ON THE LOWER CENTRAL SERIES OF THE IA-AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract. In this paper, we study the Johnson homomorphisms τ'_k of the automorphism group of a free group of rank n, which are defined on the graded quotients of the lower central series of the IA-automorphism group. In particular, we determine the cokernel of τ'_k for any $k \geq 2$ and $n \geq k+2$.

1. Introduction

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\,F_n$ the automorphism group of F_n. Let denote $\rho : \mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ the natural homomorphism induced from the abelianization $F_n \to H$. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The subgroup IA_n reflects many richness and complexity of the structure of $\mathrm{Aut}\,F_n$, and plays important roles on various studies of $\mathrm{Aut}\,F_n$.

Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [14] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, any presentation for IA_n is not known in general. Nielsen [19] showed that IA_2 coincides with the inner automorphism group, hence, is a free group of rank 2. For $n \geq 3$, however, IA_n is much larger than the inner automorphism group $\mathrm{Inn}\,F_n$. Krstić and McCool [13] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

Because of the complexity of the group structure of IA_n as mentioned above, it would be sometimes not suitable to handle whole IA_n directly. In order to study IA_n with a phased approach, we consider the Johnson filtration of $\mathrm{Aut}\,F_n$. The Johnson filtration is one of descending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of normal subgroups of $\mathrm{Aut}\,F_n$, which first term is IA_n. (For detail, see Subsection 2.4.) Each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n, \mathbf{Z})$-module structure, and from it we can extract some valuable information for IA_n. For example, $\mathrm{gr}^1(\mathcal{A}_n)$ is just the abelianization of IA_n due to Andreadakis [1], and $\mathrm{gr}^2(\mathcal{A}_n)$ is applied to determine the image of the cup product $\cup_{\mathbf{Q}} : \Lambda^2 H^1(\mathrm{IA}_n, \mathbf{Q}) \to H^2(\mathrm{IA}_n, \mathbf{Q})$ by Pettet [20].

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To understand the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ more closely, we use the Johnson homomorphisms

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

(For detail, see Subsection 2.4.) One of the most fundamental properties of the Johnson homomorphism is that τ_k is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism for each $k \geq 1$. Hence, we can consider $\mathrm{gr}^k(\mathcal{A}_n)$ as a submodule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ which module structure is easy to handle. Historically, the study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [10] who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [11]. Now, there is a broad range of remarkable results for the Johnson homomorphisms of the mapping class group. (For example, see [9] and [16], [17], [18].) These works also inspired the study of the Johnson homomorphism of $\mathrm{Aut}\, F_n$. Recently, it achieved good progress through the works of many authors, for example, [4], [5], [6], [12], [16], [17], [18] and [20].

In general, from a viewpoint of computation, it seems that to determine the structure of the cokernel of the Johnson homomorphism is inclined to be more simpler and easier to handle than that of the image of the Johnson homomorphism. For $1 \leq k \leq 3$, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of the cokernel $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ of the rational Johnson homomorphism $\tau_{k,\mathbf{Q}} := \tau_k \otimes \mathrm{id}_{\mathbf{Q}}$ has been determined so far. (See [1], [20] and [22] for $k = 1$, 2 and 3 respectively.) Furthermore, by a recent remarkable work of Morita, it is known that there appears the symmetric tensor product $S^k H_{\mathbf{Q}}$ of $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$ in $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$. (See [18].) In general, however, it is quite difficult problem to determine $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ for arbitrary $k \geq 4$. One reason for it is that we cannot obtain an explicit generating system of $\mathrm{gr}^k(\mathcal{A}_n)$ easily.

To avoid this difficulty, we consider the lower central series $\mathcal{A}'_n(1) = \mathrm{IA}_n$, $\mathcal{A}'_n(2)$, $\ldots$ of IA_n. Since the Johnson filtration is central, we have $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ for each $k \geq 1$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$ in [1]. Now, it is known that $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [4, 5], Farb [6] and Kawazumi [12], and that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [20].

