Ming-Lun Hsieh Lecture IV

<u>Recall</u> $\displaystyle\int_{G} \hat{\chi}\, d\mathcal{E}_{2,1} = e\,\hat{E}(\chi|1)$

$\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times \qquad \chi(z_\infty) = z_\infty^k$

$\hat{\chi} : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathcal{O}_{\mathbb{C}_p}^\times$ ← p-adic avatar

$G = \mathrm{Gal}(K(Np^\infty)/K)$

⟶ ordinary $\mathcal{O}[\![G]\!]$-adic Eisenstein Series on $U(2,1)$

⇓

$$\tilde{\mathcal{E}}_{2,1}^{ord} = \sum_{n>0} \Theta_n q^n + \underset{\substack{\uparrow \\ \text{Kubota-Leopoldt}}}{L_{K\text{-}L}} \cdot \underset{\substack{\uparrow \\ \text{Katz-Yager}}}{L_{K\text{-}Y}}$$

$\Theta_n \in H^0(\underset{\substack{(\\ \text{certain CM elliptic curves by } K}}{Z_{/\mathcal{O}}}, \mathcal{L}^{\otimes n}) \otimes_{\mathcal{O}} \mathcal{O}[\![G]\!]$

Description of Z in a simple situation

$\underline{B}$ family of CM elliptic curves by K

$\downarrow$ (if $\underline{B} = (B, \bar{\lambda}, \imath, \bar{\eta}, \mathfrak{J})$)

$S_{U(1)}$ Suppose λ is principally polarized

$$Z = B \qquad \begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \times B \\ x & \longmapsto & (x, x) \end{array}$$

$\mathcal{P}$ ← Poincaré line bundle

Fix branch character $\psi : \mathrm{Gal}(K'/K) \longrightarrow \mathbb{C}_p^\times$

K'/K : finite abelian disjoint from $K_\infty = K_\infty^+ K_\infty^-$

$\Gamma = \mathrm{Gal}(K_\infty/K)$

$$\pi_\psi : \mathcal{O}[\![G]\!] \longrightarrow \Lambda = \mathcal{O}[\![\Gamma]\!]$$
$$g \longmapsto \psi(g)\cdot g|_{K_\infty}$$

Let $\mathcal{E}^{\mathrm{ord}}_{2,1} = \pi_\psi(\widetilde{\mathcal{E}}^{\mathrm{ord}}_{2,1}) \rightsquigarrow$ Λ-adic Eisenstein series

<u>Assumption</u>

$$\begin{cases} \psi \text{ is unramified at } \bar{\mathfrak{p}} & (p = \mathfrak{p}\cdot\bar{\mathfrak{p}}) \\ \psi\,\omega_K^{-a} \text{ is unramified at } \mathfrak{p} \text{ for some } a \not\equiv 2 \ (p-1) \end{cases}$$

$\Rightarrow \exists\, \chi$ Galois char s.t. $\chi(z_\infty) = z_\infty^k$ $\begin{pmatrix} k \equiv a & (p-1) \\ k > 2 \end{pmatrix}$

$$\hat{\chi} : \mathrm{Gal}(K'\cdot K_\infty/K) \longrightarrow \mathbb{C}_p^\times$$

$$\hat{\chi}\big|_{\mathrm{Gal}(K'/K)} = \psi$$

$\Rightarrow \chi$ is unramified at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$

To prove $\mathcal{E}^{\mathrm{ord}}_{2,1}$ is nonvanishing modulo $\mathfrak{m}_\Lambda$

$\Leftrightarrow$ $e\cdot E(\chi|1)$ is 〃 modulo p

<u>Recall</u>

$$e\cdot E(\chi|1) \longleftarrow e(\phi_\chi^{\gamma,*})^{pb} = \bigotimes_{v\neq p} (\phi^*_{\chi,v})^{pb} \otimes \bigotimes_{v=p} (e\cdot\phi^{\gamma,*}_{\chi,p})^{pb}$$

$$E^\circ(\chi|1) \longleftarrow\!\!\sim \bigotimes_{v\neq p} (\phi^*_{\chi,v})^{pb} \otimes \bigotimes_{v=p} f^*_{\chi,p}$$

$f^\circ_{\chi,p}$ is the spherical section in $I_p(\chi,1,0)$

$$f^*_{\chi,p} = L_p(0,\chi)\cdot L_p(-1,\chi + \tau_{K/\mathbb{Q}})\cdot f^\circ_{\chi,p}$$

