A note on Samelson products in the exceptional Lie groups

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1 Introduction

Samelson products have been studied extensively for the classical groups ([5], [9], [10]), but few results are known for exceptional Lie groups. In [13], Ōshima determines the Samelson product

$$\pi_n(G_2) \times \pi_{11}(G_2) \to \pi_{n+11}(G_2)$$

for $n = 3, 11$. Let $G(l)$ be the compact, simply connected, exceptional simple Lie group of rank l, where $l = 2, 4, 6, 7, 8$. Define the set of integers $N(l)$ and the prime $r(l)$ as in the following table.

l	$G(l)$	$N(l)$	$r(l)$
2	G_2	$\{2, 6\}$	7
4	F_4	$\{2, 6, 8, 12\}$	13
6	E_6	$\{2, 5, 6, 8, 9, 12\}$	13
7	E_7	$\{2, 6, 8, 10, 12, 14, 18\}$	19
8	E_8	$\{2, 8, 12, 14, 18, 20, 24, 30\}$	31

If p is a prime and $p \geq r(l)$, then $G(l)$ is p-regular (see [14]), that is, there is a homotopy equivalence

$$\prod_{j \in N(l)} S^{2j-1}_{(p)} \xrightarrow{\simeq} G(l)_{(p)},$$

where $-_{(p)}$ stands for the localization at the prime p in the sense of Bousfield and Kan [3]. For $k \in N(l)$ define $\epsilon_{2k-1} \in \pi_{2k-1}(G(l)_{(p)}) \cong \mathbf{Z}_{(p)}$ by the composition

$$S^{2k-1} \xrightarrow{i} \prod_{j \in N(l)} S^{2j-1}_{(p)} \xrightarrow{\simeq} G(l)_{(p)},$$

where i is the canonical inclusion. The purpose of this paper is to show:

Theorem 1.1. *If $k_1, k_2 \in N(l)$ satisfy $k_1+k_2 = r(l)+1$, then the Samelson product $\langle \epsilon_{2k_1-1}, \epsilon_{2k_2-1}\rangle \neq 0$ in $\pi_{2r(l)}(G(l)_{(r(l))})$.*

Theorem 1.2. *If k_1 and $k_2 \in N(l)$ satisfy $k_1 + k_2 = r(l) + 1$, then the Samelson product $\langle\langle \epsilon_{2k_1-1}, \epsilon_{2k_2-1}\rangle, \epsilon_{2r(l)-3}\rangle \neq 0$ in $\pi_{4r(l)-3}(G(l)_{(r(l))})$.*

Corollary 1.1. *The nilpotency class of the localized self homotopy group $[G(l), G(l)]_{r(l)}$ is greater than or equal to 3.*

In order to prove Theorem 1.1, we use the following lemma which will be proved in §3 and §4. Let p be a prime greater than 5. Then we have

$$H^*(BG(l); \mathbf{F}_p) = \mathbf{F}_p[y_{2j}; j \in N(l)],\ |y_{2j}| = 2j. \tag{1.1}$$

Lemma 1.1. *Modulo $(\tilde{H}^*(BG(l); \mathbf{Z}/p))^3$, we have*

$$\mathcal{P}^1 y_4 \equiv \begin{cases} \xi_1 y_4 y_{60} + \xi_2 y_{16} y_{48} + \xi_3 y_{24} y_{40} + \xi_4 y_{28} y_{36} & (l,p) = (8,31) \\ \xi_1 y_4 y_{36} + \xi_2 y_{12} y_{28} + \xi_3 y_{16} y_{24} + \xi_4 y_{20}^2 & (l,p) = (7,19) \\ \xi_1 y_4 y_{24} + \xi_2 y_{12} y_{16} + \xi_3 y_{10} y_{18} & (l,p) = (6,13) \\ \xi_1 y_4 y_{24} + \xi_2 y_{12} y_{16} & (l,p) = (4,13) \end{cases}$$

for $\xi_j \in (\mathbf{Z}/p)^\times$.

Proof of Theorem 1.1. The proof for $l = 2$ is done in [13]. Put $l = 4, 6, 7$. Then we follow the proof of [8, Theorem 1.1]. Consider the map $\epsilon'_{2k} : S^{2k} \to BG(l)_{(p)}$ which is the adjoint of ϵ_{2k-1}. Suppose that the Whitehead product $[\epsilon'_{2k_1}, \epsilon'_{2k_2}] = 0$ for $k_1, k_2 \in N(l)$ and $k_1 + k_2 = p + 1$ ($p = r(l)$). Then we have a homotopy commutative diagram:

$$\begin{tikzcd} S^{2k_1} \vee S^{2k_2} \arrow[r, "\epsilon'_{2k_1} \vee \epsilon'_{2k_2}"] \arrow[d] & BG(l)_{(p)} \arrow[d, equal] \\ S^{2k_1} \times S^{2k_2} \arrow[r, "\theta"] & BG(l)_{(p)}, \end{tikzcd}$$

where the left vertical arrow is the inclusion. It is clear that $\mathcal{P}^1\theta^*(y_4) = 0$. On the other hand, we have $\theta^*(\mathcal{P}^1 y_4) \neq 0$. Then we obtain $[\epsilon'_{2k_1}, \epsilon'_{2k_2}] \neq 0$ and thus, by adjointness of Whitehead products and Samelson products, we have established $\langle \epsilon_{2k_1-1}, \epsilon_{2k_2-1}\rangle \neq 0$. □

