Ming-Lun Hsieh Lecture II

§ Shimura varieties

K = imag quad field

$G = U(2,1)_{/\mathbb{Q}} := U(V)_{/\mathbb{Q}}$

$(V, \theta_{2,1}) = \left(Kx' \oplus Kw' \oplus Ky', \begin{pmatrix} & & -1 \\ & \theta & \\ 1 & & \end{pmatrix} \right)$

$\theta \in K$, $i\,\iota_\infty(\theta) > 0$ $\left(\overline{\mathbb{Q}} \xrightarrow{\iota_\infty} \mathbb{C},\ \overline{\mathbb{Q}} \xrightarrow{\iota_p} \overline{\mathbb{Q}_p} \right)$ $v_p(\iota_p(\theta)) = 0$

$(\mathrm{Im}\, \iota_\infty(\theta) < 0)$

<u>Def</u> $K \subset G(\mathbb{A}_f)$ open compact subgroup

$K_p = GL_3(\mathbb{Z}_p)$

p: split, > 2

$\mathcal{O}_K \otimes \mathbb{Z}_p \cong \mathbb{Z}_p e_+ \oplus \mathbb{Z}_p e_-$ e_+ : the idempotent w.r.t. ι_p

$G(\mathbb{Q}_p) \approx GL_3(\mathbb{Q}_p)$

$g \longmapsto g|_{e_+(V \otimes_{\mathbb{Q}} \mathbb{Q}_p)}$

$M = \mathcal{O}_K x' \oplus \mathcal{O}_K w' \oplus \mathcal{O}_K y'$

- $S_G(K)_{/\mathcal{O}}$ is a model of unitary Shimura variety ass. to $(\underline{V}, M)$

$\mathcal{O} = \mathcal{O}_{K,p}[\zeta_N]$ $p \nmid N$

$\left(K = \{ g \mid M(g-1) \subset NM \},\ N > 3 \right)$

$S_G(K)(S) = \{ [(A, \overline{\lambda}, \iota, \overline{\eta})_{/S}] \}$

- A : abelian scheme$_{/S}$ of dim 3 (3d in totally real case)
- $\overline{\lambda} = \mathbb{Z}_{(p),+} \cdot \lambda$ $\mathbb{Z}_{(p),+}$-orbit of a polarization $\lambda_{/S}$
- $\iota : \mathcal{O}_K \hookrightarrow \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}$

- $\bar{\eta}$ is $\pi_1(S, s)$-orbit of isomorphisms of

$$\eta : M \otimes \hat{\mathbb{Z}}^{(p)} \simeq H_1(A_s, \hat{\mathbb{Z}}^{(p)})$$

$$k \cdot \eta(m) = \eta(mk) \qquad k \in K$$

$$\bar{\eta} = K\eta$$

as skew-Hermitian $\mathcal{O}_{K,(p)}$-modules $(\xi : \mathbb{A}_f \approx \mathbb{A}_f(1))$

(det) $\det\left(X - \imath(b)\big|_{\mathrm{Lie}\,A}\right) = (X - \bar{b})^2 (X - b)$

$b \in \mathcal{O}_K$

- $I_G(K^n)_{/\mathcal{O}}$: a model of Igusa schemes

$$K^n = \left\{ g \in K \;\middle|\; g_p \equiv \left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array}\right) \bmod p^n \right\}$$

$$K_1^n = \left\{ g \in K \;\middle|\; g \equiv N_3(\mathbb{Z}_p) \bmod p^n \right\}$$

$$K_0^n = \{ \quad \text{"} \qquad \text{"} \quad B_3(\mathbb{Z}_p) \quad \text{"} \quad \}$$

$$\begin{array}{l} I_G(K^n)_{/\mathcal{O}} := \underline{\mathrm{Inj}}\left(\mu_{p^n} \otimes M^\circ, \mathcal{A}[p^n]\right) \\ \quad \big\downarrow \text{ étale (not finite!)} \\ S_G(K)_{/\mathcal{O}} \end{array}$$

$\mathcal{A}$: the universal abelian scheme$/ S_G(K)$

$$M^\circ = (\mathcal{O}_K \otimes \mathbb{Z}_p) x^1 \oplus \mathbb{Z}_p e_+ w^1$$

$$M^{-1} = (\mathcal{O}_K \otimes \mathbb{Z}_p) y^1 \oplus \mathbb{Z}_p e_- w^1$$

Then, $\{M^{-1}, M^\circ\}$ is a polarization of $M \otimes \mathbb{Z}_p$ w.r.t. $\theta_{2,1}$

