Growth of Casson handles and transversality for ASD moduli spaces

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Introduction

Casson handles are open four manifolds which arise when topologically decomposing simply connected oriented four manifolds with respect to their intersection forms. Let us describe how Casson handles appear inside smooth four manifolds. Let M be a simply connected smooth closed oriented four manifold of even type. Let us fix an isomorphism of the forms:

$$\Phi : (H_2(M; \mathbf{Z}), (\quad , \quad)) \mapsto (\mathbf{Z}^{8k} \oplus \mathbf{Z}^{2l}, -kE_8 \oplus lH)$$

where H is the hyperbolic matrix. It is called a marking.

Freedman constructed a simply connected topological closed orientable four manifold $|-E_8|$ whose intersection form is isomorphic to $-E_8$. Moreover any Φ above is induced from a homeomorphism, $\Phi = \varphi_*$:

$$\varphi : M \cong k|-E_8| \sharp l(S^2 \times S^2)$$

where $k|-E_8|$ is k times connected sums of $|-E_8|$ ([Fr]).

With respect to the above decomposition, there exists an open subset $S \subset M$ homeomorphic to $l(S^2 \times S^2) \setminus \mathrm{pt}$, which is compatible with the decomposition of the forms. In particular S admits an induced smooth structure. This smoothing on the end is far from the standard product $S^3 \times [0, \infty)$ in general.

Casson and Freedman constructed the end of S explicitly. It admits a smooth decomposition:

$$S \cong D^4 \cup_{i=1}^{2l} CH(T_i)$$

by end connected sums, and $CH(T_i)$ are called *Casson handles.* We say that $CH(T_i)$ are *embedded Casson handles.*

Each Casson handle can be represented by T_i, an infinite tree with one end point, whose edges are assigned with signs $\pm$. We will call T_i infinite signed trees. When two signed trees admit an embedding $T \subset T'$, then correspondingly there is a smooth embedding of Casson handles $CH(T') \subset CH(T)$ with the same attatching circle. Thus growth of the signed trees parametrizing S above measures complexity of smooth structures of $S \subset M$.

There are sufficiently many Casson handles. In fact for each infinite signed tree, one can associate a Casson handle. It is known that there are uncountably many Casson handles which are mutually non diffeomorphic (see refs in [K1]). In fact even the simplest CH, periodic Casson handles are known to be exotic ([BG]). On the other hand, they are all homeomorphic to the standard open 2 handle $(D^2 \times D^2, S^1 \times D^2)$ ([Fr]).

In [K1], [K2], we have defined a subclass of infinite trees which is called *homogeneous trees of bounded type*. These are constructed by iterating to attatch infinitely many another half periodic trees. In precise, let $\mathbf{R}_+$ be the positive real line equipped with the same signs on each edge $+$ or $-$. The corresponding $CH(\mathbf{R}_+)$ is the periodic Casson handles. There are two types of them with respect to the signs $\pm$. The set of vertices in $\mathbf{R}_+$ are $\mathbf{N} \subset \mathbf{R}_+$. Attatch another infinite number of $\mathbf{R}_+$ on each vertex in $\mathbf{R}_+$ as:

$$T_2^0 = \mathbf{R}_+ \cup_{n \in \mathbf{N}} \mathbf{R}_+.$$

Again let us take infinite T_2^0 and attatch the base vertex of each T_2^0 with each $n \in \mathbf{N} \subset \mathbf{R}_+$ and one obtains:

$$T_3^0 = \mathbf{R}_3 = \mathbf{R}_+ \cup_{n \in \mathbf{N}} T_2^0.$$

One can iterate the same process and obtain T_m^0, $m = 1, 2, \ldots$. There are 2^m numbers of type T_m^0 with respect to choices of signs, and these are the homogeneous trees of bounded type.

Definition 0.1 *A connected infinite subtree of a homogeneous tree of bounded type is called a tree of bounded type.*

A Casson handle of a tree of bounded type is also called bounded type.

Thus a Casson handle of bounded type can be embedded into the one of homogeneously bounded type preserbing the attatching circle. It is known that they are all exotic (see [K1]). Any tree of bounded type grows polynomially. So for example regular infinite trees of valency more than two, like the binary trees are not included in this class.

Let M be as above, and take an $SO(3)$ bunfle $E \to M$. With respect to any marking $H_2(M : \mathbf{Z}) \cong \mathbf{Z}^{8k} \oplus \mathbf{Z}^{2l}$, the second SW class of E w_2 decomposes as $w_2 = w_2^1 \oplus w_2^2$. Let us say that a marking is *generic* with respect to the $SO(3)$ bundle E, if both $w_2^1 \neq 0$ and $w_2^2 \neq 0$ do not vanish.

In this paper we show the following:

Theorem 0.1 *Let M be K3 surface or logarithmic transforms X_p for odd p. Then there is an $SO(3)$ bundle $E \to M$ so that for any generic marking, the corresponding embedded Casson handles can not be all of bounded type.*

There are many logarithmic transforms which are homotopically equivalent to K3, but non diffeomorphic ([FriM],[GM],[Kr]).

For this bundle, there are many generic markings. In fact the construction of E above shows that with respect to the decomposition of the form over Kummer surface, there is some $\mathbf{Z}^{12} \subset \mathbf{Z}^8 \oplus \mathbf{Z}^8 \subset H_2(M : \mathbf{Z})$ on $-2E_8$ side so that w_2 does not vanish on each generating element in $\mathbf{Z}^{12}$. By the results on the unimodular forms, there are automorphisms of the form so that any generator in $-2E_8$ side can be transformed into $3H$ side.

One can obtain more general result. Suppose M admits a marking of the form isomorphic to $k(-E_8) \oplus lH$ which satisfies the followings. Correspondingly H_2 decomposes as $H_2(M : \mathbf{Z}) \cong \mathbf{Z}^8 \oplus \cdots \oplus \mathbf{Z}^8(k \text{ times}) \oplus \mathbf{Z}^2 \oplus \cdots \oplus \mathbf{Z}^2(l \text{ times})$. Suppose an $SO(3)$ bundle E satisfing $|p_1(E)| = 2l$ has $w_2 = w_2^1 \oplus w_2^2 = u_1 \oplus \dots u_k \oplus v_1 \oplus \cdots \oplus v_l$ with respect to the above decomposition so that these satisfy (1) both $w_2^1, w_2^2 \neq 0$ and (2) at least $l-1$ elements in $\{u_1, \dots, u_k, v_1, \dots, v_l\}$ do not vanish. Then if the Donaldson's invariant $Q(M) \neq 0$ does not vanish, then with respect to the marking, the embedded Casson handles are not all of bounded type. The above condition (2) is assumed so that bubbling of connections do not occurr as below.

The analytic tool to study such structure of Casson handles is the Yang-Mills gauge theory and Donaldson's invariant. The above theorem follows from our main theorem below and the following. Let Φ be any marking of the K3 surface, and for any $w \in H_2(M : \mathbf{Z})$, let $w = w_1 + w_2$ be the corresponding decomposition of the second homology group.

Proposition 0.1 (Kr) *Let M be a K3 surface and X_p be the logarithmic trans-*

form. Then there is an $SO(3)$ bundle E with $|p_1(E)| = 6$ and a marking over M or X_p satisfying $w_2^1, w_2^2 \neq 0$ so that the Donaldson's invariant satisfies:

$$Q(M) = \pm 1, \quad Q(X_p) = \pm p.$$

We say that an open four manifold S has *tree-like end* if there is a finite family of signed trees $T_1, \dots, T_l$, such that S is diffeomorphic to $D^4 \cup_j^l CH(T_j)$, where every $CH(T_j)$ is attatched to the zero handle D^4 along the attatching S^1 of the first stage kinky handle (which corresponds to the end point in T_j). Notice that there is a natural smooth structure on it induced from the handle body.

We have the following:

Theorem 0.2 (K1) *Let $S = D^4 \cup_j^l CH(T_j)$ have a tree-like end whose trees are homogeneous of bounded type. Then there are a complete Riemannian metric g of bounded geometry and weight function w on S so that the bounded complex:*

$$0 \ \to W_w^{k+2}(S; \Lambda^0) \ \xrightarrow{d} W_w^{k+1}(S; \Lambda^1) \ \xrightarrow{d_+} W_w^k(S; \Lambda_+^2) \ \to 0$$

is Fredholm with $H^0 = 0$, $H^1 = 0$ and $\dim H_+^2(S : \mathbf{R}) \leq \dim H^2 \leq 2l = \dim H^2(S : \mathbf{R})$.

One can choose Riemannian metrics explicitly on each building block, kinky handle, and will attach them isometrically.

Let S be an oriented open four manifold which is simply connected and also simply connected at inifinity. A pair of a complete Riemannian manifold g and a weight function w of bounded geometry on S is called *admissible*, if the triplet (S, g, w) satisfies the conclusion of theorem 0.2.

The proof of theorem 0.1 relies on only admissibility of metrics and do not use other special properties. Thus we have the following:

Theorem 0.3 *Let $S \subset M$ be embedded Casson handles in the theorem 0.1. Then there are no admissible pair of metrics and weight function (g, w) on S.*

Thus the Fredholm theory for the AHS complex cannot work on such Casson handles in K3 surface and its logarithmic transforms.

This complex is the linearization of the *Yang-Mills moduli space* (ASD moduli space) which is one of the most important subjects in four dimensional geometry. Let $E \to S$ be an $SO(3)$ bundle whose trivialization near the end is fixed.

Notice that both the end of S and S itself are simply connected. A connection A over E is called the *anti self dual* (ASD), if its curvature F_A satisfies the equation:

$$F_A^+ \equiv F_A + *F_A = 0.$$

Let A_0 be an ASD connection over (E, S, g) whose curvature is of L^2 finite, $||F_{A_0}||L^2(S,g) < \infty$. Then the ASD moduli space $\mathfrak{M}(A_0)_w$ is given by the set of the ASD solutions in $A_0 + W^k_w(S; E \otimes T^*S)$ modulo gauge transformation. It is a smooth finite dimensional manifold near $A \in A_0 + W^k_w(S; E \otimes T^*S)$, when d_A^+ is surjective. Notice that the base connection A_0 has a meanining only when the space S is open.

Let $\Sigma_1, \ldots, \Sigma_n \subset S$ be embedded surfaces. Then as in the construction of the Donaldson's invariant, one obtains the restricted ASD moduli space $\mathfrak{M}(A_0)_w(\Sigma_1, \ldots, \Sigma_n)$. Its formal dimension is less than $2n$ to the one of $\mathfrak{M}(A_0)_w$.

In order to realize transversality, we use the perturbed ASD equation by use of holonomy. In this paper it will play one of the most important role. The holonomy perturbation is local, and it has metrits when one considers families of solutions over an infinite sequences of spaces. In $2.B.6$, we have used properties of the holonomy perturbation that it gives enough supplementary sections. As far as these properties are satisfied, basically our argument can be applicable for other local perturbations.

Let M be an oriented closed smooth four manifold of even type with $\pi_1(M) = 1$, equipped with a marking:

$$\varphi : (H_2(M; \mathbf{Z}), <, >) \cong (\oplus^{8k}\mathbf{Z} \oplus^{2l} \mathbf{Z}, \quad k(-E_8) \oplus lH)$$

where $k \geq 2$ and $l \geq 3$. In this paper we always assume l is odd.

By Casson-Freedman, one finds an open four manifold S with tree-like end, homeomorphic to the interior of $l(S^2 \times S^2) \backslash D^4$, and finds a smooth embedding $S \hookrightarrow M$ which induces an embedding of the form:

$$(H^2(S : \mathbf{Z}), < \quad , \quad >) \cong (\oplus^{2l}\mathbf{Z}, lH) \hookrightarrow (H^2(M : \mathbf{Z}), < \quad , \quad >).$$

Let $E \to M$ be an $SO(3)$ bundle and take a generic marking Φ. Correspondingly let $S \subset M$ be as above.

with the second SW class $\hat{\alpha} \in H^2(M; \mathbf{Z}_2)$. Then one has the corresponding decomposition $\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2$ with respect to the form $k(-E_8) \oplus lH$.

Definition 0.2 *The triplet (M, E, Φ) is non degenerate, if there are exausing compact subsets $K_0 \subset K_1 \subset \cdots \subset S$ and a family of generic Riemannian metrics $\{g_i\}_i$ so that:*

(1) the $SO(3)$ Donaldson's invariant $Q(E; [\Sigma_1], \ldots, [\Sigma_n])$ is non zero for some $[\Sigma_1], \ldots, [\Sigma_n] \in H_2(S; \mathbf{Z})$,

(2) $||g_i|K_i - g|K_i||C^l \to 0$ as $i \to \infty$ for all $l = 0, 1, \ldots$

(3) For any family of (perturbed) ASD connections $A_i \in \mathfrak{M}(E; \Sigma_1, \ldots, \Sigma_n)$ over (M, g_i), a subsequence converges to an ASD connection A over (S, g) without bubbling on S on each compact subset in C^∞.