For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Since IA_n is finitely generated as mentioned above, each $\mathrm{gr}^k(\mathcal{A}'_n)$ is also finitely generated as an abelian group. Then we can define a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

by the same way as τ_k. We also call τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$. In our research, we are interested in the study of the cokernel of τ'_k for the following three reasons. First, we can directly obtain information about the cokernel of τ'_k using finitely many generators of $\mathrm{gr}^k(\mathcal{A}'_n)$. Second, using the representation theory, we can consider $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n, \mathbf{Z})$-submodule of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$. Hence, we can give an upper bound on $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$. Finally, by the conjecture that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$, these research would be applied to the study of the difference between the Johnson filtration and the lower central series of IA_n.

For $1 \leq k \leq 3$, we have $\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) = \mathrm{Coker}(\tau_{k,\mathbf{Q}})$, and hence they have been completely determined. In our previous paper [23], we give the irreducible decomposition of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ for $n \geq 6$. Furthermore, in [22], we showed that $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ is large in its own way. More precisely, let $T(H)$ be the tensor algebra of H, and $T(H)^{\mathrm{ab}}$ its abelianization as a Lie algebra. Then $T(H)^{\mathrm{ab}}$ naturally has a graded $\mathrm{GL}(n,\mathbf{Z})$-module structure. We denote by $\mathcal{C}_n(k)$ the degree k part of $T(H)^{\mathrm{ab}}$ for each $k \geq 1$. In [22], we have essentially shown that $\mathcal{C}_n^{\mathbf{Q}}(k)$ appears in $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n,\mathbf{Z})$-equivariant submodule. In particular, we have seen that $\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) = \mathcal{C}_n^{\mathbf{Q}}(k)$ for $1 \leq k \leq 4$ and $n \geq k+2$ from our results.

In this paper, we determine the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$ for $k \geq 2$ and $n \geq k+2$. Our main theorem is

Theorem 1. *(= Theorem 3.1.) For any $k \geq 2$ and $n \geq k+2$,*

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) = \mathcal{C}_n^{\mathbf{Q}}(k).$$

This paper consists of four sections. In Section 2, we recall the IA-automorphism group, the free Lie algebra and the Johnson homomorphisms of the automorphism group of a free group. In Section 3, we discuss the cokernel of the rational Johnson homomorphisms $\tau'_{k,\mathbf{Q}}$.

Contents

2. Preliminaries

In this section, we recall the IA-automorphism group, the free Lie algebra and the Johnson filtration of $\mathrm{Aut}\,F_n$.

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\,G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\,G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.

- For any $\mathbf{Z}$-module M, we denote $M \otimes_{\mathbf{Z}} \mathbf{Q}$ by the symbol obtained by attaching a subscript $\mathbf{Q}$ to M, like $M_{\mathbf{Q}}$ or $M^{\mathbf{Q}}$. Similarly, for any $\mathbf{Z}$-linear map $f : A \to B$, the induced $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ is denoted by $f_{\mathbf{Q}}$ or $f^{\mathbf{Q}}$.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **IA-automorphism group.**

In this paper, we fix a basis $x_1, \ldots, x_n$ of F_n. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n and $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ the natural homomorphism induced from the abelianization of F_n. In the following, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. It is clear that the inner automorphism group $\mathrm{Inn}\, F_n$ of F_n is contained in IA_n. In general, however, IA_n for $n \geq 3$ is much larger than $\mathrm{Inn}\, F_n$. In fact, Magnus [14] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} {x_j}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct i, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct i, j, $l \in \{1, 2, \ldots, n\}$ such that $j < l$. Recently, Cohen-Pakianathan [4, 5], Farb [6] and Kawazumi [12] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual group of H.

2.3. **Free Lie algebra.**

In this subsection we recall the free Lie algebra. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n, and by $\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ the associated graded sum. Since the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{L}_n(k)$ for each $k \geq 1$, and since IA_n acts on it trivially, the action of $\mathrm{GL}(n, \mathbf{Z})$ on each $\mathcal{L}_n(k)$ is well-defined. Furthermore, the graded sum $\mathcal{L}_n$ naturally has a graded Lie algebra structure induced from the commutator bracket on F_n, and called the free Lie algebra generated by H. (See [21] for basic material concerning the free Lie algebra.) It is classically well known due to Witt [25] that each $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n(k) := \frac{1}{k} \sum_{d | k} \mu(d) n^{\frac{k}{d}} \tag{2}$$

where μ is the Möbius function. For example, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathcal{L}_n(k)$ for $1 \leq k \leq 3$ is given by

$$\mathcal{L}_n(1) = H, \quad \mathcal{L}_n(2) = \Lambda^2 H,$$
$$\mathcal{L}_n(3) = (H \otimes_{\mathbf{Z}} \Lambda^2 H) \big/ \langle x \otimes y \wedge z + y \otimes z \wedge x + z \otimes x \wedge y \mid x, y, z \in H \rangle.$$