§. Complex uniformization of $I_G(K^n)$ (Shimura)

$\mathbb{C}^{2,1} = \mathbb{C}(\Sigma^c) \oplus \mathbb{C}(\Sigma^c) \oplus \mathbb{C}(\Sigma)$
is 3-dim'l $\mathbb{C}$-vect sp. with K-action given by

$$b \cdot (z_1, z_2, z_3) = (\bar{b} z_1, \bar{b} z_2, b z_3)$$

$$C_{2,1} : \mathbb{C}^{2,1} \longrightarrow \mathbb{C}^{2,1}$$
$$(z_1, z_2, z_3) \longmapsto (\bar{z}_1, \bar{z}_2, z_3)$$

$$X^+_{2,1} = \left\{ \iota \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \,\middle|\, i(\bar{x} - x) > -i \bar{y} \theta^{-1} y \right\}$$

$$\alpha \in G(\mathbb{R}), \quad \alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix}$$

$$\alpha \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by + c \\ gx + ey + f \end{bmatrix} \cdot (hx + ly + d)^{-1}$$

$$J\left(\alpha, \begin{bmatrix} x \\ y \end{bmatrix}\right) = hx + ly + d$$

$$I_G(K^n)(\mathbb{C}) = {}_{G(\mathbb{Q})}\backslash X^+_{2,1} \times G(\mathbb{A}_f) / K$$

$$\underline{\mathcal{A}_g(\tau)} \longleftarrow [(\tau, g)]$$

$$\underline{\mathcal{A}_g(\tau)} = (\underset{\text{abel var}}{\mathcal{A}_g(\tau)}, \underset{\text{pol.}}{\overline{(\,,\,)}_{2,1}}, \underset{\text{endom}}{\iota_V}, \underset{\text{level outside } p}{\bar{\eta}_g}, \underset{\text{level at } p}{\bar{J}_g})$$

$$\mathcal{A}_g(\tau) = \mathbb{C}^{2,1} / p(\tau)(M g^{-1})$$

$(\overline{(\,,\,)}_{2,1}, \iota_V)$ is induced from V via

$$p(\tau) : V_{\mathbb{R}} \longrightarrow \mathbb{C}^{2,1}$$
$$v \longmapsto C_{2,1}(v B(\tau))$$

$$B(\tau) = \begin{bmatrix} \bar{x} & \bar{y} & x \\ 0 & \theta & y \\ 1 & 0 & 1 \end{bmatrix} \qquad M = \mathcal{O}_K y' \oplus \mathcal{O}_K w' \oplus \mathcal{O}_K x'$$

$$\eta_g : M \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} H_1(\mathcal{A}_g(\tau), \hat{\mathbb{Z}}^{(p)}) \cong Mg^{-1}$$
$$m \longmapsto mg^{-1}$$

$$j_g : M^\circ \otimes \mu_{p^n} \hookrightarrow \mathcal{A}_g(\tau)[p^n] = Mg^{-1} \otimes \mathbb{Z}/p^n\mathbb{Z}$$
$$m \longmapsto mg^{-1}$$

◎ Morphisms between Igusa schemes

$$\underline{V} = (V, \theta_{2,1}, M, M^{-1} \oplus M^\circ)$$
$$\underline{-W} = (W = Kw^1, -\theta, R, R^\circ \oplus R^1)$$
$$R = \mathcal{O}_K w^1 \quad R^\circ = (\mathcal{O}_K \otimes \mathbb{Z}_p) \cdot e_- w^1$$
$$R^{-1} = (\mathcal{O}_K \otimes \mathbb{Z}_p) \cdot e_+ w^1$$

$$\underline{\mathbb{W}} = \underline{V} \oplus \underline{-W}$$

We have a canonical morphism between Igusa schemes in an obvious way

$$I_{U(V)} \times I_{U(-W)} \longrightarrow I_{U(\mathbb{W})/\mathcal{O}}$$
$$\mathbb{x}^1 = x^1 \quad \mathbb{y}^1 = y^1 \quad \mathbb{y}^2 = \frac{1}{2} w^{+,1} - \frac{1}{2} w^{-,1}$$
$$\mathbb{x}^2 = \theta^{-1} w^{+,1} + \theta^{-1} \cdot w^{-,1}$$

Put

$$\mathbb{X} = \sum_{i=1}^{2} (\mathcal{O}_K \otimes \mathbb{Z}_p)\, \mathbb{x}^i$$
$$\mathbb{Y} = \sum_{i=1}^{2} (\mathcal{O}_K \otimes \mathbb{Z}_p)\, \mathbb{y}^i$$