By the Uhlenbeck's compactness, any family of such connections contain a convergent subsequence, but in general bubbling may occur.

The main theorem is the following:

Theorem 0.4 *Suppose M has a non degenerate triplet (M, E, Φ). Then with respect to Φ, the embedded Casson handles cannot be all of bounded type.*

This gives an answer to the continuity problem addressed in [K2] under the non degeneracy condition.

When $|p_1(E)|$ is small, bubbling does not occurr as below:

Lemma 0.1 *Let E be an $SO(3)$ bundle with $|p_1| \leq 6$ and equipp with a generic marking. Suppose the $SO(3)$ Donaldson's invariant $Q(E)$ does not vanish. Then the condition (3) is satisfied.*

Proof: The conclusion follows except bubbling statement. By the Uhlenbeck's compactness, a subsequence of the family $\{A_i\}_i$ converges to A on each compact subset except bubbling points in C^∞. One can assume that the family $\{A_i\}_i$ itself converges by replacing the indices and $\{K_i\}_i$ by $\{K_{l_i}\}_i$ if necessarily.

As the Riemannian metrics are deformed by the family, E splits as E_1 and E_2 over S and S^c respectively. By genericity, both bundles E_1 and E_2 are non trivial. Since the base spaces S and S^c are both simply connected, they are not flat bundles.

If bubbling occur, then the estimate should hold:

$$|p_1(E_1)| \leq 6 - 4 - 2 = 0.$$

Thus E_1 becomes flat. This is a contradiction. This completes the proof.

Let us describe the idea of the proof of the main theorem. For simplicity, assume the generic ASD moduli space $\mathfrak{M}(E)$ over M has 0 dimension.

Suppose there could exist an admissible triplet (S, g, w). We induce a contradiction by the following argument.

Let us take exaustion on S by compact subsets as:

$$K_0 \subset K_1 \subset \cdots \subset S \subset M.$$

Choose a family of generic Riemannian metrics $\{g_i\}_i$ such that $g_i|K_i \sim g|K_i$ are sufficiently near each other in C^∞. By the condition (3), there exist a family of ASD connections A_i with respect to (M, g_i) which converges to an L^2 ASD connection A with respect to (S, g). By taking A as a base connection, one can construct the ASD moduli spaces $\mathfrak{M}(A, g)_w$, where we use weighted Sobolev spaces. Its formal dimension is of finite, and so if g could be generic with respect to (S, A), then it should be a non empty finite dimensional smooth manifold. But its dimension is is in fact negative, and so this will be a contradiction.

Thus in that case, one can say that these Casson handles should grow much more than bounded type so that the Fredholm theory breaks under non degeneracy condition.

In this paper we show that in fact this argument essentially works. Because we use infintely many spaces at the same time, the perturbation is required to be local. Here we use holonomy perturbation by Floer ([F]).

In order to describe difficulty of transversality condition, let us try to follow a standard argument of generic perturbation of metrics. The Freed-Uhlenbeck's generic perturbation method of Riemannian metrics also works over the open manifold S. Let $\mathfrak{C}$ be the set of sufficiently small perturbation of g by automorphisms of the tangent spaces. Then there exists a Baire set $\mathfrak{C}(A) \subset \mathfrak{C}$ so that for any $g' \in \mathfrak{C}(A)$, the corresponding weighted ASD moduli spaces $\mathfrak{M}(A, g')_w$ will be a smooth finite dimensional manifold, if it is non empty.

Now let us choose any g' as above. One can choose another generic family $\{g'_i\}_i$ with $g'_i \to g'$ as $i \to \infty$ and ASD connections $\{A'_i\}_i$ over (M, g'_i). Then again a subsequence converges to another A' which is ASD with respect to (S, g'). The point is that A' may not be an element in $\mathfrak{M}(A, g')_w$, since apriori, their difference:

$$A - A' \in L^2((S, g'); \Lambda^1 \otimes AdP)$$

is just in L^2. In order for A' in $\mathfrak{M}(A, g')_w$, the difference should be in the weighted L^2 space $L^2_w((S,g');\Lambda^1 \otimes AdP)$. On the other hand if one chooses A' as a base connection, then the moduli space $\mathfrak{M}(A',g')_w$ may not be a smooth manifold, since $g' \in \mathfrak{C}(A)$ depends on A and is generic only for $\mathfrak{M}(A, \quad)_w$.

In this paper we overcome this difficulty by using projection method of function spaces so that irrelevant parts of connections which might affect to estimate norms of connections are eliminated, and we pick up only restrictions of connections on $S \subset M$. More precisely even though $A_j|K_j^c$ will degenerate very badly, we show that $A_j|K_j$ part behaves very much like the ordinary convegence which can apply the above idea. For this we use the quotient Sobolev spaces $\bar{W}^*_w(K_i,g) \equiv W^*_w(S,g)/W^*_w(K_i^c,g)_0$. By use of holonomy perturbation, essentially we show that if we project perturbed connections $\bar{A}'_j$, then they converge to some A' which is uniformly bounded from A in $W^{k+1}_w(S,g)$.

Following contradiction argument, if embedded CH could be of boundede type, then using the Fredholm property of the AHS complex with respect to some pair (g,w) of complete Riemannian metrics and weight function over S, one would be able to obtain the non empty perturbed ASD moduli space $\mathfrak{M}_b(A)_w$ over S for a generic element b of the perturbation. Thus it becomes a regular manifold, but its formal dimension should be negative, which gives a contradiction.

In order to follow the above idea using these function spaces, we will use mostly more general arguments of functional analysis which are not special to ASD equations, and will work also for general operatros of Fredlolm type. In fact in order to find out A' above in the proof of theorem 0.4 in section $2.B$, we use a particular property of ASD only in $2.B.7$ and $2.B.8$, that Donaldson's invariant does not vanish.

We believe that detailed analysis of bubbling phenomenon will allow to extend our method to the case of higher dimensional moduli spaces.

We conjecture the following: Let M be a simply connected smooth closed oriented four manifold of even type. Suppose an $SO(3)$ bundle over M has non vanishing Donaldson's inavriant on $H_2(S:\mathbf{Z})$ as above. Then for any generic marking, the corresponding embedded Casson handles will not be all of bounded type.

1 ASD moduli spaces

Let (S, g) be a complete Riemannian open four manifold of bounded geometry. We always assume that both the end of S and S itself are simply connected. Let $E \to S$ be an $SO(3)$ vector bundle such that except some compact subset $K \subset S$, the trivialization of E on $S \backslash K$ is fixed. One denotes the corresponding principal $SO(3)$ bundle by P and the adjoint bundle by AdP for clarity. E is determined by $w_2(E) \in H^2(S; \mathbf{Z}_2)$ and $p_1(E)$.

Let us fix an L^2 ASD connection A_0 over E, $F_{A_0} + *F_{A_0} = 0$:

$$||F_{A_0}||L^2(S, g) < \infty.$$

By gauge transformation, one may assume that for $A_0 = d + u$ $||u||L^2 < \infty$ holds near the end, where u is the section of $AdP \otimes T^*S$.

Later on we denote the weighted Sobolev k spaces by $W^k_w(S, g)$. Then one defines an affine Hilbert space as:

$$\mathfrak{A}_k(P)_0 = \{A_0 + a | a \in W^k_w(S; AdP \otimes T^*S)\}, \quad k \geq 3.$$

Let us take $g \in C^1_{\text{loc}}(Y; AutE)$. By embedding as $g \in C^1_{\text{loc}}(S; Hom(E, E))$, one may consider $\nabla_{A_0} g \in C^0(Y; Hom(E, E) \otimes T^*S)$. Notice that if g is locally W^4, then it is of C^1 class. Now one defines the weighted Sobolev gauge group $(l \geq 4)$:

$$\begin{aligned} \mathfrak{G}_l(P) &= \{h \in W^l_{loc}(S; Aut(E)) | \nabla_{A_0} h \in W^{l-1}_w(S; Hom(E, E) \otimes T^*S)\}, \\ \mathfrak{G}_l(P)_0 &= \{h \in W^l_{loc}(S; Aut(E)) | h - id \in W^{l-1}_w(Y; Hom(E, E) \otimes T^*S)\}. \end{aligned}$$

The Lie algebras of $\mathfrak{G}_l(P)$ and $\mathfrak{G}_l(P)_0$ are correspondingly as follows:

$$\begin{aligned} \mathfrak{g}_l(P) &= \{h \in W^l_{loc}(Y; Ad(P)) | \nabla_{A_0} h \in W^{l-1}_w(Y; Ad(P) \otimes T^*Y)\}, \\ \mathfrak{g}_l(P)_0 &= W^l_w(Y; Ad(P)). \end{aligned}$$

These spaces all admit structure of Banach manifolds. $\mathfrak{G}_{k+1}(P)_0$ acts on $\mathfrak{A}_k(P)_0$ by $g^*(A_0 + a) = g^{-1}\nabla_{A_0} g + g^{-1} a g$. Notice that $\mathfrak{G}_{k+1}(P)$ may not act on $\mathfrak{A}_k(P)_0$, since A_0 may live only in $L^2(S, g)$.

Let us put:

$$\hat{\mathfrak{M}}_k(A_0)_w = \{A \in A_0 + W^k_w((S, g); T^*S \otimes AdP) : F^+(A) = 0\}$$

and take any $A \in \hat{\mathfrak{M}}_k(A_0)_w$. Then one has the AHS complex coefficient in A:

$$0 \longrightarrow W_w^{k+1}((S,g');\Lambda^0 \otimes AdP) \xrightarrow{d_A} W_w^k((S,g');\Lambda^1 \otimes AdP)$$
$$\xrightarrow{d_A^+} W_w^{k-1}((S,g');\Lambda_+^2 \otimes AdP) \longrightarrow 0.$$

Proposition 1.1 *Suppose* $d_A^+ : W_w^k(S; AdP \otimes T^*Y) \mapsto W_w^{k-1}(S; AdP \otimes \Lambda_+^2)$ *is surjection for* $A \in \hat{\mathfrak{M}}_k(A_0)_w$. *Then* $\mathfrak{M}_k(A_0)_w = \hat{\mathfrak{M}}_k(A_0)_w / \mathfrak{G}_{k+1}(P)_0$ *is a finite dimensional smooth manifold at* $[A]$. *Its tangent space is naturally isomorphic to* $\mathrm{Ker}\ \mathrm{d}_\mathrm{A}^+ / \mathrm{im}\ \mathrm{d}_\mathrm{A}$.

Now let M be a compact four manifold as in the introduction with $b_2^+(M) = 2l$, and E be an $SO(3)$ bundle over M with $-p_1(E) = s$. Then the dimension of the generic ASD moduli space $\mathfrak{M}(E)$ is $d \equiv 2s - 3(1+l)$.

Suppose an ASD connection A over $E_1 \to (S,g)$ is obtained as in the introduction.

Lemma 1.1 *The formal dimension* D *of* $\mathfrak{M}(A)_w$ *satisfies the inequality:*

$$D \leq d - 4 + 3 = d - 1.$$

Proof: Since the form is even, p_1 takes an even value. 3 comes from the constant gauge transformation near the infinity. This complets the proof.

In particular if $d = 0$, then D is negative, and so the generic ASD moduli space $\mathfrak{M}(A)_w$ will be empty. This will be a cruicial point in our argument of the proof of the main theorem.

1.B Holomony perturbation: Perturbation of the equations often uses Banach spaces. In this paper we use a variant of the Floer's holomony perturbation. In our analytic setting, we need to use Hilbert spaces as perturbation spaces, and for this, we will use larger spaces as Hilbert spaces B which contain the Floer's Banach space P densely. For our construction of perturbed connections, this extension is auxiliary, and finally we will choose generic paths inside P.

Let us take a fixed point $p_0 \in K_0$, and fix a trivialization of $E|U$ on some small neighbourhood U of p_0. Let $(l_1, \dots, l_K)$ be a family of smooth loops in K_0 with $l_i(0) = p_0$. For any $SO(3)$ connection A on M or on S, let:

$$\mathrm{Hol}(A) = (\mathrm{Hol}_{l_1}(A), \dots, \mathrm{Hol}_{l_K}(A)) \in SO(3)^K$$

be the set of holonomy around these loops.