Proof of Theorem 1.2. If $p = r(l)$, $G(l)$ is p-regular and then

$$\pi_{2p}(G(l)_{(p)}) \cong \bigoplus_{j \in N(l)} \pi_{2p}(S^{2j-1}_{(p)}) \cong \pi_{2p}(S^3_{(p)})$$

for a dimensional reason (see [15]). If $k_1, k_2 \in N(l)$ and $k_1 + k_2 = p + 1$, there is an integer $\xi'_{k_1,k_2} \in \mathbf{Z}^{\times}_{(p)}$ satisfying a homotopy commutative diagram

$$\begin{tikzcd} S^{2p} \arrow[r, "\alpha_1"] \arrow[dr, "\langle \epsilon_{2k_1-1}, \epsilon_{2k_2-1} \rangle"'] & S^3 \arrow[d, "\xi'_{k_1,k_2}\epsilon_3"] \\ & G(l)_{(p)}, \end{tikzcd}$$

where α_1 is a generator of the p-primary component of $\pi_{2p}(S^3)$ which is isomorphic to $\mathbf{Z}/p$. In particular, we have a homotopy commutative diagram:

$$\begin{tikzcd} S^{2p} \arrow[r, "\alpha_1"] \arrow[dr, "\langle \epsilon_3, \epsilon_{2p-1} \rangle"'] & S^3 \arrow[d, "\xi'_{2,p-1}\epsilon_3"] \\ & G(l)_{(p)} \end{tikzcd}$$

Then there is a homotopy commutative diagram:

$$\begin{tikzcd} S^{2p} \wedge S^{2p-3} \arrow[r, "\alpha_1 \wedge 1_{S^{2p-3}}"] \arrow[d, "\langle \epsilon_{2k_1-1}, \epsilon_{2k_2-1} \rangle \wedge \epsilon_{2p-3}"'] & S^3 \wedge S^{2p-3} \arrow[r, equal] \arrow[d, "\xi'_{k_1,k_2}\epsilon_3 \wedge \epsilon_{2p-3}"'] & S^{2p} \arrow[r, "\alpha_1"] \arrow[d, "\xi'_{k_1,k_2}\langle \epsilon_3, \epsilon_{2p-3} \rangle"'] & S^3 \arrow[dl, "\xi'_{k_1,k_2}\xi'_{2,p-1}\epsilon_3"] \\ G(l)_{(p)} \wedge G(l)_{(p)} \arrow[r, equal] & G(l)_{(p)} \wedge G(l)_{(p)} \arrow[r, "\gamma"] & G(l)_{(p)}, & \end{tikzcd}$$

where γ is the commutator map of $G(l)_{(p)}$. Since $\alpha_1 \circ (\alpha_1 \wedge 1_{S^{2p-3}}) \neq 0$ in $\pi_{4p+1}(S^3_{(p)})$ (see [14] and [15]), we have established Theorem 1.2. $\square$

Proof of Corollary 1.1. Define $\theta_i \in [G(l)_{(p)}, G(l)_{(p)}]$ for $i \in N(l)$ by the composition

$$G(l)_{(p)} \xrightarrow{q} S^{2k_1-1} \xrightarrow{\epsilon_{2k_1-1}} G(l)_{(p)},$$

where q is the projection. If $l \geq 4$, there are $k_1, k_2 \in N(l)$ satisfying $k_1 < k_2 < p = r(l)$ and $k_1 + k_2 = p + 1$. Then it follows from Theorem 1.2 that the commutator $[[\theta_{k_1}, \theta_{k_2}], \theta_{p-1}]$ in $[G(l)_{(p)}, G(l)_{(p)}] \cong [G(l), G(l)]_{(p)}$ is nontrivial ([3]) and therefore the proof is completed. $\square$

2 $W(E_8)$-invariant polynomials

Let p be a prime > 5. Then we have

$$H^*(BG(l); \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}[\tilde{y}_{2j}; j \in N(l)], \ \rho(\tilde{y}_i) = y_i,$$

where ρ is the modulo p reduction and y_i are as in (1.1). Let T be a maximal torus of $G(l)$ and let $W(G(l))$ be the Weyl group of $G(l)$. Consider the fibre sequence:

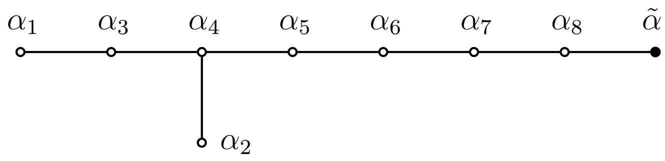
$$G(l)/T \to BT \xrightarrow{\lambda_l} BG(l)$$

Then $\lambda_l^*(\tilde{y}_{2j})$ is $W(G(l))$-invariant and the sequence $\{\lambda_l^*(\rho(\tilde{y}_{2j})); j \in N(l)\}$ is a regular sequence.