Then, $(\mathbb{Y}, \mathbb{X})$ gives another polarization of $\mathbb{W}$

In terms of complex uniformization,

$$I_{U(V)}(\mathbb{C}) \times I_{U(W)}(\mathbb{C}) \longrightarrow I_{U(\mathbb{W})}(\mathbb{C})$$

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, g\right) \times (\{pt\}, h) \longmapsto \left(\begin{bmatrix} x & 0 \\ y & -\frac{1}{2}\theta \end{bmatrix}, \begin{pmatrix} g & \\ & h \end{pmatrix}_{\Delta} \gamma \right) \quad 4\times 4$$

$$\begin{pmatrix} g & \\ & h \end{pmatrix}_{\Delta} = \Delta^{-1} \begin{pmatrix} g & \\ & h \end{pmatrix} \Delta$$

$$\Delta = \begin{bmatrix} 1 & & & \\ & 1 & & -2^{-1}\theta \\ & & 1 & \\ & -1 & & -2^{-1}\theta \end{bmatrix} \in GL_4(K)$$

$$\gamma = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & -\frac{1}{2} \\ & & 1 & \\ & \theta^{-1} & & \theta^{-1} \end{bmatrix} \in GL_4(\mathbb{Q}_p) \overset{\text{ex.}}{\approx} U(\mathbb{W})(\mathbb{Q}_p)$$

$$\Delta\gamma = 1 \quad \text{in } GL_4(\mathbb{Q}_p)$$

§ Construction of Siegel-Eisenstein measure on $U(2,2) = U(\mathbb{W})$

$$\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times, \quad \chi(z_\infty) = z_\infty^k \quad k > 2$$

- v is a place of $\mathbb{Q}$

$$\mathbb{G} = U(2,2) = \left\{ g \in GL_4(K) \;\middle|\; g \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix} {}^*g = \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix} \right\}$$

$${}^*g = {}^t\overline{g}$$

induced rep :

$$I_v(\chi, s) = \left\{ \text{smooth funct } f : \mathbb{G}(\mathbb{Q}_v) \to \mathbb{C} \;\middle|\; f\left(\begin{pmatrix} {}^*A^{-1} & * \\ 0 & A \end{pmatrix} g \right) = \chi^{-1}(\det A) \left|\det({}^*A A)\right|^{-s} f(g) \right\}$$

$$I(\chi, s) = \bigotimes' I_v(\chi, s)$$

Take $\phi \in I(\chi, s)$; $\mathbb{P} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \cap G$

$$E_{\mathbb{A}}(g, \phi) = \sum_{\gamma \in \mathbb{P} \backslash G} \phi(\gamma g) \text{, whenever this is convergent}$$

<u>Fourier coefficient</u>

If $\beta \in \mathbb{Q}$,

$$W_\beta(g, E_{\mathbb{A}}) = \int_{\mathcal{H}_2(\mathbb{Q}) \backslash \mathcal{H}_2(\mathbb{A})} E_{\mathbb{A}}\left(\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} g, \phi \right) \underline{\psi}_\beta(X)\, dX$$

where $\mathcal{H}_2$: 2×2 symmetric matrices in $M_2(K)$

ψ is the additive char $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$

s.t. $\psi(x_\infty) = \exp(2\pi i\, x_\infty)$

For $X \in \mathcal{H}_2(\mathbb{A})$, $\psi_{-\beta}(X) := \psi(-\beta \operatorname{Tr}(\beta X))$

<u>Key</u> If $\phi = \bigotimes_v' \phi_v$

For some finite v_0, $\operatorname{Supp} \phi_{v_0} \subseteq \mathbb{P}(\mathbb{Q}_{v_0})\, w \cdot \mathbb{P}(\mathbb{Q}_{v_0})$

$$w = \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix}$$

Then, $W_\beta(g, E_{\mathbb{A}}) = \prod_v W_\beta(g_v, \phi_v)$

$$W_\beta(g_v, \phi_v) = \int_{\mathcal{H}_2(\mathbb{Q}_v)} \phi_v\left(w \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} g_v \right) \underline{\psi}_\beta(X)\, dX$$