Next, we consider an embeddings of the free Lie algebra into the tensor algebra. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then $T(H)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\iota : \mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective graded Lie algebra homomorphism. We denote by ι_k be the homomorphism of degree k part of ι, and consider $\mathcal{L}_n(k)$ as a submodule $H^{\otimes k}$ through ι_k.

2.4. **Johnson homomorphisms.**

In this subsection, we recall the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. To begin with, we consider a descending filtration of $\mathrm{Aut}\, F_n$ called the Johnson filtration. For each $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on the nilpotent quotient group $F_n/\Gamma_n(k+1)$ of F_n induces a homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote its kernel by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = \mathrm{IA}_n$. (See [1] for details.) It is called the Johnson filtration of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$ on the graded quotients $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. The graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n.

In order to study the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n)$, we consider the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ as follows. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \ \mapsto \ (x \mapsto x^{-1} x^{\sigma}), \quad x \in H.$$

Then the kernel of $\tilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

The homomorphisms $\tilde{\tau}_k$ and τ_k are called the k-th Johnson homomorphisms of $\mathrm{Aut}\, F_n$. It is known that each τ_k is $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism. Therefore, to determine the image (or equivalently, the cokernel) of τ_k is an important problem on the study of the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

For the Magnus generators of IA_n, their images by τ_1 are given by

$$\tau_1(K_{ij}) = x_i^* \otimes [x_i, x_j], \quad \tau_1(K_{ijl}) = x_i^* \otimes [x_j, x_l]. \tag{3}$$

Furthermore, we remark that τ_1 is an isomorphism and nothing but the abelianization of IA_n. (See [4, 5, 6, 12].)

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is considered as $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, and we identify them in this paper. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k : \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}\,(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\xi$ the element of $\mathrm{Der}\,(\mathcal{L}_n)$ corresponding to an element $\xi \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$, and write the action of $\partial\xi$ on $X \in \mathcal{L}_n$ as $X^{\partial\xi}$ then we have

$$\tau_{k+l}([\sigma, \sigma']) = \tau_k(\sigma)^{\partial\tau_l(\sigma')} - \tau_l(\sigma')^{\partial\tau_k(\sigma)}.$$

for any $\sigma \in \mathcal{A}_n(k)$ and $\sigma' \in \mathcal{A}_n(l)$. This formula is very useful to study the image of the Johnson homomorphism inductively. In general, however, to determine the structure of the image and the cokernel of τ_k is quite difficult.

Let $\mathcal{A}_n'(k)$ be the lower central series of IA_n with $\mathcal{A}_n'(1) = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathcal{A}_n'(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. Set $\mathrm{gr}^k(\mathcal{A}_n') := \mathcal{A}_n'(k)/\mathcal{A}_n'(k+1)$ and $\mathrm{gr}(\mathcal{A}_n') := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n')$. Then $\mathrm{gr}(\mathcal{A}_n')$ is also a graded Lie algebra induced from the commutator bracket on IA_n, and $\mathrm{GL}(n, \mathbf{Z})$ naturally acts on each of $\mathrm{gr}^k(\mathcal{A}_n')$. Moreover, since IA_n is finitely generated by the Magnus generators K_{ij} and K_{ijl}, each $\mathrm{gr}^k(\mathcal{A}_n')$ is also finitely generated by commutators of weight k among the components K_{ij}s and K_{ijl}s.