Let α_i $(i = 1, \ldots, 8)$ be the simple roots E_8. Then, as is well known, the dominant root $\tilde{\alpha}$ of E_8 is given by

$$\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.$$

Recall that the completed Dynkin diagram of E_8 is:



Let e_i be the dual of $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \in \mathbf{R}^8$ for $i = 1, \ldots, 8$. Then we can put

$$\begin{aligned}\alpha_1 =& \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7)\\ \alpha_2 =& e_1 + e_2\\ \alpha_i =& e_{i-2} - e_{i-3} \ (3 \leq i \leq 8).\end{aligned}$$

Hence we have $\tilde{\alpha} = e_7 + e_8$ (see [7]). Put $t_1 = -e_1, t_8 = -e_8$ and $t_i = e_i$ for $i = 2, \ldots, 7$. are generator of $H^*(BT)$. Let φ_i and $\tilde{\varphi}$ be the reflections on the hyperplanes $\alpha_i = 0$ and $\tilde{\alpha} = 0$ respectively. Then it is well known that $W(E_8)$ is generated by $\varphi_1, \ldots, \varphi_8$. Let W' be the subgroup of $W(E_8)$ generated by $\varphi_2, \ldots, \varphi_8, \tilde{\varphi}$. Namely, W' is the Weyl group of Ss(16) in E_8 which is a compact connected simple Lie group of type D_8. Put $\varphi = \varphi_1$. Then it is straight forward to check

$$\varphi(t_i) = t_i - \tfrac{1}{4}(t_1 + \cdots + t_8) \tag{2.1}$$

for $i = 1, \ldots, 8$.

We can regard $t_1, \ldots, t_8$ are a basis of $H^2(BT; \mathbf{Z}_{(p)})$. Define polynomials c_i and p_i by

$$\prod_{i=1}^{8}(1 + t_i) = \sum_{i=0}^{8} c_i$$

and

$$\prod_{i=1}^{8}(1 - t_i^2) = \sum_{i=0}^{8}(-1)^i p_i$$

respectively. Then since W' is the Weyl group of Ss(16) in E_8 as is noted above, we have

$$H^*(BT; \mathbf{Z}_{(p)})^{W'} = \mathbf{Z}_{(p)}[p_1, \ldots, p_7, c_8].$$

It follows from (2.1) that

$$\sum_{i=0}^{8}\varphi(c_i)=\prod_{i=1}^{8}(1+\varphi(t_i))=\prod_{i=1}^{8}(1-\tfrac{1}{4}c_1+t_i)=\sum_{k=0}^{8}(1-\tfrac{1}{4}c_1)^{8-k}c_k$$

and then we obtain

$$\varphi(c_1)=-c_1,\varphi(c_2)=c_2 \text{ and } \varphi(c_i)\equiv c_i-\tfrac{1}{4}(8-k+1)c_{i-1}c_1 \mod (c_1^2) \tag{2.2}$$

for $i=3,\ldots,8$. In particular, we have $\varphi(p_1)=p_1$ and the ideals $(c_1),(c_1^2,p_1)=(c_1^2,c_2),(c_1^2,p_1^2)=(c_1^2,c_2^2)$ are $W(E_8)$-invariant. Then it follows from (2.2) that

$$\varphi(p_j)\equiv p_j+h_jc_1 \mod (c_1^2)$$

for $i=2,\ldots,8$, where

$$\begin{aligned}
&h_2=\tfrac{3}{2}c_3, && h_3=-\tfrac{1}{2}(5c_5+c_3c_2), && h_4=\tfrac{1}{2}(7c_7+3c_5c_2-c_4c_3),\\
&h_5=-\tfrac{1}{2}(5c_7c_2-3c_6c_3+c_5c_4), && h_6=-\tfrac{1}{2}(5c_8c_3-3c_7c_4+c_6c_5), && h_7=\tfrac{1}{2}(3c_8c_5-c_7c_6).
\end{aligned}$$

Summarizing, we have established:

Lemma 2.1. *Modulo (c_1^2), we have*

$$\begin{aligned}
\varphi(p_4)&\equiv p_4+\frac{1}{2}(c_7c_1+3c_5c_2c_1-c_4c_3c_1), & \varphi(p_2^2)&\equiv p_2^2+6c_4c_3c_1+3c_3c_2^2c_1,\\
\varphi(c_8)&\equiv c_8-\frac{1}{4}c_7c_1, & \varphi(p_3p_1)&\equiv p_3+5c_5c_2c_1+c_3c_2^2c_1,\\
\varphi(p_2p_1^2)&\equiv p_2+6c_3c_2^2c_1.
\end{aligned}$$

Corollary 2.1. *If $f\in H^{16}(BT;\mathbf{Z}_{(p)})$ satisfies $\varphi(f)\equiv f \mod (c_1^2)$, then there exist $\alpha,\alpha'\in\mathbf{Z}_{(p)}$ such that*

$$f=\alpha\tilde{f}_{16}+\alpha' p_1^4,$$

where $\tilde{f}_{16}=120p_4+10p_2^2+1680c_8-36p_3p_1+p_2p_1^2$.

