Francesco Lemma

A norm compatible system of Galois cohomology classes for $GSp(4)$

I. Motivation
construction of p-adic L-function from Euler systems

II. The elliptic polylogarithm pro-sheaf

III. The norm relation

p: odd prime

I. Motivations

Basic example: $a, b > 0$ prime to p

$(\zeta_n)_{n \geq 1}$ system of p^n-th roots of unity $\zeta_n^p = \zeta_{n-1}$

Then,

$$u_n = \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}} \in \mathbb{Z}[\zeta_n]^\times$$

for $m | n$,

$$\mathrm{Norm}(u_n) = u_m \iff (u_n)_{n \geq 1} \in \varprojlim_n H^1(\mathbb{Q}(\zeta_n), \mathbb{Z}_p(1))$$

$$(u_n) \xrightarrow{\text{Coleman map}} \text{(pseudo-) measure on } \mathbb{Z}_p^\times$$

such that $d\zeta_p$

$$\int_{\mathbb{Z}_p^\times} x^k \, d\zeta_p = (b^k - a^k)(1 - p^{k-1}) \zeta(1-k)$$

$\forall k > 0$ even

In general: $G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$

$$G_\infty = \varprojlim_n G_n \cong \mathbb{Z}_p^\times$$

$\mathbb{Z}_p[\![G_\infty]\!]$: algebra of measures over $\mathbb{Z}_p^\times$

V : p-adic repn of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = G_{\mathbb{Q}_p}$

Def V is ordinary if it has a filtration $F^{\bullet}V$ stable by $G_{\mathbb{Q}_p}$
s.t. the inertia group acts on $Gr^{\cdot}V$ via a power of the cyclotomic char.

$T \subset V$ stable lattice

Thm (Coleman, Perrin-Riou)
let $\eta \in D_{cris}(V^*(1)) = (B_{crys} \otimes V^*(1))^{G_{\mathbb{Q}_p}}$
assume V ordinary
Then, $\exists!$ homomorphism

$$\mathcal{L}_\eta : \varprojlim_n H^1(\mathbb{Q}_p(\zeta_n), T) \otimes \mathbb{Q}_p \longrightarrow \mathbb{Z}_p[[G_\infty]] \otimes \mathbb{Q}_p$$

that interpolates Bloch-Kato dual exponentials.

Other example (Kato)
f : elliptic modular form (ordinary at p)
$V(f)$: associated Galois rep
$\forall h, r \geq 0 \quad \exists\, z^{BK} \in \varprojlim H^1(\mathbb{Q}(\zeta_n), V(f)(h-r))$
choose $\alpha \in \overline{\mathbb{Q}}_p^\times$ s.t.

$$(1-\alpha X) \mid (1 - a_p X + \varepsilon(p) p^{k-1} X^2)$$

$$\mathrm{ord}_p(\alpha) < k-1, \quad \mathrm{ord}_p(p) = 1$$

Then, for a good $\eta \in D_{crys}(V^*(r-h+1))$

$$\int_{\mathbb{Z}_p^\times} x^r \mathcal{L}_\eta(z^{BK}) = (r-1)!\,(2\pi i)^{h-r-1} \cdot \frac{1}{\Omega_\pm}(1-p^{r-1}\alpha^{-1})(1-\varepsilon(p)p^{h-r-1}\alpha^{-1}) \cdot L(f,r)$$

for $1 \leq r \leq k-1$, $\pm = (-1)^{h-r-1}$ $\quad \Omega_\pm$: complex period

<u>Goal</u> : provide another example of norm compatible system for $GSp(4)$

<u>Rem</u> : we construct global classes

$\rightsquigarrow$ 1 st step towards the construction of an Euler system for $GSp(4)$

II. The elliptic polylog pro-sheaf

S : connected scheme of char 0.

$$\begin{array}{l} E \leftarrow \text{elliptic curve} \\ \pi \downarrow \uparrow e \ \text{zero section} \\ S \end{array} \qquad \begin{tikzcd} E \arrow[rr, "{[p^j]}"] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd}$$

