(y)\,] \in \mathrm{Der}(V) \tag{5.2}$$

for simplicity. Then $\exp(tD(x, y)) \in \mathrm{Aut}(V)$ for all $t \in \mathbb{R}$.

Proposition 5.1. *Let $a \in \mathfrak{I}_p$ and $x \in V_{1/2}(a)$. Then the curve $\gamma_{a,x}(t) := (\exp tD(x, a))a$ is a geodesic in $\mathfrak{I}_p$ such that $\gamma_{a,x}(0) = a$, $\dot{\gamma}_{a,x}(0) = x$.*

Proof. We set $D = D(x, a)$ and $\gamma(t) = \gamma_{a,x}(t) = e^{tD}a$ for brevity. An easy computation yields

$$Da = x, \qquad Dx = 2(x^2 - 2ax^2). \tag{5.3}$$

Now it is obvious that $\gamma(t)$ is a curve in $\mathfrak{I}_p$ with $\gamma(0) = a$. Since $\dot{\gamma}(t) = e^{tD}Da = e^{tD}x$, we have $\dot{\gamma}(0) = x$. Let us show $E_{1/2}(\gamma(t))(\ddot{\gamma}(t)) = 0$ for all $t \in \mathbb{R}$. Since $\ddot{\gamma}(t) = e^{tD}Dx = 2e^{tD}(x^2 - 2ax^2)$, the expression (1.9) for $E_{1/2}(\gamma(t))$ together with the fact that $e^{tD} \in \mathrm{Aut}(V)$ for all $t \in \mathbb{R}$ gives

$$\begin{aligned} E_{1/2}(\gamma(t))(\ddot{\gamma}(t)) &= 8\left(L(e^{tD}a) - L(e^{tD}a)^2\right)\left(e^{tD}(x^2 - 2ax^2)\right) \\ &= 8e^{tD}(I - L(a))(ax^2 - 2a(ax^2)). \end{aligned}$$

Here it is clear that $ax^2 - 2a(ax^2) \in V_1(a)$, so $L(a) = I$ on this element. Hence we obtain $E_{1/2}(\gamma(t))(\ddot{\gamma}(t)) = 0$. ∎

To express $\gamma_{a,x}(t)$ in a convenient form we need two lemmas.

Lemma 5.2. *Let a, x be as in Proposition 5.1 and set $D = D(x, a)$. Then D^2 leaves both $V_0(a) + V_1(a)$ and $V_{1/2}(a)$ invariant, and one has*

$$D^2 y = \begin{cases} 4\left[L(x)^2 - L(x^2)\right] y & \text{for } y \in V_0(a) + V_1(a), \\ -4L(x)^2 y & \text{for } y \in V_{1/2}(a). \end{cases}$$

Proof. First, we get by a direct calculation

$$Dy = \begin{cases} (-1)^{1-k} 2xy \in V_{1/2}(a) & \text{for } y \in V_k(a),\ k = 0, 1, \\ 2(xy - 2a(xy)) \in V_0(a) + V_1(a) & \text{for } y \in V_{1/2}(a), \end{cases} \tag{5.4}$$

from which follows immediately the assertion concerning the invariance of the subspaces under D^2. Suppose now $y \in V_1(a)$. Then $Dy = 2xy$ by (5.4), so that

$$D^2 y = 2D(xy) = 2(Dx)y + 2x(Dy) = 4(x^2 - 2ax^2)y + 4x(xy).$$

Since $x^2 \in V_0(a) + V_1(a)$ and since $y \in V_1(a)$, we have $(x^2 - 2ax^2)y = -x^2 y$. Hence, $D^2 y = 4\left[L(x)^2 - L(x^2)\right] y$. The case $y \in V_0(a)$ is treated similarly. Finally

if $y \in V_{1/2}(a)$, then $xy \in V_0(a)+V_1(a)$, and so we obtain $D(xy) \in V_{1/2}(a)$ by (5.4). Therefore, $2aD(xy) = D(xy)$. We thus arrive at

$$\begin{aligned} D^2y = D(Dy) &= 2D(xy) - 4(Da)(xy) - 4aD(xy) \qquad \text{(by (5.4))} \\ &= -4(Da)(xy) = -4x(xy) \qquad \text{(by (5.3)).} \ \blacksquare \end{aligned}$$