We also have established:

Lemma 2.2. *Modulo $(c_1^2,c_2^2)=(c_1^2,p_1^2)$, we have*

$$\begin{aligned}
\varphi(p_0)&\equiv p_6-\tfrac{5}{2}c_8c_3c_1+\tfrac{3}{2}c_7c_4c_1-\tfrac{1}{2}c_6c_5c_1,\\
\varphi(p_5p_1)&\equiv p_5p_1-3c_6c_3c_2c_1+c_5c_4c_2c_1,\\
\varphi(p_4p_2)&\equiv p_4p_2+3c_8c_3c_1+7c_7c_4c_1+3c_6c_3c_2c_1+3c_5c_4c_2c_1,-3c_5c_3^2c_1+\tfrac{1}{2}c_4^2c_3c_1,\\
\varphi(c_8p_2)&\equiv c_8p_2+\tfrac{3}{2}c_8c_3c_1-\tfrac{1}{2}c_7c_4c_1,\\
\varphi(p_3^2)&\equiv p_3^2+10c_6c_5c_1+2c_6c_3c_2c_1+10c_5c_4c_2c_1-5c_5c_3^2c_1-c_3^3c_2c_1,\\
\varphi(p_3p_2p_1)&\equiv p_3p_2p_1+6c_6c_3c_2c_1+10c_5c_4c_2c_1-3c_3^3c_2c_1,\\
\varphi(p_2^3)&\equiv p_3^3+18c_4^2c_3c_1.
\end{aligned}$$

Corollary 2.2. *If $f \in H^{24}(BT;\mathbf{Z}_{(p)})$ satisfies $\varphi(f) \equiv f \mod (c_1^2, p_1^2)$, there exists $\beta \in \mathbf{Z}_{(p)}$ such that $f \equiv \beta\tilde{f}_{24} \mod (p_1^2)$, where*

$$\tilde{f}_{24} = 60p_6 - 5p_5p_1 - 5p_4p_2 + 110c_8p_2 + 3p_3^2 - p_3p_2p_1 + \tfrac{5}{36}p_2^3.$$

Now we make a choice of generators of $H^*(BE_8;\mathbf{Z}_{(p)})$ as follows:

Theorem 2.1. *We can choose $\tilde{y}_{16}$ and $\tilde{y}_{24}$ such that $\lambda_8^*(\tilde{y}_{16}) = \tilde{f}_{16}$ and $\lambda_8^*(\tilde{y}_{24}) = \tilde{f}_{24} \mod (p_1^2)$.*

Proof. First of all, we can choose $\tilde{y}_4$ such that $\lambda_8^*(\tilde{y}_4) = p_1$. Then since $\varphi(\lambda_8^*(\tilde{y}_{16})) = \lambda_8^*(\tilde{y}_{16})$, it follows from Corollary 2.1 that we can choose $\tilde{y}_{16}$ such that $\lambda_8^*(\tilde{y}_{16}) = \alpha\tilde{f}_{16}$ for some $\alpha \in \mathbf{Z}_{(p)}$. Suppose that $(\alpha, p) = p$. Then $\{\lambda_8^*(\rho(\tilde{y}_4)), \lambda_8^*(\rho(\tilde{y}_{16}))\}$ is not a regular sequence and this is a contradiction. Thus we obtain $(\alpha, p) = 1$. The case of $\tilde{y}_{24}$ is quite similar. □

3 Proof of Lemma 1.1 for $l = 8$

We abbreviate the modulo 31 reduction of t_i, c_i, p_i by the same t_i, c_i, p_i respectively. We write the modulo 31 reduction of $\tilde{f}_i$ by f_i for $i = 16, 24$. Put $T_n = t_1^{2n} + \cdots + t_8^{2n}$. Then, by Girard's formula ([11]), we have:

$$(-1)^k T_k = k \sum_{i_1+2i_2+\cdots+8i_8} (-1)^{i_1+\cdots+i_8} \frac{(i_1+\cdots i_8 - 1)!}{i_1!\cdots i_8!} p_1^{i_1}\cdots p_8^{i_8}. \tag{3.1}$$