<u>Key observation</u>

$$\begin{cases} (1) & e\,E^\circ(\chi|1) = \boxed{C}\, e\,E(\chi|1) \\ (2) & e\,E^\circ(\chi|1) \equiv E^\circ(\chi|1) \mod p \quad k>2 \end{cases}$$

?

$$\left(\begin{aligned} \underline{\text{e.g.}}\ \ e\cdot E_k(z) &= E_k(z) - p^{k-1}E_k(pz) \\ &\equiv E_k(z) \mod p \quad \text{if } k>1 \end{aligned} \right)$$

Recall

$$e(\phi^{*,\gamma}_{\chi,p})^{pb} = E_p(-1,\chi_+)\cdot E_p(0,\chi_1)\cdot\phi^{ord}\cdot(\text{p-unit})$$

$$= \frac{L(-1,\chi_+)}{L(2,\chi_+^{-1})}\cdot\frac{L(0,\chi_1)}{L(1,\chi_+^{-1})}\cdot\phi^{ord}$$

$$\chi\big|_{(K\otimes\mathbb{Q}_p)^\times} = (\chi_1,\chi_2) \qquad \chi_1:\mathbb{Q}_p^\times\to\mathbb{C}^\times \quad \chi_2$$

Compute By Prop,

$$e(f^\circ_{\chi,p}) = M_{w_2} f^\circ_{\chi,p}(1)\cdot\phi^{ord}$$

Prop [Gindikin – Karpelavič] $\longrightarrow$ Casselman Spherical funct. I, II

$$M_{w_2} f^\circ_\chi(1) = \frac{L(-1,\chi_2)}{L(0,\chi_1)}\cdot\phi^{ord}$$

Combined with these formulae, we find

$$C = [L(-1,\chi_2)\cdot L(1,\chi_1^{-1})\cdot L(2,\chi_+^{-1})]^{-1}$$

$$= (1-\chi_2(p)p)(1-\chi_1^{-1}(p)p^{-1})(1-\chi_+^{-1}(p)p^{-2}) \text{ is } \underline{\text{p-unit}}!!$$

$$\chi_+ = \chi_1\chi_2 \qquad v_p(\chi_1(p)) = -k \qquad k>2$$

$$v_p(\chi_2(p)) = 0$$

(2) $e\cdot E^\circ(\chi|1) \equiv E^\circ(\chi|1)$ (Hida, Yellow book, Chap 8)

We have reduced the problem to the non-vanishing of $E^\circ(\chi|1)$

$$E^\circ(\chi|1) = \sum_{n\geq 0}\Theta_n q^n$$

$$\begin{array}{c} B_K \\ \downarrow \\ S_{U(1)/K} \end{array}$$

$$\Theta_n \in H^0(B, \mathcal{L}^n) = \varinjlim_K H^0(B_K, \mathcal{L}^n)$$

$U(1) \circlearrowright$; $\circlearrowleft$ Metaplectic rep of $U(1)$

Murase - Sugano gives a complete description of the spectral decomposition of $H^0(B, \mathcal{L}^n)$ for metaplectic rep. of $U(1)$.

$$H^0(B, \mathcal{L}^n) = \bigoplus_{\kappa} \mathbb{C} \cdot \Theta^{prim}_{\kappa, n}$$ (Generalized Shintani's theory)

κ runs over Hecke characters of K

s.t. $\kappa|_{\mathbb{A}^\times_{\mathbb{Q}}} = \tau_{K/\mathbb{Q}} \cdot |\cdot|^{-1}_{\mathbb{A}_{\mathbb{Q}}}$, $\kappa(z_\infty) = z_\infty^{-1}$

with $\underbrace{\text{correct}}_{\text{root}}$ #'s.

$\Theta^{prim}_{\kappa,n} \in H^0(E', \mathcal{L}')$ s.t. $\chi(\mathcal{L}') = 1$

$$\Theta^{prim}_{\kappa,n} = \boxed{\text{period}} \cdot \Theta^{p\text{-primitive}}_{\kappa,n}$$

To show $E^0(x|1)$ is <u>non-vanishing</u>

$$\Theta^0 = \Theta^{prim}_{\kappa,n}$$

Consider $$l_{\Theta^0}\left(E^0(x|1)\right) = \frac{(\Theta_n, \Theta^0)}{(\Theta^0, \Theta^0)} \cdot I(\Theta^0) \in \overline{\mathbb{Z}_p}$$