Lemma 5.3. *For $n = 1, 2, \dots$, one has*

$$D^{2n}a = 2(-4)^{n-1}x^{2n} + (-4)^n L(x^2)^n a.$$

Proof. By Lemma 5.2, we have $D^{2n}a = 4^n[L(x)^2 - L(x^2)]^n a$ $(n = 1, 2, \dots)$. Since $D^2a = Dx = 2x^2 - 4L(x^2)a$ by (5.3), we get for $n \geqq 2$

$$\begin{aligned} D^{2n}a &= 4^{n-1}[L(x)^2 - L(x^2)]^{n-1}(2x^2 - 4L(x^2)a) \\ &= -4^n[L(x)^2 - L(x^2)]^{n-1}L(x^2)a = -4L(x^2)D^{2(n-1)}a, \end{aligned}$$

where the last equality is a consequence of $[L(x), L(x^2)] = 0$ implied by (1.1). Now the lemma follows easily. $\blacksquare$

Let ϕ be an entire function on $\mathbb{C}$ with zero of order k at $z = 0$. Then for any $T \in \boldsymbol{B}(V)$, we define

$$\frac{\phi(T)}{T^l} := \sum_{n=k}^{\infty} a_n T^{n-l} \qquad \text{for } l \leqq k,$$

where $\phi(z) = \sum_{n=k}^{\infty} a_n z^n$ is the Taylor expansion of ϕ at $z = 0$. The expressions $(\sin T)/T$ and $(I - \cos T)/T^2$ for $T = 2tL(x^2)^{1/2} \in \boldsymbol{B}(V)$ in the following proposition should be understood in this sense.

Proposition 5.4. *Let $\gamma_{a,x}$ be as in Proposition 5.1. Then one has*

$$\gamma_{a,x}(t) = \left[\cos(2tL(x^2)^{1/2})\right] a + \frac{1}{2}\frac{\sin(2tL(x^2)^{1/2})}{L(x^2)^{1/2}}\, x + \frac{1}{2}\frac{I - \cos(2tL(x^2)^{1/2})}{L(x^2)}\, x^2.$$

Proof. By definition, we have

$$\begin{aligned}\gamma_{a,x}(t) &= \sum_{n=0}^{\infty} \frac{t^n D^n}{n!}\, a = a + \sum_{n=1}^{\infty} \frac{t^{2n-1} D^{2n-2}}{(2n-1)!}\, x + \sum_{n=1}^{\infty} \frac{t^{2n} D^{2n}}{(2n)!}\, a \qquad \text{(by (5.3))} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2t)^{2n} L(x^2)^n}{(2n)!}\, a + \sum_{n=1}^{\infty} \frac{t^{2n-1}(-4L(x)^2)^{n-1}}{(2n-1)!}\, x \\ &\qquad + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2t)^{2n}}{(2n)!}\, x^{2n} \qquad \text{(by Lemmas 5.2 and 5.3).}\end{aligned}$$

Rewriting $L(x)^{2n-2}x = L(x^2)^{n-1}x$ and $x^{2n} = L(x^2)^{n-1}x^2$ in the second and the third summation respectively, we obtain the formula in the proposition by noting that the operator $L(x^2)$ is positive semi-definite. ∎

For $a \in \mathfrak{I}_p$ and $x \in V_{1/2}(a)$, we decompose the subalgebra $V[a,x]$ as in (3.16) : $V[a,x] = \mathbb{R}a_\natural \oplus \left[\bigoplus_{j=1}^{\upsilon} W_j\right]$ with $a = a_\natural + \sum_{j=1}^{\upsilon} a_j$ $(a_j := ad_j^2)$, $x = \sum_{j=1}^{\upsilon} \xi_j d_j$ the corresponding decompositions of a, x respectively. Here, each d_j is a non-zero tripotent and $\xi_j > 0$. Then we get the following corollary.