Lemma 1.2 *There is a large K and a family of smooth loops $\{l_1, \dots, l_K\}$ so that for all large $i \geq i_0$, the corresponding holonomy maps:*

$$Hol(A_i), Hol(A) \in SO(3)^K$$

have no isotropy subgroups with respect to the diagonally adjoint $SO(3)$ action respectively.

This implies that the induced map $\mathfrak{B}(M,E) = \mathfrak{A}(E)/\mathfrak{G}(E) \to SO(3)^K/SO(3)$ is embedding near $[A_i]$ and $[A]$.

Let us extend these embedded loops as $l_i : D^3 \times S^1 \hookrightarrow M$, and take a smooth compactly supported function $\psi \in C^\infty(D^3)_c$ with $\int_{D^3} \psi = 1$. Then we extend the holonomy maps to $\mathrm{Hol}'_i(A_i) \in SO(3)$ which are small perturbations of:

$$\int_{D^3} \psi(x) \mathrm{Hol}_{l_i(\quad ,x)}(A_i).$$

By this way one obtains a smooth map:

$$\mathfrak{B}(M,E) = \mathfrak{A}(E)/\mathfrak{G}(E) \to SO(3)^K/SO(3)$$

which is embedding near $[A_i]$ and $[A]$ for all large i.

Floer constructed a Banach space P as a perturbation space by use of a subspace of C^∞ functions $C^{\bar{\beta}}(U) \subset C^\infty(U)_0$. In this paper we construct a Hilbert space B as the perturbation space which contains P by use of Sobolev spaces. It has a metrit to be a Hilbert space, by wasting smoothness.

When one chooses $b \in B \backslash P$, then the corresponding perturbed moduli spaces will be C^l manifolds rather than C^∞ for some l. This will still work enough for our method of application. But here in fact we will see that at last one can take $b \in P$ and so the corresponding moduli spaces are smooth, since $P \subset B$ is dense.

Let us choose a sufficiently large $m >> k$. For any $u \in C^0_c(U)^9$ and a connection A, let us assign:

$$\Psi(u, A) \in C^0(M; \Lambda^2_+ \otimes AdP)$$

by identifying u in this space by use of A. Let us identify $\Lambda^2_+ \mathbf{R}^4 \otimes so(3) \cong \mathbf{R}^9$, and let:

$$\gamma = SO(3)^K \times_{SO(3)} \mathbf{R}^9$$

be the vector bundle over $SO(3)^K/SO(3)$. Then we define B by:

$$B = W^m(\gamma \times U)_0.$$

It is a subspace of $W^m(SO(3)^K; \mathbf{R}^9)$ consisted by all $SO(3)$ equivariant Sobolev maps.

This embeds into $C^l(SO(3)^K \times \bar{U}; \mathbf{R}^9)^{SO(3)}$ by the Sobolev embedding for some l. The perturbation space P is a dense subspace of $C_0^\infty(SO(3)^K \times U; \mathbf{R}^9)^{SO(3)}$. In particular P is dense in our space B.

The maps Ψ induces a perturbation:

$$\begin{aligned} &s_0 : \mathfrak{B}_{k+1}(E) \times B \to W_0^k(M; \Lambda_+^2 \otimes AdP), \\ &\Psi([A], \Sigma_i u_i \otimes v_i) = \Psi(\Sigma_i u_i(\mathrm{Hol}A) \otimes v_i, A). \end{aligned}$$

Let us choose a sufficiently small $\epsilon_0 > 0$ and consider the perturbed map:

$$F_s^+ = F^+ + s \equiv F^+ + \epsilon_0 s_0 : \mathfrak{A}_k(E) \times B \to W^k(M; \Lambda_+^2 \otimes AdP).$$

Let g_i be a Riemannian metric on M, and denote its differential:

$$D_A^+ : W^{k+1}((M, g_i) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W^k((M, g_i) : \Lambda_+^2 \otimes AdP).$$

For any chioce $b \in B$, we denote the perturbed ASD moduli space by:

$$\mathfrak{M}_b(M, g_i) = \{(A, b) : F^+(A) + s(A, b) = 0\}/\mathfrak{G}_l(E).$$

Let $P \subset B$ be the Banach space by Floer. By the restriction, it also gives a perturbed map $F_s^+ : \mathfrak{A}_k(E) \times P \to W^k((M, g_i); \Lambda_+^2 \otimes AdP)$. It follows by the unique continuation property of elliptic equations, that the restriction $D_A^+ | W^{k+1}(M, g_i) \oplus T_0 P$ is surjective.

The essential point for this is that the restrictions on U of any nonzero elements in the cokernel are still nonzero. Once one has this property, then it is immediate to see surjectivity. For convenience, we recall its proof. Suppose not, and choose non zero element $u \in \mathrm{Ker}(D_A^+)^*_{w_i}$. As above, the restriction $u|U$ on the open subset is still non zero. Thus one can choose some $v \in C_c^\infty(U)$ so that the $L^2_{w_i}$ inner product with u is still non zero $< u, v > L^2_{w_i} \neq 0$. By lemma 1.2, there are no isotropy subgroup for $\mathrm{Hol}A \in SO(3)^K$, and so one can choose some $\psi \in P$ so that $\psi(\mathrm{Hol}A, \quad) = v$. In particular $< ds(A, \psi), u > L^2_{w_i} \neq 0$ which cannot happen, since u lies in the cokernel of D_A^+.

Now in particular $D_A^+ : W^{k+1}((M, g_i) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W^k((M, g_i) : \Lambda_+^2 \otimes AdP)$ is also surjective, since it is already so under the restriction.

Below we verify this by a different argument. Notice that when one uses Sobolev spaces for perturbation, the unique continuity becomes quite delicate matter. Let A be a smooth L^2 ASD connection obtained as $\lim_i A_i$ as before, and consider $F_s^+ : A + W_w^{k+1}(S, g) \oplus T_0 B \to W_w^k(S, g)$.

Proposition 1.2 *For a sufficiently small $\epsilon_0 > 0$, the linealization map:*

$$D_A^+ \equiv d_A^+ + ds : W_w^{k+1}((S, g); \Lambda^1 \otimes AdP) \oplus T_0 B \to W_w^k((S, g); \Lambda_+^2 \otimes AdP)$$

is surjective.

Proof: Let $H \subset W_w^k((S, g) : \Lambda_+^2 \otimes AdP)$ be the L_w^2 orthogonal complement to im d_A^+. By the unique continuity, the restriction $u|U$ of any non zero $u \in H$ on an open subset $U \subset S$ is still non zero.

Let U be the open subset used in the perturbation, and put $\delta = \inf\{||u||L_w^2(U) : u \in H, ||u||L_w^2(S, g) = 1\}$. Let us choose $0 < \epsilon_0 << \delta$. Then with respect to this, let $H' \subset W_w^k((S, g) : \Lambda_+^2 \otimes AdP)$ be the cokernel to D_A^+, L_w^2 orthogonal to the image. In particular H' is orthogonal to $D_A^+(W_w^{k+1}(S, g), 0)$. Since we have chosen a small $\epsilon_0 > 0$, for any $u' \in H'$, $||u||L_w^2(S, g) = 1$, the distance $||u' - H||L_w^2$ is sufficiently small. In order to verify the proposition, it is enough to see that for non zero $u' \in H'$, the restriction $u'|U$ is still non zero. With respect to the decomposition $L_w^2((S, g) : \Lambda_+^2 \otimes AdP) = H \oplus H^\perp$, let us decompose any $u = u_1 + u_2 \in H'$, $||u||L_w^2(S, g) = 1$. Then one may assume $||u_1||L_w^2(S, g) \geq 1 - \mu$ for a small $\delta >> \mu > 0$ and $||u_2||L_w^2(S, g) \leq \mu$. Then one has the estimates:

$$\begin{aligned} ||u||L_w^2(U) &\geq ||u_1||L_w^2(U) - ||u_2||L_w^2(U) \\ &\geq (1 - \mu)\delta - \mu > 0. \end{aligned}$$

This complets the proof.

Notice that one has surjectivity only near $A \in A + W_w^{k+1} \times T_0 B$. Later on, we fix such small $\epsilon_0 > 0$ and also fix the perturbation s as above.

In later sections, we will use W_w^*-orthogonal complements of spaces, rather than L_w^2.

Let us put:

$$V = \mathrm{im} d_A, \quad W = V^\perp \subset W_w^{k+1}((S, g); \Lambda^1 \otimes AdP)$$

where $V^{\perp}$ is the $W_w^{k+1}(S,g)$ orthogonal complement. Then we consider the restriction:

$$D_A^+ : W \oplus T_0B \to W_w^k((S,g);\Lambda_+^2 \otimes AdP).$$

The same proof of the above gives the following:

Corollary 1.1 *The above restriction is still surjective.*

Later on we choose and fix a small $\epsilon > 0$ so that ϵ ball $D_\epsilon \subset T_0B$ is identified with an open subset in B near zero.

By the Sard-Smale transversality, one obtains the following. There are some positive $\delta_1, \delta_2 > 0$ so that for the δ_i balls $D_{\delta_1} \subset W_w^{k+1}(S,g)$ and $D_{\delta_2} \subset B$, one obtains the following:

Proposition 1.3 *There is a Bair set $B(S,g) \subset D_{\delta_2}$, so that for any $b \in B(S,g)$, the perturbed moduli spaces $\mathfrak{M}_b(S,g) \cap (A + D_{\delta_1})/\mathfrak{G}$ is a finite dimensional manifold.*

One obtains similar statements by replacing (S,g) by (M,g_i).

Since $P \subset B$ is dense, for any $b \in B$ and small $\epsilon > 0$, there is some $b' \in P$ so that $||b - b'||B < \epsilon$. Since surjectivity is an open condition, one can choose any generic element $b \in P$ inside P. Later on we will use this property.

Now let g_i be a generic Riemannian metric on M. When one uses a smooth perturbation, then the smooth transversality follows by Floer:

Lemma 1.3 (F) *There is a Baire set $P(M,g_i) \subset P$, so that the perturbed moduli space $\mathfrak{M}_b(M,g_i)$ is a smooth manifold of finite dimension for any $b \in P(M,g_i)$. Moreover if g_i is already generic, then there is a smooth cobordism between $\mathfrak{M}(E,g_i)$ and $\mathfrak{M}_b(E,g_i)$.*

Let g_i be as in the introduction. Now the countable union of Baire sets is dense. Thus by putting $P' = \cap_i P(M,g_i) \subset P$, one obtains a dense subset in P so that for any $b \in P'$, the perturbed moduli spaces $\mathfrak{M}_{\mathfrak{b}}(M,g_i)$ and $\mathfrak{M}_w(A)_b$ are all finite dimensional manifolds if non empty. Moreover in that case, there are cobordisms between $\mathfrak{M}(M,g_i)$ and $\mathfrak{M}_b(M,g_i)$ by a generic smooth path b_t.

For our application, we need families of moduli spaces by two dimensional parametrization in $2.B.8$. Let $c : [0,1]^2 \to P$ be a smooth map and put $\mathfrak{M}_c(M,g_j) = \{([A], c(s,t)) : F_s^+([A], c(s,t)) = 0, 0 \leq s,t \leq 1\}$.

Let b_t and b'_t be two generic paths. The same argument as cobordisms states:

Lemma 1.4 *Let $c : [0,1]^2 \to P$ be a smooth map with $c(0,t) = b_t$ and $c(1,t) = b'_t$. Then for a Baire set of paths c as above, $\mathfrak{M}_c(M, g_j)$ give smooth finite dimensional manifolds with corners.*

2 Proof of theorem

Let E be the $SO(3)$ bundle over M, and choose a generic marking Φ with respect to E. Below we assume that the triplet (M, E, Φ) is non degenerate, and the corresponding $S \subset M$ has an admissible triple (S, g, w). Under this condition, we will induce a contradiction. Our final aim is to construct perturbed moduli spaces whose formal dimension is negative.

We verify the theorem when the generic ASD moduli space $\mathfrak{M}(E)$ has 0 dimension. If it has positive dimension, then by taking embedded surfaces $\Sigma_1, \dots, \Sigma_n \subset M$, one can consider the restricted 0 dimensional moduli space $\mathfrak{M}(E; \Sigma_1, \dots, \Sigma_n)$, as the construction of the Donaldson's invariant. The argument is parallel for these cases.

Let us choose an exaustion by compact subsets $K_0 \subset K_1 \subset \cdots \subset S$. Choose a generic family of Riemannian metrics $\{g_i\}_i$ and ASD connections A_i with respect to (M, g_i) so that $||g_i|K_i - g|K_i||C^k \to 0$ for all $k = 0, 1, \dots,$ and A_i converges to an $L^2(S, g)$ ASD connection A in C^∞ on each compact subset. By the assumption, bubbling does not occur on the family $\{A_i\}_i$. Let us put $\mathfrak{A}(A) = A + W^{k+1}_w((S, g); \Lambda^1 \otimes AdP)$.

