Ming-Lun Hsieh, Lecture III

§. Eisenstein series on $U(2,1)/\mathbb{Q}$

$$\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times \qquad \chi(z_\infty) = z_\infty^k \qquad k>2$$

$$\left(\chi_\infty(z) = z^{k+a}\,\bar{z}^{-a},\ k>1,\ a\geq 0 \right)$$

$$G = U(2,1)/\mathbb{Q}$$

$$I(\chi, \underline{1}, s) := \left\{ \overset{\text{smooth}}{f : G(\mathbb{A}) \to \mathbb{C}} \;\middle|\; f\left(\begin{pmatrix} \bar{a}^{-1} & * & * \\ & b & * \\ & & a \end{pmatrix} g\right) = \chi^{-1}(a)\,|\bar{a}a|_{\mathbb{A}}^{-s} f(g) \right\}$$

$$f \in I(\chi, 1, s) \rightsquigarrow E(g,f) = \sum_{\gamma \in P\backslash G} f(\gamma g)$$

$$P = \text{standard parabolic} = \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \cap G$$

$$\chi \rightsquigarrow f_\chi$$

s.t.
- $E(g, f_\chi)$ is "arithmetic" and can be p-adically interpolated.
- the "constant term" of $E(g, f_\chi)$ is $L_{D\text{-}R} \cdot L_{\text{Katz}}$ (p-units)
- $E(g, f_\chi)$ is <u>ordinary</u> ✓

§. Pull-back formula

Igusa schemes

$$I_{U(2,1)}(\mathbb{C}) \times I_{U(1)}(\mathbb{C}) \longrightarrow I_{U(2,2)}(\mathbb{C})$$

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, g\right) \times \left(\{pt\}, h\right) \longmapsto \left(\begin{bmatrix} x & 0 \\ y & 2^{-1}\theta \end{bmatrix}, (g,h)_\Delta \gamma\right)$$

- $E(\chi, N)$ Siegel-Eisenstein series on $U(2,2)$ (in the morning)

<u>Define</u>

$$E(\chi|1)\left(\begin{bmatrix} x \\ y \end{bmatrix}, g\right) := \sum_{h \in U(1)_{\mathbb{Q}} \backslash U(1)_{\mathbb{A}} / D_0} E(\chi, N)\left(\begin{bmatrix} x & 0 \\ y & 2^{-1}\theta \end{bmatrix}, (g,h)_\Delta \gamma\right) \chi(h)$$

$$\left(\underset{\text{norm } 1}{U(1)} \subset K^\times \right)$$

Thm (Pull-back formula) [Shimura]

$E(\chi|1)$ is an Eisenstein series on $U(2,1)$ attached to the pull-back section of

$$\phi^{*,\gamma}_{\chi,s}(g) = \phi^{*}_{\chi,s}(g\gamma), \quad \gamma = \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & & -\frac{1}{2} \\ & & 1 & \\ & \theta^{-1} & & \theta^{-1} \end{pmatrix}$$

$\phi^{*,\gamma}_{\chi,s} \in I(\chi, 1, s)$, $\gamma \in GL_4(\mathbb{Q}_p) \simeq U(2,2)(\mathbb{Q}_p)$

Def (Pull-back section)

if $f \in I(\chi, s)$

Define $f^{pb} \in I(\chi, 1, s)$ by $f^{pb}(g) := \int_{U(1)(\mathbb{A})} f((g,h)_\Delta)\chi(h)\,dh$

If $f = \bigotimes_v f_v$, then $f^{pb} = \bigotimes_v f_v^{pb}$

- $E(\chi|1)$ is p-integral and can be put into a measure.

So we have to apply Hida's idempotent e

$\Rightarrow$ $eE(\chi|1)$ is the one we want.

- We want to compute the ideal of $\mathcal{O}[\![\mathcal{G}]\!]$ $p \nmid N$, $N \neq 1$ $\left(\mathcal{G} = \mathrm{Gal}(K(Np^\infty)/K)\right)$ generated by constant terms of $eE(\chi|1)$ at ALL cusps.

e.g. $X_0(p)$ $\quad E_k^\circ(z) = E_k(z) - p^{k-1}E_k(pz)$

$E_k^\circ(z)$ has no constant term at cusp 0.

$\rightarrow$ can be generalized to $U(r,1)$!!

