Loop structure on equivariant K-theory of semi-infinite flag manifolds*

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Abstract

We explain that the Pontryagin product structure on the equivariant K-group of an affine Grassmannian considered in [Lam-Schilling-Shimozono, Compos. Math. **146** (2010)] coincides with the tensor structure on the equivariant K-group of a semi-infinite flag manifold considered in [K-Naito-Sagaki, Duke Math. **169** (2020)]. Then, we construct an explicit isomorphism between the equivariant K-group of a semi-infinite flag manifold and a suitably localized equivariant quantum K-group of the corresponding flag manifold. These exhibit a new framework to understand the ring structure of equivariant quantum K-groups and the Peterson isomorphism.

Introduction

Let G be a simply connected simple algebraic group over $\mathbb{C}$ with a maximal torus H. Let Gr denote its affine Grassmannian and let $\mathcal{B}$ be its flag variety.

Following the seminal work of Peterson [49] (on the quantum cohomology, see also Lam-Shimozono [42]), there were many efforts to understand the (small) quantum K-group $qK(\mathcal{B})$ of $\mathcal{B}$ in terms of the K-group $K(\mathrm{Gr})$ of affine Grassmannians (see [41, 40] and the references therein). One of its form, borrowed from Lam-Li-Mihalcea-Shimozono [40], is a (conjectural) ring isomorphism:

$$K_H(\mathrm{Gr})_{\mathrm{loc}} \cong qK_H(\mathcal{B})_{\mathrm{loc}}, \tag{0.1}$$

where subscript H indicate the H-equivariant version and the subscript loc denote certain localizations. Here the multiplication in $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is the *Pontryagin product*, that differs from the usual action of the K-group of the thick affine Grassmannian (that one may internalize using the perfect pairing [39] or the identification with the topological K-group [37]), while the multiplication of $qK_H(\mathcal{B})_{\mathrm{loc}}$ is standard in quantum K-theory [19, 43].

On the other hand, we have another version $\mathbf{Q}_G^{\mathrm{rat}}$ of affine flag variety of G, called the semi-infinite flag variety ([14, 16, 13]). Almost from the beginning [18], it is expected that $\mathbf{Q}_G^{\mathrm{rat}}$ have some relation with the quantum cohomology of $\mathcal{B}$. In fact, we can calculate the equivariant K-theoretic J-function of $\mathcal{B}$

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using $\mathbf{Q}_G^{\mathrm{rat}}$ ([20, 6]), and the reconstruction theorem [44, 24] tells us that they essentially recover the ring structure of the (big) quantum K-group of $\mathcal{B}$.

In [32], we have defined and calculated the equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$, that is also expected to have some relation to $qK_H(\mathcal{B})$, and hence also to $K_H(\mathrm{Gr})$. The goal of this paper is to tell the exact relations as follows:

Theorem A ($\doteq$ Theorem 2.12)**.** *The both of $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ and $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ admits actions of a variant $\mathcal{H}$ of the double affine Hecke algebra and the coroot lattice $Q^\vee$ of G. It gives rise to a dense embedding*

$$\Phi : K_H(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

of $(\mathcal{H}, Q^\vee)$-bimodules that sends the Pontryagin product on the LHS to the tensor product on the RHS.

Here we note that the topology of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ arises from the Schubert stratification of $\mathbf{Q}_G^{\mathrm{rat}}$, and its role in Theorem A is minor. By transplanting the path model of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$, Theorem A yields multiplication formulas of the classes in $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ ([32, 48]).

Our strategy to prove Theorem A is as follows: the $\mathcal{H} \otimes \mathbb{C}Q^\vee$-module $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ is cyclic. Hence, its $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism is determined by the image of a cyclic vector. Moreover, the tensor product action of an equivariant line bundle on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ yields a $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism. These make it possible to identify important parts of the Pontryagin action on the LHS that gives a $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism with the tensor product action on the RHS.

The other part of the exact relation we exhibit is:

Theorem B ($\doteq$ Corollary 3.13 and Theorem 4.12)**.** *We have a $K_H(\mathrm{pt})$-module isomorphism*

$$\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \xrightarrow{\cong} K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends the quantum product of a primitive anti-nef line bundle to the tensor product of the corresponding line bundle.
Moreover, Ψ sends a Schubert class of the LHS to a Schubert class in the RHS, and intertwines the Novikov variable twist in the LHS to the right translation of the Schubert classes in the RHS. In particular, the topology of $qK_H(\mathcal{B})_{\mathrm{loc}}$ with respect to the Novikov variables is compatible with the topology of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ through Ψ.

Here we remark that a priori $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is not a ring (see Remark 1.24).

Unlike Theorem A, Theorem B is best understood by its completed topological form as the inverse quantum multiplication of an anti-nef line bundle corresponds to the tensor product of a nef line bundle through Ψ. The latter tensor product action on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is quite natural, and its structure constants with respect to the Schubert classes are positive ([32, Theorem 5.11]). However, it lives genuinely in the completions in general (as the sum is infinite; see §2.4).

Such tensor products (with infinitely many nonzero structure constants) play a central role in our proof of Theorem B. To analyze them, we need to include an extra q-variable that is responsible for the $\mathbb{G}_m$-action on a curve $\mathbb{P}^1$ in the both sides. In particular, our proof of Theorem B is in fact the $q = 1$ specialization of an isomorphism

$$\Psi_q : \mathbb{C}[q^{\pm 1}] \otimes qK_H(\mathcal{B})_{\mathrm{loc}} \cong K_{\mathbb{G}_m \times H}(\mathbf{Q}_G^{\mathrm{rat}}),$$

that intertwines shift operators (of line bundles on $\mathcal{B}$) and line bundle twists (on $\mathbf{Q}_G^{\mathrm{rat}}$). Combining Theorems A and B, we conclude:

Corollary C ($\doteq$ Corollary 4.15)**.** *We have a commutative diagram, whose bottom arrow is a natural embedding of rings:*

$$\begin{array}{ccc} & K_H(\mathbf{Q}_G^{\mathrm{rat}}) & \\ {\scriptstyle\Phi}\nearrow & & \nwarrow{\scriptstyle\Psi}\ {\scriptstyle\cong} \\ K_H(\mathrm{Gr})_{\mathrm{loc}} & \xhookrightarrow{(0.1)} & qK_H(\mathcal{B})_{\mathrm{loc}} \end{array}.$$

Here the uncompleted version of $qK_H(\mathcal{B})_{\mathrm{loc}}$ is isomorphic to $K_H(\mathrm{Gr})_{\mathrm{loc}}$, the map Φ is an injective $K_H(\mathrm{pt}) \otimes \mathbb{C}Q^\vee$-module homomorphism, and $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ acquires the structure of a ring from $K_H(\mathrm{Gr})$ or $qK_H(\mathcal{B})$.

The explicit nature of Corollary C verifies conjectures in [40] (Corollary 4.15). In the same vein, we find that the structure constants of quantum multiplications, as well as the shift operator actions, on $qK_H(\mathcal{B})$ are finite (Corollary 4.16 and Corollary 4.14; see also Anderson-Chen-Tseng [2]). Therefore, this paper also provides an indispensable step in the proof [2] of the finiteness of the multiplication of quantum K-groups of partial flag manifolds, that stood as a fundamental problem in the quantum K-theoretic Schubert calculus from the beginning.

