T. Ohshita The Euler systems of cyclotomic units & the higher Fitting ideals

§1. Introduction

IMC ... $\mathrm{char}_\Lambda(\underbrace{X}_{\text{certain Iwasawa module}})$

this talk ... $\{\mathrm{Fitt}_i(X)\}_{i \geq 0}$ $\quad \mathrm{Fitt}_0(X) \subseteq \mathrm{Fitt}_1(X) \subseteq \cdots \subset \Lambda$

$\mathrm{Fitt}_0(X) = \mathrm{char}(X)$; $\mathrm{Fitt}_1(X)$, ... : to be defined

Kurihara's work

K_∞

| cyclo $\mathbb{Z}_p$-ext

K_0

Δ

k : tot real # field

X : the minus part of Iwasawa modules of ideal class group

He determined $\mathrm{Fitt}_{\Lambda_\chi, i}(X_\chi)$ for $^\forall i \geq 0$, $\chi \in \hat{\Delta}$: odd

$\mathrm{Fitt}_{\Lambda_\chi, i}(X_\chi)$ = "the i-th Stickelberger ideals"

This talk

X : the plus part

We determine "upper bounds" of $\mathrm{Fitt}_{\Lambda_\chi, i}(X_{\chi})$ for $^\forall i \geq 0$ (χ : even; $\subset$ ideal)

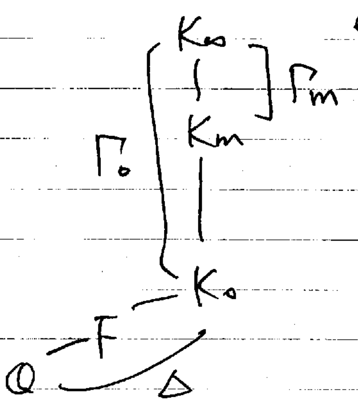
Notation

p : odd prime #

F : tot. real abel $/\mathbb{Q}$

$p \nmid D_F \cdot [F:\mathbb{Q}]$

$K_m := F(\mu_{p^{m+1}})^+$



$$\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/\mathbb{Q})]] = \bigoplus_{\chi \in \hat{\Delta}/\sim} \Lambda_\chi$$

$$\Delta \overset{\chi}{\curvearrowright} \Lambda_\chi \cong \underbrace{\mathbb{Z}_p[\mathrm{Im}\,\chi]}_{\mathcal{O}_\chi}[[\Gamma_0]] \cong \mathcal{O}_\chi[[T]]$$

A_m : the p-Sylow subgp of the ideal class gp of K_m

$$X := \varprojlim_{\substack{m \\ \text{norm}}} A_m = \bigoplus_{\chi \in \hat{\Delta}/\sim} X_\chi$$

$$X' = X / \underbrace{X_{fin}}_{\text{the max. submod of finite order}}$$

In §3, we define "the cyclotomic ideals" $\{\mathfrak{C}_{i,\chi}\}_{i \geq 0}$ by using the Kolyvagin derivatives of cyclotomic units.

<u>Thm</u> (Done for $F = \mathbb{Q}$ / general case ... in progress)

Let $\chi \in \hat{\Delta}$

(1) $\exists\, I_{0,\chi} \subseteq \Lambda_\chi$: an ideal of finite index

s.t. $\underbrace{I_{0,\chi}\, \mathfrak{C}_{0,\chi}}_{\text{the lower bound}} \subseteq \mathrm{Fitt}_{\Lambda_\chi, 0}(X'_\chi)$ ← for $i=0$

(2) For $\forall i \geq 0$, $\exists\, J_{i,\chi} \subseteq \Lambda_\chi$: fin. index ← for $i \geq 0$

$\exists\, a \in \mathbb{Z}_{\geq 0}$

s.t. $\underbrace{(\gamma - 1)^a \cdot J_{i,\chi}}_{\text{error term}} \cdot \mathrm{Fitt}_{\Lambda_\chi, i}(X'_\chi) \subset \underbrace{\mathfrak{C}_{i,\chi}}_{\text{upper bound}}$

<u>Rem</u> • When $F = \mathbb{Q}$ and $\chi \neq 1$, we can take

$$\begin{cases} I_{0,\chi} = (1) & \boxed{a=0} \\ J_{i,\chi} = \mathrm{ann}_{\Lambda_\chi}(X_{fin,\chi}) \end{cases}$$

• Thm for $i=0$ $\Leftrightarrow$ IMC

• We use IMC in the proof of Thm

So we do not give a new proof of IMC

$$\mathcal{C}_0 \doteq \mathrm{char}_\Lambda(E_\infty / C_\infty)$$

§2.

<u>Def</u> (higher Fitting ideals)

R : comm ring

M : R-mod of fin. pres.