A restriction of $\tilde{\tau}_k$ to $\mathcal{A}_n'(k)$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau_k' : \mathrm{gr}^k(\mathcal{A}_n') \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

and the sum

$$\tau' := \bigoplus_{k \geq 1} \tau_k' : \mathrm{gr}(\mathcal{A}_n') \to \mathrm{Der}\,(\mathcal{L}_n)$$

is also a graded Lie algebra homomorphism. Furthermore, we have

$$\tau_{k+l}'([\sigma, \sigma']) = \tau_k'(\sigma)^{\partial\tau_l(\sigma')} - \tau_l'(\sigma')^{\partial\tau_k(\sigma)}.$$

for any $\sigma \in \mathcal{A}_n'(k)$ and $\sigma' \in \mathcal{A}_n'(l)$. Using this formula recursively, we can easily compute $\tau_k'(\sigma)$ for any $\sigma \in \mathcal{A}_n'(k)$ from (3). We should remark that in general, it is not known whether τ_k' is injective or not. In this paper, we study the cokernel of the rational Johnson homomorphism $\tau_{k,\mathbf{Q}}' = \tau_k' \otimes \mathrm{id}_{\mathbf{Q}}$. We remark that for $1 \leq k \leq 4$, the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}}')$ have already determined as follows:

k	$\mathrm{Coker}(\tau_{k,\mathbf{Q}}')$	
1	0	Andreadakis [1]
2	$S^2 H_{\mathbf{Q}}$	Pettet [20]
3	$S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}}$	Satoh [22]
4	$S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}$	Satoh [23]

Here, for any $k \geq 1$, H^λ denotes the Schur-Weyl module of H corresponding to the partition λ of k. In particular, the modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product

S^kH and the exterior product Λ^kH respectively. (See [8] for basic material concerning the Schur-Weyl module for example.)

3. The cokernel of $\tau'_{k,\mathbf{Q}}$

In this section, we determine the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}}$ for $n \geq k+2$. In Subsection 3.1, we consider a lower bound on $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ using trace maps. In Subsections 3.2 and 3.3, we give an upper bound on $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$

3.1. **Contractions and trace maps.**

Let

$$T(H) = \bigoplus_{k \geq 0} H^{\otimes k}$$

be the tensor algebra generated by H over $\mathbf{Z}$. The algebra $T(H)$ is isomorphic to the non-commutative polynomial ring $\mathbf{Z}\langle x_1, \ldots, x_n \rangle$, and on which $\mathrm{GL}(n, \mathbf{Z})$ naturally acts. The abelianization $T(H)^{\mathrm{ab}}$ of $T(H)$ as a Lie algebra is also graded $\mathrm{GL}(n, \mathbf{Z})$-module. We write $\mathcal{C}_n(k)$ for the degree k part of $T(H)^{\mathrm{ab}}$. Namely, $\mathcal{C}_n(k)$ is a quotient module of $H^{\otimes k}$ by a submodule of $H^{\otimes k}$ generated by elements type of

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} - x_{i_2} \otimes \cdots \otimes x_{i_k} \otimes x_{i_1} \in H^{\otimes k}, \quad 1 \leq i_l \leq n.$$

Each of $\mathcal{C}_n(k)$ is also $\mathrm{GL}(n, \mathbf{Z})$-module. For $1 \leq k \leq 3$, the irreducible decomposition of $\mathcal{C}_n^{\mathbf{Q}}(k)$ is given by

$$\mathcal{C}_n^{\mathbf{Q}}(1) = H_{\mathbf{Q}}, \quad \mathcal{C}_n^{\mathbf{Q}}(2) = S^2 H_{\mathbf{Q}}, \quad \mathcal{C}_n^{\mathbf{Q}}(3) = S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}}.$$

In this subsection, we define trace maps which are used to detect $\mathcal{C}_n^{\mathbf{Q}}(k)$ in the cokernel of $\tau'_{k,\mathbf{Q}}$. To begin with, we consider contraction maps.

For $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi^k : H^* \otimes_{\mathbf{Z}} H^{\otimes (k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_n^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes (k+1)}$, we obtain a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\Phi^k = \varphi^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call Φ^k a contraction map.