Corollary 5.5. *One has*

$$\gamma_{a,x}(t) = a_\natural + \sum_{j=1}^{\upsilon} \left[(\cos 2t\xi_j) a_j + (\sin 2t\xi_j) d_j/2 + (1 - \cos 2t\xi_j) d_j^2/2 \right].$$

Proof. By (3.13), we have $x^2 = \sum \xi_j^2 d_j^2$. Then it suffices to note that for $1 \leqq j \leqq \upsilon$, the operator $L(d_j^2)$ is the identity operator on $W_j = V[a_j, d_j]$ as can be seen by using the isomorphism $\Xi_j : W_j \to \mathrm{Sym}(2,\mathbb{R})$ defined by (3.17). ∎

Using the geodesic $\gamma_{a,x}$ in Proposition 5.1, we put

$$\mathrm{Exp}_a(x) := \gamma_{a,x}(1).$$

We call the mapping Exp_a the *exponential mapping* at $a \in \mathfrak{I}_p$. We recall here that the Peirce 1/2-space $V_{1/2}(a)$ is a Hilbert JTS of rank $p < \infty$ (Proposition 3.4). Let $|\cdot|_{1/2}$ be the spectral norm of $V_{1/2}(a)$. We set

$$B_a := \{x \in V_{1/2}(a)\,;\, |x|_{1/2} < \pi/2\}.$$

Then the $V_{1/2}(a)$-valued real analytic function $x \mapsto \tan x$ is defined on B_a by means of (3.7). Moreover, its inverse function $x \mapsto \mathrm{Arctan}\, x$ is defined and real analytic on the whole of $V_{1/2}(a)$.

Lemma 5.6. *If* $|x|_{1/2} < \pi/2$, *then* $\mathrm{Exp}_a(x) = \eta_a(\tan x)$, *where* η_a *is the mapping defined in* (4.3).

Proof. Let the notation be as in Corollary 5.5. We have $x = \sum_{j=1}^{\upsilon} \xi_j d_j$. The assumption $|x|_{1/2} < \pi/2$ implies that $0 < \xi_j < \pi/2$ for $1 \leqq j \leqq \upsilon$ and we have $\tan x = \sum_{j=1}^{\upsilon} (\tan \xi_j) d_j$. Using the isomorphism (3.17), we see easily that

$$\begin{aligned}\eta_a(\tan x) &= a_{\natural} + \sum_{j=1}^{\upsilon} \left[(\cos 2\xi_j) a_j + (\sin 2\xi_j) d_j/2 + (1 - \cos 2\xi_j) d_j^2/2 \right] \\ &= \gamma_{a,x}(1) = \mathrm{Exp}_a(x) \qquad \text{(by Corollary 5.5).}\ \blacksquare\end{aligned}$$

Theorem 5.7. *The exponential mapping* Exp_a *is a diffeomorphism of* B_a *onto the open subset* N_a *of* $\mathfrak{I}_p$ *in* (4.2). *Its inverse, denoted by* Log_a, *is given by*

$$\mathrm{Log}_a(b) = \mathrm{Arctan}(\xi_a(b)) \qquad (b \in N_a),$$

where ξ_a is the mapping in (4.3).