Let us choose a generic element $b \in B$ with respect to all the family $\{A_i\}_i$ and A. We choose it so that its norm $||b||B$ is small.

We seek for a pair (A', b) with $F^+_s(A', b) = 0$ and $A - A' \in W^{k+1}_w((S, g); \Lambda^1 \otimes AdP)$. Then one obtains a non empty and generically weighted and perturbed ASD modulis space over S:

$$\mathfrak{M}_w(A, b)$$

whose formal dimension should be negative by lemma 1.1. This is enough to verify the theorem.

Let $\mathfrak{M}(M, g_i)$ be a regular ASD moduli space.

Lemma 2.1 *There is an element $[A_i] \in \mathfrak{M}(M, g_i)$ and a parametrization b_t, $b_0 = 0$ and $b_1 = b$ of small norms $||b_t|| << 1$ so that for any $t \in [0,1]$ and i, there are solutions (A^t_i, b_t) with $F^+_s(A^t_i, b_t) = 0$ over $E \to M$ and $A^0_i = A_i$.*

Proof: Since both g_i and b are generic and the Donaldson's invariant does not vanish, there is a cobordism between the ASD moduli space $\mathfrak{M}(M, g_i)$ and its perturbed one $\mathfrak{M}_b(M, g_i)$ in $\mathfrak{B}(E) \times B$. From this, the result follows. This completes the proof.

Let us take a family of C^∞ weight functions:

$$w_i : M \to [0, \infty)$$

so that $w_i|K_i = w|K_i$ hold, where w is the fixed one on S. Then using weighted Sobolev spaces $W^k_{w_i}((M, g_i); \Lambda^*)$, instead of the usual ones $W^k((M, g_i); \Lambda^*)$, one can obtain the same ASD moduli spaces over (M, g_i), since M is compact.

2.A Test case: Let $\mathfrak{M}(M, g_i)$ be a regular moduli space, and choose an element $[A_i]$ in it. Then we consider the surjective differential:

$$d^+_{A_i} : W^{k+1}_{w_i}((M, g_i); \Lambda^1) \to W^k_{w_i}((M, g_i); \Lambda^2_+).$$

Let

$$(d^+_{A_i})^*_{w_i} : W^{k+1}_{w_i}((M, g_i); \Lambda^2_+) \to W^k_{w_i}((M, g_i); \Lambda^1)$$

be the $L^2_{w_i}$ weighted adjoint operator. In the case when uniform estimates of spectral radii:

$$||(d^+_{A_i})^*_{w_i}|| \geq c > 0$$

might hold, where c is independent of i, then for any smooth path b_t with sufficiently small norms $||b_t|| << c$ and any $t \in [0, 1]$, the solutions:

$$F^+_s(A^t_i, b_t) = 0$$

will exist with respect to (M, g_i) by the implicit function theorem. Moreover uniform estimates:

$$||A_i - A^t_i||W^{k+1}_{w_i}(M, g_i : \Lambda^1 \otimes AdP) \leq C$$

hold. Then a subsequence $\{A^1_{i_m}\}_m$ converges to some A' over (S, g) which satisfies the perturbed ASD solution $F^+_s(A, b) = 0$. Moreover the estimate

$$||A - A'||W^{k+1}_w \leq C$$

holds. In particular one obtains an element in $\mathfrak{M}(A)_w$. This will lead a contradiction as we have already explained.

Of course it is too much to expect such situation, and we will have a refined argument below.

2.B Proof of the theorem 0.4**:** Now let us treat the general case, where we have no more uniformity of the spectrum from below with respect to i.

Let $H \subset W^k_w((S,g) : \Lambda^2_+ \otimes AdP)$ be the finite dimensional subspace of cokernel of d^+_A. Later on, we will assume that H has positive dimension. Otherwise we have nothing to do with, since the moduli space $\mathfrak{M}(A)_w$ would be regular when $H = 0$.

The idea is that the spectrum as above are certainly uniformly bounded from below for particular directions in Hilbert spaces. Firstly let us describe what are the situations we will encounter. When $b_t \in B$ is a generic path between 0 and b, then since the Donaldson's invariant does not vanish, both moduli spaces $\mathfrak{M}(M, g_i)$ and $\mathfrak{M}_b(M, g_i)$ admit non empty cobordisms. In particular there are continuous paths A^t_i satisfying the equations $F^+_s(A^t_i, b_t) = 0$. On the other hand $||A_i - A^t_i||W^{k+1}_{w_i}(K_i)$ may have unbounded norms. Recall that one has obtained an L^2 ASD connection A over (S, g) by taking a sebsequence of $\{A_i\}_i$. Thus one cannot expect $A - \lim_{i\to\infty} A^1_i|S$ might lie on $W^{k+1}_w(S, g; \Lambda^1 \otimes AdP)$.

The main point here is that connections on the complements of K_i will affect for our analysis. We will use a method to eliminate them on such domains by use of quotient function spaces.

2.B.2 Quotient spaces: Recall that the family of weight functions $\{w_i\}_i$ is fixed so that they satisfy $w_i|K_i = w|K_i$. Thus they are independent of the metrics.

Later on we fix a family of cut off functions $\varphi_i \in C^\infty_c(K_{i+1})$ satisfying $\varphi_i|K_i \equiv 1$.

Let us fix i and $j \geq i$. Thus $w_j|K_i = w_i|K_i$ hold. Let $K^c_i \subset M$ be the complement of K_i, and let us decompose:

$$\begin{aligned}
&W^k_{w_j}((M, g_j); \Lambda^* \otimes AdP) \\
&= W^k_{w_j}((K^c_i, g_j); \Lambda^* \otimes AdP)^\perp_0 \oplus W^k_{w_j}((K^c_i, g_j); \Lambda^* \otimes AdP)_0 \\
&\equiv W^k_{w_j}((K_i, g_j); \Lambda^* \otimes AdP) \oplus W^k_{w_j}((K^c_i, g_j); \Lambda^* \otimes AdP)_0.
\end{aligned}$$

We denote the quotient spaces as:

$$\bar{W}^k_{w_j}((K_i, g_j); \Lambda^* \otimes AdP)$$
$$= W^k_{w_j}((M, g_j); \Lambda^* \otimes AdP)/W^k_{w_j}((K_i^c, g_j); \Lambda^* \otimes AdP)_0.$$

Let h_j be complete Riemannian metrics on S so that $h_j|K_j = g_j|K_j$ and $h_j \to g$ uniformly in C^∞. Then for $j \geq i$, one also obtains similar spaces:

$$W^k_w((S, h_j); \Lambda^* \otimes AdP)$$
$$= W^k_w((K_i^c, h_j); \Lambda^* \otimes AdP)_0^\perp \oplus W^k_w((K_i^c, h_j); \Lambda^* \otimes AdP)_0,$$
$$\bar{W}^k_w((K_i, h_j); \Lambda^* \otimes AdP)$$
$$= W^k_w((S, h_j); \Lambda^* \otimes AdP)/W^k_w((K_i^c, h_j); \Lambda^* \otimes AdP)_0.$$

We equip with the metrics on them by use of orthogonal decomposition. Thus $\bar{W}^*_w(K_i, h_j)$ and $\bar{W}^*_{w_j}(K_i, g_j)$ are the same spaces, but they are different as metric spaces.

We show that $\bar{W}^k_{w_j}((K_i, g_j); \Lambda^* \otimes AdP)$ and $\bar{W}^k_w((K_i, h_j); \Lambda^* \otimes AdP)$ are uniformly equivalent as metric spaces. In fact they are asymptotically isometric in the following sense:

Lemma 2.2 *For any small $\epsilon > 0$, there is a constant $N >> 0$ independent of choice of i, j, so that for $j \geq i + N$, the metrics on them are $(1-\epsilon, 1+\epsilon)$-equivalent:*

$$(1-\epsilon)||\bar{u}||\bar{W}^k_{w_j}(K_i, g_j) \leq ||\bar{u}||\bar{W}^k_w(K_i, h_j) \leq (1+\epsilon)||\bar{u}||\bar{W}^k_{w_j}(K_i, g_j)$$

hold for all u.

Proof: Let $u \in W^k_{w_j}((K_i, g_j); \Lambda^* \otimes AdP)$ and $u' \in W^k_w((K_i, h_j); \Lambda^* \otimes AdP)$ be representatives of $\bar{u}$. Then for some $i \leq i_0 \leq j-1$, the estimate:

$$||u||W^k_{w_j}(K_{i_0+1} \backslash K_{i_0}, g_j) \leq \epsilon ||u||W^k_{w_j}(M, g_j)$$

should hold, where $\epsilon = 1/N$. Let $\varphi_{i_0} \in C^\infty_c(K_{i_0+1})$ be the cut off function. Then $||\varphi_{i_0} u||W^k_{w_j}(M, g_j) \leq (1 + C\epsilon)||u||W^k_{w_j}(M, g_j)$ holds where C is independent of i, j and ϵ. At the same time one may regard $\varphi_{i_0} u \in W^k_w(S, h_j)$. Since $\bar{\varphi_{i_0} u} = \bar{u}$ have the same quotient element, this implies the estimates:

$$\begin{aligned} ||\bar{u}||\bar{W}^k_w(K_i, h_j) &= ||u'||W^k_w(S, h_j) \leq ||\varphi_{i_0} u||W^k_w(S, h_j) = ||\varphi_{i_0} u||W^k_w(M, g_j) \\ &\leq (1 + C\epsilon)||u||W^k_{w_j}(M, g_j) = (1 + C\epsilon)||u||W^k_{w_j}(M, g_j) \\ &= (1 + C\epsilon)||\bar{u}||\bar{W}^k_{w_j}(K_i, g_j). \end{aligned}$$

By the same argument, one obtains another inequality $||\bar{u}||\bar{W}^k_{w_j}(K_i, g_j) \leq (1 + C\epsilon)||\bar{u}||\bar{W}^k_w(K_i, h_j)$. Since ϵ can be chosen arbitrarily small by taking large $j >> i$, this completes the proof.

Since metrics are all uniformly equivalent, later on one can choose any of them.

Let us consider the perturbed self dual curvature operators:

$$F^+_s : A_j + W^{k+1}_{w_j}((M, g_j) : \Lambda^1 \otimes AdP) \to W^k_{w_j}((M, g_j) : \Lambda^2_+ \otimes AdP)$$

by $F^+_s(A_j + \alpha, b) \equiv F^+(A_j + \alpha) + s(A_j + \alpha, b)$.

Lemma 2.3 *Suppose $F^+_s(A_j, b) = 0$ over (M, g_j) holds. Then the above operator induces:*

$$\bar{F}^+_s(\quad, b) : A_j + \bar{W}^{k+1}_{w_j}((K_i, g_j); \Lambda^1 \otimes AdP) \to \bar{W}^k_{w_j}((K_i, g_j); \Lambda^2_+ \otimes AdP).$$

Proof: It is enough to check that for $\alpha \in W^{k+1}_{w_j}((K_i^c, g_j); \Lambda^1 \otimes AdP)_0$, the element $F^+_s(A_j + \alpha, b)$ lies in $W^k_{w_j}((K_i^c, g_j); \Lambda^2_+ \otimes AdP)_0$. By the condition,

$$F^+_s(A_j + \alpha, b) = d^+_{A_j}(\alpha) + P_+(\alpha \wedge \alpha)$$

holds, since the support of s lies in K_0, where P_+ is the projection to the self dual part. From the Sobolev inequality, the result follows. This completes the proof.

Corollary 2.1 *Let $\alpha \in W^{k+1}_{w_j}((M, g_j) : \Lambda^1 \otimes AdP)$. If $F^+_s(A_j + \alpha, b) = 0$ holds, then still $\bar{F}^+_s(A_j + \bar{\alpha}, b) = 0$ holds, where $\bar{\alpha} \in \bar{W}^{k+1}_{w_j}((K_i, g_j) : \Lambda^1 \otimes AdP)$ is the projection.*

Let $^-$ be the projection onto $\bar{W}^*_{w_i}(K_i, g_j)$.

Lemma 2.4 *Let us take A_j on (M, g_j) and suppose the equality $F^+(A_j) = 0$ holds. Then the projection induces maps:*

$$d_{\bar{A}_j} : \bar{W}^{k+1}_{w_j}((K_i, g_j); AdP) \to \bar{W}^k_{w_j}((K_i, g_j); \Lambda^1 \otimes AdP),$$
$$d^+_{\bar{A}_j} : \bar{W}^k_{w_j}((K_i, g_j); \Lambda^1 \otimes AdP) \to \bar{W}^{k-1}_{w_j}((K_i, g_j); \Lambda^2_+ \otimes AdP).$$

In particular, if $d^+_{A_j}$ is surjective, then $d^+_{\bar{A}_j}$ is still the same.