$j' \geq j$ $\quad [p^{j'-j}]_! [p^{j'-j}]^! \mathbb{Q}_p \xrightarrow{tr} \mathbb{Q}_p$

$$= [p^{j'-j}]_* [p^{j'-j}]^* \mathbb{Q}_p$$

$$[p^j]_* tr : [p^{j'}]_* \mathbb{Q}_p \longrightarrow [p^j]_* \mathbb{Q}_p$$

<u>Def</u> the <u>logarithm pro-sheaf</u> is the smooth étale pro-sheaf $\left([p^j]_* \mathbb{Q}_p\right)_{j \geq 1} =: \mathcal{L}og_E$

<u>Rem</u> proper base change thm

$\Rightarrow$ for $f : E' \longrightarrow E$ (over S) $\quad f^* \mathcal{L}og_E = \mathcal{L}og_{E'}$

$$R = e^* \mathcal{L}og_E \qquad J = \mathrm{Ker}(R \xrightarrow{\text{augm.}} \mathbb{Q}_p)$$

Prop (Beilinson - Levin)

$$R^n \pi_{U*} \mathcal{L}og_E\big|_U = \begin{cases} 0 & \text{if } n \neq 1 \\ \mathcal{J}(-1) & \text{if } n = 1 \end{cases}$$

$$\begin{tikzcd} U = E \setminus e(S) \arrow[r, hook, "j"] \arrow[dr, "\pi_U"'] & E \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

Cor $\mathrm{Ext}^1_U(\pi_U^* \mathcal{J}, \mathcal{L}og|_U(1)) \cong \mathrm{Hom}_S(\mathcal{J}, \mathcal{J})$

Def the polylog. prosheaf is the ext mapping to the Id under the above iso.

$$\mathcal{P}ol_E$$

Lemma for $f: E' \longrightarrow E$ $\quad f^* \mathcal{P}ol_E = \mathcal{P}ol_{E'}$

$$\begin{tikzcd} E' \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd}$$

proof

$$\begin{tikzcd} \mathrm{Ext}^1_U(\pi_U^* \mathcal{J}, \mathcal{L}og(1)) \arrow[r, "f^*"] \arrow[d, "\wr"] & \mathrm{Ext}^1_{U'}(\pi_{U'}^* \mathcal{J}, \mathcal{L}og_E|_{U'}(1)) \arrow[d, "\wr"] \\ \mathrm{Hom}(\mathcal{J}, \mathcal{J}) \arrow[r, "f^*"] & \mathrm{Hom}(\mathcal{J}', \mathcal{J}') \end{tikzcd}$$

Norm compatibility

$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd} \quad \text{isogeny}$$

$$N_f : \mathrm{Ext}^1_{U'}(\pi_{U'}^* \mathcal{J}', \mathcal{L}og_{E'}|_{U'}(1)) \longrightarrow \mathrm{Ext}^1_U(\pi_U^* \mathcal{J}, \mathcal{L}og|_U(1))$$

Prop (Wildeshaus) $N_f(\mathcal{P}ol_{E'}) = \mathcal{P}ol_E$

Pullback along torsion sections

let $t : S \hookrightarrow E$ be torsion section $\neq 2$

Lemma $t^* \mathcal{L}og_E = e^* \mathcal{L}og_E = R$

~~proof~~ follows from functoriality of $\mathcal{L}og_E$

$$t^* \mathcal{P}ol_E \in \mathrm{Ext}^1_{R,S}(\mathcal{I}, R(1)) \xrightarrow{2} \mathrm{Ext}^1_{R,S}(R, R(1)) = \mathrm{Ext}^1_{\mathbb{Q}_p, S}(\mathbb{Q}_p, R(1)) = H^1_{et}(S, R(1))$$

$$R \cong \prod_{k \geq 1} \mathrm{Sym}^k \mathcal{H}$$

where $\mathcal{H} = \underline{\mathrm{Hom}}(R^1\pi_* \mathbb{Q}_p, \mathbb{Q}_p)$