The constant $\underbrace{\text{term is}}_{\text{of } E_{\mathbb{A}}(g,\phi)}$ given by $\phi(g) + M_w \phi(g)$

$$M_w \phi(g) = \prod_v W_0(g_v, \phi_v)$$

We are going to choose a "very nice" $\phi = \bigotimes_v \phi_v$

Let $\mathfrak{n}$ be the conductor of χ

$$N := \begin{cases} N_{K/\mathbb{Q}}(\mathfrak{n}) & \text{if } \mathfrak{n} \neq \mathcal{O}_K \\ \ell & \text{an auxiliary prime of } \mathbb{Q} \end{cases}$$

$$D(N) := \left\{ g \in G(\mathbb{A}_f) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

If $v \nmid pN\infty$, $\phi_{\chi,s,v}$ is the spherical section

If $v = \infty$, $\phi_{\chi,s,\infty}(g) = J(g,i)^{-k} \left| J(g,i) \right|^{-2s} \left| \det g \right|^{s}$

$v \mid N$, $v \nmid p$

$$f_{N,v}(u) = 1 \quad \text{if } u \in D(N)$$

and $\operatorname{Supp} f_{N,v} \subseteq \mathbb{P}(\mathbb{Q}_v) \cdot D(N)$

Then, $\phi_{\chi,s,v}(g) = f_{N,v}(gw')$, $w' = \begin{pmatrix} & -1 \;\; \xi \\ 1 \;\; \xi^{-1} & \end{pmatrix} \in G(\mathbb{Q}_v)$ $\quad \xi = -\frac{1}{2}\theta$

Check : $\operatorname{Supp} \phi_{\chi,s,v} \subseteq \mathbb{P}(\mathbb{Q}_v)\, w\, \mathbb{P}(\mathbb{Q}_v)$

$\boxed{\text{If } v = p}$

$$\chi|_{K_p^\times} = (\chi_1, \chi_2) : \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$$

$$\chi_1 = \chi|_{(e^+K_p)^\times = \mathbb{Q}_p^\times}$$

$$\mathcal{P}_1 = \begin{bmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p^\times & \mathbb{Z}_p^\times \end{bmatrix} \subset M_2(\mathbb{Z}_p)$$

$$\mathcal{P}_2 = \begin{bmatrix} \mathbb{Z}_p^\times & p^m\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix} \subset M_2(\mathbb{Z}_p)$$

$$m = \max\{1,\ v_p(\mathrm{cond}(\chi_1))\}$$

char. funct

- $\Phi_1(X) = \mathbb{I}_{\mathcal{P}_1}(X) \cdot \chi_2^{-1}(p X_{21}) \qquad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$
- $\Phi_2(Y) = \mathbb{I}_{\mathcal{P}_2}(Y) \cdot \chi_1\chi_2(Y_{11}) \cdot \chi_1(Y_{22})$

$$\Phi(X,Y) := \Phi_1(X) \cdot \hat{\Phi}_2(Y)$$

$$\left(\hat{\Phi}(Y) = \int_{M_2(\mathbb{Q}_p)} \Phi(Z) \cdot \psi(\mathrm{Tr}({}^tZY))\, dZ \right)$$

Define $/\!/\ \phi_{\chi,s,p}$

$$f_\Phi(g) = \chi_2(\det g)\,|\det g|^s \int_{GL_2(\mathbb{Q}_p)} \Phi((0,z)g)\, \underbrace{\chi_1\chi_2(\det z)\,|\det z|^{2s}}_{d^\times z}$$

Reason: $[t]\cdot f(g) = f\left(g \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & t_4 \end{pmatrix} \right)$

$$\longrightarrow [t]\cdot f = (\underset{0}{1},\ \underset{a}{\chi_2},\ \underset{k+a}{\chi_1^{-1}},\ \underset{k+2a}{\chi_1^{-1}\chi_2})(t) \cdot f(g)$$

$$\chi(z_\infty) = z^{k+a} \cdot \bar{z}^{-a} \qquad (k>1,\ a \geq 0)$$

(cf Katz's paper, p-adic L for CM fields)

- Fourier coefficients

(1) $v = \infty$

(Shimura)

$$J(p, \mathring{i})^{-k} \cdot W_\beta(p, \phi_{\chi,s,\infty})$$

$$= \begin{cases} \Lambda_{2,\infty}(0,\chi)^{-1} \cdot (\det\beta)^{k-2} \cdot \exp(2\pi i \operatorname{Tr}(\beta Z)) & \text{if } \beta > 0 \\ & Z \in X_{2,2}^{+} \\ 0 \quad \text{if } \beta < 0 \end{cases}$$