(Fourier-Jacobi coeff)

$$\Theta_n = \sum_{\kappa} \alpha_\kappa \cdot \Theta^{prim}_{\kappa,n}, \quad (\Theta_1, \Theta_2) = \int_{U(1)_{\mathbb{Q}} \backslash U(1)_{\mathbb{A}}} \Theta_1(r)\overline{\Theta_2(r)}\,dr$$

$$I(\Theta^0) = \sum_{x \in S_{U(1)}(K)} \Theta^0(B_x)(0)$$

$l_{\Theta^\circ}(E^\circ(\chi|1))$

$\downarrow$

Computed by Murase–Sugano

$$\boxed{(NV) \quad \frac{L(0, \chi\kappa)}{\Omega_\kappa^{k-1}} \cdot \frac{L(1,\kappa)}{\Omega_\kappa} \cdot h_K \not\equiv 0 \quad \text{for a good choice of } K}$$

(NV) is OK if $(2, D_K) = 1$ (Hida, Hsieh)

(By density of CM points in Hilbert modular variety mod p)

$$\left(\underline{\text{cf.}} \quad E_{2k}(\sqrt{-1}) = \zeta_K(k) \quad \begin{array}{c} \downarrow \\ GL_2 \end{array} \right)$$

If K is imaginary quadratic, (NV) is obtained by Finnis in Annals (2006) without ANY condition except for root # = 1

§. Construction of elements in Selmer groups using Eisenstein congruence

By Hida theory for unitary groups

$$\left(\text{In } GL(2), \quad 0 \longrightarrow M^\circ_{\text{cusp.}}(K, \Lambda) \longrightarrow M(K,\Lambda) \xrightarrow{\Phi} \bigoplus_{q \in Cusp(X_0(N))} \Lambda \cdot [q] \longrightarrow 0 \right) \quad (!!)$$

We can construct $\lambda_{\mathcal{F}} : h^{\text{ord}}(K, \Lambda) \longrightarrow \Lambda/(\mathcal{L})$

$\mathcal{E}^{\text{ord}}_{2,1}$ Eisenstein component of ordinary Λ-adic cuspidal Hecke algebra

$$\mathcal{L} = L_{D\text{-}R} \cdot L_K$$

Deligne–Ribet Katz

(c : cpx conj)

s.t. $\lambda_{\mathcal{F}_1} \equiv \lambda_{\mathcal{E}_{2,1}^{ord}} \equiv \Psi^{-c}\varepsilon + 1 + \Psi\varepsilon^{-1} \pmod{\mathcal{L}}$

$$\Psi : G_K \longrightarrow \Lambda^\times \qquad \varepsilon : G_K \longrightarrow \mathbb{Z}_p^\times$$
$$g \longmapsto \psi(g)\, g|_{K_\infty} \qquad \text{the cyclotomic character}$$

By the existence of Galois rep attached to cusp forms on $U(2,1)/\mathbb{Q}$

By the lattice construction (generalized Ribet's lemma)

$$\Rightarrow \boxed{\rho_{\mathcal{F}_1} \equiv \begin{pmatrix} \Psi^{-c}\varepsilon & * & * \\ 0 & 1 & * \\ 0 & 0 & \Psi\varepsilon^{-1} \end{pmatrix} \pmod{\mathcal{L}}}$$

Heuristic picture

(D-R) $Sel_K(\Psi^{-1-c}\varepsilon^2)$ (K) $Sel_K(\Psi^{-1}\varepsilon)$

If $\mathfrak{P}$ is "trivial zero" for the Gal. rep.

$$\Psi^{-1-c}\varepsilon^2\Big|_{I_{\mathfrak{p}}} \equiv 1 \pmod{\mathfrak{P}}$$

$$\Leftrightarrow \quad \mathfrak{P} = \left((1+T_+)u^{-2}\zeta - 1 \right)$$

$$\Rightarrow \quad L_{Katz}(x_k) = L(k+1, \theta_k) \neq 0$$

$(x_k \in (\mathfrak{P}))$ (Shimura)

θ_k is Hilbert mod. form (CM) of parallel weight $2k+1$

→ No "triv. zero" phenomena for Gal rep.