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0 \quad \text{exact}$$

$\mathrm{Fitt}_{R,i}(M)$: the ideal of R generated by $(n-i)\times(n-i)$ minors of f

$$\mathrm{Fitt}_0 \subset \mathrm{Fitt}_1 \subset \mathrm{Fitt}_2 \subset \cdots \subset \mathrm{Fitt}_n = R = \cdots$$

<u>ex</u> $M_1 = \Lambda_\chi/(f^2)$ $\quad (f^2)$

$M_2 = (\Lambda_\chi/(f))^2$ $\quad \begin{pmatrix} f & \\ & f \end{pmatrix}$

$$\begin{array}{ccccc} \mathrm{Fitt}_0(M_1) & = & \mathrm{Fitt}_0(M_2) & = & (f^2) \\ \| & & \| & & \\ \mathrm{char}(M_1) & & \mathrm{char}(M_2) & & \end{array}$$

$\mathrm{Fitt}_1(M_1) = 1$, $\mathrm{Fitt}(M_2) = (f)$

<u>In fact</u> when $R = \Lambda_\chi$, higher Fitt ideals determine the pseudo-isom class

§3. cyclotomic units & cyclotomic ideals

$e \in \mathbb{Z}$: fixed top gen. of $\mathbb{Z}_p^\times$, $N \in \mathbb{Z}_{>0}$

$S_N := \{ \ell \mid \text{prime } \#, \ell \nmid e, \ell \text{ splits in } K_0, \ell \equiv 1 \pmod{p^N} \}$

$\mathcal{N}_N := \{ \prod_{i=1}^{r} \ell_i \mid \ell_i \in S_N, \ell_i \neq \ell_j \text{ if } i \neq j \} \cup \{1\}$

- For each prime power ℓ^ν, we fix primitive $\zeta_{\ell^\nu} \in \mu_{\ell^\nu}$ s.t. $\zeta_{\ell^{\nu+1}}^{\ell} = \zeta_{\ell^\nu}$
- For $n = \prod_{\ell : \text{prime}} \ell^{\nu_\ell}$, we put $\zeta_n := \prod_\ell \zeta_{\ell^{\nu_\ell}}$

Let $n = \prod_{i=1}^{r} \ell_i \in \mathcal{N}_N$

$K_m(n) := F(\mu_{p^{m+1} \cdot n})^+$

(In particular, $K_m(1) = K_m$)

- For $\nu \in \mathbb{Z}_{>0}$ with $\nu \mid D_F$, we define

$$\underset{K_m(n)^\times}{\overset{\cap}{\eta_\nu(n)}} := \begin{cases} N_{\underbrace{\mathbb{Q}(\mu_{\nu \cdot n \cdot p^{m+1}})}_{\mathbb{Q}_{\nu,n,m}} / \mathbb{Q}_{\nu,n,m} \cap K_m(n)} \left(1 - \zeta_{\nu \cdot n \cdot p^{m+1}}\right) & \text{if } \nu \neq 1 \\ \dfrac{\zeta^{-e/2} - \zeta^{e/2}}{\zeta^{-1/2} - \zeta^{1/2}} & \text{if } \nu = 1 \end{cases}$$

$(\zeta = \zeta_{n \cdot p^{m+1}})$

- $H_n := \mathrm{Gal}(K_m(n)/K_m) \cong \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$

$\cong H_{\ell_1} \times \cdots \times H_{\ell_r}$

$$D_{l_i} = \sum_{k=1}^{l_i-2} k\,\sigma_{l_i}^{k} \in \mathbb{Z}[H_{l_i}]$$

where $\sigma_{l_i} \in H_{l_i}$ (fixed generator)

$$D_n = \prod_{i=1}^{r} D_{l_i} \in \mathbb{Z}[H_n]$$

<u>Def</u> Let $n \in \mathcal{N}_N$, $\nu \mid D_F$

$$K_m^\times / (K_m^\times)^{p^N} \xrightarrow{\sim} \left[K_m(n)^\times / (K_m(n)^\times)^{p^N} \right]^{H_n}$$

$$\exists!\ \kappa_{m,N}(\eta_{\nu,n}) \longmapsto \eta_\nu(n)^{D_n}$$

<u>Def</u> $W_{m,N}^{n} \subset K_m^\times / (K_m^\times)^{p^N}$

the $R_{m,N} := \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]$-submod

generated by $\{ \kappa_{m,N}(\eta_{\nu,n}) \mid \nu \mid D_F \}$

$\mathcal{C}_{i,m,N}$: an ideal of $R_{m,N}$ gen. by
<u>all</u> images of
$$f \in \mathrm{Hom}_{R_{m,N}}(W_{m,N}^{n}, R_{m,N})$$
for $\forall n$ satisfying $\varepsilon(n) \leq i$

the # of prime divisors

$$\mathcal{C}_i := \varprojlim_{\substack{(m,N)\\ N > m+1}} \mathcal{C}_{i,m,N} \subset \Lambda$$

the <u>i-th cyclotomic ideal</u>