Proof. By Proposition 4.2 and Lemma 5.6, we have $\xi_a(\mathrm{Exp}_a(x)) = \tan x$ for every $x \in B_a$. Hence $x = \mathrm{Arctan}\,\xi_a(\mathrm{Exp}_a(x))$. Conversely, if $b \in N_a$, then clearly we have $\|\mathrm{Arctan}\,\xi_a(b)\|_{1/2} < \pi/2$. Therefore $\mathrm{Arctan}\,\xi_a(b) \in B_a$ and we get

$$\mathrm{Exp}_a(\mathrm{Arctan}\,\xi_a(b)) = \eta_a(\xi_a(b)) = b. \quad \blacksquare$$

For $a, b \in \mathfrak{I}_p$, we denote by $\mathrm{dist}(a, b)$ the *Riemannian distance* between a, b. We have by definition

$$\mathrm{dist}(a, b) = \inf\{\mathrm{length}(\gamma)\,;\ \gamma \text{ a curve from } a \text{ to } b\}.$$

The function 'dist' defines a metric on $\mathfrak{I}_p$ compatible with the initial norm topology inherited from V. The following proposition plays a vital role in deriving an explicit expression of the Riemannian distance on $\mathfrak{I}_p$.

Proposition 5.8. *Let $a, b \in \mathfrak{I}_p$ with p as in* (2.1) *and suppose $b \in N_a$. Then there is a geodesic γ of the form $\gamma_{a,x}$ in Proposition 5.1 with some $x \in V_{1/2}(a)$ such that $\gamma(0) = a$, $\gamma(1) = b$. This geodesic $\gamma_{a,x}$ joining a and b is a minimizing one.*

Proof. Put $x = \mathrm{Log}_a(b) \in B_a$ and take the geodesic $\gamma_{a,x}$ in Proposition 5.1. It has the form described in Corollary 5.5. Then $b = \mathrm{Exp}_a(x) = \gamma_{a,x}(1)$. Moreover, transferring each component of $P(a)\gamma_{a,x}(t)$ to $\mathrm{Sym}(2, \mathbb{R})$ by means of the isomorphism Ξ_j defined by (3.17), we see that

$$P(a)\gamma_{a,x}(t) = a_\natural + \sum_{j=1}^{v} (\cos^2 t\xi_j)a_j,$$

cf. (4.4). Since $x \in B_a$, we have $|t\xi_j| < \pi/2$ for $0 \leqq t \leqq 1$ and $1 \leqq j \leqq v$. Hence, $\gamma_{a,x}(t) \in N_a = \mathrm{Exp}_a(B_a)$ if $t \in [0,1]$. Therefore the usual argument relying on the Gauß lemma [**14**, 1.9] on Levi-Civita connections proves the minimality of $\gamma_{a,x}$. ∎

By introducing the triple product (3.8), the *JH*-algebra V itself also becomes a Hilbert JTS. Let $\| \cdot |$ denote the spectral norm of V. Then the V-valued real analytic function $v \mapsto \mathrm{Arcsin}(v)$ is defined on the domain $\{v \in V\,;\, \|v| < \pi/2\}$ by (3.7). We extend it to the closure $\{v \in V\,;\, |v\| \leqq \pi/2\}$ by continuity.

Theorem 5.9. *Let $a, b \in \mathfrak{I}_p$. Then* $\mathrm{dist}(a,b) = \|\mathrm{Arcsin}(a-b)\|$.

Proof. First we suppose $b \in N_a$. Let $\gamma = \gamma_{a,x}$ be the geodesic in Proposition 5.8 such that $\gamma(0) = a$, $\gamma(1) = b$. Since $\|\dot\gamma(t)\| = \|e^{tD(x,a)}x\| = \|x\|$, we have

$$\mathrm{dist}(a,b) = \int_0^1 \|\dot\gamma(t)\|\,dt = \|x\|. \tag{5.5}$$

We recall that $x \in B_a$ is given by $x = \mathrm{Log}_a(b) = \mathrm{Arctan}(\xi_a(b))$. Now let $V[a,b] = V_* \oplus \Big[\bigoplus_{j=1}^{\tau} V_j\Big]$ be the decomposition (2.9) of the subalgebra $V[a,b]$. Note the assumption $b \in N_a$ implies that V_0 is absent and that b has the expression (2.11) with b_0 missing. Then Step 2 of the proof of Proposition 4.2 shows that $\xi_a(b) = \sum_{j=1}^{\tau} (\tan\theta_j) d_j$ with θ_j and d_j as in (2.11). Thus $x = \sum_{j=1}^{\tau} \theta_j d_j$. On the other hand, letting p_j denote the common rank of the idempotents a_j, b_j, we have $\|d_j\|^2 = 2p_j$, so that we get $\|x\|^2 = \sum_{j=1}^{\tau} 2p_j\theta_j^2$. Moreover, recall the isomorphism Θ_j described in Proposition 2.5. Using the matrices in (2.8), we have

