MEAN DIMENSION OF THE UNIT BALL IN ℓ^p

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Abstract. We prove that the mean dimension of the unit ball in $\ell^p(\Gamma)$ is zero ($1 \leq p < \infty$ and Γ is a finitely generated infinite amenable group). This is the answer to a question proposed by M. Gromov.

1. Main result

In this note we give a solution to a problem proposed by M. Gromov in [1, p. 340]. Let Γ be a finitely generated amenable group (cf. Gromov [1, p. 335]). In this paper we always assume that Γ is an infinite group. Let V be a finite dimensional $\mathbb{R}$-vector space with a norm $\|\cdot\|$. Let p be a real number such that $1 \leq p < \infty$, and set

$$\ell^p(\Gamma, V) := \{x = (x_\gamma)_{\gamma\in\Gamma} \in V^\Gamma |\ \|x\|_p := \left(\sum_{\gamma\in\Gamma} \|x_\gamma\|^p\right)^{1/p} < \infty\}.$$

We consider the natural right action of Γ on $\ell^p(\Gamma, V)$:

$$(x\cdot\delta)_\gamma := x_{\delta\gamma} \quad \text{for } x = (x_\gamma)_{\gamma\in\Gamma} \in \ell^p(\Gamma, V) \text{ and } \delta \in \Gamma.$$

Let $B(\ell^p(\Gamma, V))$ be the unit ball in $\ell^p(\Gamma, V)$:

$$B(\ell^p(\Gamma, V)) := \{x \in \ell^p(\Gamma, V) |\ \|x\|_p \leq 1\}.$$

We give the product topology to V^Γ, and we consider the restriction of this topology to $B(\ell^p(\Gamma, V)) \subset V^\Gamma$. Then $B(\ell^p(\Gamma, V))$ becomes a compact topological space (and it is metrizable). If $p > 1$, then this topology is equal to the restriction of the weak topology of $\ell^p(\Gamma, V)$. In this paper we always consider this topology on $B(\ell^p(\Gamma, V))$. $B(\ell^p(\Gamma, V))$ is Γ-invariant, and the action of Γ on $B(\ell^p(\Gamma, V))$ is continuous. Then we can consider the mean dimension $\dim(B(\ell^p(\Gamma, V)) : \Gamma)$ (cf. Gromov [1]). Our main result determines this value:

Theorem 1.1.

$$\dim(B(\ell^p(\Gamma, V)) : \Gamma) = 0.$$

Date: June 6, 2007.

2000 *Mathematics Subject Classification.* 46B20.

Key words and phrases. ℓ^p-spaces, mean dimension.

*JSPS Research Fellow; I am supported by Grant-in-Aid for JSPS Fellows (19·1530) from Japan Society for the Promotion of Science.

This is the answer to the question of Gromov proposed in [1, p. 340]. It makes a sharp contrast with the following (cf. Gromov [1, p. 340] and Lindenstrauss-Weiss [3, Proposition 3.3]):

$$\dim(B(\ell^\infty(\Gamma, V)) : \Gamma) = \dim V.$$

Remark 1.2. Actually the argument in Section 3 shows the following more general result:

Let Γ be a countable infinite group (not necessarily finitely generated nor amenable), and let $\{\Omega_i\}_{i\geq 1}$ be a sequence of finite sets in Γ such that $|\Omega_i| \to \infty$. Then we have

$$\dim(B(\ell^p(\Gamma, V)) : \{\Omega_i\}) = 0.$$

For the definition of $\dim(B(\ell^p(\Gamma, V)) : \{\Omega_i\})$, see Gromov [1, p. 338].

2. Preliminary constructions

Let n be a positive integer, and let $d_\infty(\cdot,\cdot)$ be the sup-distance on $\mathbb{R}^n$:

$$d_\infty(x, y) := \max_{1\leq i\leq n} |x_i - y_i| \quad \text{for } x = (x_1, \cdots, x_n) \text{ and } y := (y_1, \cdots, y_n).$$

Let S_n be the n-th symmetric group. We define the group G by

$$G := \{\pm 1\}^n \rtimes S_n.$$

The multiplication in G is given by

$$((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \cdot ((\varepsilon'_1, \cdots, \varepsilon'_n), \sigma') := ((\varepsilon_1\varepsilon'_{\sigma^{-1}(1)}, \cdots, \varepsilon_n\varepsilon'_{\sigma^{-1}(n)}), \sigma\sigma')$$

where $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon'_1 \cdots, \varepsilon'_n \in \{\pm 1\}$ and $\sigma, \sigma' \in S_n$. G acts on $\mathbb{R}^n$ by

$$((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \cdot (x_1, \cdots, x_n) := (\varepsilon_1 x_{\sigma^{-1}(1)}, \cdots, \varepsilon_n x_{\sigma^{-1}(n)})$$

where $((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \in G$ and $(x_1, \cdots, x_n) \in \mathbb{R}^n$. The action of G on $\mathbb{R}^n$ preserves the sup-distance $d_\infty(\cdot,\cdot)$.