Lemma 3.1. *For any $1 \leq i, i_1, \ldots, i_k \leq n$ such that $i_1 \neq i$, we have*

$$\begin{aligned} &\Phi^k(x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}]) \\ &\qquad = x_{i_1} \otimes \cdots \otimes x_{i_k} - \sum_{l=2}^{k} \delta_{i,i_l} [x_i, x_{i_1}, \ldots, x_{i_{l-1}}] \otimes x_{i_{l+1}} \otimes \cdots \otimes x_{i_k}. \end{aligned}$$

Proof. We prove the lemma by the induction on k. For $k=1$, since

$$x_i^* \otimes [x_i, x_{i_1}] = x_i^* \otimes (x_i \otimes x_{i_1} - x_{i_1} \otimes x_i),$$

we have $\Phi^1(x_i^* \otimes [x_i, x_{i_1}]) = x_{i_1}$. For any $k > 1$, we have

$$x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}] = x_i^* \otimes ([x_i, x_{i_1}, \ldots, x_{i_k-1}] \otimes x_{i_k} - x_{i_k} \otimes [x_i, x_{i_1}, \ldots, x_{i_{k-1}}]).$$

Hence, using the inductive hypothesis, we obtain

$$\begin{aligned}
\Phi^k(x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}]) \\
&= \Phi^{k-1}(x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_{k-1}}]) \otimes x_{i_k} - \Phi^1(x_i^* \otimes x_{i_k}) \otimes [x_i, x_{i_1}, \ldots, x_{i_{k-1}}], \\
&= x_{i_1} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_{i_k} \\
&\qquad - \sum_{l=2}^{k-1} \delta_{i,i_l}[x_i, x_{i_1}, \ldots, x_{i_{l-1}}] \otimes x_{i_{l+1}} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_{i_k} \\
&\qquad - \delta_{i,i_k}[x_i, x_{i_1}, \ldots, x_{i_{k-1}}].
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *For $n \geq k+1$, the contraction map Φ^k is surjective.*

Proof. For any $1 \leq i_1, \ldots, i_k \leq n$, there exists a some $1 \leq i \leq n$ such that $i \neq i_l$ for $1 \leq l \leq k$ since $n \geq k+1$. Then, from Lemma 3.1, we have

$$\Phi^k(x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}]) = x_{i_1} \otimes \cdots \otimes x_{i_k}.$$

Since $H^{\otimes k}$ is generated by elements type of $x_{i_1} \otimes \cdots \otimes x_{i_k}$, the homomorphism Φ^k is surjective. This completes the proof of Lemma 3.2. □

Now, let $\mathrm{Tr}(k)$ be the composition of the contraction map Φ^k and the natural projection $H^{\otimes k} \to \mathcal{C}_n(k)$. We call it the trace map for $\mathcal{C}_n(k)$. From Lemma 3.2, for $n \geq k+1$, the trace map $\mathrm{Tr}(k)$ is surjective. On the other hand, in our previous paper [22], we showed that

$$\mathrm{Tr}(k) \circ \tau'_k \equiv 0 \tag{4}$$

for any $n \geq k \geq 2$. Hence we see

Proposition 3.1. *For $n \geq k+1$ and $k \geq 2$,*

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \supset \mathcal{C}_n^{\mathbf{Q}}(k).$$

Here we remark that the inclusion in Proposition 3.1 means that $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ contains a $\mathrm{GL}(n, \mathbf{Z})$-submodule which is isomorphic to $\mathcal{C}_n^{\mathbf{Q}}(k)$.

3.2. The image of $\Phi^k \circ \tau'_k$.

Let $\mathcal{U}_n(k)$ be the kernel of the natural projection $H^{\otimes k} \to \mathcal{C}_n(k)$. From (4), we see that $\mathrm{Im}(\Phi^k \circ \tau'_k) \subset \mathcal{U}_n(k)$. In this subsection, we show that $\mathrm{Im}(\Phi^k \circ \tau'_k)$ coincides with $\mathcal{U}_n(k)$ for any $n \geq k+2$ and $k \geq 1$.

Lemma 3.3. *For $n \geq 3$ and $k \geq 1$, if $i_1, \ldots, i_{k+1} \neq i$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathrm{Im}(\tau'_k).$$

Proof. We show the lemma by induction on k. For $k = 1$, we have $\tau'_1(K_{ii_1i_2}) = x_i^* \otimes [x_{i_1}, x_{i_2}]$. Assume $k \geq 2$. By the inductive hypothesis, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}].$$

On the other hand, we have $\tau_1(K_{ii_{k+1}}) = x_i^* \otimes [x_i, x_{i_{k+1}}]$. Then

$$\tau'_k([K_{ii_{k+1}}, \sigma]) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}].$$