§. Computation of $e(\phi^{\gamma}_{\chi,s})^{pb}$ $\left((\phi^{*,\gamma}_{\chi,s})^{pb} = c(\chi,s)\cdot(\phi^{\gamma}_{\chi,s})^{pb} \right)$

$v \nmid p$ — certain "ordinary" p-stabilization

$v \nmid pN$, $\phi^{pb}_{\chi,s,v}$ can be computed by "the doubling method"

Piatetski-Shapiro, Rallis, Gelbart LNM "Explicit construction of automorphic L-functions"

Jian Shu, Li

$$\left(\begin{array}{ccc} U(2,1)\times U(1) & \hookrightarrow & U(2,2) \\ \cup & & \cup \\ U(1)\times U(1) & \hookrightarrow & U(1,1) \end{array} \right)$$

$$\phi^{pb}_{\chi,s,v} = \frac{L(s,\chi)}{L(2s,\chi_+)}\cdot f^{\circ}_{\chi,1,s,v}$$

spherical section

$v \mid N$ $\quad \phi^{pb}_{\chi,s,1} = * \cdot f^{N}_{\chi,1,s,v}$

$f^{N}_{\chi,1,s,v}$ is the unique section supported in

$$P(\mathbb{Q}_v)\, D_v(N)\, w_\ell^{-1}$$

$$D_v(N) = \left\{ g \equiv \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & 1 \end{pmatrix} \bmod N \right\}$$

s.t. $f^{N}_{\chi,1,s,v}(u w_\ell^{-1}) = 1$, $u \in D_v(N)$

$$\left(w_\ell = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix} \right)$$

§. $v=p$, $e(\phi^{\gamma}_{\chi,s})^{pb}$

<u>Def</u> (Hida's idempotent)

$$f \in I_p(\chi, 1, s)$$

$$\alpha_i = \begin{pmatrix} 1_{3-i} & \\ & p\cdot 1_i \end{pmatrix} \qquad i=1,2$$

$$f|_{U_p(\alpha_i)}(g) = \frac{p^k}{p^{3-i}} \sum_{u \in N_3(\mathbb{Z}_p)/\alpha_i^{-1}N_3(\mathbb{Z}_p)\alpha_i} f(gu\alpha_i^{-1})$$

$$U_p = \prod_{i=1}^{2} U_p(\alpha_i)$$

$$e = \underbrace{\lim_{n\to\infty} U_p^{n!}}_{\text{this is convergent !!}} \qquad (\iota_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p)$$

• Construction of <u>ordinary section</u>.

$$I_p(\chi, 1, s) = I(\chi_2|\cdot|^{-1}, 1, \chi_1^{-1}|\cdot|)$$

$$\underbrace{(a+1,\ 0,\ k+a-1)}_{\text{regular weight}} \qquad \text{unitarily induced rep}$$

$$\chi|_{(K\otimes\mathbb{Q}_p)^\times} = (\chi_1, \chi_2) : \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$$

$$\left(\chi_\infty(z) = z_\infty^{k+a}\,\bar{z}_\infty^{-a} \qquad k>1,\ a\geq 0\right)$$

$$k+a-1 \geq a+1 > 0$$

Let $\phi^{w_\ell} \in I(\chi_1^{-1}|\cdot|, \chi_2|\cdot|^{-1}, 1)$

be the unique funct. supp in $P(\mathbb{Q}_p)w_\ell N(\mathbb{Z}_p)$

s.t. $\phi^{w_\ell}(w_\ell n) = 1 \quad \forall n \in N(\mathbb{Z}_p)$

It is easy to check that ϕ^{w_ℓ} is ordinary.

$$\left(\text{i.e.}\ \ \phi^{w_\ell}\big|_{U(\alpha_i)} = \underset{\substack{\downarrow \\ p\text{-unit (w.r.t. } \iota_p)}}{c_i}\, \phi^{w_\ell} \quad (i=1,2) \right)$$

We apply the intertwining operator M_{s_1} — $s_1 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$

$$M_{s_1} : I(\chi_1^{-1}|\cdot|,\ \chi_2|\cdot|^{-1},\ 1) \longrightarrow I(\chi_2|\cdot|^{-1},\ 1,\ \chi_1^{-1}|\cdot|)$$

$$\phi^{\mathrm{ord}} := M_{s_1}\cdot \phi^{w_\ell}(g) = \int_{N\backslash N_{s_1}^{-1}} \phi^{w_\ell}(s_1\cdot n\cdot g)\, dn$$

$$\begin{cases} \operatorname{Lie} N = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \\ \operatorname{Lie} N_{s_1}^{-1} = \bigoplus_{\substack{\alpha>0 \\ s_1\alpha<0}} \mathfrak{g}_\alpha \end{cases}$$