We define $\mathbb{R}^n_{\geq 0}$ and Δ_n by

$$\mathbb{R}^n_{\geq 0} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | \, x_i \geq 0 \ (1 \leq i \leq n)\},$$
$$\Delta_n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | \, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

The following can be easily checked:

Lemma 2.1. *For $\varepsilon \in \{\pm 1\}^n$ and $x \in \mathbb{R}^n_{\geq 0}$, if $\varepsilon x \in \mathbb{R}^n_{\geq 0}$, then $\varepsilon x = x$. For $\sigma \in S_n$ and $x \in \Delta_n$, if $\sigma x \in \Delta_n$, then $\sigma x = x$. For $g = (\varepsilon, \sigma) \in G$ and $x \in \Delta_n$, if $gx \in \Delta_n$, then $gx = \varepsilon(\sigma x) = \sigma x = x$.*

Let m, n be positive integers such that $1 \leq m < n$. We define the continuous map $f_0 : \Delta_n \to \Delta_n$ by

$$f_0(x_1, \cdots, x_n) := (x_1 - x_{m+1}, x_2 - x_{m+1}, \cdots, x_m - x_{m+1}, \underbrace{0, 0, \cdots, 0}_{n-m}).$$

The following is the key fact for our construction:

Lemma 2.2. *For $g \in G$ and $x \in \Delta_n$, if $gx \in \Delta_n$ ($\Rightarrow gx = x$), then we have*

$$f_0(gx) = gf_0(x).$$

Proof. First we consider the case of $g = \varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$. If $x_{m+1} = 0$, then

$$f_0(\varepsilon x) = (\varepsilon_1 x_1, \cdots, \varepsilon_m x_m, 0, \cdots, 0) = \varepsilon f_0(x).$$

If $x_{m+1} > 0$, then $\varepsilon_i = 1$ $(1 \le i \le m+1)$ because $\varepsilon_i x_i = x_i \ge x_{m+1} > 0$ $(1 \le i \le m+1)$. Hence

$$f_0(\varepsilon x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0) = f_0(x) = \varepsilon f_0(x).$$

Next we consider the case of $g = \sigma \in S_n$. $gx \in \Delta_n$ implies $x_{\sigma(i)} = x_i$ $(1 \le i \le n)$. Set $y := f_0(x)$. Let r $(1 \le r \le m+1)$ be the integer such that

$$x_{r-1} > x_r = x_{r+1} = \cdots = x_{m+1}.$$

From $x_{\sigma(i)} = x_i$ $(1 \le i \le n)$, we have

$$1 \le i < r \Rightarrow 1 \le \sigma(i) < r \Rightarrow y_{\sigma(i)} = x_{\sigma(i)} - x_{m+1} = y_i,$$
$$r \le i \Rightarrow r \le \sigma(i) \Rightarrow y_{\sigma(i)} = 0 = y_i.$$

Hence we have $f_0(\sigma x) = f_0(x) = \sigma f_0(x)$.

Finally we consider the case of $g = (\varepsilon, \sigma) \in G$. Since $gx \in \Delta_n$, we have $gx = \varepsilon(\sigma x) = \sigma x = x \in \Delta_n$ (see Lemma 2.1). Hence

$$f_0(gx) = f_0(\varepsilon(\sigma x)) = \varepsilon f_0(\sigma x) = \varepsilon \sigma f_0(x) = gf_0(x).$$

□

We define a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ as follows; For any $x \in \mathbb{R}^n$, there is a $g \in G$ such that $gx \in \Delta_n$. Then we define

$$f(x) := g^{-1} f_0(gx).$$

From Lemma 2.2, this definition is well-defined. Since $\mathbb{R}^n = \bigcup_{g \in G} g\Delta_n$ and $f|_{g\Delta_n} = gf_0g^{-1}$ $(g \in G)$ is continuous on $g\Delta_n$, f is continuous on $\mathbb{R}^n$. Moreover f is G-equivariant.