This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *For $n \geq 3$ and $k \geq 1$, if $i, j \neq i_2, \ldots, i_{k+1}$ and $i \neq j$, we have*

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \ldots, x_{i_k}] \in \mathrm{Im}(\tau'_k).$$

Proof. If $k = 1$, it is clear from the fact that τ'_1 is surjective. Suppose $k \geq 2$. From Lemma 3.3, there exists a certain $\sigma' \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma') = x_j^* \otimes [x_i, x_{i_2}, \ldots, x_{i_k}].$$

Then, we obtain

$$\tau'_k([K_{iji_{k+1}}, \sigma']) = x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \ldots, x_{i_k}].$$

This completes the proof of Lemma 3.4. $\square$

Proposition 3.2. *For $n \geq k+2$ and $k \geq 1$, the map $\Phi^k \circ \tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to \mathcal{U}_n(k)$ is surjective.*

Proof. If $k = 1$, it is clear since $\mathcal{U}_n(1) = 0$. suppose $k \geq 2$. For any $1 \leq i_1, \ldots, i_k \leq n$, there exists distinct i and j such that i, $j \neq i_l$ for $1 \leq l \leq k$ since $n \geq k+2$. Then from Lemma 3.4,

$$x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}] - x_j^* \otimes [x_j, x_{i_k}, x_{i_1}, \ldots, x_{i_{k-1}}] \in \mathrm{Im}(\tau'_k).$$

Using Lemma 3.1, we have

$$x_{i_1} \otimes \cdots \otimes x_{i_k} - x_{i_k} \otimes x_{i_1} \otimes \cdots \otimes x_{i_{k-1}} \in \mathrm{Im}(\Phi^k \circ \tau'_k).$$

Since $\mathcal{U}_n(k)$ is generated by all elements type of the above, we see that $\mathrm{Im}(\Phi_1^k \circ \tau'_k) = \mathcal{U}_n(k)$. This completes the proof of Proposition 3.2. $\square$

3.3. **The Kernel of Φ^k.**

In this subsection we show that $\mathrm{Ker}(\Phi^k) \subset \mathrm{Im}(\tau'_k)$ for $n \geq k+2$ and $k \geq 2$. (It is clear for the case where $k = 1$ since $\tau'_1 = \tau_1$ is surjective.) Here we use $\equiv$ for the equality in $\mathrm{Ker}(\Phi^k)$ modulo $\mathrm{Ker}(\Phi^k) \cap \mathrm{Im}(\tau'_k)$.

Take any $X \in \mathrm{Ker}(\Phi^k)$. We show $X \equiv 0$. Since $X \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, X is written as a linear combination of elements type of

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

for $1 \leq i$, $i_l \leq n$. We fix one of such expressions. In the following, we reduce such linear combination observing some elements in $\mathrm{Ker}(\Phi^k)$.

First, considering Lemma 3.3, we may assume that in each of $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ in the linear combination above, $i_l = i$ for at least one $1 \leq l \leq k+1$. Next, take any $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ such that $i_{l_1} = i_{l_2} = i$ for some $l_1 \neq l_2$. Since $n \geq k+1$, there exists a certain $1 \leq j \leq n$ such that $j \neq i, i_l$ for $1 \leq l \leq k+1$. If we set

$$\sigma_1 := \begin{cases} K_{iji_{k+1}}, & i \neq i_{k+1}, \\ K_{ij}^{-1}, & i = i_{k+1}, \end{cases}$$

we have

$$\tau'_1(\sigma_1) = x_i^* \otimes [x_j, x_{i_{k+1}}].$$

From Lemma 3.3, there exists a certain $\sigma_2 \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma_2) = x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_k}].$$

Then,

$$\begin{aligned} \tau'_k([\sigma_1, \sigma_2]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \\ &\qquad - \sum_{l=1}^{k} \delta_{i,i_l} x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, [x_j, x_{i_{k+1}}], x_{i_{l+1}}, \dots, x_{i_k}], \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \\ &\qquad + \sum_{l=1}^{k} \delta_{i,i_l} x_j^* \otimes [x_j, x_{i_{k+1}}, [x_{i_1}, \dots, x_{i_{l-1}}], x_{i_{l+1}}, \dots, x_{i_k}], \end{aligned} \tag{5}$$