On the other hand, we have $\lambda_8^*(y_4) = p_1 = T_1, \mathcal{P}^1 T_1 = 2T_{16}$ and then

$$\lambda_8^*(\mathcal{P}^1 y_4) = 2T_{16} \tag{3.2}$$

We denote the subalgebra $\mathbf{F}_{31}[p_1, \ldots, p_7, c_8]$ of $H^*(BT;\mathbf{F}_{31})$ by R. Note that $\mathbf{Im}\lambda_8^* \subset R$. Define an algebra homomorphism $\pi_1 : R \to \mathbf{F}_{31}[x_1, x_5]/(x_1^2)$ by

$$\pi_1(p_i) = 0 \ (i = 2, 3, 4, 7),\ \pi_1(p_1) = x_1,\ \pi_1(p_5) = x_5,\ \pi_1(p_6) = \tfrac{1}{12}x_1x_5,\ \pi_1(c_8) = 0.$$

Put $\phi_1 = \pi_1 \circ \lambda_8^*$. Then we have $\phi_1(y_4^2) =$ and, by Theorem 2.1, $\phi_1(y_{16}) = \phi_1(y_{24}) = 0$. Hence, for a degree reason, we can put

$$\mathcal{P}^1 y_4 = \xi_1 y_4 y_{60} + \gamma_1$$

for $\xi_1 \in \mathbf{F}_{31}$ and $\gamma_1 \in \mathbf{Ker}\phi_1$. It follows from (3.1) that $2T_{16} \equiv p_1p_5^3 - p_5^2p_6 \mod \mathbf{Ker}\pi_1$ and hence we obtain $\pi_1(2T_{16}) = \frac{11}{12}x_1x_5^3 \neq 0$. Thus, by (3.2), we have established $\phi_1(\mathcal{P}^1 y_4) = \pi_1(2T_{16}) \neq 0$ which implies $\xi_1 \neq 0$.

Define an algebra homomorphism $\pi_2 : R \to \mathbf{F}_{31}[x_4, x_4']$ by

$$\pi_2(p_i) = 0 \ (i = 1, 2, 3, 5, 6, 7),\ \pi_2(p_4) = x_4,\ \pi_2(c_8) = x_4'.$$

Put $\phi_2 = \pi_2 \circ \lambda_8^*$. Then we have $\phi_2(y_4) = 0$ and, for a degree reason, $\phi_2(y_{24}) = \phi_2(y_{28}) = 0$. Thus, for a dimensional reason, we can put

$$\mathcal{P}^1 y_4 = \xi_2 y_{16} y_{48} + \xi_2' y_{16}^4 + \gamma_2$$

for $\xi_2, \xi_2' \in \mathbf{F}_{31}$ and $\gamma_2 \in \mathbf{Ker}\phi_2$. Now, by (3.1), we have $2T_{16} = \frac{1}{4}p_4^4 - p_4^2 p_8 + \frac{1}{2}p_8^2 \mod \mathbf{Ker}\pi_2$ and then $\pi_2(2T_{16}) = \frac{1}{4}x_4^4 - {x_4'}^2 x_4^2 + \frac{1}{2}{x_4'}^4$, where $p_8 = c_8^2$. This implies that if $\xi_2 = 0$, then $\xi_2' \neq 0$. On the other hand, it follows from Theorem 2.1 that $\phi_2(y_{16}) = 120(x_4 + 14x_4')$. Suppose that $\xi_2 = 0$. Then $\xi_2' \neq 0$ as above and thus, by (3.2),

$$\phi(\mathcal{P}^1 y_4) = \xi_2'(120(x_4 + 14x_4'))^4 = \tfrac{1}{4}x_4^4 - {x_4'}^2 x_4^2 + \tfrac{1}{2}{x_4'}^4.$$

This is a contradiction and therefore $\xi_2 \neq 0$.

Define an algebra homomorphism $\pi_3 : R \to R_3 = K[x_5, x_6]$ by

$$\pi_3(p_i) = 0\ (i = 1,2,3,4,7),\ \pi_3(p_5) = x_5,\ \pi_3(p_6) = x_6,\ \pi_3(c_8) = 0.$$

Put $\phi_3 = \pi_3 \circ \lambda_8^*$. Then it follows that $\phi_3(y_4) = \phi_3(y_{16}) = \phi_3(y_{28}) = 0$ and hence we can put

$$\mathcal{P}^1 y_4 = \xi_3 y_{24} y_{40} + \gamma_3$$

for $\xi_3 \in \mathbf{F}_{31}$ and $\gamma_3 \in \mathbf{Ker}\phi_3$. By (3.1), we have $2T_{16} = -p_5^2 p_6 \mod \mathbf{Ker}\pi_3$ and then $\pi_3(2T_{16}) = -x_5^2 x_6 \neq 0$ which implies $\xi_3 \neq 0$ by (3.2).

