Stability of self-similar solutions of the Dafermos regularization of a system of conservation laws

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Consider a system of viscous conservation laws in one space dimension,

$$u_T + f(u)_X = (B(u)u_X)_X, \quad -\infty < X < \infty, \tag{1}$$

with constant boundary conditions $(-\infty, T) = u^\ell$, $\quad u(+\infty, T) = u^r$, and some initial condition $u(X, 0) = u^0(X)$. It is believed that as $T \to \infty$, solutions typically approach Riemann solutions $\hat{u}(x)$, $x = \frac{X}{T}$, for the system of conservation laws

$$u_T + f(u)_X = 0. \tag{2}$$

Discontinuities (shock waves) in the Riemann solution are allowed if they correspond to traveling waves of (1).

In the variables $x = \frac{X}{T}$, $t = \ln T$, (1) becomes

$$u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x. \tag{3}$$

If we "freeze time" in (3) by setting $\epsilon = e^{-t}$, we obtain

$$u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x. \tag{4}$$

Equation (4) is the Dafermos regularization of the system of conservation laws (2) associated with the viscosity $B(u)$.

It is known in many cases that near a Riemann solution $\hat{u}(x)$ of (2), there are, for small $\epsilon > 0$, steady-state solutions $u_\epsilon(x)$ of (4). Their time-asymptotic stability as solutions of (4) can be studied by linearization. The information thus obtained should be useful in understanding Riemann solutions as time-asymptotic states of (1).

For the case $B(u) \equiv I$ and a Riemann solution consisting of n Lax shocks, I shall explain how geometric singular perturbation theory can be used to construct the $u_\epsilon(x)$ and to find eigenvalues and eigenfunctions of the linearization of (4) at $u_\epsilon(x)$. The eigenvalues are related to (i) eigenvalues of the traveling waves that correspond to the individual shock waves, which have been studied by a number of authors using Evans function methods; and (ii) inviscid stability conditions that have been obtained by various authors for the underlying Riemann solution.