Proof: This follows since the inclusions:

$$d^+_{A_j}(W^{k+1}_{w_j}((K_i^c, g_j); \Lambda^1 \otimes AdP)_0) \subset W^k_{w_j}((K_i^c, g_j); \Lambda^2_+ \otimes AdP)_0$$

holds. This completes the proof.

Let A be a connection satisfying the perturbed equation $F_s^+(A, b) = 0$ on (K_i, g_j). Then one has the AHS complex at (A, b):

$$\begin{aligned} 0 \longrightarrow \bar{W}^{k+1}_w((K_i, g_j); AdP) &\xrightarrow{d_{(A,b)}} \bar{W}^k_w((K_i, g_j); \Lambda^1 \otimes AdP) \\ &\xrightarrow{d^+_{(A,b)}} \bar{W}^{k-1}_w((K_i, g_j); \Lambda^2_+ \otimes AdP) \longrightarrow 0. \end{aligned}$$

Even though we donot use for the proof of our theorem, we include the following basic Fredholm property of the quotient function spaces:

Proposition 2.1 *The above complex is of Fredholm.*

Proof: For the first statement, because K_i is compact, it is enough to see that the AHS complex without coefficient is of Fredholm.

Let us consider the compex:

$$\begin{aligned} 0 \longrightarrow \bar{W}^{k+1}_w(K_i, g_j) &\xrightarrow{d} \bar{W}^k_w((K_i, g_j); \Lambda^1) \\ &\xrightarrow{d^+} \bar{W}^{k-1}_w((K_i, g_j); \Lambda^2_+) \longrightarrow 0. \end{aligned}$$

Firstly both differentials have closed range.

The following compex is of Fredholm:

$$\begin{aligned} 0 \longrightarrow W^{k+1}_{w_j}(K_i^c, g_j)_0 &\xrightarrow{d} W^k_{w_j}((K_i^c, g_j); \Lambda^1)_0 \\ &\xrightarrow{d^+} W^{k-1}_{w_j}((K_i^c, g_j); \Lambda^2_+)_0 \longrightarrow 0. \end{aligned}$$

We choose any basis of H^1 of the above complex $\{\tilde{u}_1, \dots, \tilde{u}_l\} \subset W^k_{w_j}((K_i^c, g_j); \Lambda^1)_0$.

Let us consider H^0 of the former complex, and take $\bar{f} \in H^0$. Thus $df \in W^k_{w_j}((K_i^c, g_j); \Lambda^1)_0$, and there exist some $\tilde{g} \in W^{k+1}_{w_j}(K_i^c, g_j)_0$ and numbers $b_1, \dots, b_l$ so that the equality $df = dg + \Sigma_{j-1}^l b_j \tilde{u}_j$ holds. Thus the dimension of H^0 is at most $l+1$. In particular it is of finite dimension.

Let us see that H^2 is of finite dimension. Let $\bar{u} \in \bar{W}^{k-1}_w((K_i, g_j); \Lambda^2_+)$. Then by definition, it is the image of the projection of some element $u \in W^{k-1}_{w_j}((M, g_j); \Lambda^2_+)$. Since $d^+ : W^k_{w_j}((M, g_j); \Lambda^1) \to W^{k-1}_{w_j}((M, g_j); \Lambda^2_+)$ is of finite dimension, there are elements $\{v_1, \dots, v_m\} \subset W^{k-1}_{w_j}((M, g_j); \Lambda^2_+)$ so that any element $u \in W^{k-1}_{w_j}((M, g_j); \Lambda^2_+)$ can be decomposed as:

$$u = d^+(w) \oplus \Sigma_{i=1}^m a_i v_i.$$

Thus $\bar{u} = d^+(\bar{w}) \oplus \Sigma_{i=1}^m a_i \bar{v}_i$.

Finally let us see H^1 is also of finite dimension. Let $\bar{u} \in \bar{W}^k_{w_j}((K_i, g_j); \Lambda^1)$ be $d^+(\bar{U}) = 0$. Take a representative $u \in W^k_{w_j}((M, g_j); \Lambda^1)$. This implies $d^+(u) \in W^{k-1}_{w_j}((K_i^c, g_j); \Lambda^1)_0$. By a similar way as the cae of H^0, there are finite $\{w_1, \dots, w_m\} \subset W^{k-1}_{w_j}((K_i^c, g_j); \Lambda^2_+)_0$, and some $u' \in W^k_{w_j}((K_i^c, g_j); \Lambda^1)_0$ so that the equality $d^+(u) = d^+(u') + \Sigma_{i=1}^m a_i w_i$ holds.

Now let $\bar{u}_1, \dots, \bar{u}_l$ be linear independent elements in H^1. Then if $l \geq m+1$, then by above, there are some numbers $c_1, \dots, c_l$ so that the equality $\Sigma_{i=1}^l c_i d^+(\bar{u}_i) = d^+(u')$ holds for some $u' \in W^k_{w_j}((K_i^c, g_j); \Lambda^1)_0$. In particular by choosing another representatives, one may assume $\Sigma_{i=1}^l c_i d^+(\bar{u}_i) = 0$.

Then again there are some basis $\{v_1, \dots, v_r\} \subset W^k_{w_j}((M, g_j); \Lambda^1)$ of H^1, numbers $b_1, \dots, b_r$ and $f \in W^{k+1}_{w_j}(M, g_j)$ so that the equality holds:

$$\Sigma_{i=1}^l c_i u_i = df + \Sigma_{j=1}^r b_j v_j.$$

In particular the equality $\Sigma_{i=1}^l c_i \bar{u}_i = d\bar{f} + \Sigma_{j=1}^r b_j \bar{v}_j$ implies that dim H^1 is less than $\max(m+1, r)$. This completes the proof.

Let A_j be a connection on (M, g_j) satisfying $F^+(A_j) = 0$. Let us consider the differentials $d^+_{A_j} : W^{k+1}_{w_j}((M, g_j); \Lambda^1 \otimes AdP) \to W^k_{w_j}((M, g_j); \Lambda^2_+ \otimes AdP)$. Let us put $W^j = (\text{ Ker} d^+_{A_j})^{\perp} \subset W^{k+1}_{w_j}((M, g_j); \Lambda^1 \otimes AdP)$, and $U^j = \text{ im } d^+_{A_j} \subset W^k_{w_j}((M, g_j) : \Lambda^2_+ \otimes AdP)$.

$d^+_{A_j} : W^j \to U^j$ gives an isomorphism, and for all $u \in W^j$, the estimates $||d^+_{A_j}(u)||W^k_{w_j} \geq C_j ||u|| W^{k+1}_{w_j}$ hold for some positive constant $C_j > 0$. Let:

$$d^+_{\bar{A}_j} : \bar{W}^{k+1}_{w_j}((K_i, g_j); \Lambda^1 \otimes AdP) \to \bar{W}^k_{w_j}((K_i, g_j); \Lambda^2_+ \otimes AdP)$$

be the induced map, and denote $\bar{W}^j = (\text{Ker} d^+_{\bar{A}_j})^{\perp}$.

Lemma 2.5 *For $\bar{u} \in \bar{W}^j$, the estimate:*

$$||d^+_{\bar{A}_j}(\bar{u})||\bar{W}^k_{w_j} \geq C_j ||\bar{u}|| \bar{W}^{k+1}_{w_j}$$

still holds, where C_j is the same constant as above.

Proof: $d^+_{A_j} : W^j \cong U^j$ implies that the above estimate is equivalent to that the image of the unit ball contains the $\epsilon = C_j$ ball:

$$d^+_{A_j}(B_1) \supset D_\epsilon$$

where $B_1 \subset W^j$ and $D_\epsilon \subset U^j$.

Now for any $u \in W^j$, decompose $u = u_1 + u_2$ where $u_1 \in W^{k+1}_{w_j}((K_i^c, g_j); \Lambda^1 \otimes AdP)_0^\perp$ and $u_2 \in W^{k+1}_{w_j}((K_i^c, g_j); \Lambda^1 \otimes AdP)_0$. Since $d^+_{A_j}(u_2) \in W^k_{w_j}((K_i^c, g_j); \Lambda^2_+ \otimes AdP)_0$ holds, the result follows. This completes the proof.

If one follows the above proof, then the constants C_j above will depend on i and j. Below we will have a *comparison method* which induces, in fact a uniform bound of constants, where we will compare K_i part of functions between M and S.

Let A be an L^2 ASD connection over (S, g), obtained as $\lim_i A_i$ as before, and cosider the differential $d^+_A : W^{k+1}_w((S,g); AdP) \to W^k_w((S,g); \Lambda^2_+ \otimes AdP)$. Similarly as before, we put $W = (\text{ Ker} d^+_A)^\perp \subset W^{k+1}_w((S,g); \Lambda^1 \otimes AdP)$, and $U = \text{ im } d^+_A \subset W^k_{w_j}((S,g) : \Lambda^2_+ \otimes AdP)$.

As above, there is a constant $C > 0$ so that $||d^+_A(u)||W^k_w(S,g) \geq C||u||W^{k+1}_w(S,g)$ hold for all $u \in W$.

Let $\bar{W}^*_w(S,g) = W^*_w(S,g)/W^*_w(K_i^c, g)_0$ be the quotient spaces as before. Then by the same argument as lemma 2.5, one still obtains the bound:

$$||d^+_{\bar{A}}(\bar{u})||\bar{W}^k_w(K_i, g) \geq C||\bar{u}||\bar{W}^{k+1}_w(K_i, g)$$

hold for all $\bar{u} \in \bar{W} \equiv (\text{Ker } d^+_{\bar{A}})^\perp$.

By lemma 2.2, if j are sufficiently large, then the metrics are uniformly equivalent $\frac{1}{2}|| \quad ||\bar{W}^*_w(K_i, g) \leq || \quad ||\bar{W}^*_{w_j}(K_i, g_j) \leq 2|| \quad ||\bar{W}^*_w(K_i, g)$.

Thus one obtains:

Corollary 2.2 *There are constants $C > 0$ and N independent of i, j so that for $|i - j| \geq N$, the uniform estimates hold:*

$$||d^+_{\bar{A}_j}(\bar{u})||\bar{W}^k_{w_j} \geq C||\bar{u}||\bar{W}^{k+1}_{w_j}$$

for all $\bar{u} \in \bar{W}^j$.

Remark 2.1: This method does not depend on particular choice of differential operators. In fact the conclusion follows as far as the operatros have closed range. This will be used in $2.B.4$.

Let us consider $d_{A_j} : W^{k+2}_{w_j}((M, g_j); AdP) \to W^{k+1}_{w_j}((M, g_j); \Lambda^1 \otimes AdP)$, and put $\bar{H}^j_i = (\text{Ker } d_{\bar{A}_j})^{\perp} \cap \bar{W}^{k+2}_{w_j}((K_i, g_j) : AdP)$.

Corollary 2.3 *For any $\bar{f} \in \bar{H}^j_i$, the uniform estimates hold:*

$$||d_{\bar{A}_j}(\bar{f})||\bar{W}^{k+1}_{w_j}(K_i, g_j) \geq C||\bar{f}||\bar{W}^{k+2}_{w_j}(K_i, g_j)$$

2.B.3 Convergence of image spaces: Let us consider a family of closed subspaces:

$$\bar{V}^j_i \subset \bar{W}^{k+1}_w((K_i, g); \Lambda^1 \otimes AdP), \quad j \geq i$$

and also a closed subspace $V \subset W^{k+1}_w((S, g); \Lambda^1 \otimes AdP)$. For each i, we denote its quotient by $\bar{V} \subset \bar{W}^{k+1}_w((K_i, g) : \Lambda^1 \otimes AdP)$.

