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A p-adic limit of Siegel Eisenstein series of degree 2

§1. Introduction

$E^{(2)}_{k,\psi}$: Siegel Eisenstein series of degree 2

In this talk, we treat the following

- p-stabilization of $E^{(2)}_{k,\psi}$ $(= G^{(2)}_{k,\psi})$
- An explicit formula for Fourier coefficients of $G^{(2)}_{k,\psi}$
- the p-adic $\underbrace{\text{family}}_{\text{analytic}}$ which interpolates $G^{(2)}_{k,\psi}$

§2. Siegel-Eisenstein series and Hecke operator $U(p)$

$n \in \mathbb{Z}_{\geq 1}$

$\mathbb{H}_n := \{ Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}\, Z > 0 \}$

$Sp_{2n}(\mathbb{Z}) := \{ \alpha \in GL_{2n}(\mathbb{Z}) \mid {}^t\alpha\, \eta\, \alpha = \eta \}$ $\eta := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$

$\Gamma^{(n)}_0(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z}) \;\middle|\; C \equiv 0 \mod N \right\}$

$\Gamma_\infty := \{ \quad \text{〃} \quad \mid C = 0 \}$

Def Let $k \in \mathbb{Z}_{>n+1}$, ψ : Dirichlet char mod N and assume $\psi(-1) = (-1)^k$

$$E^{(n)}_{k,\psi}(Z) := \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\psi}(\det D) \cdot \det(CZ+D)^{-k}$$

$E^{(n)}_{k,\psi} \in M_k(\Gamma^{(n)}_0(N), \psi)$

$\forall f \in M_k(\Gamma^{(n)}_0(N), \psi)$ f has the following Fourier expansion

$$f(Z) = \sum_{h \in L_{\geq 0}} a(f,h) \exp(2\pi i \,\mathrm{Tr}\, hZ)$$

here

$$L := \{ h = (h_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid 2h_{ij},\ h_{ii} \in \mathbb{Z} \}$$

$$L_{\geq 0} := \{ h \in L \mid h \geq 0 \}$$

<u>Def</u> For p: prime and $f \in M_k(\Gamma_0^{(n)}(N), \psi)$

$$f|_{U(p)} := \sum_{h \in L_{\geq 0}} a(f, ph) \exp(2\pi i \operatorname{Tr} hZ)$$

$$f|_{U(p)} \in \begin{cases} M_k(\Gamma_0^{(n)}(N), \psi) & p \mid N \\ M_k(\Gamma_0^{(n)}(Np), \psi) & p \nmid N \end{cases}$$

§3. p stabilization of $E^{(2)}_{k,\psi}$ and the p-adic analytic family of Siegel-Eisenstein series of degree 2

Fix a prime p and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$

Let ψ : primitive Dirichlet char mod N

$$E'_{k,\psi} := 2^{-1} \cdot L(1-k, \psi) \cdot \underset{\uparrow}{L^{(N)}}(3-2k, \psi^2) \times E^{(2)}_{k,\psi}$$

(Euler factors at q $(q \mid N)$ removed)

<u>Def</u>

(i) $p \mid N$

$$G^{(2)}_{k,\psi} := E'_{k,\psi}\Big|_{\prod_{q \mid N} V(q)}$$

$$V(q) := \frac{1}{1 - \overline{\psi}^2(q) \cdot q^{3-2k}} \left(U(q)^2 - \overline{\psi}^2(q) \cdot q^{3-2k} \cdot U(q)^3 \right)$$

$V(q)$ simplifies the Euler factor of $a(h, E'_{k,\psi})$ $(q \mid N)$

(ii) $p \nmid N$

$$G^{(2)}_{k,\psi} := \left(E'_{k,\psi}\Big|_{\prod_{q \mid N} V(q)} \right)\Big|_{W(p)}$$

here $W(p) := \left(U(p) - \psi(p)\, p^{k-1}\right)\left(U(p) - \psi^2(p)\, p^{2k-3}\right)$

By next thm, $G^{(2)}_{k,\psi}\Big|_{U(p)} = G^{(2)}_{k,\psi}$

<u>Thm 1</u> (explicit formula for $a(h, G^{(2)}_{k,\psi})$)

ψ : as before, $k > 3$

$a(h) = a(h, G^{(2)}_{k,\psi})$

(i) rank $h = 0$

$$a(h) = \frac{1}{2} L^{(p)}(1-k, \psi)\, L^{(Np)}(3-2k, \psi^2)$$

(ii) rank $h = 1$

$$a(h) = L^{(Np)}(3-2k, \psi^2) \cdot \prod_{q \nmid Np} F^{(1)}_q(\varepsilon(h); \psi(q) q^{k-2})$$

$$\varepsilon(h) := \max\left\{ m \in \mathbb{Z}_{\geq 1} \;\middle|\; m^{-1} h \in L \right\}$$

$$F^{(1)}_q(n; T) = 1 + (qT) + \cdots + (qT)^{\mathrm{ord}_q(n)}$$

(iii) rank $h = 2$

$$a(h; G^{(2)}_{k,\psi}) = L^{(Np)}(2-k, \chi_h \psi) \cdot \prod_{q \nmid Np} F^{(2)}_q(h; \psi(q) q^{k-3})$$

Here,

χ_h : quad. char $\chi_h \longleftrightarrow \mathbb{Q}(\sqrt{-\det h})/\mathbb{Q}$

$$F^{(2)}_q(h; T) = \sum_{i=0}^{\alpha_1} (q^2 T)^i \left(\sum_{j=0}^{\alpha - i} (q^3 T^2)^j - \chi_h(q) \cdot qT \sum_{j=0}^{\alpha - i - 1} (q^3 T^2)^j \right)$$

$\cap$
$\mathbb{Z}[T]$

$\alpha_1 = \mathrm{ord}_q(\varepsilon(h))$

$\alpha_q = \alpha = \frac{1}{2}\left(\mathrm{ord}_q(\det 2h) - \mathrm{ord}_q \mathrm{Cond}(\chi_h)\right)$

$\deg F^{(2)}_q(h; T) = 2\alpha$

By thm 1, we can prove the following

Thm 2 Assume $\psi(-1) = 1$

$u := 1 + 2p$

$\forall h \in L_{\geq 0}$, $\exists\, a(h;T) \in \mathrm{Frac}(\mathbb{Z}_p[\psi][[T]])$

s.t. $a(h; \varepsilon(u)u^k - 1) = a(h; G^{(2)}_{k, \varepsilon\psi\omega^{-k}})$

$\forall \varepsilon : 1 + 2p\mathbb{Z}_p \longrightarrow \overline{\mathbb{Q}}_p^\times$ finite order character $\quad (-1)^k = \varepsilon\psi\omega^{-k}(-1)$

$\forall k \in \mathbb{Z}_{\geq 3}$, ω : Teichmüller char

$q \nmid N$ Euler product of $a(h, E^{(2)}_{k,\psi})$ is known

$q \mid N$ If q is odd prime
Gunji calculated Euler factor at q