The idea of the construction of Ψ in Theorem B is to compare the structure of the both sides via the asymptotic behavior of the cohomology of quasi-map spaces with respect to the degree of curves. Adapting the technicality on the topology and the q-variables discussed above, it is rather natural to consider such a thing if we know the "cohomological invariance" between two models of semi-infinite flag manifolds proved in [7, 32], the reconstruction theorem in the form of [24], and the J-function calculations in [20, 6]. In order to show that Ψ respects products (Theorem 4.12), we need to analyze the geometry of graph spaces and quasi-map spaces. Such an analysis is based on the identification of natural subvarieties of quasi-map spaces with the scheme-theoretic intersection of orbit closures in $\mathbf{Q}_G^{\mathrm{rat}}$ with respect to two mutually opposite (Iwahori) subgroups $\mathbf{I}$ and $\mathbf{I}^-$ of $G(\!(z)\!)$, that we call Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$ ([29]). With this in hands, the core of the proof of our assertion reduces to a generalization of the following fact from the case of $\mathcal{B}$ to the case of $\mathbf{Q}_G^{\mathrm{rat}}$: the singularity types of Richardson varieties come from the singularity types between two strata in the Bruhat stratification. However, there are some pit-falls to carry this out (see Remark 1.11 and §3.2 for related accounts), and our proof employs the factorizaton property to identify the transversal slices arising from Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$ and the Bruhat stratification of $\mathbf{Q}_G^{\mathrm{rat}}$. The outcome of our analysis includes a proof that Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$ have rational singularities and are Cohen-Macaulay (Theorem 4.9)[1],which might be of its own interest. We note that Theorem A, and hence Corollary C, also have $\mathbb{G}_m$-equivariant versions by supplementing cosmetic arguments to the results presented in this paper that we exhibit in [28] together with its representation-theoretic consequences.

[1]Previous versions of this paper contained proofs of Theorem 4.12 with gaps. To clarify the whole point, the author decided to separate out the proof of the normality and other related technical results into [29] (see Theorem 4.4).

Note that $\mathbf{Q}_G^{\mathrm{rat}}$ is the universal indscheme associated to the formal loop space of $\mathcal{B}$ ([29] see also [6, 32]). Hence, it is tempting to spell out the following, that unifies the proposals by Givental [18, §4] (cf. Iritani [23]), Peterson [49] (cf. [40]), and Arkhipov-Kapranov [3, §6.2]:

Conjecture D. *Let X be a smooth projective Fano variety with an action of an algebraic group H. Let $\mathcal{L}X$ be the formal loop space of X (see [3]). Then, we have an inclusion that intertwines the quantum product and tensor product of primitive anti-nef line bundles:*

$$\Psi_X : qK_H(X) \hookrightarrow K_H(\mathcal{L}X),$$

where we define $K_H(\mathcal{L}X)$ as the $q = 1$ specialization of a subspace of $K_{\mathbb{G}_m \times H}(\mathcal{L}X)$ (cf. §1.5 and [32]).

The organization of this paper is as follows: In section one, we recall some basic results from previous works (needed to formulate Theorems A and B), and prove some complementary results including the definition of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In section two, we formulate and prove the precise version of Theorem A and exhibit its $SL(2)$-example. In section three, we make recollections on quasi-map spaces and J-functions, and construct the map Ψ following ideas of [20, 6, 7, 24] using results from [32]. At the same time, we prove Theorem B modulo the behavior of Schubert basis (Corollary 3.13). In section four, we first prove that Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$ have only rational singularities, using a detailed analysis of the transversal slices. Then, we prove Theorem 4.12 about the behavior of bases under Ψ. In conjunction with Corollary 3.13, this completes the proof of Theorem B in its precise from. After that, we exhibit the consequences of our results in §4.5, including the proof of the Lam-Li-Mihalcea-Shimozono conjecture, the finiteness of the quantum multiplications for $\mathcal{B}$, and some properties of shift operators.

The results of this paper is supported by our previous works on semi-infinite flag manifolds and representation theory of current algebras, including [31, 26, 32, 29]. An overview of the whole project, as well as brief accounts on this paper and [27, 28], can be found in the survey [30].

Finally, a word of caution is in order. The equivariant K-groups dealt in this paper are *not* identical to these dealt in [41] and [32] in the sense that both groups are just dense subset (or intersects with a dense subset) in the original K-groups (the both groups are suitably topologized). The author does not try to complete this point as he believes it not essential.

1 Preliminaries

A vector space is always a $\mathbb{C}$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above. Tensor products are taken over $\mathbb{C}$ unless stated otherwise. We define the graded dimension of a graded vector space as

$$\mathrm{gdim}\, M := \sum_{i \in \mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}(\!(q^{-1})\!).$$

For a (possibly operator-valued) rational function $f(q)$ on q, we set $\overline{f(q)} := f(q^{-1})$. In this paper, a variety is a separated integral scheme of finite type over $\mathbb{C}$.

1.1 Groups, root systems, and Weyl groups

Basically, material presented in this subsection can be found in [12, 38].

Let G be a connected, simply connected simple algebraic group of rank r over $\mathbb{C}$, and let B and H be a Borel subgroup and a maximal torus of G such that $H \subset B$. We set N $(= [B, B])$ to be the unipotent radical of B and let N^- be the opposite unipotent subgroup of N with respect to H. We set $B^- := HN^-$. We denote the Lie algebra of an algebraic group by the corresponding German small letter. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E, we denote its set of $\mathbb{C}[z]$-valued points by $E[z]$, its set of $\mathbb{C}[\![z]\!]$-valued points by $E[\![z]\!]$, and its set of $\mathbb{C}(z)$-valued points by $E(z)$. Let $\mathbf{I} \subset G[\![z]\!]$ be the preimage of $B \subset G$ via the evaluation of $G[\![z]\!]$ at $z = 0$ (the Iwahori subgroup of $G[\![z]\!]$). We also define a subgroup $\mathbf{I}^- \subset G[z^{-1}]$ as the preimage of B^- via the evaluation of $G[z^{-1}]$ at $z = \infty$. Here we warn that while $E[\![z]\!]$ can be understood as a(n infinite type) group scheme, the group $E[z]$ is a group ind-scheme in general (we refer [38, Chap. IV] for basics on ind-schemes).

Let $P := \mathrm{Hom}_{gr}(H, \mathbb{G}_m)$ be the weight lattice of H, let $\Delta \subset P$ be the set of roots, let $\Delta_+ \subset \Delta$ be the set of roots that yield root subspaces in $\mathfrak{b}$, and let $\Pi \subset \Delta_+$ be the set of simple roots. We set $\Delta_- := -\Delta_+$ and $Q_+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha \subset P$. Let $Q^\vee$ be the dual lattice of P with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times P \to \mathbb{Z}$. We define $\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q^\vee_+ \subset Q^\vee$ be the set of non-negative integer span of $\Pi^\vee$. For $\beta, \gamma \in Q^\vee$, we define $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^\vee_+$. For $\lambda, \mu \in P$, we define $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. We set $P_+ := \{\lambda \in P \mid \langle \alpha^\vee, \lambda \rangle \geq 0, \ \ \forall \alpha^\vee \in \Pi^\vee\}$. Let $\mathtt{I} := \{1, 2, \ldots, r\}$. We fix bijections $\mathtt{I} \cong \Pi \cong \Pi^\vee$ such that $i \in \mathtt{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i \in W$ corresponding to α_i. We also have a reflection $s_\alpha \in W$ corresponding to $\alpha \in \Delta_+$. Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset P_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{i,j}$) and we set $\rho := \sum_{i \in \mathtt{I}} \varpi_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in P_+$.

Let $\Delta_{\mathrm{af}} := \Delta \times \mathbb{Z}\delta \cup \{m\delta\}_{m \neq 0}$ be the untwisted affine root system of Δ with its positive part $\Delta_+ \subset \Delta_{\mathrm{af},+}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, and $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$, where ϑ is the highest root of Δ_+. We set $W_{\mathrm{af}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element in $W \subset W_{\mathrm{af}}$. Together with the normalization $t_{-\vartheta^\vee} := s_\vartheta s_0$ (for the coroot $\vartheta^\vee$ of ϑ), we introduce the translation element $t_\beta \in W_{\mathrm{af}}$ for each $\beta \in Q^\vee$.

For each $i \in \mathtt{I}_{\mathrm{af}}$, we have a subgroup $SL(2, i) \subset G(\!(z)\!)$ that is isomorphic to $SL(2, \mathbb{C})$ corresponding to $\alpha_i \in \mathtt{I}_{\mathrm{af}}$. We set $B_i := SL(2, i) \cap \mathbf{I}$, that is a Borel subgroup of $SL(2, i)$. For each $i \in \mathtt{I}$, we denote the minimal parabolic subgroup $SL(2, i)B$ of G corresponding to $i \in \mathtt{I}$ by P_i. For each $w \in W$ or $w \in W_{\mathrm{af}}$, we find a representative $\dot{w}$ in $N_G(H)$ or $N_{G(\!(z)\!)}(H(\!(z)\!))$, respectively.