Let us say that a bounded sequence $\{\bar{u}^j_i\}_{j \geq i}$ in $\{\bar{V}^j_i\}_{j \geq i}$ *lies inside V at infinity*, or $\lim_{i,j \to \infty} \{\bar{u}^j_i\} \in V$, if for any small $\epsilon > 0$, there are large i_0 and larger $i' \geq i_0$ so that for all $j \geq i \geq i'$, there are some $\bar{v}^j_i \in \bar{V}$ such that the estimates hold:

$$||\bar{u}^j_i - \bar{v}^j_i||\bar{W}^{k+1}_w((K_{i_0}, g); \Lambda^1 \otimes AdP) < \epsilon$$

where we regard $\bar{u}^j_i$ are elements in $\bar{W}^{k+1}_w((K_{i_0}, g); \Lambda^1 \otimes AdP)$ by taking another quotients:

$$^{-}: \bar{W}^*_w(K_i, g) \to \bar{W}^*_w(K_{i_0}, g).$$

We say that $\{\bar{V}^j_i\}$ lies inside V at infinity, if any bounded sequence $\{\bar{u}^j_i\}$ lies inside V as above. The family of spaces is said to *converge* to V, if they lie inside V at infinity, and for any $v \in V$, there is a sequence $\{\bar{u}^j_i\}$ so that for any small $\epsilon > 0$ and any large i_0, there is a larger $i' \geq i_0$ so that for all $j \geq i \geq i'$, the estimates hold:

$$||\bar{u}^j_i - \bar{v}||\bar{W}^{k+1}_w((K_{i_0}, g); \Lambda^1 \otimes AdP) < \epsilon.$$

A closed space $T \subset W^{k+1}_w((S, g) : \Lambda^1 \otimes AdP)$ is said to be *orthogonal* to $\lim_{i,j \to \infty} \bar{V}^j_i$, if for any small $\epsilon > 0$ and any element $v \in T$, there are large i_0 and larger $i' \geq i_0$ so that for all $j \geq i \geq i'$, the estimate:

$$||\bar{u}^j_i - \bar{v}||\bar{W}^{k+1}_{w_j}((K_{i_0}, g_j); \Lambda^1 \otimes AdP) \geq (1 - \epsilon)||\bar{v}||$$

hold for all $\bar{u}^j_i \in \bar{V}^j_i$, where $\bar{v}$ is its quotient.

Let $\{h_j\}_j$ be a family of complete Riemannian metrics on S converging to g uniformly in C^∞ as before. Let $\{A_j\}_j$ be a family of ASD connections over

$\{(M, g_j)\}_j$ respectively which converges to A over (S, g) on each compact subsets in C^∞.

Let us fix i and take sufficiently large $j >> i$. We consider the quotient spaces $\bar{W}^*_{w_j}((K_i, g_j); \Lambda^* \otimes AdP)$. Let $d_{\bar{A}_j} : \bar{W}^{k+2}_{w_j}((K_i, g_j); AdP) \to \bar{W}^{k+1}_{w_j}((K_i, g_j); \Lambda^1 \otimes AdP)$ be the differentials. Let us put $\bar{V}^j_i = \mathrm{im} d_{\bar{A}_j}$. Similarly let $V = \mathrm{im}\ d_A \subset W^{k+1}_w((S, g); \Lambda^1 \otimes AdP)$, and denote its quotient by $\bar{V} \subset \bar{W}^{k+1}_w((K_i, g); \Lambda^1 \otimes AdP)$.

Let us put:

$$U = \mathrm{Ker}\ d^+_A \subset W^{k+1}_w((S, g) : \Lambda^1 \otimes AdP).$$

The next is the cruitial for our later discussion:

Proposition 2.2 *The orthogonal subspace $T \subset U$ to $\lim_{i,j\to\infty} \bar{V}^j_i$ is of finite dimension.*

For this, we have the following lemma.

Lemma 2.6 *The family $\{\bar{V}^j_i\}_{j\geq i}$ converges to V.*

Proof: Let $\bar{u}^j_i \in \bar{V}^j_i$ be a bounded sequence, $||\bar{u}^j_i||\bar{W}^{k+1}_{w_j}(K_i, g_j) \leq C$. We choose representatives $u^j_i \in W^{k+1}_{w_j}(M, g_j)$ with $||\bar{u}^j_i||\bar{W}^{k+1}_{w_j}(K_i, g_j) = ||u^j_i||W^{k+1}_{w_j}(M, g_j)$.

Let us take any i_0. Then for any $\epsilon > 0$, there is a larger i' so that for all $i \geq i'$, the following property holds; for each $\bar{u}^j_i$, there is some $i_0 \leq i'' \leq i$ so that $||u^j_i||W^{k+1}_{w_j}(K_{i''+1}\backslash K_{i''}, g_j) \leq \epsilon$. Let $\varphi_{i''} \in C^\infty_c(K_{i''+1})$ be the cut off function. Then one has the estimate:

$$||d^+_{A_j}(\varphi_{i''}u^j_i)||W^k_{w_j}(M, g_j) \leq C\epsilon.$$

Now regard $\varphi_{i''}u^j_i \in W^{k+1}_w(S, g)$ and decompose

$$\varphi_{i''}u^j_i = d_A(f^j_i) + z^j_i$$

with respect to $W^{k+1}_w(S, g) = V \oplus V^\perp$, where $||z^j_i||W^{k+1}_w(S, g) \leq C\epsilon$ hold. Since the metrics $||\quad||\bar{W}^*_{w_j}(K_{i_0}, g_j)$ and $||\quad||\bar{W}^*_w(K_{i_0}, g)$ are uniformly equivalent, it follows that $v^j_i = d_A(f^j_i)$ is the desired family. Thus one has shown that $\lim_{i,j} \bar{V}^j_i \subset V$.

Next we show surjectivity, that the orthogonal subspace $T \subset V$ to $\lim_{i,j\to\infty} \bar{V}^j_i$ is equal to zero.

Let $u = d_A(f) \in W^{k+1}_w((S, g); \Lambda^1 \otimes AdP)$ be a unit element. Then since d_A has closed range with zero kernel ([K1]), there is a positive constant $C > 0$ so that the estimate $||f||W_w^{k+2} \leq C||d_A(f)||W_w^{k+1} = C$ holds. Thus for any small $\epsilon > 0$, there are sufficiently large i so that the estimate:

$$||u - d_A(\varphi_i f)||W_w^{k+1} < \epsilon$$

holds. Since one can assume $\varphi_i f \in W_{w_j}^{k+2}((M, g_j) : AdP)$, the assertion holds. This completes the proof.

Proof of proposition: Since $H^1(A) = \mathrm{Ker} d_A^+ / d_A(W_w^{k+2}((S, g); AdP)$ is of finite dimension by theorem 0.2, the above lemma already implies that $T \subset U$ is of finite dimension. This completes the proof.

Let us choose a bounded sequence $\bar{v}_i^j \in (\mathrm{im}\ d_{\bar{A}_j})^\perp$ with $||\bar{v}_i^j||W_{w_j}^{k+1} \leq C$.

Corollary 2.4 *If* $\lim_{i,j\to\infty} d_{\bar{A}_j}^+(\bar{v}_i^j) = 0$ *holds, then the above sequence lies inside a finite dimensional subspace* T *at infinity.*

2.B.4 Uniformity in implicit function theorems: So far we have three types of spaces by use of Sobolev spaces over them:

$$W_w^k(S, g; \Lambda^*), \quad W_{w_i}^k(M, g_i; \Lambda^*), \quad \bar{W}_{w_j}^k(K_i, g_j; \Lambda^*).$$

Let A_j be ASD connections over (M, g_j) which converges to A over (S, g) as before. Let $F_s^+ : A_j + W_{w_j}^{k+1}((M, g_j) : \Lambda^1 \otimes AdP) \times B \to W_{w_j}^k((M, g_j) : \Lambda_2^+ \otimes AdP)$ be the parametrized self dual curvature operators, and denote its differentials at A_j:

$$D_{A_j}^+ : W_{w_j}^{k+1}((M, g_j) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W_{w_j}^k((M, g_j) : \Lambda_2^+ \otimes AdP).$$

Similarly one obtains $D_A^+ : W_w^{k+1}((S, g) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W_w^k((S, g) : \Lambda_2^+ \otimes AdP)$. These are all surjective maps by proposition 1.2.

By putting $A \equiv A_\infty$, one obtains a family of operators $\{D_{A_j}^+\}_{0 \leq j \leq \infty}$.

Let us fix i, and for any $i \leq j \leq \infty$, one obtains projections:

$$^- : W_{w_j}^k((X, h) : \Lambda^* \otimes AdP) \to \bar{W}_{w_j}^k((K_i, h) : \Lambda^* \otimes AdP)$$

where $(X, h) = (M, g_j)$ or (S, g).

Let us take $\epsilon > 0$ so that the image of the unit ball contains the ϵ ball:

$$D_A^+(B_1) \supset D_\epsilon$$

where $B_1 \subset W_w^{k+1}((S,g) : \Lambda^1 \otimes AdP) \oplus T_0 B$ and $D_\epsilon \subset W_w^k((S,g) : \Lambda_+^2 \otimes AdP)$. This is equivalent to the estimate:

$$||D_A^+(u,c)|| \geq \epsilon ||(u,c)||$$

for all $u \in (\mathrm{Ker} D_A^+)^\perp$. We put $C = \frac{\epsilon}{2}$. Let:

$$D_{\bar{A}}^+ : \bar{W}_w^{k+1}((K_i,g) : \Lambda^1 \otimes AdP) \oplus T_0 B \to \bar{W}_w^k((K_i,g) : \Lambda_+^2 \otimes AdP)$$

be the induced map. Similarly there are surjective maps $D_{A_j}^+ : W_{w_j}^{k+1}((M,g_j) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W_{w_j}^k((M,g_j) : \Lambda_+^2 \otimes AdP)$. For sufficiently large $j \geq i$, let $^-$be the quotient on $\bar{W}_{w_j}^*((K_i,g_j) : \Lambda^* \otimes AdP)$, and consider the induced map:

$$\bar{D}_{\bar{A}_j}^+ : \bar{W}_w^{k+1}((K_i,g_j) : \Lambda^1 \otimes AdP) \oplus T_0 B \to \bar{W}_w^k((K_i,g_j) : \Lambda_+^2 \otimes AdP).$$

By lemma 2.4, they are all surjective.

Let us fix i, and$^-$be as above. By corolalry 2.2 and remark 2.1, the following holds:

Corollary 2.5 *For $C = \frac{\epsilon}{2} > 0$ above and for all large $i << j \leq \infty$, the estimates:*

$$||D_{\bar{A}_j}^+(\bar{u},c)||\bar{W}_w^k \geq C||(\bar{u},c)||$$

hold for all $(\bar{u},c) \in (\ \mathit{Ker} D_{\bar{A}_j}^+)^\perp$.

Let us consider the surjective map:

$$D_A^+ : W_w^{k+1}((S,g) : \Lambda^1 \otimes AdP) \oplus T_0 B \to W_w^k((S,g) : \Lambda_+^2 \otimes AdP).$$

Let $V = \mathrm{Ker}\ D_A^+$ and $W = (\mathrm{Ker} D_A^+)^\perp$ be the kernel and its orthogonal complement respectively. Then by the infinite dimensional implicit function theorem, there are open balls $B_1 \subset V$ and $B_2 \subset W$, and a smooth map $G : B_1 \to B_2$, $G(0,0) = 0$ so that $F_s^+(A',b') = 0$ holds for $(A',b') \in B_1 + B_2$, if and only if $(A',b') = (B,c) + G(B,c)$ for some $(B,c) \in B_1$. Moreover $dG_{(A',b')} = -[(D_{(A',b')}^+|W)^{-1} \circ D_{(A',b')}^+]$ holds. Notice that there are positive $\delta_1, \delta_2 > 0$ so that for δ_i balls $D_{\delta_1} \subset W_w^{k+1}((S,g) : \Lambda^1 \otimes AdP)$ and $D_{\delta_2} \subset T_0 B$, the inclusion holds:

$$D_{\delta_1} \times D_{\delta_2} \subset B_1 + B_2.$$

Let $(D_A^+)^{-1} : W_w^k((S,g) : \Lambda_+^2 \otimes AdP) \to W$ be the inverse. The next follows from the proof of the abstract implicit function theorem and is elementary:

Lemma 2.7 *The diameters of B_1 and B_2 depends only on the operator norms $||(D_A^+)^{-1}||$ and $||F_s^+(A+\quad,\quad)||$.*

Let us fix i. Let $\bar{V}^j = \mathrm{Ker} D^+_{\bar{A}_j}, \bar{W}^j = (\mathrm{Ker} D^+_{\bar{A}_j})^\perp \subset \bar{W}^{k+1}_{w_j}((K_i, g_j) : \Lambda^1 \otimes AdP)$. Then the implicit function theorem again states that there are smooth maps $G_j : \bar{B}_1^j \to \bar{B}_2^j$, where $\bar{B}_2^j \subset \bar{W}^j$ and $\bar{B}_1^j \subset \bar{V}^j$, so that $F^+_{(\bar{A}', b')} = 0$ holds for $(\bar{A}', b') \in \bar{B}_1^j + \bar{B}_2^j$, if and only if $(\bar{A}', b') = (\bar{B}, c) + G_j(\bar{B}, c)$ for some $(\bar{B}, c) \in \bar{B}_1^j$.