- $\phi^{\mathrm{ord}}(w_2) = 1$, $w_2 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$
- Use multiplicity one theorem for ordinary sections in regular principal series.
  (Hida, yellow book, Chap. 5)

Prop (Hsieh, Fourier 2010)
For $f \in I_p(\chi, 1, 0)$, $e\cdot f = \underline{M_{w_2}(f)(1)}\cdot \phi^{\mathrm{ord}}$

Prop $M_{w_2}\left((\phi^{\gamma}_{\chi_1,s,p})^{pb}\right)(1) = \mathrm{vol}(\mathbb{Z}_p^\times, d^\times z)\cdot (p-1)\cdot \underbrace{E_p(2s-1, \chi_+)}_{E_p(s,\chi_1)}$

$$\| $$

$$\int_F (\phi^{\gamma}_{\chi_1,s,p})^{pb}\left(w_2 \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) dx$$

<u>Def</u> (Modified Euler factor at p)

$\mu : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$

<u>Define</u>

$$E_p(s,\mu) = \frac{L(s,\mu)}{L(1-s,\mu^{-1})\cdot \underbrace{\varepsilon(s,\mu)}_{\text{Tate's } \varepsilon\text{-factor}}}$$

<u>Conjecture</u> of Coates and Perrin-Riou

$\rho : G_F \longrightarrow GL_d(V)$, $\tilde{\rho} : G_F \longrightarrow GL_d(V \otimes \mathcal{O}[[T]])$

<u>There</u> exists a "p-adic L-fct for $\tilde{\rho}$"

<u>Assumption</u>

ρ is ordinary

$0 \longrightarrow V_+ \longrightarrow \rho|_{G_{\mathbb{Q}_p}} \longrightarrow V_- \longrightarrow 0$ as $G_{\mathbb{Q}_p}$-modules

$\begin{cases} V_+ \text{ is Hodge-Tate with arith. HT wts } > 0 \\ V_- \quad " \quad\quad\quad\quad\quad " \quad \leq 0 \end{cases}$

$$\text{"}\mathcal{L}(0,\rho)\text{"} = L^{(p)}(0,\rho)\cdot \underbrace{E_p(0,\rho)}\cdot \underbrace{E_\infty(0,\rho)}_{\substack{\text{powers of } \pi \\ \Gamma\text{-factors}}}$$

p-adic L-funct specialized at ρ

Modified Euler facto

$$E_p(0,\rho) = \frac{L_p(0,\rho|_{V_+})}{\varepsilon(0,\rho|_{V_+})\cdot L_p(1,(\rho|_{V_+})^\vee)}$$

$\boxed{E_p(2s-1,\chi_+)\cdot E_p(s,\chi_1)}$

$\chi(z_\infty) = z^k$, $\chi_+ = \chi|_{\mathbb{A}_\mathbb{Q}}$, $\chi_+(r) = r^k$ $\underline{\underline{k > 2}}$

$E_p(-1,\chi_+)\cdot E_p(0,\chi_{1 \to k})$

(Deligne-Ribet $\rightsquigarrow$ $L^{(p)}(2, \chi_+^{-1}) = E_p(-1, \chi_+) \cdot L^{\{p\}}(-1, \chi_+)$)

~~proof~~
$$M_{w_2}\left((\phi^{\gamma}_{\chi_1, s, p})^{pb} \right)(1) = \int_{\mathbb{Q}_p} dx \int_{U(1)(\mathbb{Q}_p)} f_{\Phi}\left((w_2 u(x), h)_{\Delta} \gamma \right) \chi_1^{-1} \chi_2(h) dh$$

$$\| \qquad \Delta^{-1}(g,h)\Delta\gamma \qquad \Delta\gamma = 1$$

$$\left(u(x) = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= \int_{\mathbb{Q}_p} \int_{U(1)(\mathbb{Q}_p)} \int_{GL_2(\mathbb{Q}_p)} \Phi\left(Z \begin{pmatrix} 0 & 0 \\ 1 & x \end{pmatrix}, Z \begin{bmatrix} 1 \\ h \end{bmatrix} \right) Q(Z) P(h) \underbrace{d^{\times}Z,\ dh\ dX}$$

where
$$P(h) = \chi_1(h) \cdot |h|^s, \quad Q(Z) = \chi_1\chi_2(\det Z) |\det Z|^{2s}$$

$$Z \longmapsto Z \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \ ; \ h \longmapsto xh$$

The integral equals
$$\int_{\mathbb{Q}_p} \chi_2^{-1}(x) |x|^{-s} dx \int_{U(1)(\mathbb{Q}_p)} P(h)\, dh \int_{GL_2(\mathbb{Q}_p)} Q(Z)\, \Phi_1\left(Z \begin{bmatrix} 0 & 0 \\ x^{-1} & 1 \end{bmatrix}\right) \hat{\Phi}_2\left(Z \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}\right)$$