Define $\pi_4 : R \to \mathbf{F}_{31}[x_3, x_7]$ by

$$\pi_4(p_i) = 0\ (i = 1,2,4,5),\ \pi_4(p_3) = x_3,\ \pi_4(p_6) = -\tfrac{1}{20}x_3^2,\ \pi_4(p_7) = x_7,\ \pi_4(c_8) = 0.$$

Put $\phi_4 = \pi_4 \circ \lambda_8^*$. Then we have $\phi_4(y_4) = \phi_4(y_{16}) = 0$ and, by Theorem 2.1, $\phi_4(y_{28}) = 0$. Thus, for a dimensional reason, we can put

$$\mathcal{P}^1 y_4 = \xi_4 y_{28} y_{36} + \gamma_4$$

for $\xi_4 \in \mathbf{F}_{31}$ and $\gamma_4 \in \mathbf{Ker}\phi_4$. It follows from (3.1) that $2T_{16} = -2p_3 p_6 p_7 + p_3^3 p_7 \mod \mathbf{Ker}\pi_4$ and then $\pi_4(2T_{16}) = \frac{2}{20}x_3^3 x_7 + x_3^3 x_7 = \frac{11}{10}x_3^3 x_7 \neq 0$. Thus, for (3.2), we obtain $\xi_4 \neq 0$. Now we have established Lemma 1.1 for $l = 8$.

4 Proof of Lemma 1.1 for $l = 4, 6, 7$

Let us first recall the construction of $G(l)$ for $l = 4, 6, 7$. Consider the following commutative diagram of the natural inclusions:

$$\begin{tikzcd}
\mathrm{SU}(2) \arrow[r] \arrow[d] & \mathrm{SU}(3) \arrow[r] \arrow[d] & G_2 \arrow[d] & \\
\mathrm{Spin}(4) \arrow[r] & \mathrm{Spin}(6) \arrow[r] & \mathrm{Spin}(7) \arrow[r] & \mathrm{Spin}(16)
\end{tikzcd}$$

Note that we have the canonical map $i : \mathrm{Spin}(16) \to \mathrm{Ss}(16) \subset E_8$ as in the previous section. Then $i(G_2), i(\mathrm{SU}(3)), i(\mathrm{SU}(2))$ are the closed subgroups of E_8 and we know that F_4, E_6, E_7 are the identity component of the centralizers of the image of the above $i(G_2), i(\mathrm{SU}(3)), i(\mathrm{SU}(2))$ in E_8 respectively (see [1]). Consider the natural inclusion $\mathrm{Spin}(k) \times \mathrm{Spin}(16-k) \to \mathrm{Spin}(16)$ for $k = 4, 5, 6, 7$. Then we obtain a commutative diagram of inclusions:

$$\begin{tikzcd}
\mathrm{Spin}(9) \arrow[r, "j_1"] \arrow[d, "i_1"'] & \mathrm{Spin}(10) \arrow[r, "j_2"] \arrow[d, "i_2"'] & \mathrm{Spin}(11) \arrow[r] \arrow[d, "i_3"'] & \mathrm{Spin}(12) \arrow[d, "i_4"'] \\
F_4 \arrow[r, "k_1"] & E_6 \arrow[r, "k_2"] & E_7 \arrow[r, equal] & E_7
\end{tikzcd}$$

Next we recall classical results due to [2], [6] and [16] on the cohomology of homogeneous spaces given by the above inclusions:

Lemma 4.1. *The integral cohomology of E_6/F_4, $E_6/\mathrm{Spin}(10)$ and E_7/E_6 are given as follows.*

1. $H^*(E_6/F_4; \mathbf{Z}) = \Lambda(x_9, x_{17})$, $|x_j| = j$.
2. $H^*(E_6/\mathrm{Spin}(10); \mathbf{Z}) = \mathbf{Z}[x_8]/(x_8^3) \otimes \Lambda(x_{17})$, $|x_j| = j$.
3. $H^*(E_7/E_6; \mathbf{Z}) = \mathbf{Z}\{1, z_{10}, z_{18}, z_{37}, z_{45}, z_{55}\} \oplus \mathbf{Z}/2\{z_{28}\}$, $|z_j| = j$, *where $R\{a_1, a_2, \ldots\}$ stands for a free module with a basis $a_1, a_2, \ldots$ over a ring R.*

Hereafter, we let p be a prime greater than 5. Then the mod p cohomology of $BG(l)$ is given by (1.1). Then, by the standard spectral sequence argument together with 4.1, we obtain:

Lemma 4.2. 1. *We can choose generators $y_{2i} \in H^*(BE_6; \mathbf{F}_p)$ such that*

$$k_1^*(y_{2i}) = \begin{cases} 0 & i = 5, 9 \\ y_{2i} & i \in N(4) \end{cases}$$

2. *We can choose generators $y_{2i} \in H^*(BE_7; \mathbf{F}_p)$ such that*

$$k_2^*(y_{2i}) = \begin{cases} y_{10}^2 & i = 10 \\ y_{10}y_{18} & i = 14 \\ y_{2i} & i \in N(4). \end{cases}$$