Def The *Eisenstein classes* are the

$$E_t^k = (i\, t^* \mathcal{P}ol_E)^k \in H^1_{et}(S, \mathrm{Sym}^k \mathcal{H}(1))$$

III The norm relations

$M, N \geq 3$

$$\begin{array}{c} E \\ \pi \downarrow \\ Y(N) \\ \circlearrowleft\, GL_2(\mathbb{Z}/N\mathbb{Z}) \end{array}$$

$Y(N)$ modular curve of full level N

L = common multiple of M and N

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/L\mathbb{Z}) \;\middle|\; \begin{array}{ll} a \equiv 1\ (M) & c \equiv 0\ (N) \\ b \equiv 0\ (M) & d \equiv 1\ (N) \end{array} \right\}$$

$$Y(M,N) = Y(L)/_G$$

$$Y(M,N)(S) = \left\{ \begin{array}{c} E \\ \downarrow \\ S \end{array} \quad (\mathbb{Z}/_M \times \mathbb{Z}/_N)_S \subset E \right\} \Big/ \sim$$

$$Y(M,M) = Y(M) \qquad (e_1, e_2) \text{ basis}$$

$(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 - \{(0,0)\}$ define $E^h_{\alpha,\beta}$ as follows:

let N s.t. $N\alpha = N\beta = 0$

$$(\alpha, \beta) = \left(\frac{a}{N}, \frac{b}{N}\right) \in (\mathbb{Q}/\mathbb{Z})^2$$

$$E^h_{\alpha,\beta} := E^h_{(ae_1 + be_2)} \in H^1_{ét}(Y(N), \mathrm{Sym}^h \mathcal{H}(1))$$

For $M|N$, we have an étale cover

$$Y(N) \longrightarrow Y(M)$$

in the $\bigcup_N H^1_{et}(Y(N), \mathrm{Sym}^h \mathcal{H}(1))$, the class $E^h_{\alpha,\beta}$ does not depend on N.

<u>Lemma</u>

(1) (functoriality) $\forall \sigma \in GL_2(\mathbb{Z}/N\mathbb{Z})$ $\sigma^* E^h_{\alpha,\beta} = E^h_{\alpha',\beta'}$
where $(\alpha', \beta') = (\alpha, \beta)\sigma$

(2) (distribution relation)
$$\forall a \geq 1, \quad E^h_{\alpha,\beta} = \sum_{\substack{a\alpha' = \alpha \\ a\beta' = \beta}} E^h_{\alpha',\beta'} \in \bigcup_N H^1_{ét}(Y(N), \mathrm{Sym}^h \mathcal{H}(1))$$

(V, ψ) symplectic space of dim 4 $/\mathbb{Z}$

$$GSp(4)/_{\mathbb{Z}} = GSp(V,\psi) = \left\{ g \in GL(V) \,\middle|\, \begin{array}{l} \psi(gv, gw) = \nu(g)\psi(v,w) \\ \exists\, \nu(g) \in \mathbb{G}_m \end{array} \right\}$$

$N \geq 3$, $S(N)/_{\mathbb{Q}(\zeta_N)}$: Siegel modular variety
smooth and quasi-projective
$\mathbb{Q}(\zeta_N)$ - scheme.

$$S(N)(S) = \left\{ \begin{array}{l} \begin{matrix} A \\ \downarrow \\ S \end{matrix} \text{ abel scheme}, \ \lambda : A \to \check{A} \text{ principal polarization} \\ \text{level } N \text{ structure } (V/_{NV}) \xrightarrow{\sim} A[N] \end{array} \right\} \Big/ \sim$$

fix symplectic basis s.t. $\psi = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$
and the following embedding

$$GL_2 \underset{\mathbb{G}_m}{\times} GL_2 \hookrightarrow GSp(4)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

then we have

$$Y(N) \underset{\mathbb{Q}(\zeta_N)}{\times} Y(N) \overset{\iota}{\hookrightarrow} S(N) \quad \text{closed of codimension } 1$$