Let p be a real number such that $1 \le p < \infty$, and define the ℓ^p-norm $\|\cdot\|_p$ by

$$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p} \quad \text{for } x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

Let $B_{\ell^p}(\mathbb{R}^n)$ be the ℓ^p-unit ball:

$$B_{\ell^p}(\mathbb{R}^n) := \{x \in \mathbb{R}^n | \ \|x\|_p \le 1\}.$$

$B_{\ell^p}(\mathbb{R}^n)$ is G-invariant.

Lemma 2.3. *For any $x \in B_{\ell^p}(\mathbb{R}^n)$, we have*

$$d_\infty(x, f(x)) \le \left(\frac{1}{m+1}\right)^{1/p}.$$

Note that the right-hand side does not depend on n.

Proof. Since f is G-equivariant and d_∞ is G-invariant, we can suppose $x \in B_{\ell^p}(\mathbb{R}^n) \cap \Delta_n$, i.e. $x = (x_1, x_2, \cdots, x_n)$ with $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. We have

$$f(x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0).$$

Hence

$$d_\infty(x, f(x)) = \max(x_{m+1}, x_{m+2}, \cdots, x_n) = x_{m+1}.$$

From $\|x\|_p \leq 1$,

$$(m+1)x_{m+1}^p \leq x_1^p + \cdots + x_{m+1}^p \leq 1.$$

Thus

$$d_\infty(x, f(x)) = x_{m+1} \leq \left(\frac{1}{m+1}\right)^{1/p}.$$

□

Proposition 2.4. *For any positive number ε, let m be a positive integer satisfying*

$$2\left(\frac{1}{m+1}\right)^{1/p} < \varepsilon.$$

Then we have

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_\infty) \leq m \quad \textit{for any } n \geq 1.$$

For the definition of $\mathrm{Widim}_\varepsilon$, *see Gromov* [1, p. 332].

Proof. If $n \leq m$, then the statement is trivial. Hence we suppose $m < n$. We have

$$f(\mathbb{R}^n) = \bigcup_{g \in G} g f(\Delta_n).$$

Note that $f(\Delta_n) \subset \mathbb{R}^m := \{(x_1, \cdots, x_m, 0, \cdots, 0) \in \mathbb{R}^n\}$. Lemma 2.3 implies that

$$f|_{B_{\ell^p}(\mathbb{R}^n)} : (B_{\ell^p}(\mathbb{R}^n), d_\infty) \to \bigcup_{g \in G} g \cdot \mathbb{R}^m \text{ is a } 2\left(\frac{1}{m+1}\right)^{1/p}\text{-embedding.}$$

Thus we get the conclusion. □

3. Proof of Theorem 1.1

First we consider the case of $V = \mathbb{R}$ with the natural norm. Set $\ell^p(\Gamma) := \ell^p(\Gamma, \mathbb{R})$ and $X := B(\ell^p(\Gamma))$. Let $w : \Gamma \to \mathbb{R}_{>0}$ be a positive function satisfying

$$\sum_{\gamma \in \Gamma} w(\gamma) \leq 1. \tag{1}$$

We define the distance $d(\cdot, \cdot)$ on X by

$$d(x, y) := \sum_{\gamma \in \Gamma} w(\gamma)|x_\gamma - y_\gamma| \quad \text{for } x = (x_\gamma)_{\gamma \in \Gamma} \text{ and } y = (y_\gamma)_{\gamma \in \Gamma} \text{ in } X.$$

This distance gives the topology introduced in Section 1. For a finite set $\Omega \subset \Gamma$ we define the distance $d_\Omega(\cdot,\cdot)$ on X by

$$d_\Omega(x,y) := \max_{\gamma\in\Omega} d(x\gamma, y\gamma).$$

Let ε be a positive number. We want to evaluate $\mathrm{Widim}_\varepsilon(X, d_\Omega)$.