A uniformity of scales of domains follows from lemma 2.7 and corollary 2.5:

Corollary 2.6 *There are positive $\delta_1, \delta_2 > 0$, independent of $j >> i$ so that $\bar{B}_m^j$ contains δ_m balls for $m = 1, 2$.*

In particular there are some potitive $\delta, \mu > 0$ independent of i, j such that for δ ball $\bar{D}^j \subset \bar{W}^{k+1}_{w_j}((K_i, g_j) : \Lambda^1 \otimes AdP)$ and μ ball $D_\mu \subset T_0 B$, $\bar{B}_1^j + \bar{B}_2^j$ contains $\bar{D}^j \times D_\mu$.

Proposition 2.3 *For all sufficiently large j, there are uniform estimates of norms:*

$$||dG_j|| \leq C$$

where C is independent of j.

Proof: By corollary 2.5, one obtains uniform estimates of operator norms:

$$||(D_{\bar{A}}^+ | \bar{W}^j)^{-1}|| \leq C^{-1}.$$

Notice that the norms of $D_{\bar{A}}^+ | \bar{W}^{k+1}_w(K_i, g)$ are less than $C' = ||D_A^+||$ on $W^{k+1}_w(S, g)$. Thus if j are sufficiently large, then $||D^+_{\bar{A}_j} | \bar{W}^{k+1}_w(K_i, g_j)||$ are less than $2C'$. Thus one obtains the desired estimates $||dG_j|| \leq 2C^{-1}C'$ from the formula for G_j. This completes the proof.

2.B.5 Restriction of the differentials: Let A be an ASD connection obtained as $\lim_{i \to \infty} A_i$ as before, where A_i are ASD connections on (M, g_i). Let us consider $d_A^+ : W^{k+1}_w((S, g) : \Lambda^1 \otimes AdP) \to W^k_w((S, g) : \Lambda^2_+ \otimes AdP)$, and denote:

$$V = \mathrm{im} d_A, \quad W = V^\perp \subset W^{k+1}_w((S, g) : \Lambda^1 \otimes AdP).$$

By corollary 1.1, the restriction of the differential of F_s^+ at $(A, 0)$:

$$D_A^+ : W \oplus T_0 B \to W^k_w((S, g) : \Lambda^2_+ \otimes AdP)$$

is still surjective.

Later on, we will restrict on the closed subspace W. This becomes cruicial at $2.B.6$.

Let us put:

$$U_1 \equiv (W \oplus T_0 B) \cap \text{ Ker } D_A^+, \quad U_2 = (W \oplus T_0 B) \cap (\text{ Ker } D_A^+)^{\perp}.$$

Then by the implicit function theorem again, there are balls $B \subset U_1$ and $D \subset U_2$, and a smooth function $G : B \to D$ so that any solution $(A, b) \in B + D$ is on the graph of G.

Now for each $j \geq i$, let us put

$$\bar{V}_i^j = \text{im } d_{\bar{A}_j}, \bar{W}_i^j = (\text{im } d_{\bar{A}_j})^{\perp} \subset \bar{W}_w^{k+1}((K_i, g_j) : \Lambda^1 \otimes AdP).$$

We consider the restrictions:

$$D_{\bar{A}_j}^+ : \bar{W}_i^j \oplus T_0 B \to \bar{W}_w^k((K_i, g_j) : \Lambda_+^2 \otimes AdP).$$

As above these are also surjective.

Let us put:

$$\bar{U}_1 \equiv (\bar{W}_i^j \oplus T_0 B) \cap \text{ Ker } D_{\bar{A}_j}^+, \quad \bar{U}_2 = (\bar{W}_i^j \oplus T_0 B) \cap (\text{ Ker } D_{\bar{A}_j}^+)^{\perp}.$$

Then there are balls $\bar{B}_i^j \subset \bar{U}_1$ and $\bar{D}_i^j \subset \bar{U}_2$, and a smooth function $G_i^j : \bar{B}_i^j \to \bar{D}_i^j$ so that any solution $(A, b) \in \bar{B}_i^j \oplus \bar{D}_i^j$ is on the graph of G_i^j.

By use of these restriction, one has the same statements as corollary 2.5, 2.6 and proposition 2.3. Throughout the rest of the paper, we use these restrictions.

2.B.6 Dimension comparisons by (in)finiteness: Here we choose carefully generic path $b_t \in B$ as follows.

Firstly let:

$$T = \text{Ker} d_A^+ \cap (\text{im } d_A)^{\perp} \subset W_w^{k+1}((S, g) : \Lambda^1 \otimes AdP)$$

be the finite dimensional subspace as in proposition 2.2. Let $D \subset T$ and $B_1 \subset T_0 B$ be the unit balls. We put the projection by $pr : W_w^{k+1}((S, g) : \Lambda^1 \otimes AdP) \oplus T_0 B \to T_0 B$, and Gr be the graph of G. Let us put:

$$\tilde{G}r(D, 0) = Gr \cap D \times B_1 \subset W \oplus T_0 B.$$

Lemma 2.8 *There is some constant $C > 0$ and smooth path $b_t \in B$ with $b_0 = 0$ so that the estimate hold:*

$$||b_t - pr \circ \tilde{G}r(D, 0)|| \geq Ct$$

for all $0 \leq t \leq 1$.

Proof: Let us put the linear space:

$$\tilde{V} = \text{ Ker } D_A^+ \cap T_0 B.$$

By the construction of perturbation, $\tilde{V} \subset T_0 B$ has infinite codimension. In fact the image $ds(0, T_0 B)$ is already infinite dimensional. Then we put the map:

$$\tilde{G}_0 : \tilde{V} \to \tilde{V}^\perp \subset T_0 B$$

by $c \to pr \circ Gr(0, c)$. Let $\tilde{G}_0 r$ be the graph of $\tilde{G}_0$. Then since $||dG|| \leq C$ is bounded, and $\tilde{V}^\perp$ is infinite dimensional, there is a smooth path b_t which satisfies the lower bound:

$$||b_t - \tilde{G}_0 r \cap B|| \geq Ct$$

for all t as follows. Let $T\tilde{G}_o r = \{v + d\tilde{G}_0 r(v)\} \subset T_0 B$ be the closed linear subspace. Then since $\tilde{V}^\perp$ is infinite dimensional, its orthogonal complement $(T\tilde{G}_0 r)^\perp$ is also infinite dimensional. In particular one can choose a smooth path $b_t \in B$ so that $\frac{d}{dt} b_t|_{t=0} \in (T\tilde{G}_0 r)^\perp$. Since $||dG|| \leq C$ is bounded, this gives a path satisfying the above estimate.

In order to obtain the desired path, one will basically follow a parallel argument. Let $B_1 \subset T_0 B$, $B_1' \subset W$ be the unit balls. For each $c \in B_1$, let us put a set:

$$S(c) = \{pr \circ Gr(m, c) : (m, c) \in U_1 \cap B_1' \times B_1\}.$$

This defines a multi-valued map:

$$\tilde{G} : B_1 \cap (pr \circ U_1) \to (pr \circ U_2).$$

We claim that each $\tilde{G}(c)$ has at most finite dimension bounded by dim T. In order to verify this, it is enough to see the set:

$$dS(c) \equiv \{pr \circ dGr(m, c) : (m, c) \in T_0 U_1\}$$

is finite dimensional.

For $(m,c) \in T_0U_1$, $D_A^+(m,c) = 0$ holds, since $U_1 = \text{Ker } D_A^+ \cap (W \oplus T_0B)$. Let us put $dGr(m,c) = (m,c) + dG(m,c)$. Suppose $pr \circ dGr(m,c) = pr \circ dGr(m',c)$ hold for some $(m,c), (m',c) \in T_0U_1$. Then $m - m' \in \text{Ker } D_A^+ \cap W$. We see that this space is of finite dimension. Let us decompose $W = T \oplus (\text{Ker} d_A^+)^\perp$. Then there is a positive constant $C > 0$ independent of perturbation so that for any $v \in (\text{Ker} d_A^+)^\perp$, $||d_A^+(v)|| \geq C||v||$ holds. Since perturbation is sufficiently small, one may assume the estimate $||D_A^+(v)|| \geq C||v||$ still holds. In particular the projection:

$$\text{Ker} D_A^+ \cap W \hookrightarrow T$$

is injective, where the latter space is finite dimensional. This verifies the claim.

Now we show that $pr \circ U_1 \subset T_0B$ has infinite codimension. Then it follows from uniformly finite dimensionality of $\tilde{G}(c)$ for each $c \in pr \circ U_1$ that one can obtain the desired b_t by the same argument as the case of $\tilde{G}_0$.

For any closed subspace $L \subset W_w^k((S,g) : \Lambda_+^2 \otimes AdP)$ and any $u \in W_w^k((S,g) : \Lambda_+^2 \otimes AdP)$, we denote $||u - L|| = \inf\{||u - v||W_w^k(S,g) : v \in L\}$.

Let $H \subset W_w^k((S,g) : \Lambda_+^2 \otimes AdP)$ be the set $\{u : (d_A^+)_w^*(u) = 0\}$ which is finite dimensional subspace of cokernel of d_A^+ over (S,g). We claim that there are sufficiently small and positive $c, c' > 0$ so that the subspace

$$J \equiv \{v \in T_0B : c'||v|| \geq ||ds(0,v) - H^\perp|| \geq c||v|| > 0\}$$

has infinite dimension (notice that one has assumed H has positive dimension).

In fact by the construction of perturbation, the image $ds(0, T_0B)$ is infinite dimensional. Since ds is compact, it has closed range. It is easy to see that $\text{Ker} ds \cap T_0B$ is infinite dimensional. In fact any elements $\psi \in B$ which vanishes on small neighbourhoods of $\text{Hol}A$ lies in this subspsace. Thus it is enough to see that the projection of $ds(0, T_0B)$ on H is surjective.

Then certainly it follows by the construction of perturbation, since restriction of any non zero element of H on any open subset is still non zero (section 2). In fact if there could exist some linear space $L \subset H$ so that the projection of $ds(0, T_0B)$ is contained in $L^\perp$, then the restriction of any $v \neq 0 \in L$ on U is still non zero, $v|U \neq 0$. Then one can choose some element $u \in ds(0, T_0B)$ with $< v, u > \neq 0$, where $U \subset S$ is the open subset in the construction of the holonomy perturbation. This is a contradiction. This verifies the claim.

Now we claim that $pr \circ U_1 \subset T_0B$ has infinite codimension, if $(W \times J) \cap \text{Ker } D_A^+$ is finite dimensional. In fact suppose not and assume it is of finite codimension of $l-1<\infty$ under the above condition. Notice that J consistes of an open cone at0. Let us choose linear independent vectors $(m_i,c_i)\in W\times T_0J$ for $i=1,\dots,l$. Each (m_i,c_i) decomposes as (m_i^j,c_i^j), $j=1,2$ with respect to Ker $D_A^+\oplus(\text{Ker } D_A^+)^\perp$. By the assumption, there are non zero familiy of real numbers $\{a_i\}_i$ so that the equality $\Sigma_i a_i c_i^2=0$ should hold. Then the inclusion:

$$\Sigma_i a_i(m_i,c_i)\in(W\times T_0J)\cap \text{Ker } D_A^+\oplus W$$

hold. By the condition, the former space is of finite dimension $l-1'$. This implies that if one takes linear independent vectors $(m_i,c_i)\in W\oplus T_0J$, $i=1,\dots,ll'$, then for some real numbers $\{a_j\}_j$, the linear combination $\Sigma_{j=1}^{ll'}a_j(m_j,c_j)\in W$. In particular this contradicts to infinite dimensionality of J.

Now we show that $(W\times J)\cap\text{Ker } D_A^+$ is finite dimensional. For this firstly we show $(T^\perp\times J)\cap\text{Ker}D_A^+=0$. if the pair $m\in W$ and $v\in J$ satisfies the equality $D_A^+(m,v)=0$, then one may assume the estimate $||m||\geq||v||$, since the projection of $D_A^+(m,0)$ on H is less than $c||m||$ for a small $c>0$, where c depends only on the perturbation.