Let W_{af}^- denote the set of minimal length representatives of W_{af}/W in W_{af}. We set

$$Q^\vee_< := \{\beta \in Q^\vee \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in \mathtt{I}\}.$$

Let $\leq$ be the Bruhat order of W_{af}. In other words, $w \leq v$ holds if and only if a subexpression of a reduced decomposition of v yields a reduced decomposition of w (see [4]). We define the generic (semi-infinite) Bruhat order $\le_{\frac{\infty}{2}}$ as:

$$w \le_{\frac{\infty}{2}} v \Leftrightarrow wt_\beta \le vt_\beta \quad \text{for every } \beta \in Q^\vee \text{ such that } \langle \beta, \alpha_i \rangle \ll 0 \text{ for } i \in \mathtt{I}. \tag{1.1}$$

By [45], this defines a preorder on W_{af}. Here we remark that $w \le v$ if and only if $w \ge_{\frac{\infty}{2}} v$ for $w, v \in W$. See also [32, §2.2].

For each $\lambda \in P_+$, we denote a finite-dimensional simple G-module with a B-eigenvector with its H-weight λ by $L(\lambda)$. We understand that $L(\lambda) = \{0\}$ for $\lambda \notin P_+$. Let $R(G)$ be the (complexified) representation ring of G. We have an identification $R(G) = (\mathbb{C}P)^W \subset \mathbb{C}P$ by taking characters. For a semi-simple H-module V, we set

$$\mathrm{ch}\, V := \sum_{\lambda \in P} e^\lambda \cdot \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathbb{C}_\lambda, V).$$

If V is a graded H-module in addition, then we set

$$\mathrm{gch}\, V := \sum_{\lambda \in P, n \in \mathbb{Z}} q^n e^\lambda \cdot \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathbb{C}_\lambda, V_n).$$

Let $\mathcal{B} := G/B$ and call it the flag manifold of G. It is equipped with the Bruhat decomposition

$$\mathcal{B} = \bigsqcup_{w \in W} \mathbb{O}_{\mathcal{B}}(w)$$

into B-orbits such that $\dim \mathbb{O}_{\mathcal{B}}(w) = \ell(w_0) - \ell(w)$ for each $w \in W \subset W_{\mathrm{af}}$. Namely, we have $\mathbb{O}_{\mathcal{B}}(w) = B\dot{w}\dot{w}_0 B/B$. We set $\mathcal{B}(w) := \overline{\mathbb{O}_{\mathcal{B}}(w)} \subset \mathcal{B}$. We also define $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(w) := B^- \dot{w}\dot{w}_0 B/B$ and $\mathcal{B}^{\mathrm{op}}(w) := \overline{\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(w)}$.

For each $\lambda \in P$, we have a line bundle $\mathcal{O}_{\mathcal{B}}(\lambda)$ on $\mathcal{B}$ such that

$$\mathrm{ch}\, H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\lambda)) = \mathrm{ch}\, L(\lambda), \quad \mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(-\mu) \cong \mathcal{O}_{\mathcal{B}}(\lambda - \mu) \qquad \lambda, \mu \in P_+.$$

The line bundle $\mathcal{O}_{\mathcal{B}}(\lambda)$ is usually referred to as $G \times^B (w_0\lambda)$.

We have a notion of H-equivariant K-group $K_H(\mathcal{B})$ of $\mathcal{B}$ with coefficients in $\mathbb{C}$ (see e.g. [37]). Explicitly, we have

$$K_H(\mathcal{B}) = \bigoplus_{w \in W} \mathbb{C}P\, [\mathcal{O}_{\mathcal{B}(w)}] = \mathbb{C}P \otimes_{R(G)} \bigoplus_{\lambda \in P} \mathbb{C}[\mathcal{O}_{\mathcal{B}}(\lambda)]. \tag{1.2}$$

The map ch extends to a $\mathbb{C}P$-linear map

$$\chi : K_H(\mathcal{B}) \to \mathbb{C}P,$$

that we call the H-equivariant Euler-Poincaré characteristic. The group $K_H(\mathcal{B})$ is equipped with the product structure $\cdot$ induced by the tensor product of line bundles. For each $i \in \mathtt{I}$, we have

$$[\mathcal{O}_{\mathcal{B}(s_i)}] = [\mathcal{O}_{\mathcal{B}}] - e^{\varpi_i} [\mathcal{O}_{\mathcal{B}}(-\varpi_i)] \in K_H(\mathcal{B}) \tag{1.3}$$

coming from the B-equivariant short exact sequence

$$0 \to \mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathcal{B}}(-\varpi_i) \to \mathcal{O}_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}(s_i)} \to 0. \tag{1.4}$$

Here the B-equivariant map $\mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathcal{B}}(-\varpi_i) \to \mathcal{O}_{\mathcal{B}}$ is unique up to scalar.

1.2 Level zero nil-DAHA

Definition 1.1. The level zero nil-DAHA $\mathcal{H}$ of type G is a $\mathbb{C}$-algebra generated by $\{e^\lambda\}_{\lambda \in P} \cup \{D_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ subject to the following relations:

1. $e^{\lambda+\mu} = e^\lambda \cdot e^\mu$ for $\lambda, \mu \in P$;
2. $D_i^2 = D_i$ for each $i \in \mathtt{I}_{\mathrm{af}}$;
3. For each distinct $i, j \in \mathtt{I}_{\mathrm{af}}$, we set $m_{i,j} \in \mathbb{Z}_{>0}$ as the minimum number such that $(s_i s_j)^{m_{i,j}} = 1$. Then, we have
$$\overbrace{D_i D_j \cdots}^{m_{i,j}\text{-terms}} = \overbrace{D_j D_i \cdots}^{m_{i,j}\text{-terms}};$$
4. For each $\lambda \in P$ and $i \in \mathtt{I}$, we have
$$D_i e^\lambda - e^{s_i \lambda} D_i = \frac{e^\lambda - e^{s_i \lambda}}{1 - e^{\alpha_i}};$$
5. For each $\lambda \in P$, we have
$$D_0 e^\lambda - e^{s_\vartheta \lambda} D_0 = \frac{e^\lambda - e^{s_\vartheta \lambda}}{1 - e^{-\vartheta}}.$$

Let $\mathcal{S} := \mathbb{C}P \otimes \mathbb{C}W_{\mathrm{af}}$ be the smash product algebra, whose multiplication reads as:
$$(e^\lambda \otimes w)(e^\mu \otimes v) = e^{\lambda + w\mu} \otimes wv \qquad \lambda, \mu \in P, w, v \in W_{\mathrm{af}},$$
where s_0 acts on P as s_ϑ. Let $\mathbb{C}(P)$ denote the fraction field of (the Laurent polynomial algebra) $\mathbb{C}P$. We have a scalar extension
$$\mathcal{A} := \mathbb{C}(P) \otimes_{\mathbb{C}P} \mathcal{S} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}}.$$

Theorem 1.2 ([41] §2.2)**.** *We have an embedding of algebras $\imath^* : \mathcal{H} \hookrightarrow \mathcal{A}$:*
$$e^\lambda \mapsto e^\lambda \otimes 1,\ D_i \mapsto \frac{1}{1 - e^{\alpha_i}} \otimes 1 - \frac{e^{\alpha_i}}{1 - e^{\alpha_i}} \otimes s_i, \qquad \lambda \in P, i \in \mathtt{I}$$
$$D_0 \mapsto \frac{1}{1 - e^{-\vartheta}} \otimes 1 - \frac{e^{-\vartheta}}{1 - e^{-\vartheta}} \otimes s_0.$$

Since we have a natural action of $\mathcal{A}$ on $\mathbb{C}(P)$, we obtain an action of $\mathcal{H}$ on $\mathbb{C}(P)$, that we call the polynomial representation.