fix $k \geq k' \geq 0$ and W an algebraic irred rep of $GSp(4)$

s.t. $\iota^* W \supset (\mathrm{Sym}^k V \boxtimes \mathrm{Sym}^{k'} V) \otimes \det^{\otimes 3}$

where V is standard rep of GL_2

as smooth étale sheaves on $Y(N)$, we have $V \cong \mathcal{H}$

$$H^1(Y(N), \mathrm{Sym}^k V(1)) \otimes H^1(Y(N), \mathrm{Sym}^{k'} V(1))$$

$$\xrightarrow{\cup} H^2(Y(N) \underset{\mathbb{Q}(\zeta_N)}{\times} Y(N), (\mathrm{Sym}^k V \boxtimes \mathrm{Sym}^{k'} V)(2))$$

$$\hookrightarrow H^2(Y(N) \underset{\mathbb{Q}(\zeta_N)}{\times} Y(N), \iota^* W(1)) \xrightarrow[\iota_*]{\text{Gysin}} H^4_{et}(S(N), W)$$

$$E_N^{k,k'} := \iota_*(E^k_{1/N,0} \cup E^{k'}_{0,1/N})$$

Prop let $M \mid N$ with the same prime factors
we have

$$\mathrm{Norm} : H^4_{ét}(S(N), W) \longrightarrow H^4_{ét}(S(M), W)$$

then

$$\mathrm{Norm}(E_N^{kk'}) = d_{NM}^2 \, E_M^{kk'}$$

where d_{NM} is the degree of the étale cover

$$Y(M,N) \longrightarrow Y(M)$$

proof:
- the Gysin morphism and the Norm commute
- $\mathrm{Norm}(- \cup -) = \mathrm{Norm}(-) \cup \mathrm{Norm}(-)$

so it is enough to show:

$$\mathrm{Norm}(E^k_{1/N,0}) = d_{NM} \cdot E^k_{1/M,0}$$

$$\begin{tikzcd} Y(N) \arrow[rr] \arrow[dr, "f"] & & Y(M) \\ & Y(N,M) \arrow[ur, "g"'] & \end{tikzcd}$$

H

$$H = \left\{ s_{xy} = \begin{pmatrix} 1+Nu & Nv \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z}) \;\middle|\; \left.\begin{array}{l} u \equiv x \pmod{\frac{N}{M}} \\ v \equiv y \pmod{\frac{N}{M}} \end{array}\right\}_{x,y \in \mathbb{Z}/\frac{N}{M}\mathbb{Z}} \right.$$

$$\mathrm{Norm}_f(E^k_{1/N,0}) = \sum_{x,y \in \mathbb{Z}/\frac{N}{M}\mathbb{Z}} s^*_{xy} \, E^k_{1/N,0} = E^k_{\frac{1}{M},0}$$

(by the lemma)

$$E^k_{\frac{1}{M},0} = g^* E^k_{\frac{1}{M},0} \qquad \in H^1_{ét}(Y(N,M), \mathrm{Sym}^k V(1))$$

$$\mathrm{Norm}_g(E^k_{\frac{1}{M},0}) = d_{N,M} \, E^k_{\frac{1}{M},0}$$

Write $\mathcal{E}^{kk'}_{Np^t} = \dfrac{1}{\prod_{i=1}^{t} d^2_{Np^{i-1}, Np^i}} E^{kk'}_{Np^t} \in H^4_{et}(S(Np^t), W)$

<u>Cor</u> $(\mathcal{E}^{kk'}_{Np^t}) \in \varprojlim_t H^4_{et}(S(Np^t), W)$

<u>Prop</u> $H^0(\mathbb{Q}(\zeta_N), H^4_{et}(S(N)\otimes\overline{\mathbb{Q}}, W)) = 0$ if $k > k' > 0$