By $\rho_{\mathcal{F}_1}^c = \rho_{\mathcal{F}_1}^\vee$ (self-duality)

$$\Rightarrow (D\text{-}R) \subset Sel_K(\Psi^{-1-c}\varepsilon^2)^{c=-1} = Sel_{\mathbb{Q}}(\Psi_+^{-1}\varepsilon^2\tau_{K/\mathbb{Q}})$$

$$\left(\Psi_+ : G_{\mathbb{Q}}^{ab} \xrightarrow[\text{ver}]{V} G_K^{ab} \xrightarrow{\Psi} \Lambda^\times \right)$$

$$L_{D\text{-}R}(\hat{\chi}) = L(-1, \chi_+ \tau_{K/\mathbb{Q}}) = L(2, \chi_+^{-1}\tau_{K/\mathbb{Q}})$$

$$L_K(\hat{\chi}) = L(0, \chi)$$

$\ell_P(*) = \mathrm{length}_{\Lambda_P}(* \otimes_\Lambda \Lambda_P)$ P is a height one prime of Λ

We have $\ell_P(\mathrm{Sel}_K(\Psi^{-1}\varepsilon)) = \ell_P(\mathrm{Sel}_K(\Psi))$

(using Ochiai's result (specialization principle) + Poitou-Tate duality)

We can prove $\ell_P(L_K) \leq \ell_P(\mathrm{Sel}_K(\Psi))$ if $\underline{\mathrm{ord}_P(L_{D\text{-}R}) = 0}$

$$\begin{pmatrix} 1 & * & \circledast \\ (*) & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

If we assume $\underline{\text{the root \# of } L(s,\psi) \text{ is } \equiv +1}$ $(\mathrm{mod}\ P)$

$[K':K]$ prime to p

$\Rightarrow \mu^-(L_K) = 0$ (under some technical cond. ...)

∘ Katz p-adic L funct

$$L_K \in \mathcal{O}[[\underset{\text{cyclotomic}}{T_+}, \underbrace{S_1, \cdots, S_d}_{\text{anticyclotomic}}]]$$

$L_K^- = L_K \pmod{T_+} \in \mathcal{O}[[S_1, \cdots, S_d]]$

$\mu(L_K^-) = 0$ if the root # of $\Psi \equiv +1 \pmod{P}$

$\mu(L_K^-) = 0$ ~~if~~ true if ψ is obtained from some CM elliptic curve E, $w_E = +1$, $(N_E, 6) = 1$

(in progress (Hsieh))

(Hida)

This relates to computation of Fourier coeff of certain Eisenstein series E.

We find that

χ : CM char, attached to E

$$a_\beta(E) = \prod_v \left(W_\beta(\chi_v) + \chi_v(2\theta) \right) \times \cdots$$

$$W_\beta(\chi) = \pm \chi(2\theta)$$

(corresponds to "theta dichotomy mod p" ?)

We assume $\psi = \psi_E$ obtained from CM elliptic curves E

$$\psi_E : \mathrm{Gal}(K(E[p^\infty])/K) \longrightarrow \mathrm{Aut}\, E[p^\infty] \hookrightarrow \mathbb{Z}_p^\times$$

By Perrin-Riou control thm for Selmer groups of E (Compositio 43 (1981))

$$L_p(E) \in \Lambda^+ = \mathcal{O}[\![\mathrm{Gal}(F_\infty/F)]\!]$$

$$\left(L_p(E)\right) \supseteq \left(\mathrm{char}_{\Lambda^+} \mathrm{Sel}_{F_\infty}(E)\right) \quad \text{if } w_E = +1$$

if
- $w_E = +1$
- the conductor of E is a product of split primes in K $((N_E, 6) = 1)$
- $p \nmid h_K/h_F \cdot 6 \cdot [\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot D_F$
- E has good ordinary reduction at p

$D_{K/F} = 1$

If we had the other divisibility for all E/F with $w_E = +1$
(or p-adic class # formula) for $F(\sqrt{D})$
By a base change trick,
we can prove the equality $\big(L_p(E)\big) = \big(\mathrm{char}_\Lambda\, Sel_{F_\infty}(E)\big)$
without condition on conductor !! for $w_E = +1$

If $K(E[\mathfrak{p}])$ E/F : CM by $M = \mathbb{Q}(\sqrt{-D})$
$\big|$ abelian $K = F(\sqrt{-D})$, $p = \mathfrak{p}\cdot\bar{\mathfrak{p}}$
M

$\Rightarrow \big(L_p(E)\big) = \big(\mathrm{char}_{\Lambda_+}\, Sel_{F_\infty}(E)\big)$

cf. Rubin's main conj.
$\Rightarrow$ "=" if $K(\underset{\substack{\uparrow \\ \text{ALL torsion pts}}}{\underline{E_{tors}}})/M$ abelian