For $w \in W_{\mathrm{af}}$, we find a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ $(i_1, \ldots, i_\ell \in \mathtt{I}_{\mathrm{af}})$ and set
$$D_w := D_{s_{i_1}} D_{s_{i_2}} \cdots D_{s_{i_\ell}} \in \mathcal{H}.$$
By Definition 1.1 3), the element D_w is independent of the choice of a reduced expression. By Definition 1.1 2), we have $D_i D_{w_0} = D_{w_0}$ for each $i \in \mathtt{I}$, and hence $D_{w_0}^2 = D_{w_0}$. We have an explicit form
$$D_{w_0} = 1 \otimes \left(\sum_{w \in W} w \right) \cdot \frac{e^{-\rho}}{\prod_{\alpha \in \Delta^+} (e^{-\alpha/2} - e^{\alpha/2})} \otimes 1 \in \mathcal{A} \tag{1.5}$$
obtained from the (left W-invariance of the) Weyl character formula.

1.3 Affine Grassmannians

We define our (thin) affine Grassmannian and (thin) affine flag variety by

$$\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!] \quad \text{and} \quad \mathrm{Fl}_G := G(\!(z)\!)/\mathbf{I},$$

respectively. We have a natural fibration map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$ whose fiber is isomorphic to $\mathcal{B}$. For each $w \in W_{\mathrm{af}}$, we set $\mathbb{O}^{\mathrm{Fl}}_w = \mathbf{I}\dot{w}\mathbf{I}/\mathbf{I}$. For each $\beta \in Q^\vee$, we find $w \in t_\beta W$ and set $\mathbb{O}^{\mathrm{Gr}}_\beta := \pi(\mathbb{O}^{\mathrm{Fl}}_w)$. The sets $\mathbb{O}^{\mathrm{Fl}}_w$ and $\mathbb{O}^{\mathrm{Gr}}_\beta$ do not depend on the choices involved.

Theorem 1.3 (Bruhat decomposition, [38] Corollary 6.1.20)**.** *We have* $\mathbf{I}$*-orbit decompositions*

$$\mathrm{Gr} = \bigsqcup_{\beta \in Q^\vee} \mathbb{O}^{\mathrm{Gr}}_\beta \quad \textit{and} \quad \mathrm{Fl}_G = \bigsqcup_{w \in W_{\mathrm{af}}} \mathbb{O}^{\mathrm{Fl}}_w$$

such that we have $\mathbb{O}^{\mathrm{Fl}}_v \subset \overline{\mathbb{O}^{\mathrm{Fl}}_w}$ *if and only if* $v \le w$*.*

Let us set $\mathrm{Gr}_\beta := \overline{\mathbb{O}^{\mathrm{Gr}}_\beta}$ and $\mathrm{Fl}_w := \overline{\mathbb{O}^{\mathrm{Fl}}_w}$ for $\beta \in Q^\vee$ and $w \in W_{\mathrm{af}}$. For $w \in W^-_{\mathrm{af}}$, we also set $\mathrm{Gr}_w := \mathrm{Gr}_\beta$ for a unique $\beta \in Q^\vee$ such that $w \in t_\beta W$.

We set

$$K_H(\mathrm{Gr}) := \bigoplus_{\beta \in Q^\vee} \mathbb{C}P\,[\mathcal{O}_{\mathrm{Gr}_\beta}] \quad \text{and} \quad K_H(\mathrm{Fl}_G) := \bigoplus_{w \in W_{\mathrm{af}}} \mathbb{C}P\,[\mathcal{O}_{\mathrm{Fl}_w}].$$

Theorem 1.4 (Kostant-Kumar [37])**.** *The vector space* $K_H(\mathrm{Fl}_G)$ *affords a regular representation of* $\mathcal{H}$ *such that:*

1. *the subalgebra* $\mathbb{C}P \subset \mathcal{H}$ *acts by the multiplication as* $\mathbb{C}P$*-modules;*
2. *we have* $D_i[\mathcal{O}_{\mathrm{Fl}_w}] = [\mathcal{O}_{\mathrm{Fl}_{s_i w}}]$ $(s_i w > w)$ *or* $[\mathcal{O}_{\mathrm{Fl}_w}]$ $(s_i w < w)$*.* □

Being a regular representation, we sometimes identify $K_H(\mathrm{Fl}_G)$ with $\mathcal{H}$ (through $e^\lambda[\mathcal{O}_{\mathrm{Fl}_w}] \leftrightarrow e^\lambda D_w$ for $\lambda \in P, w \in W_{\mathrm{af}}$) and consider product of two elements in $\mathcal{H} \cup K_H(\mathrm{Fl}_G)$, that results in an element of $K_H(\mathrm{Fl}_G) \cong \mathcal{H} \subset \mathcal{A}$.

Theorem 1.5 (Kostant-Kumar [37])**.** *The pullback defines a map* $\pi^* : K_H(\mathrm{Gr}_G) \hookrightarrow K_H(\mathrm{Fl}_G)$ *such that*

$$\pi^*[\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Fl}_{t_\beta}}]D_{w_0} = D_{t_\beta}D_{w_0} \qquad \beta \in Q^\vee.$$

In particular, $\mathrm{Im}\,\pi^* = \mathcal{H}D_{w_0}$ *is a* $\mathcal{H}$*-submodule, that can be regarded as a left ideal of* $\mathcal{H}$*.* □

Let $\mathcal{C} := \mathbb{C}(P) \otimes \mathbb{C}Q^\vee \subset \mathcal{A}$ be a subalgebra generated by elements of the form $f \otimes t_\beta$ ($f \in \mathbb{C}(P)$, $\beta \in Q^\vee$). By our convention on the W_{af}-action on P, we deduce that $\mathcal{C}$ is commutative. We have a projection map

$$\mathsf{pr} : \mathcal{A} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}} \longrightarrow \mathbb{C}(P) \otimes \mathbb{C}Q^\vee = \mathcal{C}$$

defined as $\mathsf{pr}(f \otimes t_\beta w) = f \otimes t_\beta$ for each $f \in \mathbb{C}(P), w \in W, \beta \in Q^\vee$.

Theorem 1.6 (Lam-Schilling-Shimozono)**.** *The composition map* $\mathsf{pr} \circ \imath^* \circ \pi^*$ *defines an embedding*

$$K_H(\mathrm{Gr}) \hookrightarrow K_H(\mathrm{Fl}_G) \to \mathcal{C} \quad (\subset \mathcal{A})$$

whose image is equal to $K_H(\mathrm{Fl}_G) \cap \mathcal{C}$*. It descends to a* $\mathbb{C}P$*-module isomorphism*

$$r^* : K_H(\mathrm{Gr}) \hookrightarrow K_H(\mathrm{Fl}_G) \cap \mathcal{C} \quad (\subset \mathcal{A}).$$

This equips $K_H(\mathrm{Gr})$ *with a subalgebra structure of a commutative algebra* $\mathcal{C}$*.*

Proof. By [40, Proposition 2], we deduce that the image of D_v under the map pr is the same for each $v \in t_\beta W$. Therefore, the assertion follows from the description of [41, §5.2]. □

Thanks to Theorem 1.6, we obtain a commutative product structure of $K_H(\mathrm{Gr})$ inherited from $\mathcal{C}$, that we denote by $\odot$. We call it the *Pontryagin product*. This is the same product as in [41, §5.2], and its relation with the Pontryagin product in the topological K-group of the based loop space of the maximal compact subgroup of G is explained in [41, §5.1].