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \Rightarrow z_4 \in \mathbb{Z}_p^{\times}$$

$$Z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ z_3 & z_4 \end{bmatrix}$$

$$d^{\times}Z = |\det Z|^{-2} dZ = |a^{-1}|\, d^{\times}a\, dy\, dz_3\, d^{\times}z_4$$

$$\Rightarrow \int_{\mathbb{Q}_p} d^{\times}a\, \chi_+(a) |a|^{2s-1} \hat{\Phi}_{\chi_+}(a) \times \int_{U(1)(\mathbb{Q}_p)} dh\, P(h)\, \hat{\Phi}_{\chi_1}(h) \times \int \cdots dX$$

[Key observation] (Hida-Tilouine)

For $\mu : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$

$$\Phi_\mu(x) = \mathbb{I}_{\mathbb{Z}_p^\times}(x) \cdot \mu(x)$$

$$\hat{\Phi}_\mu(x) = \begin{cases} \mu^{-1}(x) \cdot \mathbb{I}_{p^{-c(\mu)}\mathbb{Z}_p^\times}(x) \cdot \frac{1}{\varepsilon(0,\mu)} \\ \mathbb{I}_{\mathbb{Z}_p}(x) - |p| \cdot \mathbb{I}_{p^{-1}\mathbb{Z}_p}(x) \end{cases}$$

$$Z(\hat{\Phi}_\mu, \mu, s) = \int_{\mathbb{Q}_p^\times} \hat{\Phi}_\mu(x) \cdot \mu(x) \cdot |x|^s \, d^\times x = E_p(s, \mu)$$

The constant term of Eisenstein series attached to

$$e(\phi^{*,\gamma}_{\chi_1, s})^{pb}\Big|_{s=0}$$

$$= \underbrace{\frac{\Gamma(k-1)}{(2\pi i)^{(k-1)}} \cdot E_p(-1, \chi_+) \cdot L(-1, \chi_+ \tau_{K/\mathbb{Q}})}_{\rightsquigarrow\ L^{\{p\}}(2, \chi_+^{-1} \tau_{K/\mathbb{Q}})}$$

$$\times \underbrace{\frac{\Gamma(k)}{(2\pi i)^k} \cdot E_p(0, \chi_1) \cdot L^{\{p\}}(0, \chi_1)}_{\text{Katz p-adic L funct}} \times \cdots\cdots$$

at the cusp W_2

$$\hat{E}(\chi|1)(\underline{A}) = \sum_{h \in U(1)/\mathbb{Q} \backslash U(1)(\mathbb{A})/U_0} \hat{E}(\chi, N)(\underline{A} \times \mathcal{B}_h) \cdot \chi(h)$$

complex unif

- $E(\chi, N)(z, g) = E(\chi, N)([z, g], 2\pi i dz)$
- $E(\chi, N)\left(\begin{bmatrix} x & 0 \\ y & 2^{-1}\theta \end{bmatrix}, (g, h)_\Delta \gamma\right)$

$$= E(\chi, N)([z, g] \times [h], (2\pi i dz_p, 2\pi i dz_W))$$

Take a <u>Néron differential</u> of $[h]/\mathcal{O}$ $\quad \omega_{[h]}$

$$\Omega_p \cdot \underbrace{w(j)}_{\text{p-adic 1-form induced from level } p^\infty\text{-structures } j} = \omega_{[h]} = \Omega_\infty \, dz_w$$

To sum up, we constructed a ordinary Eisenstein measure on $\mathfrak{G}$ $\quad d\mathcal{E}_{2,1}$

s.t. $$\int_{\mathfrak{G}} \hat{\chi} \, d\mathcal{E}_{2,1} = e\,E(\chi \mid 1)$$

constant term $= \boxed{L_{(L\text{-}K)}}\, L_{\text{Katz-Yager}}$

$\uparrow$ Leopoldt-Kubota

tempered endoscopy $U(1,1)$

$L_{(K-L)} \rightsquigarrow$ congruence between Eisenstein series and <u>endoscopic form of type $U(1,1)$</u>

(Kudla lift) $\leftarrow$ tempered

$p \mid L_{(K-L)} \Rightarrow$ E.S. on $GL(2) \equiv F$ cusp form on $GL(2)$

$\downarrow$ <u>Kudla lift</u>

$\mathcal{L}(F)$ endoscopic on $U(2,1)$