Recall that we have

$$H^*(B\mathrm{Spin}(2l+1); \mathbf{F}_p) = \mathbf{F}_p[p_1, \ldots, p_l],$$
$$H^*(B\mathrm{Spin}(2l); \mathbf{F}_p) = \mathbf{F}_p[p_1, \ldots, p_{l-1}, c_l],$$

where p_i is the i-th universal Pontrjagin class and c_l is the Euler class. Let T^l be the standard maximal torus of $\mathrm{Spin}(2l+\epsilon)$ for $\epsilon = 0, 1$ and let $t_1, \ldots, t_l$ be the standard generators of $H^2(BT^l; \mathbf{F}_p)$. Then the canonical map $\lambda' : BT^l \to B\mathrm{Spin}(2l+1)$ satisfies

$$\sum_{j=0}^{l}(-1)^j \lambda'^*(p_j) = \prod_{i=1}^{l}(1-t_i^2), \tag{4.1}$$

where $p_0 = 1$ and $c_l^2 = p_l$ (see [4]). Specializing to our case, we have

$$j_1^*(p_k) = p_k \ (k=1,2,3,4),\ j_2^*(p_k) = p_k \ (k=1,2,3,4),\ j_1^*(c_5) = 0,\ j_2^*(p_5) = c_5^2.$$

It follows from [4, (3) in §19] that we can choose $y_{2i} \in H^*(BF_4; \mathbf{F}_p)$ for $i = 2, 6, 8$ such as

$$i_1^*(y_4) = p_1,\ i_1^*(y_{12}) = -6p_3 + p_2p_1,\ i_1^*(y_{16}) = 12p_4 + p_2^2 - \tfrac{1}{2}p_1^2p_2. \tag{4.2}$$

4.1 The case of $l = 7$

We put $l = 7$ and $p = 19$ in the above observation. Put $T_n = t_1^{2n} + \cdots + t_5^{2n}$. Then we have Girard's formula (3.1). By (4.1), we have $i_3^*y_4 = p_1 = T_1$ and then

$$i_3^*(\mathcal{P}^1 y_4) = \mathcal{P}^1 i_3^*(y_4) = \mathcal{P}^1 T_1 = 2T_{10}. \tag{4.3}$$

Define an algebra homomorphism $\pi_1 : \mathbf{F}_{19}[p_1, \ldots, p_5] \to \mathbf{F}_{19}[x_1, x_2, x_5]/(x_1^2, x_2^3, x_5^2)$ by

$$\pi_1(p_i) = x_i \ (i = 1, 2, 5),\ \pi_1(p_3) = \tfrac{1}{6}x_2x_1,\ \pi_1(p_4) = -\tfrac{1}{12}x_2^2$$

Then we have $\pi_1(p_4 1^2) = \pi_1(p_1p_3) = 0$. Put $\phi_1 = \pi_1 \circ i_3^*$. Then it follows from Lemma 4.2, (4.2) and a degree reason that $\phi_1(y_4^2) = \phi_1(y_{12}) = \phi(y_{16}) = \phi_1(y_{20}) = 0$. Hence, for a degree reason, we can put

$$\mathcal{P}^1 y_4 = \xi_1 y_4 y_{36} + \gamma_1$$

for $\xi_1 \in \mathbf{F}_{19}$ and $\gamma_1 \in \mathbf{Ker}\phi_1$. On the other hand, by (4.3), we have $2T_{10} \equiv 3p_1p_2^2p_5 - p_1p_4p_5 - p_2p_3p_5 \mod \mathbf{Ker}\pi_1$ and then $\pi_1(2T_{10}) = \frac{35}{12}x_1x_2^2x_5 \neq 0$. Thus, by (4.3), we have obtained $\xi_1 \neq 0$.

Define an algebra homomorphism $\pi_2 : \mathbf{F}_{19}[p_1, \ldots, p_5] \to \mathbf{F}_{19}[x_2, x_3, x_5]/(x_2^2, x_3^2, x_5^2)$ by

$$\pi_2(p_i) = x_i \ (i = 2, 3, 5),\ \pi_2(p_i) = 0 \ (i = 1, 4).$$

Put $\phi_1 = \pi_2 \circ i_3^*$. Then it follows from Lemma 4.2, (4.2) and a degree reason that $\phi_2(y_4) = \phi_2(y_{16}) = \phi_2(y_{20}^2) = 0$ and hence we can put

$$\mathcal{P}^1 y_4 = \xi_2 y_{12} y_{28} + \gamma_2$$

for $\xi_2 \in \mathbf{F}_{19}$ and $\gamma_2 \in \mathbf{Ker}\phi_2$. Now, by (3.1), we have $\pi_2(2T_{10}) = -x_2x_3x_5$ and then, by (4.3), $\xi_2 \neq 0$.