$$\Theta_j(a_j - b_j) = A - B(\theta_j) = (\sin\theta_j)\begin{pmatrix} \sin\theta_j & -\cos\theta_j \\ -\cos\theta_j & -\sin\theta_j \end{pmatrix} =: (\sin\theta_j)\,U_j.$$

Put $u_j := \Theta_j^{-1}(U_j) \in V_j$. Then u_j is a tripotent because $U_j^3 = U_j$. Let us set $y := \sum_{j=1}^{\tau} \theta_j u_j \in V[a,b]$. This is the Schmidt expansion, since the θ_j are distinct

positive numbers. Hence we have $\sin y = \sum_{j=1}^{\tau} (\sin\theta_j)u_j = a-b$, so that $\|a-b\| < \pi/2$. Therefore $y = \mathrm{Arcsin}(a-b)$ and we obtain

$$\begin{aligned}\|\mathrm{Arcsin}(a-b)\|^2 = \|y\|^2 &= \sum_{j=1}^{\tau} \theta_j^2 \|u_j\|^2 = \sum_{j=1}^{\tau} p_j \theta_j^2 \|U_j\|^2 \\ &= \sum_{j=1}^{\tau} 2p_j\theta_j^2 = \|x\|^2 = \mathrm{dist}(a,b)^2 \qquad (\text{by } (5.5)).\end{aligned}$$

In order to deal with the general case where a_0, b_0 are present, we take $d_0 \in V_{1/2}(a_0) \cap V_{1/2}(b_0)$ with $\|d_0\|^2 = 2\|a_0\|^2$ such that $\mathbb{R}a_0 \oplus \mathbb{R}d_0 \oplus \mathbb{R}b_0 \cong \mathrm{Sym}(2,\mathbb{R})$ according to Proposition 2.8. Setting $\theta_0 = \pi/2$, we consider the curve

$$\gamma(t) := a_* + \sum_{j=0}^{\tau} \left[(\cos 2t\theta_j)a_j + (\sin 2t\theta_j)d_j/2 + (1-\cos 2t\theta_j)d_j^2/2 \right],$$

where $0 \leqq t \leqq 1$. Then we have

$$P(a)\gamma(t) = a_* + \sum_{j=0}^{\tau} (\cos^2 t\theta_j)a_j.$$

Since $|t\theta_j| < \pi/2$ for $0 \leqq t < 1$ and $0 \leqq j \leqq \tau$, we see that $\gamma(t) \in N_a$ for such t. The previous argument then shows $\mathrm{dist}(a,\gamma(t)) = \|\mathrm{Arcsin}(a-\gamma(t))\|$ for $0 \leqq t < 1$. Letting $t \to 1$, we get the formula by continuity. ∎

Remark 5.10. If $p = 1$, then $a - b = (\sin\theta)u$ with $0 \leqq \theta \leqq \pi/2$ and a primitive tripotent u. Here $\|u\| = \sqrt{2}$, so that $\|a-b\| = \sqrt{2}\sin\theta$. Hence

$$\mathrm{dist}(a,b) = \|\mathrm{Arcsin}(a-b)\| = \|\theta u\| = \sqrt{2}\,\theta = \sqrt{2}\,\mathrm{Arcsin}\,\frac{\|a-b\|}{\sqrt{2}},$$

which is the formula in Theorem 4.5 of [**19**].