Below, we might think of an element of $K_H(\mathrm{Gr})$ as an element of $K_H(\mathrm{Fl}_G)$ through π^*, an element of $\mathcal{A}$ through $\imath^* \circ \pi^*$, and as an element of $\mathcal{C}$ through r^* interchangeably. The next two results are natural extensions of the results from [42, §9] (originally due to Peterson):

Theorem 1.7. *Let* $w \in W_{\mathrm{af}}^-$ *and let* $\beta \in Q_<^\vee$*. We have*

$$[\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}].$$

Proof. By our assumption on β, we have $\ell(t_\beta) = \ell(w_0) + \ell(w_0 t_\beta)$ (see [46, (2.4.1)]). In particular, the element $[\mathcal{O}_{\mathrm{Gr}_\beta}]$, viewed as an element of $\mathcal{A}$ through $\imath^* \circ \pi^*$, is of the form $(\sum_{v \in W} v)\xi$ for some $\xi \in \mathcal{A}$ by (1.5). Hence, it is invariant by the left action of W. Since the effect of the map pr is to twist by elements of W from the right in a term by term fashion, we deduce the equality

$$[\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}] = \mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}])[\mathcal{O}_{\mathrm{Gr}_\beta}]$$

of multiplications in $\mathcal{A}$ (multiplication in a non-commutative algebra). By examining the definition of pr, we further deduce

$$\mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}]) = \mathsf{pr}(\mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}])[\mathcal{O}_{\mathrm{Gr}_\beta}]) = \mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]). \tag{1.6}$$

Since $w \in W_{\mathrm{af}}^-$, we have $\ell(w) + \ell(t_\beta) = \ell(w t_\beta)$ (see [49, Lecture 8, page12]). Consequently, we have $D_{wt_\beta} = D_w D_{t_\beta}$. Therefore, (1.6) and Theorem 1.6 implies that

$$[\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] = [\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}] \in K_H(\mathrm{Gr})$$

as required. □

Since $t_\beta \in W_{\mathrm{af}}^-$ for each $\beta \in Q_<^\vee$, Theorem 1.7 implies that the set

$$\{[\mathcal{O}_{\mathrm{Gr}_\beta}] \mid \beta \in Q_<^\vee\} \subset (K_H(\mathrm{Gr}), \odot)$$

forms a multiplicative system. We denote by $K_H(\mathrm{Gr})_{\mathrm{loc}}$ its localization. The action of an element $[\mathcal{O}_{\mathrm{Gr}_\beta}]$ on $K_H(\mathrm{Gr})$ in Theorem 1.7 is torsion-free, and hence we have an embedding $K_H(\mathrm{Gr}) \hookrightarrow K_H(\mathrm{Gr})_{\mathrm{loc}}$.

Corollary 1.8. *Let $i \in \mathtt{I}$. For $\beta \in Q^\vee_<$, we set*

$$\mathbf{h}_i := [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1}.$$

Then, the element $\mathbf{h}_i$ is independent of the choice of β.

Proof. By Theorem 1.7, we have

$$\begin{aligned}[\mathcal{O}_{\mathrm{Gr}_{s_i t_{\gamma+\beta}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{\gamma+\beta}}]^{-1} &= [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{-1} \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1} \\ &= [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1}\end{aligned}$$

for $\gamma \in Q^\vee_<$. Hence, we conclude the assertion. □

For each $\gamma \in Q^\vee$, we can write $\gamma = \beta_1 - \beta_2$, where $\beta_1, \beta_2 \in Q^\vee_<$. Using this, we define an element

$$\mathtt{t}_\gamma := [\mathcal{O}_{\mathrm{Gr}_{\beta_1}}] \odot [\mathcal{O}_{\mathrm{Gr}_{\beta_2}}]^{-1}.$$

Lemma 1.9. *For each $\gamma \in Q^\vee$, the element $\mathtt{t}_\gamma \in K_H(\mathrm{Gr})_{\mathrm{loc}}$ is independent of the choices involved.*

Proof. Similar to the proof of Corollary 1.8. The details are left to the reader. □

1.4 Semi-infinite flag manifolds

The main reference of this subsection is [29]. We define the semi-infinite flag manifold as the reduced scheme associated to:

$$\mathbf{Q}_G^{\mathrm{rat}} := G(\!(z)\!)/(H \cdot N(\!(z)\!)).$$

This is a pure ind-scheme of ind-infinite type. Note that the group $Q^\vee \subset H(\!(z)\!)/H$ acts on $\mathbf{Q}_G^{\mathrm{rat}}$ from the right. We have an embedding

$$\Upsilon : \mathbf{Q}_G^{\mathrm{rat}} \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z)\!)), \tag{1.7}$$

which is $\mathbb{G}_m \ltimes G(\!(z)\!)$-equivariant by enhancing the $\mathbb{G}_m$-action dilating z prolonged trivially along the component $L(\varpi_i)^*$ for each $i \in \mathtt{I}$, and the G-action on $L(\varpi_i)^*$ prolonged trivially along the component $\mathbb{C}(\!(z)\!)$ ([29, Theorem 4.18]). Note that the RHS of (1.7) is not a scheme by itself, but it acquires the structure of a scheme if we additionally impose the z-degree bound from the below on each factor. For $w \in W_{\mathrm{af}}$, we set $\mathbb{O}(w) := \mathbf{I}\dot{w}\dot{w}_0 HN(\!(z)\!)/HN(\!(z)\!)$ and $\mathbf{Q}_G(w) := \overline{\mathbb{O}(w)}$. Note that we can take the closure either in $\mathbf{Q}_G^{\mathrm{rat}}$ or the RHS of (1.7) since Υ restricts to a closed embedding of schemes

$$\Upsilon_m : \mathbf{Q}_G(w) \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}[\![z]\!] z^{-m}) \quad \left(\subset \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z)\!)) \right) \tag{1.8}$$

for each $w \in W_{\mathrm{af}}$ under a suitable choice of $m \in \mathbb{Z}$. We refer $\mathbf{Q}_G(w)$ as a Schubert variety of $\mathbf{Q}_G^{\mathrm{rat}}$.

Theorem 1.10 ([16, 13, 32]). *We have an* $\mathbf{I}$*-orbit decomposition*

$$\mathbf{Q}_G^{\mathrm{rat}} = \bigsqcup_{w \in W_{\mathrm{af}}} \mathbb{O}(w)$$

with the following properties:

1. *each* $\mathbb{O}(w)$ *has infinite dimension and infinite codimension in* $\mathbf{Q}_G^{\mathrm{rat}}$*;*
2. *each* $\mathbb{O}(w)$ *contains a unique* $(\mathbb{G}_m \times H)$*-fixed point* p_w*;*
3. *the right action of* $\gamma \in Q^\vee$ *on* $\mathbf{Q}_G^{\mathrm{rat}}$ *yields the translation* $\mathbb{O}(w) \mapsto \mathbb{O}(wt_\gamma)$*;*
4. *we have* $\mathbb{O}(w) \subset \overline{\mathbb{O}(v)}$ *if and only if* $w \le_{\frac{\infty}{2}} v$*.* □

We may write $\mathbf{Q}_G$ instead of $\mathbf{Q}_G(e)$ for the sake of notational simplicity.

The indscheme $\mathbf{Q}_G^{\mathrm{rat}}$ is equipped with a $G(\!(z)\!)$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ for each $\lambda \in P$. This line bundle is realized as

$$\bigotimes_{i \in \mathtt{I}} \Upsilon^* \left(\mathcal{O}_{\mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z)\!))}(1)\right)^{\otimes m_i} \quad \text{when} \quad \lambda = \sum_{i \in \mathtt{I}} m_i \varpi_i.$$

In particular, the restriction $\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)$ of $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ to each $\mathbf{Q}_G(w)$ defines a line bundle. We warn that the normalization of line bundles is twisted by $-w_0$ from that of [32].

Remark 1.11 (opposite Schubert varieties). Here we discuss about opposite Schubert varieties of $\mathbf{Q}_G^{\mathrm{rat}}$. Note that (1.7) is apparently non-stable with respect to the involution $z \mapsto z^{-1}$. In particular, the group $G[\![z^{-1}]\!]$ does not act on $\mathbf{Q}_G^{\mathrm{rat}}$. Thus, our opposite Schubert subvariety of $\mathbf{Q}_G^{\mathrm{rat}}$ should be the closure of an $\mathbf{I}^-$-orbit, defined as an ind-scheme. However, such opposite Schubert subvarieties are continuously many, and hence cannot be labelled by W_{af}. Thus, we usually refer only the $\mathbf{I}^-$-orbit closures

$$\mathbf{Q}_G^-(w) := \overline{\mathbf{I}^- p_w} \subset \mathbf{Q}_G^{\mathrm{rat}} \qquad w \in W_{\mathrm{af}}$$

as the opposite Schubert varieties of $\mathbf{Q}_G^{\mathrm{rat}}$.