If $n \geq k+2$, we can take a certain $1 \leq m \leq n$ such that $m \neq j, i_l$ for $1 \leq l \leq k+1$. By Lemma 3.3, there exist some $\sigma_3 \in \mathcal{A}'_n(k-1)$ such that

$$\begin{aligned} \tau'_{k-1}(\sigma_3) &= -x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_m, x_{i_{l+1}}, \dots, x_{i_k}] \\ &= x_j^* \otimes [x_m, [x_{i_1}, \dots, x_{i_{l-1}}], x_{i_{l+1}}, \dots, x_{i_k}]. \end{aligned}$$

Then we have

$$\begin{aligned} \tau'_k([K_{mji_{k+1}}, \sigma_3]) &= x_m^* \otimes [x_m, [x_{i_1}, \dots, x_{i_{l-1}}], x_{i_{l+1}}, \dots, x_{i_k}, x_{i_{k+1}}] \\ &\qquad - x_j^* \otimes [x_j, x_{i_{k+1}}, [x_{i_1}, \dots, x_{i_{l-1}}], x_{i_{l+1}}, \dots, x_{i_k}]. \end{aligned}$$

Using this and (5), we see that an element

$$\begin{aligned} Y &:= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \\ &\qquad + \sum_{l=1}^{k} \delta_{i,i_l} x_m^* \otimes [x_m, [x_{i_1}, \dots, x_{i_{l-1}}], x_{i_{l+1}}, \dots, x_{i_k}, x_{i_{k+1}}], \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \\ &\qquad - \sum_{l=1}^{k} \delta_{i,i_l} x_m^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_m, x_{i_{l+1}}, \dots, x_{i_k}, x_{i_{k+1}}] \end{aligned} \tag{6}$$

belongs to $\mathrm{Im}(\tau'_k)$. Furthermore, from Lemma 3.1, we also see that $Y \in \mathrm{Ker}(\Phi^k)$. Considering (6), we may assume that X is a linear combination of elements

$$x_i^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_i, x_{i_{l+1}}, \dots, x_{i_{k+1}}], \quad 1 \leq l \leq k+1 \tag{7}$$

such that $i_1, \dots, i_{l-1}, i_{l+1}, \dots, i_{k+1} \neq i$.

Now, for any $l > 1$, we denote by $\mathfrak{S}_l$ the symmetric group of degree l. Then we have

Lemma 3.5. *For any $l, m \geq 1$, an element*

$$[x_{i_1}, \dots, x_{i_l}, [x_{j_1}, \dots, x_{j_m}]] \in \mathcal{L}_n(l+m)$$

is written as a linear combination of elements

$$[x_{i_1}, \dots, x_{i_l}, x_{j_{\gamma(1)}}, \dots, x_{j_{\gamma(m)}}]$$

for some $\gamma \in \mathfrak{S}_{l+m}$.

Proof. We prove this Lemma by the induction on m. For $m = 1$, it is clear. Suppose $m \geq 2$. By using the Jacobi identity, we have

$$\begin{aligned}&[x_{i_1}, \ldots, x_{i_l}, [x_{j_1}, \ldots, x_{j_m}]] \\ &\quad = -[[x_{j_1}, \ldots, x_{j_{m-1}}], [x_{j_m}, [x_{i_1}, \ldots, x_{i_l}]] - [x_{j_m}, [[x_{i_1}, \ldots, x_{i_l}], [x_{j_1}, \ldots, x_{j_{m-1}}]] \\ &\quad = -[x_{i_1}, \ldots, x_{i_l}, x_{j_m}, [x_{j_1}, \ldots, x_{j_{m-1}}]] + [x_{i_1}, \ldots, x_{i_l}, [x_{j_1}, \ldots, x_{j_{m-1}}], x_{j_m}]\end{aligned}$$

in $\mathcal{L}_n(l+m)$. Hence by the inductive hypothesis, we obtain the required result. This completes the proof of Lemma 3.5. □

Using Lemma 3.5, we see that the element

$$(7) = -x_i^* \otimes [x_i, [x_{i_1}, \ldots, x_{i_{l-1}}], x_{i_{l+1}}, \ldots, x_{i_{k+1}}],$$

and hence X, is written as a linear combination of elements type of

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}]$$

for $i_2, \ldots, i_{k+1} \neq i$.