§6. Sectional curvature

In this section, we express and estimate the sectional curvature of $\mathfrak{I}_p$ by means of the relevant Jordan structure. The tangent space $T_a(\mathfrak{I}_p)$ at $a \in \mathfrak{I}_p$ being identified with the Peirce 1/2-space $V_{1/2}(a)$, we have now a smooth family of orthogonal projections $a \mapsto E_{1/2}(a)$ given by (1.9). For $a \in \mathfrak{I}_p$, $x, y \in V_{1/2}(a)$, we let $R_a(x,y) \in \boldsymbol{B}(V_{1/2}(a))$ be the *curvature operator* derived from the curvature tensor

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \qquad (X, Y \in \mathfrak{X}(\mathfrak{I}_p))$$

of our Levi-Civita connection ∇ defined by (5.1). Recalling the operators $D(x,y)$ introduced by (5.2), we have

$$R_a(x,y) = D(x,y)|_{V_{1/2}(a)}\,. \tag{6.1}$$

The way of deriving (6.1) is exactly the same as in [**19**, §6]. The only minor modification is needed in the formula for the Fréchet derivative $E'_{1/2}(a)$.

$$E'_{1/2}(a)(x,u) = \begin{cases} 2(E_0(a) - E_1(a))(xu) & \text{if } u \in V_{1/2}(a), \\ 2(1-2k)xu & \text{if } u \in V_k(a)\ (k = 0, 1). \end{cases}$$

Details are omitted here.

For $x, y \in V_{1/2}(a)$ with $\|x\| = \|y\| = 1$ and $\langle\, x \,|\, y \,\rangle = 0$, the *sectional curvature* $k_a(x,y)$ at $a \in \mathfrak{I}_p$ is by definition

$$k_a(x,y) := \langle\, R_a(x,y)y \,|\, x \,\rangle.$$

We recall that $V_{1/2}(a)$ forms a Hilbert JTS of rank $p < \infty$ with triple product (3.8). So we have operators $x \,\square\, y$ in (3.9). We note here that the operators $P(x)$ in (1.3) are expressed as $P(x)y = \{x, y, x\}$. Let $|\cdot|_{1/2}$ denote the spectral norm of $V_{1/2}(a)$.

Theorem 6.1. *Let $a \in \mathfrak{I}_p$ and $x, y \in V_{1/2}(a)$ with $\|x\| = \|y\| = 1, \langle\, x \,|\, y \,\rangle = 0$. Then one has*

(1) $k_a(x, y) = 2\,\langle\, (y \,\square\, y - P(y))x \,|\, x \,\rangle = 2\,\langle\, (x \,\square\, x - P(x))y \,|\, y \,\rangle$,

(2) $0 \leqq k_a(x, y) \leqq 4 \min(\|x\|_{1/2}^2, \|y\|_{1/2}^2) \leqq 2$.

Proof. (1) Using (6.1), we have $k_a(x, y) = 4\,\langle\, xy^2 \,|\, x \,\rangle - 4\,\langle\, y(xy) \,|\, x \,\rangle$. Since $2y(xy) = 2L(y)^2x = (P(y) + L(y^2))x$, we get

$$k_a(x, y) = 2\,\langle\, L(y^2)x \,|\, x \,\rangle - 2\,\langle\, P(y)x \,|\, x \,\rangle.$$

But (3.9) gives $L(y^2)x = (y \,\square\, y)x$, whence the first equality in (1). For the symmetry of x, y, we note

$$\langle\, (y \,\square\, y)x \,|\, x \,\rangle = \langle\, (x \,\square\, y)y \,|\, x \,\rangle = \langle\, y \,|\, (y \,\square\, x)x \,\rangle = \langle\, y \,|\, (x \,\square\, x)y \,\rangle,$$
$$\langle\, P(y)x \,|\, x \,\rangle = \langle\, (y \,\square\, x)y \,|\, x \,\rangle = \langle\, y \,|\, (x \,\square\, y)x \,\rangle = \langle\, y \,|\, P(x)y \,\rangle.$$