If we set $\mathbf{I}^\flat$ to be the Zariski closure of $\mathbf{I}^-$ in $G[\![z^{-1}]\!]$, then we have another version of an opposite Schubert cell, namely an $\mathbf{I}^\flat$-orbit in $\prod_i \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z^{-1})\!))$ that intersects with $\mathbf{Q}_G^{\mathrm{rat}}$ in the ambient space

$$\prod_i \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z^{-1})\!)) \subset \prod_i \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}[\![z, z^{-1}]\!]) \supset \mathbf{Q}_G^{\mathrm{rat}}.$$

These ones are labelled by W_{af}, and each of their closure defines another version of an opposite Schubert variety. As we already see, it cannot be a subscheme of $\mathbf{Q}_G^{\mathrm{rat}}$. Nevertheless, each $\mathbf{Q}_G^-(w)$ defines a Zariski dense subset of an $\mathbf{I}^\flat$-orbit.

In view of [29], the intersection of a Schubert variety and an opposite Schubert variety (labelled by W_{af}) does not depend on a choice of these two versions of opposite Schubert varieties, and referred to as a Richardson variety of $\mathbf{Q}_G^{\mathrm{rat}}$ (see §4.1). Unlike the case of $\mathcal{B}$, our open Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$ are not necessarily smooth.

We set $\mathfrak{g}[z] := \mathfrak{g} \otimes \mathbb{C}[z]$ and $\mathbf{I}' := \mathbf{I} \cap G[z]$, where the latter is an ind-group whose ind-structure is induced by $G[z]$.

Theorem 1.12 (Chari-Ion [11], see also [26] §1.2)**.** *For each* $\lambda = \sum_{i \in \mathtt{I}} m_i \varpi_i \in P_+$*, we have a* $(\mathbb{G}_m \times H)$*-semisimple* $G[z]$*-module* $\mathbb{W}(\lambda)$ *with the following properties:*

1. *It is* G*-integrable, i.e. it is a direct sum of finite-dimensional* $(\mathbb{G}_m \times G)$*-modules by restriction;*
2. *It is generated by the action of* $\mathbf{I}'$ *from a unique (cyclic) vector (up to scalar) with its* $(\mathbb{G}_m \times H)$*-weight* $(w_0\lambda)$*;*
3. *It is projective in the category of* $(\mathbb{G}_m \times H)$*-semisimple* $G[z]$*-module whose graded character belongs to* $\mathbb{Z}[\![q]\!]\{\mathrm{ch}\, L(\mu) \mid \mu \le \lambda\}$*;*
4. *We have*
$$\mathrm{gch}\, \mathbb{W}(\lambda) = (\prod_{i \in \mathtt{I}} \prod_{j=1}^{m_j} \frac{1}{1-q^j}) P_\lambda, \tag{1.9}$$
where $P_\lambda \in (\mathbb{Z}[q]P)^W$ *is the symmetric Macdonald polynomial specialized to* $t = 0$*. In particular, we have:*
$$P_\lambda \equiv \mathrm{ch}\, L(\lambda) \mod \mathbb{Z}[q]\{\mathrm{ch}\, L(\mu) \mid \mu < \lambda\}. \tag{1.10}$$

Theorem 1.13 ([32] Theorem 4.30 and Corollary 4.31, see also P2439)**.** *For each* $\lambda \in P_+$*, we have*
$$\Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda))^\vee = \mathbb{W}(-w_0\lambda). \tag{1.11}$$
For each $\lambda \in P$ *and* $w \in W_{\mathrm{af}}$ *such that* $w \le_{\frac{\infty}{2}} e$*, we have*

$$\mathrm{gch}\, \Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \in (\mathbb{C}[\![q^{-1}]\!])P \quad \textit{and} \quad H^{>0}(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) = \{0\}.$$

Corollary 1.14. *The* $(\mathbb{G}_m \times H)$*-weight* $(u\varpi_i + m\delta)$*-part of*
$$\Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\varpi_i))$$
is one-dimensional for each $i \in \mathtt{I}$*,* $u \in W$*, and* $m \in \mathbb{Z}_{\le 0}$*.*

Proof. By (1.10), the monomial $q^m e^{u\varpi_i}$ appears in P_λ only if $m = 0$ and its coefficient is 1. Since the q-series appearing as the dual of the RHS of (1.11) is
$$1 + q^{-1} + q^{-2} + \cdots$$
by (1.9), we conclude the assertion. $\square$

For each $u \in W_{\mathrm{af}}$ and $i \in \mathtt{I}$, we have a $(\mathbb{G}_m \times H)$-eigenvector $\phi_{u,i} \in (L(\varpi_i)^* \otimes \mathbb{C}[\![z]\!]z^{-N})^\vee$ dual to p_u in the middle term of (1.8). It lifts uniquely to a $(\mathbb{G}_m \times H)$-weight vector in $\Gamma(\mathbf{Q}_G(t_\beta), \mathcal{O}_{\mathbf{Q}_G(t_\beta)}(\varpi_i))$ of weight $u\varpi_i$ for each $t_\beta \ge_{\frac{\infty}{2}} u$ since this $(\mathbb{G}_m \times H)$-weight space is one-dimensional by Corollary 1.14 (and Theorem 1.10 3)). We have $\phi_{u,i}(x') \neq 0$ for every point $x' \in \mathbb{O}(u)$, and $\phi_{u,i}(\bullet) = 0$ for some $i \in \mathtt{I}$ (set-theoretically) defines $\mathbf{Q}_G(u) \setminus \mathbb{O}(u)$. By the uniqueness of these liftings, this set-theoretic defining property of $\phi_{u,i}$ holds by considering it as a section of a line bundle on any of the spaces in (1.8) when $t_\beta \ge_{\frac{\infty}{2}} u$.

It follows that each $\mathbf{Q}_G(u)$ is set-theoretically defined as

$$\{x \in \mathbf{Q}_G^{\mathrm{rat}} \mid \phi_{v,i}(x) = 0, \quad \forall (v,i) \in S(u)\}$$

through (1.7), where

$$S(u) := \{(v,i) \in W_{\mathrm{af}} \times \mathtt{I} \mid \phi_{v,i}(x) = 0, \quad \forall x \in \mathbb{O}(u)\}.$$

The same is true for the closures of $\mathbf{I}'$-orbits that contain $(\mathbb{G}_m \times H)$-fixed points (see Remark 1.11 for an account on the additional constraint).

1.5 Equivariant K-groups of semi-infinite flag manifolds

We define a $\mathbb{C}[q^{\pm 1}]P$-module $\widetilde{K}(\mathbf{Q}_G)$ as:

$$\widetilde{K}(\mathbf{Q}_G) := \{ \sum_{e \geq \frac{\infty}{2} w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}[q^{\pm 1}]P\},$$

where the sum in the definition is understood to be formal. For each $\lambda \in P$ and $\sum_w a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \in \widetilde{K}(\mathbf{Q}_G)$, we define the formal sum

$$\begin{aligned}\widetilde{\Theta}(\lambda)(\sum_w a_w [\mathcal{O}_{\mathbf{Q}_G(w)}]) &= \sum_w \sum_{i \geq 0} (-1)^i a_w \mathrm{gch}\, H^i(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \\ &= \sum_w a_w \mathrm{gch}\, \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)),\end{aligned}$$

where the second equality follows from Theorem 1.13. We have $\widetilde{\Theta}(\lambda)(\bullet) = 0$ whenever $\lambda \notin P_+$. We set

$$\widetilde{K}'(\mathbf{Q}_G) := \{ \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \in \widetilde{K}(\mathbf{Q}_G) \mid \Theta(\lambda)(\sum_{w \in W_{\mathrm{af}}} |a_w| [\mathcal{O}_{\mathbf{Q}_G(w)}]) \in (\mathbb{R}(\!(q^{-1})\!))P\},$$

where the absolute value $|a_w|$ of a_w is taken coefficientwise. This is a proper subspace of the $(\mathbb{G}_m \ltimes \mathbf{I})$-equivariant K-group $K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G)$ of $\mathbf{Q}_G$ defined in [32]. Here we need to employ their variants in order to make the $q = 1$ specialization. The subspace $\widetilde{K}'(\mathbf{Q}_G) \subset \widetilde{K}(\mathbf{Q}_G)$ is a $\mathbb{C}[q^{\pm 1}]P$-submodule.