Define algebra homomorphisms $\pi_3 : \mathbf{F}_{19}[p_1, \dots, p_5] \to \mathbf{F}_{19}[x_2]$ and $\pi_4 : \mathbf{F}_{19}[p_1, \dots, p_5] \to \mathbf{F}_{19}[x_5]$ by

$$\pi_3(p_2) = x_2,\ \pi_3(p_i) = 0\ (i \neq 2),\ \pi_4(p_i) = 0,\ \pi_4(p_5) = x_5\ (i \neq 5).$$

Put $\phi_i = \pi_i \circ i_3^*$ for $i = 3, 4$. Then it follows from Lemma 4.2, (4.2) and a degree reason that $\phi_3(y_4) = \phi_3(y_{12}) = \phi_3(y_{20}) = 0$ and $\phi_4(y_4) = \phi_4(y_{12}) = \phi_4(y_{16}) = 0$. Thus we can put

$$P^1 y_4 = \xi_3 y_{16} y_{24} + \gamma_3 = \xi_4 y_{20}^2 + \gamma_4$$

for $\xi_3, \xi_4 \in \mathbf{F}_{19}$, $\gamma_3 \in \mathbf{Ker}\phi_3$ and $\gamma_4 \in \mathbf{Ker}\phi_4$. On the other hand, by (3.1), we have $2T_{10} = -\frac{1}{5}p_2^5 \mod \mathbf{Ker}\pi_3$ and $2T_{10} = \frac{1}{2}p_5^2 \mod \mathbf{Ker}\pi_4$. Then $\pi_3(2T_{10}) = -\frac{1}{5}x_2^5 \neq 0$ and $\pi_4(2T_{10}) = \frac{1}{2}x_5^2 \neq 0$ which imply $\xi_3 \neq 0$ and $\xi_4 \neq 0$. Therefore the proof of Lemma 1.1 for $l = 7$ is completed.

4.2 The case of $l = 4, 6$

We first consider the case $l = 4$ and $p = 13$. As in the above sections, we put $T_n = t_1^{2n} + \cdots t_4^{2n}$. Then we have Girard's formula (3.1) and

$$i_1^*(\mathcal{P}^1 y_4) = 2T_7. \tag{4.4}$$

Define an algebra homomorphism $\pi_1 : \mathbf{F}_{13}[p_1, \dots, p_4] \to \mathbf{F}_{13}[x_1, x_2]/(x_1^2)$ by

$$\pi_1(p_1) = x_1,\ \pi_1(p_2) = x_2,\ \pi_1(p_3) = \tfrac{1}{6}x_1x_2,\ \pi_1(p_4) = -\tfrac{1}{12}x_2^2.$$

Put $\phi_1 = \pi_1 \circ i_1^*$. Then it follows from (4.2) that $\phi_1(y_4^2) = \phi_1(y_{12}) = \phi_1(y_{16}) = 0$ and thus we can put

$$\mathcal{P}^1 y_4 = \xi_1 y_4 y_{24} + \gamma_1$$

for $\xi_1 \in \mathbf{F}_{13}$ and $\gamma_1 \in \mathbf{Ker}\phi_1$. By (3.1), we have $2T_7 = -(3p_1p_2^3 - 2p_1p_2p_4 - 2p_2^2p_3 + p_3p_4) \mod \mathbf{Ker}\pi_1$ and then $\pi_1(T_7) \neq 0$. Thus, for (4.4), we have obtained $\xi_1 \neq 0$.

We define an algebra homomorphism $\pi_1 : \mathbf{F}_{13}[p_1, \dots, p_4] \to \mathbf{F}_{13}[x_3, x_4]$ by

$$\pi_2(p_i) = 0\ (i = 1, 2),\ \pi_2(p_i) = x_i\ (i = 3, 4).$$

Put $\phi_2 = \pi_2 \circ i_1^*$. Then, by (4.2), we have $\phi_2(y_4) = 0$ and then we can put

$$\mathcal{P}^1 y_4 = \xi_2 y_{12} y_{16} + \gamma_2$$

for $\xi_2 \in \mathbf{F}_{13}$ and $\gamma_2 \in \mathbf{Ker}\phi_2$. It follows from (3.1) that $2T_{10} = -p_3p_4 \mod \mathbf{Ker}\pi_2$ and then $\pi_2(2T_7) \neq 0$ which implies $\xi_2 \neq 0$ by (4.4). Thus the proof of Lemma (1.1) for $l = 4$ is completed.

Next we consider the case $l = 6$ and $p = 13$. By Lemma 4.2 and the above result for $l = 4$, we only have to show $\xi_3 \neq 0$ in Lemma 1.1. As is seen in [12, Theorem 5.18], we have $\mathcal{P}^1\sigma(y_4) = \xi_3\sigma(y_{28})$ in $H^*(E_7; \mathbf{F}_{13})$ for some $\xi_3 \neq 0 \in \mathbf{F}_{13}$, where σ denotes the cohomology suspension. By Lemma 4.2, we have $k_2^*(y_{28}) = y_{10}y_{18}$ and then we obtain $\mathcal{P}^1 y_4 \equiv \xi_3 y_{10}y_{18}$ mod (y_4, y_{12}, y_{16}) in $H^*(BE_6; \mathbf{F}_{13})$. Therefore the proof of Lemma 1.1 is accomplished.

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