Lemma 3.6. *For $n \geq k+2$ and $k \geq 2$, if $i \neq i_2, \ldots, i_{k+1}$, then*

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] \in \mathrm{Im}(\tau_k').$$

Proof. Since $n \geq k+2$, there exists some $1 \leq j \leq n$ such that $j \neq i, i_l$ for $2 \leq l \leq k+1$. From Lemma 3.4, there exists some $\sigma \in \mathcal{A}_n'(k-1)$ such that

$$\tau_{k-1}'(\sigma) = x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_3}, \ldots, x_{i_k}].$$

Then we have

$$\tau_k'([\sigma, K_{ii_2}]) = x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}].$$

This completes the proof of Lemma 3.6. □

Lemma 3.7. *For $n \geq k+2$ and $k \geq 2$, if $i, j \neq i_2, \ldots, i_{k+1}$ and $i \neq j$, we have*

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathrm{Im}(\tau_k') \cap \mathrm{Ker}(\Phi^k).$$

Proof. From Lemma 3.6, we have

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] \in \mathrm{Im}(\tau_k').$$

On the other hand, from 3.4, we see

$$x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] - x_j^* \otimes [x_j, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathrm{Im}(\tau_k'),$$

and hence,

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathrm{Im}(\tau_k')$$

It is clear that this element belongs to $\mathrm{Ker}(\Phi^k)$ from Lemma 3.1. This completes the proof of Lemma 3.7. □

From Lemma 3.7, we see that $x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}]$ modulo $\mathrm{Ker}(\Phi^k) \cap \mathrm{Im}(\tau_k')$ does not depend on the choice of i such that $i \neq i_l$ for $2 \leq l \leq k+1$. Since $n \geq k+2$, for any $i_2, \ldots, i_{k+1}$, we can take some i such that $i \neq i_l$ for $2 \leq l \leq k+1$. We fix such $i = i(i_2, \ldots, i_{k+1})$, and set

$$s(i_2, \ldots, i_{k+1}) := x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}].$$

Then, using Lemma 3.7, we obtain

$$X \equiv \sum_{i_2,\dots,i_{k+1}=1}^{n} a_{i_2,\dots,i_{k+1}} s(i_2,\dots,i_{k+1}) =: X'$$

for some $a_{i_2,\dots,i_{k+1}} \in \mathbf{Z}$. By the assumption, $X' \in \mathrm{Ker}(\Phi^k)$. Therefore, by Lemma 3.1, we have

$$\Phi^k(X') = \sum_{i_2,\dots,i_{k+1}=1}^{n} a_{i_2,\dots,i_{k+1}} x_{i_2} \otimes \cdots \otimes x_{i_{k+1}} = 0 \in H^{\otimes k},$$

and hence

$$a_{i_2,\dots,i_{k+1}} = 0$$

for any $1 \leq i_l \leq n$. This shows that $X \equiv 0$. Thus we conclude

Proposition 3.3. *For $k \geq 2$ and $n \geq k+2$,*

$$\mathrm{Ker}(\Phi^k) \subset \mathrm{Im}(\tau'_k).$$

Finally, we determine the cokernel of $\tau'_{k,\mathbf{Q}}$ for $n \geq k+2$. Observing a sequence

$$H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(k+1) \xrightarrow{\Phi^k_{\mathbf{Q}}} H_{\mathbf{Q}}^{\otimes k} \to \mathcal{C}_n^{\mathbf{Q}}(k)$$

of $\mathrm{GL}(n,\mathbf{Z})$-equivariant surjective homomorphisms, we see

$$H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(k+1) \cong \mathrm{Ker}(\Phi^k_{\mathbf{Q}}) \oplus \mathcal{U}_n^{\mathbf{Q}}(k) \oplus \mathcal{C}_n^{\mathbf{Q}}(k)$$

as a $\mathrm{GL}(n,\mathbf{Z})$-module. Therefore, from Propositions 3.1, 3.2 and 3.3, we conclude that

Theorem 3.1. *For $k \geq 2$ and $n \geq k+2$,*

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) = \mathcal{C}_n^{\mathbf{Q}}(k).$$

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