<u>proof</u>

$$S(N) \xhookrightarrow{j} S(N)^* \xhookleftarrow{i} \partial S(N)$$

Baily-Borel compactification

exact sequence:

$$H^4_c(S(N)\otimes\overline{\mathbb{Q}}, W) \longrightarrow H^4(S(N)\otimes\overline{\mathbb{Q}}, W)$$
$$\longrightarrow H^4(\partial S(N)\otimes\overline{\mathbb{Q}}, i^*Rj_*W)$$

<u>Thm</u> (Saper)
let S be a Shimura variety ass to a reductive group G, E: irred alg rep of G with regular highest weight

Then
$$H^i(S(\mathbb{C}), E) = 0 \qquad i < \dim S$$

By Poincaré duality and the comparison thm between singular and étale cohomology

$$\Rightarrow \quad H^4_c(S(N)\otimes\overline{\mathbb{Q}}, W) = 0$$

$\Rightarrow$ enough to show that $H^4(\partial S(N)\otimes\overline{\mathbb{Q}}, i^*Rj_*W)$ has no weight zero.

$$\partial S(N)_1 \xhookrightarrow{i_1} \partial S(N) \xhookleftarrow{i_0} \partial S(N)$$

stratification of the boundary by Shimura varieties

$\dim \partial S(N)_i = i$

$$H^4_c(\partial S(N)_1 \otimes \overline{\mathbb{Q}}, i_1^* i^* Rj_* W)$$
$$\longrightarrow H^4(\partial S(N) \otimes \overline{\mathbb{Q}}, i^* Rj_* W)$$
$$\longrightarrow H^4(\partial S(N)_0 \otimes \overline{\mathbb{Q}}, i_0^* i^* Rj_* W)$$

<u>Thm</u> (Pink)
describes $i_s i^* R^n j_* W$ in terms of group cohomology

let P be the Siegel parabolic
$$P = \left\{ \begin{pmatrix} \alpha A & AM \\ 0 & {}^t A^{-1} \end{pmatrix} \,\middle|\, \begin{array}{l} \alpha \in \mathbb{G}_m, \ A \in GL_2 \\ M \in Sym_2(\mathbb{Q}) \end{array} \right\}$$
gives rise to the strata of dimension 0

$$i_0^* i^* R^n j_* W = \bigoplus_{p+q=n} H^p(H_c, H^q(U, W))$$

H_c = arithmetic subgroup of $GL_2(\mathbb{Q}) \subset P$
U = unipotent radical of $P = Sym_2(\mathbb{Q})$
+ thm of Kostant describes the highest weight of $H^q(U, W)$

$$H^4(\partial S(N)_0 \otimes \overline{\mathbb{Q}}, i_0^* i^* Rj_* W) = H^0(\partial S(N)_0 \otimes \overline{\mathbb{Q}}, \underbrace{i_0^* i^* R^4 j_* W}_{>0 \text{ weight}})$$

$$E_2^{p,q} = H_c^p(\text{—}, i_1^* i^* R^q j_* W) \Rightarrow H_c^{p+q}(\partial S(N)_1 \otimes \overline{\mathbb{Q}}, i_1^* i^* Rj_* W)$$

$$H_c^4(\partial S(N)_1 \otimes \overline{\mathbb{Q}}, i_1^* i^* Rj_* W) = H_c^1(\partial S(N)_1 \otimes \overline{\mathbb{Q}}, i_1^* i^* R^3 j_* W)$$
$$= \underbrace{H_c^1(\text{——}, \underbrace{Sym^{k'} V}_{-k'-6}(3))}_{\text{no weight zero!}}$$

<u>Cor</u> Hochschild-Serre spectral sequence gives an isom

$$H^1(\mathbb{Q}(\zeta_N),\ H^3(S(N)\otimes\overline{\mathbb{Q}},\ W)) \xrightarrow{\sim} H^4_{ét}(S(N),\ W)$$

so we have

$$(\mathcal{E}^{kk'}_{Np^t})_{t\geq 1} \in \varprojlim_t H^1(\mathbb{Q}(\zeta_{Np^t}),\ H^3_{ét}(S(Np^t)\otimes\overline{\mathbb{Q}},W))$$