We define two $\mathbb{C}[q^{\pm 1}]P$-modules

$$\mathrm{Fun}_P := \{f : P \to (\mathbb{C}(\!(q^{-1})\!))P\}$$

and

$$\mathrm{Fun}_P^{\mathrm{neg}} := \{f : P \to (\mathbb{C}(\!(q^{-1})\!))P \mid f(\lambda) = 0 \quad \text{if} \quad \langle \alpha_i^\vee, \lambda \rangle \gg 0, \forall i \in \mathtt{I}\}.$$

In view of [32, §5], the map

$$\begin{aligned}\widetilde{\Theta} : \widetilde{K}'(\mathbf{Q}_G) \ni \sum_w a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \\ \mapsto \left[\lambda \mapsto \sum_w \sum_{i \geq 0} (-1)^i a_w \mathrm{gch}\, H^i(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \right] \in \mathrm{Fun}_P\end{aligned}$$

induces an inclusion

$$\Theta : \widetilde{K}'(\mathbf{Q}_G) \hookrightarrow \frac{\mathrm{Fun}_P}{\mathrm{Fun}_P^{\mathrm{neg}}}.$$

Let $\mathcal{E}$ be a $(\mathbb{G}_m \ltimes \mathbf{I})$-equivariant quasi-coherent sheaf on $\mathbf{Q}_G$ that satisfies the condition $(\bigstar)$ consisting of the following two:

$(\bigstar)_1$ There exists $i_0 \in \mathbb{Z}$ (that may depend on $\mathcal{E}$) such that

$$H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}} \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)) = \{0\} \quad \text{for each } \ i > i_0, \ \text{ and } \ \lambda \in P;$$

$(\bigstar)_2$ We have

$$\mathrm{gch}\, H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}} \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)) \in (\mathbb{C}(\!(q^{-1})\!))P \quad \text{for each } \ i \in \mathbb{Z}, \ \text{ and } \ \lambda \in P.$$

For the above $\mathcal{E}$ and $\lambda \in P$, we set

$$\chi_q(\mathbf{Q}_G, \mathcal{E}(\lambda)) := \sum_{i \geq 0} (-1)^i \mathrm{gch}\, H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}} \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)) \in (\mathbb{C}(\!(q^{-1})\!))P.$$

We say that the above $\mathcal{E}$ defines a class

$$[\mathcal{E}] = \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \in \widetilde{K}'(\mathbf{Q}_G), \qquad a_w \in \mathbb{C}[q^{\pm 1}]P$$

if and only if the following function on P

$$\lambda \mapsto \left(\chi_q(\mathbf{Q}_G, \mathcal{E}(\lambda)) - \sum_{w \in W_{\mathrm{af}}} a_w \chi_q(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \right)$$

belongs to $\mathrm{Fun}_P^{\mathrm{neg}}$.

Lemma 1.15. *Suppose that we have a short exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

of $(\mathbb{G}_m \ltimes \mathbf{I})$-equivariant quasi-coherent sheaves on $\mathbf{Q}_G$ that satisfy $(\bigstar)$. If two of the above three sheaves define classes in $\widetilde{K}'(\mathbf{Q}_G)$, then the remaining one also define a class in $\widetilde{K}'(\mathbf{Q}_G)$. In this case, we have

$$[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3].$$

Proof. The condition $(\bigstar)$ guarantee the existence of the functions $f_i : \lambda \mapsto \chi_q(\mathbf{Q}_G, \mathcal{E}_i(\lambda))$ $(i = 1, 2, 3)$ in Fun_P. We have $f_2 = f_1 + f_3 \in \mathrm{Fun}_P$ by the long exact sequence of cohomologies. Thus, so are their images in $\frac{\mathrm{Fun}_P}{\mathrm{Fun}_P^{\mathrm{neg}}}$. Since $\widetilde{K}'(\mathbf{Q}_G)$ is a(n abelian) group, we conclude the results. □

Since each of the coefficient of an element of $\widetilde{K}'(\mathbf{Q}_G)$ belongs to $\mathbb{C}[q^{\pm 1}]P$, the following $q = 1$ specialization make sense:

$$K_H(\mathbf{Q}_G) := \mathbb{C} \otimes_{\mathbb{C}[q^{\pm 1}]} \widetilde{K}'(\mathbf{Q}_G).$$

By abuse of notation, we denote the class of an $(\mathbb{G}_m \ltimes \mathbf{I})$-equivariant quasi-coherent sheaf $\mathcal{E}$ in $K_H(\mathbf{Q}_G)$ obtained from its class $[\mathcal{E}] \in \widetilde{K}'(\mathbf{Q}_G)$ by the same letter.

Remark 1.16. Our definition of $\widetilde{K}'(\mathbf{Q}_G)$ here and [32] depends on the $\mathbb{C}[q^{\pm 1}]P$-linear independence of the asymptotic behavior of the functions

$$\lambda \mapsto \chi_q(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \qquad e \geq_{\frac{\infty}{2}} w \in W_{\mathrm{af}}.$$

By [29, Appendix A], the push-forward of $\mathcal{O}_{\mathbf{Q}_G(w)}$ to a parabolic version of $\mathbf{Q}_G^{\mathrm{rat}}$ is the structure sheaf of a Schubert variety and have no higher direct images. This cohomological affinity makes $\widetilde{K}'(\mathbf{Q}_G)$ functorial with respect to the push-forwards to its parabolic analogues [27, 30].

Lemma 1.17. *We have*

$$K_H(\mathbf{Q}_G) = \{ \sum_{e \geq_{\frac{\infty}{2}} w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}P \},$$

where the sum in the definition is understood to be formal.

Proof. For a set of coefficients $\{a_w\}_{w \in W_{\mathrm{af}}}$ appears in the RHS, we decompose $w = ut_\beta$ ($u \in W$ and $\beta \in Q_+^\vee$) and set

$$\widetilde{a}_w := a_w q^{-\langle \rho, \beta \rangle} \in \mathbb{C}[q^{-1}]P.$$

In view of Theorem 1.13, only finitely many terms in

$$\{\widetilde{a}_w \mathrm{gch}\, \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda))\}_{w \leq_{\frac{\infty}{2}} e}$$

carries a non-zero coefficients in q^m for each $m \in \mathbb{Z}_{\leq 0}$ and $\lambda \in P$. Therefore, we have

$$\widetilde{\Theta}(\lambda)(\sum_{w \in W_{\mathrm{af}}} \widetilde{a}_w [\mathcal{O}_{\mathbf{Q}_G(w)}]) \in (\mathbb{C}[\![q^{-1}]\!])P \subset (\mathbb{C}(\!(q^{-1})\!))P$$

for each $\lambda \in P$. Thus, the assertion holds. □

As a natural extension of $K_H(\mathbf{Q}_G)$, we define

$$K_H(\mathbf{Q}_G^{\mathrm{rat}}) := \{ \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}P,\ \exists \beta_0 \in Q^\vee \text{ s.t. } a_{ut_\beta} = 0, \forall u \in W, \beta \not> \beta_0 \},$$

where the sum is understood to be formal. We have $K_H(\mathbf{Q}_G) \subset K_H(\mathbf{Q}_G^{\mathrm{rat}})$.

For each $\beta \in Q^\vee$, we have an endomorphism of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ defined by

$$a[\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto a[\mathcal{O}_{\mathbf{Q}_G(wt_\beta)}] \qquad \forall a \in \mathbb{C}P, w \in W_{\mathrm{af}} \tag{1.12}$$

induced by the right action of $Q^\vee \subset W_{\mathrm{af}}$ (see Theorem 1.10 3)). The translations of $K_H(\mathbf{Q}_G)$ with respect to this right $Q^\vee$-action equips $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ a(n open base of a linear) topology. The subset $\{t_\beta\}_{\beta \in Q_+^\vee} \subset Q^\vee$ acts on $K_H(\mathbf{Q}_G)$. We denote by $\mathcal{R}$ the ring consisting of the formal $\mathbb{C}P$-linear combinations of $\{t_\beta\}_{\beta \in Q_+^\vee}$, equipped with an induced topology from $K_H(\mathbf{Q}_G)$. We have $\mathbb{C}Q_+^\vee \subset \mathcal{R}$ spanned by the constant coefficient monomials of $Q_+^\vee$.