(2) Put $Z = V_{1/2}(a)$ for simplicity. Let $x = \sum_{j=1}^{v} \xi_j d_j$ $(\xi_1 > \xi_2 > \cdots > \xi_v > 0)$ be the Schmidt expansion of x. The orthogonal system of non-zero tripotents $\{d_j\}_{j=1}^v$ gives rise to an orthogonal decomposition $Z = \bigoplus_{0 \leqq i \leqq j \leqq v} Z_{ij}$ with Z_{ij} as in (3.4). Let $y = \sum y_{ij}$ be the corresponding decomposition of y. Setting $\xi_0 = 0$, we obtain without difficulty

$$(x \,\square\, x)y_{ij} = (\xi_i^2 + \xi_j^2)y_{ij}/2, \qquad P(x)y_{ij} = \xi_i\xi_j P(d)y_{ij},$$

where we put $d = \sum_{j=1}^{v} d_j$. We note here that the operator $P(d)$ induces an involutive Jordan algebra automorphism on the unital Jordan algebra $Z_1(d) := \bigoplus_{1 \leqq i \leqq j \leqq v} Z_{ij}$, where the Jordan product is given by $z \cdot w := \{z, d, w\}$. Each of these Z_{ij} is left invariant by $P(d)$, so we have the orthogonal decomposition $Z_{ij} = Z_{ij}^{+} \oplus Z_{ij}^{-}$ with ± 1 eigenspaces $Z_{ij}^{\pm}$ of $P(d)$. Let $y_{ij} = y_{ij}^{+} + y_{ij}^{-}$ be the corresponding decomposition of y_{ij} $(1 \leqq i \leqq j)$. Then we have

$$(6.2) \qquad (x \,\square\, x - P(x))y_{ij}^{\varepsilon} = (\xi_i - \varepsilon\xi_j)^2 y_{ij}^{\varepsilon}/2 \qquad (\varepsilon = \pm)$$

for $1 \leqq i \leqq j$. This together with the prescription $\xi_0 = 0$ leads us to the fact that the operator $x \,\square\, x - P(x)$ is positive semi-definite. Hence $k_a(x, y) \geqq 0$ by (1). Next, consideration of the operator $x \,\square\, x + P(x)$ instead gives the relation $-P(x) \leqq x \,\square\, x$, so we obtain by (1)

$$k_a(x, y) \leqq 4 \langle (x \,\square\, x)y \mid y \rangle \leqq 4 \|x \,\square\, x\| = 4 |\!| x |\!|_{1/2}^2.$$

Taking account of the symmetry of x, y in $k_a(x, y)$, we get

$$k_a(x, y) \leqq 4 \min(|\!| x |\!|_{1/2}^2, |\!| y |\!|_{1/2}^2).$$

Moreover we have $1 = \|x\|^2 = \sum \xi_j^2 \|d_j\|^2 \geqq |\!| x |\!|_{1/2}^2 \|d_1\|^2$. By Lemma 3.3 (1), we know $\|d_1\|^2 = 2\|ad_1^2\|^2 \geqq 2$, because ad_1^2 is an idempotent in the JH-algebra V. Hence $2|\!| x |\!|_{1/2}^2 \leqq 1$, completing the proof of Theorem 6.1. ∎

Remark 6.2. The formula (6.2) shows actually that we have the following

estimate of the sectional curvature $k_a(x,y)$ from below :

$$
\begin{aligned}
k_a(x,y) &= \sum_{j=1}^{v} \xi_j^2 \|y_{0j}\|^2 + \sum_{\varepsilon} \sum_{1 \leqq i \leqq j \leqq v} (\xi_i - \varepsilon \xi_j)^2 \|y_{ij}^{\varepsilon}\|^2 \\
&\geqq \xi_1^2 \|y_{01}\|^2 + 4\xi_1^2 \|y_{11}^-\|^2 = (\|y_{01}\|^2 + 4\,\|y_{11}^-\|^2)\,\|x\|_{1/2}^2.
\end{aligned} \tag{6.3}
$$