On the other hand for any $m\in T^\perp$, there is a small $c'>0$ so that the estimates hold:

$$||D_A^+(m,0)||\geq C||m||,\quad |<D_A^+(m,0),H>|\leq c'||m||.$$

where $C>0$ can be chosen independent of perturbatuion. By these estimates, for non zero $m\in T^\perp$ and $v\in J$, the equality $D_A^+(m,v)=0$ cannot happen, since the estimates hold:

$$||D_A^+(m,v)||\geq||D_A^+(m,0)||-||D_A^+(0,v)||\geq C||m||-c||v||>0.$$

Now it is enough to see $(T\times J)\cap U_1$ has finite dimensional. Suppose not. Then by compactness of unit ball of T, there is some $m\in T$ and an orthonormal sequence $\{(m_i,v_i)\}_i\in T\times T_0J\cap U_1$ so that $\lim_{i\to\infty}m_i=m$ holds, and so for any small $\epsilon>0$, $||ds(0,v_i-v_j)||=||D_A^+(0,v_i-v_j)||<\epsilon$ for all large i,j. This implies that there are familiy of vectors $u_{i,j}(t)\in J$, $0<t\leq 1$ with $u_{i,j}(0)=0$ and some positive constant $c>0$ so that for any small $\epsilon>0$, there are sufficiently large i,j and two estimates:

$$||u_{i,j}(t)||\geq ct,\quad ||ds(0,u_{i,j}(t))||\leq\epsilon t$$

should hold. This contradicts to the definition of J. This completes the proof.

2.B.7 Asymptotic solutions: So far we have not used any special properties of ASD equations, and rather we have worked within general functional analytic framework. Here we use the assumtion of non vanishing Donaldson's invariant on $E \to M$.

Let $A_j \in \mathfrak{M}(M, g_j)$ be a sequence of ASD connections $j = 1, 2, \ldots,$ which converges to an L^2 ASD connection A over (S, g). We denote by $\mathfrak{M}(M, g_j)$ the usual ASD moduli spaces, and $\mathfrak{M}(A)_w$ be the weighted moduli space in section 1. Simialrly we denote the perturbed moduli spaces $\mathfrak{M}_b(M, g_j)$ or $\mathfrak{M}_b(A)_w$. Concerning $\mathfrak{M}(M, g_j)$, we will choose the Floer's one $b \in P$ for generic values of the perturbation. In particular these are all smooth and finite dimensional manifolds.

Let us fix i, and for all $j \geq i$, denote by $\bar{A}_j$ the projection of A_j on $\bar{W}^{k+1}_{w_i}((K_i, g_j) : \Lambda^1 \otimes AdP)$.

Now recall that by the assumption and lemma 2.4, the differentials:

$$d^+_{\bar{A}_j} : \bar{W}^{k+1}_{w_i}((K_i, g_j) : \Lambda^1 \otimes AdP) \to \bar{W}^{k}_{w_i}((K_i, g_j) : \Lambda^2_+ \otimes AdP)$$

are surjective for all $j \geq i$.

Let $\bar{B}^1_\delta \subset \bar{W}^{k+1}_{w_i}((K_i, g_j) : \Lambda^1 \otimes AdP)$ be the δ ball, and $\bar{B}^2_\mu \subset T_0 B$ be the μ ball. Then by corollary 2.6, there are positive $\delta, \mu > 0$ independent of $j >> i$ so that one has obtained local charts:

$$\bar{B}^1_\delta \times \bar{B}^2_\mu \subset \bar{W}^{k+1}_{w_i}((K_i, g_j) : \Lambda^1 \otimes AdP) \times T_0 B$$

around $\bar{A}_j$ in which the implicit function theorems work for the perturbed ASD equations F^+_s.

Now we show the following:

Proposition 2.4 *There is a regular point $b \in B$ with respect to $\mathfrak{M}_b(A)_w$ with a sufficiently small norm, so that there are elements:*

$$(\bar{A}_j - \bar{C}_j, b) \in (\bar{B}^1_\delta \times \bar{B}^2_\mu)$$

satisfying the equations $F^+_s(\bar{C}_j, b) = 0$ with respect to (K_i, g_j).

In particular there is a uniform constant C independent of i and j so that the estimates: $||\bar{A}_j - \bar{C}_j|| \leq C$ hold for all $i << j$.

Proof: Here we use a property that the generic moduli spaces $\mathfrak{M}(M, g_i)$ are non empty, since the Donaldson's invariant is not zero. Even though there are cobordisms between $\mathfrak{M}(M, g_j)$ and $\mathfrak{M}_b(M, g_j)$, one has to check that the latter is non empty inside $\bar{B}^1_\delta \times \bar{B}^2_\mu$ after taking quotient.

Now let $P \subset B$ be the Floer's perturbation. Since $D^+_A|W \oplus T_0P$ is already surjective by lemma 1.3, and since the inclusion is dense, one can replace any small generic element $b \in B$ by another one $b' \in P$ with $||b - b'||$ sufficiently small. In particular for b_t in lemma 2.8, one may assume that it is a generic path inside P.

Since each ASD connection A_j is regular and so if b is generic, then there exists a one dimensional cobordism between $\mathfrak{M}(M, g_j)$ and $\mathfrak{M}_b(M, g_j)$ parametrized by A^t_j. In particular its image $\bar{A}^1_j$ under the projection will give the desired $\bar{C}_j$.

We proceed in precise below. Now by the Sard-Smale transversality theorem, there is a generic path $b_t \in B$, $b_0 = 0$ and $b_1 = b$ with respect to all $\mathfrak{M}(M, g_j)$, $j = 1, 2, \ldots$. One may assume that b_t satisfies the conclusion of lemma 2.8. Since the Donaldson's invariant does not vanish, there are non empty cobordisms between $\mathfrak{M}(M, g_j)$ and $\mathfrak{M}_b(M, g_j)$. In particular for each j, one can choose some A_j so that there is a parametrization A^t_j, $0 \leq t \leq 1$ with $A^0_j = A_j$ and satisfying the equations $F^+_s(A^t_j, b_t) = 0$.

Let us put $A_j - A^t_j = \alpha^t_j$, and put the quotients:

$$\bar{A}^t_j = A_j + \bar{\alpha}^t_j \in A_j + \bar{W}^{k+1}_{w_j}(K_i, g_j).$$

Lemma 2.9 *There is some positive constant $c > 0$ independent of i, j so that after gauge transformation by elements $\bar{W}^{k+2}_{w_j}(K_i, g_j) \cap Aut\ (E|K_i)$, one can assume that these connections satisfy the condition:*

$$\bar{A}^t_j \in (imd_{\bar{A}_j})^\perp$$

as far as $||\bar{A}^t_j - \bar{A}_j||\bar{W}^{k+1}_{w_j} < c$ hold.

Proof: This is standard for the usual ASD connection on (M, g_j). For our case, it also works since the proof uses only the implicit function theorem ([DK] $p56$). Then the result follows by corollary 2.3. This completes the proof.

Lemma 2.10 *There is some positive $t_0 > 0$ so that $(\bar{\alpha}^{t_0}_j, b_{t_0}) \in \bar{B}^1_\delta \times \bar{B}^2_\mu$ hold for all $j >> i$.*

Proof: Suppose contrary, and let us fix a small $0 < \epsilon_0 << \delta, \mu$. Then there are indices $\{t_m\}_m$ and $\{i_m, j_m\}_m$, $t_m \to 0$ and $i_m, j_m \to \infty$ so that

$$||\bar{A}_j - \bar{A}^{t_m}_{j_m}||\bar{W}^{k+1}_{w_{i_m}}((K_{i_m}, g_{j_m}); \Lambda^1 \otimes AdP) = \epsilon_0$$

should hold. Then by proposition 2.3, this implies that there are another sequence $0 < s_m \leq t_m$ so that for $\bar{\alpha}_m = \bar{A}_j - \bar{A}_{jm}^{s_m}$:

$$s_m^{-1} || \bar{\alpha}_m - \text{ Ker } d_{\bar{A}}^+ || \bar{W}_w^{k+1}(K_{i_m}, g) \to 0$$

should approach to zero, since norms of b_t are small for all t.

Since $\bar{\alpha}_m \in (\text{ im } d_{\bar{A}_{jm}})^{\perp}$, it follows from corollary 2.4 that the family $\{\bar{\alpha}_m\}_m$ converges to the finite dimensional subspace T.

Then by use of proposition 2.3, one may assume that for any small $\epsilon > 0$, the estimates $||(\bar{\alpha}_m, b_{s_m}) - \tilde{G}r(D, 0)|| \leq \epsilon s_m$ hold all large $m >> 0$. Now let $pr : W \oplus T_0 B \to T_0 B$ be the projection. Then combinig with lemma 2.8, one obtains the estimate:

$$\begin{aligned} 0 &= ||b_{s_m} - pr \circ (\bar{\alpha}_{s_m}, b_{s_m})|| \\ &\geq ||b_{s_m} - pr \circ \tilde{G}r(D, 0)|| - ||pr \circ (\bar{\alpha}_{s_m}, b_{s_m}) - pr \circ \tilde{G}r(D, 0))|| \geq \frac{Cs_m}{2}. \end{aligned}$$

This is a contradiction. This completes the proof of lemma.

Proof of proposition: Now the path may not be generic with respect to A, since one has chosen the sequence $\{A_j\}_j$ after choice of b_t. Notice that originally possible number of A may have the one of choices of sequences which is the Cantor set.

Thus let us choose again another generic path $b'_t \in P$ which is sufficiently near b_t so that it is generic with respect to all $\mathfrak{M}(M, g_j)$ for all j, and $\mathfrak{M}_{b'_t}(A)_w$ are regular for all $0 < t \leq t_0 \leq 1$ for some t_0.

Now by lemma 1.4, let $c : [0, 1]^2 \to P$ be a generic map with $c(0, t) = b_t$ and $c(1, t) = b'_t$. Thus $\mathfrak{M}_c(M, g_j)$ are smooth manifolds with corners. For each j, we have chosen A_j so that there are non empty parametrizations A_j^t with respect to b_t. Since the moduli spaces are manifolds, this implies that thier parametrizations can be extended as $A_j^{(s,t)}$ so that the equations $F_s^+(A_j^{(s,t)}, c(s, t)) = 0$ hold.

In particular there are solutions $B_j^t \equiv A_j^{(1,t)}$ with $F_s^+(B_j^t, b'_t) = 0$. By lemma 2.10, one can find another solutions $(\bar{B}_j^{t_0}, b'_{t_0})$ with $F_s^+(\bar{B}_j^{t_0}, b'_{t_0}) = 0$ which satisfy uniformity of norms:

$$||\bar{A}_j - \bar{B}_j^{t_0}|| \bar{W}_{w_j}^{k+1}(K_i, g_j) \leq c$$

for a small constant $c > 0$. Then one can put $\bar{C}_j = \bar{B}_j^{t_0}$ and $b = b'_{t_0}$. This completes the proof.

2.B.8 Completion of the proof of theorem 0.4**:** Now we replace b' by b and t_0 by 1 by reparametrization. We complete the proof of theorem 0.4.

We have a lemma concerning abstract functional analysis:

Lemma 2.11 *Let $W^{k+1}(K)$ be the Sobolev spaces on a compact $K \subset S$. Then any bounded sequence $\{u_j\}_j \subset W^{k+1}(K)$, $||u_j||W^{k+1}(K) \leq C$ admits a convergent subsequence to u in $W^k(K)$ with $||u||W^{k+1}(K) \leq C$.*

Proof: This follows from the Rellich lemma. See [GT] $p168, 169$.

In order to specify the indices i, let us write $\bar{C}_j^i$ for $\bar{C}_j$ above. Thus we have obtained a family of solutions $(\bar{C}_j^i, b)$ with $F_s^+(\bar{C}_j^i, b) = 0$ over (K_i, g_j) and $\bar{A}_j - \bar{C}_j^i \in \bar{W}_{w_i}^{k+1}((K_i, g_j) : \Lambda^1 \otimes AdP)$, which satisfy the uniform estimates:

$$||\bar{A}_j - \bar{C}_j^i||\bar{W}_{w_i}^{k+1}((K_i, g_j); \Lambda^1 \otimes AdP) \leq C.$$

Recall that as $j \to \infty$, $g_j|K_i \to g|K_i$, where g is the metric on S.

Then for each i, using the above lemma, one takes a convergent subsequence $\bar{C}_{j_m}^i$ to $\bar{C}^i$ over (K_i, g). This satisfies the equation $F_s^+(\bar{C}^i, b) = 0$ and the estimate:

$$||\bar{A} - \bar{C}^i||\bar{W}_w^{k+1}((K_i, g); \Lambda^1 \otimes AdP) \leq C.$$

Finally one takes another subsequence $\{\bar{C}^{i_m}\}_m$ which converges to a smooth connection A' over (S, g). It satisfies the equation $F_s^+(A', b) = 0$ and the estimate:

$$||A - A'||W_w^{k+1}((S, g) : \Lambda^1 \otimes AdP) \leq C.$$

This is the desired connection, and one has finished the proof of the theorem.

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