For each $\delta \in \Gamma$, there is a finite set $\Omega_\delta \subset \Gamma$ such that

$$\sum_{\gamma\in\Gamma\setminus\Omega_\delta} w(\delta^{-1}\gamma) \le \varepsilon/4.$$

Set $\Omega' := \bigcup_{\delta\in\Omega} \Omega_\delta$. Ω' is a finite set satisfying

$$\sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma) \le \varepsilon/4 \quad \text{for any } \delta\in\Omega.$$

Let $\pi : X \to B_{\ell^p}(\mathbb{R}^{\Omega'}) = \{x \in \mathbb{R}^{\Omega'} \mid \|x\|_p \le 1\}$ be the natural projection. Let $m = m(\varepsilon)$ be a positive integer satisfying

$$2\left(\frac{1}{m+1}\right)^{1/p} < \varepsilon/2.$$

From Proposition 2.4, there are an m-dimensional polyhedron K and an $\varepsilon/2$-embedding $f : (B_{\ell^p}(\mathbb{R}^{\Omega'}), d_\infty) \to K$. Then $F := f\circ\pi : (X, d_\Omega) \to K$ becomes an ε-embedding; If $F(x) = F(y)$, then $d_\infty(\pi(x), \pi(y)) \le \varepsilon/2$ and for each $\delta\in\Omega$

$$\begin{aligned} d(x\delta, y\delta) &= \sum_{\gamma\in\Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma| + \sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma|, \\ &\le \frac{\varepsilon}{2}\sum_{\gamma\in\Omega'} w(\delta^{-1}\gamma) + 2\sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma), \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $d_\Omega(x,y) \le \varepsilon$. This shows the following proposition (we don't need the amenability of Γ for this proposition):

Proposition 3.1. *For any positive number ε, there is a positive integer $m = m(\varepsilon)$ such that*

$$\mathrm{Widim}_\varepsilon(X, d_\Omega) \le m \quad \textit{for any finite set } \Omega\subset\Gamma.$$

Theorem 3.2.

$$\dim(X:\Gamma) = 0.$$

Proof. Let $\Omega_1, \Omega_2, \cdots$ ($|\Omega_n| \to \infty$ as $n\to\infty$) be an amenable sequence in Γ (cf. Gromov [1, p. 335]). For any $\varepsilon > 0$, we have

$$\frac{1}{|\Omega_n|}\mathrm{Widim}_\varepsilon(X, d_{\Omega_n}) \le \frac{m(\varepsilon)}{|\Omega_n|} \to 0 \quad (n\to\infty).$$

Hence $\mathrm{Widim}_\varepsilon(X:\Gamma) = 0$ for all $\varepsilon > 0$. Thus $\dim(X:\Gamma) = 0$. □

Proof of Theorem 1.1. Set $s := \dim V$ and take a basis $e_1, \cdots, e_s$ on V. Let $\|\cdot\|_\infty$ be the sup-norm on V:

$$\|t_1 e_1 + \cdots + t_s e_s\|_\infty := \max(|t_1|, \cdots .|t_s|) \quad \text{for } t_1, \cdots, t_s \in \mathbb{R}.$$

There is a positive constant c such that

$$c\,\|v\|_\infty \leq \|v\| \quad \text{for all } v \in V,$$

where $\|\cdot\|$ is the given norm on V (see Section 1). Let $B_c(\ell^p(\Gamma, V))$ be the ball of radius c:

$$B_c(\ell^p(\Gamma, V)) := \{x \in \ell^p(\Gamma, V) |\ \|x\|_p \leq c\}.$$

$B_c(\ell^p(\Gamma, V))$ is Γ-equivariantly homeomorphic to $B(\ell^p(\Gamma, V))$. Hence

$$\dim(B(\ell^p(\Gamma, V)) : \Gamma) = \dim(B_c(\ell^p(\Gamma, V)) : \Gamma).$$

The isomorphism $V \cong \mathbb{R}^s$ $(t_1 e_1 + \cdots + t_s e_s \mapsto (t_1, \cdots, t_s))$ defines a Γ-equivariant linear isomorphism:

$$V^\Gamma \cong \underbrace{\mathbb{R}^\Gamma \times \cdots \times \mathbb{R}^\Gamma}_{s}.$$

This defines the following Γ-equivariant topological embedding:

$$B_c(\ell^p(\Gamma, V)) \hookrightarrow B(\ell^p(\Gamma))^s.$$

Using Theorem 3.2, we get

$$\dim(B(\ell^p(\Gamma, V)) : \Gamma) = \dim(B_c(\ell^p(\Gamma, V)) : \Gamma) \leq s \dim(B(\ell^p(\Gamma)) : \Gamma) = 0.$$

□

References

[1] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, Math. Phys. Anal. Geom. **2** (1999) 323-415

[2] E. Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, Inst. Hautes Études Sci. Publ. Math. **89** (1999) 227-262

[3] E. Lindenstrauss, B. Weiss, Mean topological dimension, Israel J. Math. **115** (2000) 1-24

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