If $p = 1$ (the case of primitive idempotents), then $x = 2^{-1/2}(\sqrt{2}\,x)$ is the Schmidt expansion of the element x with $\|x\| = 1$ by [**19**, Lemma 2.3 (ii)], and the inequality in (6.3) becomes an equality with $\|x\|_{1/2} = 2^{-1/2}$. Moreover, $\langle\, x \,|\, y \,\rangle = 0$ implies $y = y_{01} + y_{11}^-$ (note $\sqrt{2}\,x$ is a *maximal* tripotent). Thus we arrive at $k_a(x,y) = (1 + 3\,\|y_{11}^-\|^2)/2$, which is exactly the formula given in [**19**, Theorem 6.4], for $Z_{11} = V_{1/2}(2x^2 - a) \cap V_{1/2}(a)$ in this case.

To take a closer look at the occurrence of $k_a(x,y) = 0$, we need the notion of flat subspaces, cf. [**16**, 3.10]. A subspace W of the Hilbert JTS $V_{1/2}(a)$ is said to be *flat* if $\{W,W,W\} \subset W$ and $\{u,v,w\} = \{v,u,w\}$ for all $u,v,w \in W$. We note that the maximal dimension of the flat subspaces of $V_{1/2}(a)$ is equal to $p < \infty$, the rank of $V_{1/2}(a)$.

Theorem 6.3. *Let $x, y \in V_{1/2}(a)$ with $\|x\| = \|y\| = 1$ and $\langle\, x \,|\, y \,\rangle = 0$. Then, $k_a(x,y) = 0$ if and only if x, y have simultaneous Schmidt expansions. In this case, x, y are contained in a flat subspace.*

Proof. Let $x = \sum_{j=1}^{v} \xi_j d_j$ $(\xi_1 > \xi_2 > \cdots > \xi_v > 0)$ be the Schmidt expansion of x and we use the notation in the proof (2) of Theorem 6.1. The first equality of (6.3) gives

$$
0 = \sum_{j=1}^{v} \xi_j^2 \|y_{0j}\|^2 + 4 \sum_{j=1}^{v} \xi_j^2 \|y_{jj}^-\|^2 + \sum_{\varepsilon} \sum_{1 \leqq i < j \leqq v} (\xi_i - \varepsilon \xi_j)^2 \|y_{ij}^{\varepsilon}\|^2.
$$

Hence y is necessarily of the form $y = y_{00} + \sum_{j=0}^{v} y_{jj}^{+}$. Then it holds that $(x \,\square\, x)y = \sum_{j=1}^{v} \xi_j^2 y_{jj}^{+}$, so that (2) of Definition 3.1 yields

$$\begin{aligned}[\, x \,\square\, x,\ y \,\square\, y\,] &= \{x,x,y\} \,\square\, y - y \,\square\, \{x,x,y\} \\ &= \sum_{j=1}^{v} \xi_j^2 (y_{jj}^{+} \,\square\, y) - \sum_{j=1}^{v} \xi_j^2 (y \,\square\, y_{jj}^{+}) = 0.\end{aligned}$$

This means that the finite rank selfadjoint operators $x \,\square\, x$ and $y \,\square\, y$ commute with each other, whence they have simultaneous eigenspace decompositions. Proceeding as in [**25**, Exercises 3, 4, p. 255], we see that x, y have simultaneous Schmidt expansions.

Conversely, if x, y have simultaneous Schmidt expansions $x = \sum \xi_j e_j$ and $y = \sum \eta_j e_j$ respectively, then it is immediate that $\{x,x,y\} = \sum \xi_j^2 \eta_j e_j = \{x,y,x\}$. Hence, $y \in \ker(x \,\square\, x - P(x))$, showing $k_a(x,y) = 0$. In this case, $W := \bigoplus \mathbb{R} e_j$ is evidently a flat subspace containing x, y. ∎

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