$p \in \mathbb{P}(\mathbb{R})$ s.t. $p\mathring{i} = Z$

$$J\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) = \det(CZ + D) \qquad \mathring{i} = \begin{bmatrix} i & 0 \\ 0 & -\frac{\theta}{2} \end{bmatrix}$$

$$\Lambda_{2,\infty}(s,\chi) = (-1)^k \cdot 2^{-2(k-1)} \cdot \pi^{-(s+k)+1} \cdot \Gamma(s+k)\Gamma(s+k-1)$$

(2) $v \nmid pN$

$$W_\beta(1, \phi_{\chi,s,v}) = \frac{R_{\beta,v}(\chi_+(\varpi)|\varpi|^{2s})}{L(2s,\chi_+) \cdot L(2s-1, \chi_+\tau_{K/\mathbb{Q}})}$$

$\chi_+ = \chi|_{\mathbb{A}_\mathbb{Q}^\times}$ $\qquad \tau_{K/\mathbb{Q}}$ = quad char attached to $K/\mathbb{Q}$

Ref
(1) Shimura, Euler products and Eisenstein series 1997
(2) " , Ann. Math 1985 Hypergeometric ...

(3) $v \mid N$, $v \nmid p$

char funct

$$W_\beta(1, \phi_{\chi,s,v}) = (p\text{-unit}) \cdot \mathbb{I}_{\mathcal{L}_v}(\beta)$$

($\mathcal{L}_v$ is a lattice in $\mathcal{H}_2(\mathbb{Q}_v)$)

(4) $v = p$, $\phi_{\chi,s,p}$

$$W_\beta(1, f_\Phi) = \begin{cases} \mathrm{vol}(\mathcal{P}_2, d^\times z)\cdot \chi_{+,p}(\det\beta) \\ \quad \cdot |\det\beta|^{2s-2} \cdot \Phi_1(\beta) & \text{if } \det\beta \neq 0 \\ 0 & \text{if } \det\beta = 0 \end{cases}$$

Normalization

$$c(\chi, s) = 2 \cdot |D_K|_{\mathbb{R}}^{1/2} \cdot \prod_{v \nmid pN} L(2s, \chi_+) \cdot L(2s-1, \chi_+ \tau_{K/\mathbb{Q}})$$
$$\cdot \mathrm{vol}(\mathcal{P}_2, d^\times z)^{-1}$$

$$\phi^*_{\chi,s} = c(\chi,s)\cdot \phi_{\chi,s}, \quad \phi_{\chi,s} = \bigotimes_v \phi_{\chi,s,v}$$

Def $g \in G(\mathbb{A}_f)$

$$E(\chi,N)(z,g) := J(g_\infty, \mathring{i})^{-k} E_{\mathbb{A}}((g_\infty, g), \phi^*_{\chi,s,v})\Big|_{s=0}$$

Claim $E(\chi, N)$ can be interpolated p-adically

Fourier coeff. of $E(\chi, N)$

$$a_{[1]}(E(\chi,N)) = \underbrace{\chi_{+,p}(\det\beta)\cdot |\det\beta|^{-2}\cdot(\det\beta)^{k-2}}_{\text{(Harmless)}}$$

$\delta_a(\hat{\chi})$

Dirac measure

Recall : $\chi(z_\infty) = z_\infty^k$

$$\hat{\chi} : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathcal{O}_{\mathbb{C}_p}^\times$$

$$\hat{\chi}(z) := \iota_p(\iota_\infty^{-1}(\chi(z)\, z_\infty^{-k}))\cdot(e_+ z_p)^k, \quad z_p \in (K\otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$$

$$\chi_{+,p}(\det\beta)\cdot|\det\beta|_p^{-2}\cdot(\det\beta)^{k-2}$$
$$= \underbrace{\hat{\chi}_+(\det\beta)}_{\text{unit}}\cdot\underbrace{|\det\beta|^{-2}\cdot(\det\beta)^{-2}}_{p\text{-unit}}$$

$$\mathfrak{G} := \mathrm{Gal}(K(Np^\infty)/\mathcal{K})$$

<u>Thm</u> There exists a unique <u>Eisenstein measure</u> dE on $\mathfrak{G}$

$$\int_{\mathfrak{G}} \hat{\chi}\, dE = E(\chi, N) \quad \text{if } \chi(z_\infty) = z_\infty^k \quad k>2$$

Use abstract Kummer congruence

$$\{\hat{\chi} \mid \chi(z_\infty) = z_\infty^k\} \underset{\text{dense}}{\subset} C(\mathfrak{G}, \mathbb{C}_p)$$