Lemma 1.18. *$K_H(\mathbf{Q}_G)$ and $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ are free modules over $\mathcal{R}$ and $\mathbb{C}Q^\vee \otimes_{\mathbb{C}Q_+^\vee} \mathcal{R}$, respectively. Moreover, their ranks are $|W|$.*

Proof. We have an explicit basis $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w \in W}$ in the both cases. □

Our $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ and $K_H(\mathbf{Q}_G)$ are the $q = 1$ specializations of subsets of the equivariant K-groups $K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G^{\mathrm{rat}})$ and $K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G)$ considered in [32]. To this end, we need to verify that the natural actions of the Demazure operators and the tensor product actions in [32] yield the corresponding actions on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. The first one is immediate from the expression:

Theorem 1.19 ([32] §6 and [26] Theorem A)**.** *The vector space $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ affords a representation of $\mathcal{H}$ with the following properties:*

1. *the subalgebra $\mathbb{C}P \subset \mathcal{H}$ acts by the multiplication as $\mathbb{C}P$-modules;*
2. *we have*

$$D_i([\mathcal{O}_{\mathbf{Q}_G(w)}]) = \begin{cases} [\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & (s_i w >_{\frac{\infty}{2}} w) \\ [\mathcal{O}_{\mathbf{Q}_G(w)}] & (s_i w <_{\frac{\infty}{2}} w) \end{cases}.$$

By Theorem 1.19, we deduce that the right $Q^\vee$-action yield $\mathcal{H}$-module endomorphisms of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$.

Theorem 1.20 (cf. [32] Theorem 5.13)**.** *For each $\mu \in P_+$, the line bundle twist by $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\mu)$ preserves the spaces $\widetilde{K}'(\mathbf{Q}_G)$. In other words, for each $w \in W_{\mathrm{af}}$ such that $w \le_{\frac{\infty}{2}} e$, there exists a collection $\{a_w^v(\mu)\}_{v \le_{\frac{\infty}{2}} w}$ of elements in $\mathbb{Z}[q^{-1}]P$ such that the function on P*

$$\lambda \mapsto \left(\chi_q(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda + \mu)) - \sum_v a_w^v(\mu) \cdot \chi_q(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(v)}(\lambda)) \right)$$

belongs to $\mathrm{Fun}_P^{\mathrm{neg}}$. In particular, we have

$$[\mathcal{O}_{\mathbf{Q}_G(w)}(\mu)] = \sum_{v \le_{\frac{\infty}{2}} w} a_w^v(\mu)[\mathcal{O}_{\mathbf{Q}_G(v)}] \in \widetilde{K}'(\mathbf{Q}_G).$$

Proof. Since the tensor product operation (= shift of functions on P) commutes with each other, we concentrate into the case $\mu = \pm\varpi_i$ for $i \in \mathtt{I}$.

We consider the case $\mu = \varpi_i$. In the Pieri-Chevalley rule [32, Theorem 5.13], the coefficients $\{a_w^v(\mu)\}_{v \le_{\frac{\infty}{2}} w}$ is given by counting the set of paths with fixed initial/final directions. The q-degrees of paths whose initial/final directions of paths are bounded from xt_β and yt_γ $(x, y \in W, \beta, \gamma \in Q_+^\vee)$ must belong to $[\langle \beta, w_0\varpi_i \rangle, \langle \gamma, w_0\varpi_i \rangle]$ by our count of q-degrees in [32]. In particular, we find $a_w^v(\varpi_i) \in \mathbb{C}[q^{-1}]P$ for each $w \in W_{\mathrm{af}}$. Thus, the assertion for $\mu = \varpi_i$ is precisely the contents of the proof of [32, Theorem 5.13].

Moreover, the set of paths with the same initial/final direction is unique (see [32, Definition 2.6]), and hence the transition matrix between $\{[\mathcal{O}_{\mathbf{Q}_G(w)}(\varpi_i)]\}_{w \in W_{\mathrm{af}}}$ and $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w \in W_{\mathrm{af}}}$ is unitriangular (up to diagonal matrix consisting of characters in P) with respect to $\le_{\frac{\infty}{2}}$. Therefore, we can invert this matrix to obtain $[\mathcal{O}_{\mathbf{Q}_G(w)}(-\varpi_i)] \in \widetilde{K}'(\mathbf{Q}_G)$ for $i \in \mathtt{I}$. Thus, we completed the proof of the assertion for $\mu = \pm\varpi_i$ $(i \in \mathtt{I})$ as required. □

Theorem 1.21 (cf. [32] Theorem 6.5 see also [26])**.** *For each $\lambda \in P$, the $\mathbb{C}P$-linear extension of the assignment*

$$[\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}}) \quad w \in W_{\mathrm{af}}$$

defines a $\mathcal{H}$-module automorphism (that we call $\Xi(\lambda)$) which commutes with the right $Q^\vee$-action. Moreover, we have $\Xi(\lambda) \circ \Xi(\mu) = \Xi(\lambda + \mu)$ for $\lambda, \mu \in P$.

Proof. The first assertion follows from Theorem 1.20, as well as [32, Proposition 6.3]. The latter assertion follows as the effect of $\Xi(\lambda)$ is just to shift functions on P. $\square$

Lemma 1.22. *For each $i \in \mathtt{I}$, we have a short exact sequence*

$$0 \to \mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathbf{Q}_G}(-\varpi_i) \to \mathcal{O}_{\mathbf{Q}_G} \to \mathcal{O}_{\mathbf{Q}_G(s_i)} \to 0, \tag{1.13}$$

that is $\mathbb{G}_m \ltimes \mathbf{I}$-equivariant.

Proof. Let $\mathbb{O}$ be the dense open $G[\![z]\!]$-orbit in $\mathbf{Q}_G$. We have $\mathbb{O} = \bigsqcup_{w \in W} \mathbb{O}(w)$ by Theorem 1.10 and the Bruhat decomposition. In particular, $\mathbb{O}$ yields an (uncountable dimensional) $G[\![z]\!]$-equivariant affine fibration over $\mathcal{B}$ by setting $z = 0$. We lift (1.4) by pulling back to obtain (1.13) on $\mathbb{O}$. Twisting by $\mathcal{O}_{\mathbb{O}}(\varpi_i)$, we can interpret the map $\mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathbf{Q}_G} \to \mathcal{O}_{\mathbf{Q}_G}(\varpi_i)$ as an unique (up to scalar) $(\mathbb{G}_m \ltimes \mathbf{I})$-equivariant section of $(\mathbb{G}_m \times H)$-weight ϖ_i in $\Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\varpi_i))$ as it restricts to an unique map (up to scalar) in (1.4) by restriction. As a consequence, the short exact sequence (1.4) yields the short exact sequence (1.13) if the natural ($\mathbb{G}_m \ltimes \mathbf{I}$-equivariant) inclusion

$$\mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathbf{Q}_G}(-\varpi_i) \hookrightarrow \ker(\mathcal{O}_{\mathbf{Q}_G} \to \mathcal{O}_{\mathbf{Q}_G(s_i)}) \tag{1.14}$$

is an isomorphism. We set $\mathcal{K} := \ker(\mathcal{O}_{\mathbf{Q}_G} \to \mathcal{O}_{\mathbf{Q}_G(s_i)})$.

Consider the map $\pi_i : P_i \times^B \mathbf{Q}_G(s_i) \to \mathbf{Q}_G$. We have a line bundle $\mathcal{O}(-1)$ on $SL(2,i) \times^{B_i} \mathbf{Q}_G(s_i)$ obtained as the pullback of $\mathcal{O}_{\mathbb{P}^1}(-D)$ through

$$P_i \times^B \mathbf{Q}_G(s_i) \to P_i/B \cong \mathbb{P}^1,$$

where D is the point $B/B \in \mathbb{P}^1$. The sheaf $\mathcal{O}_{\mathbb{P}^1}(-D)$ (and hence $\mathcal{O}(-1)$) is B-equivariant and admits a P_i-linearization after twisted by the H-character $\mathbb{C}_{\varpi_i}$ by (1.4) for $SL(2,i)$. Let infl be the functor that inflates a B-equivariant sheaf on $\mathbf{Q}_G(s_i)$ to a P_i-equivariant sheaf on $P_i \times^B \mathbf{Q}_G(s_i)$.

By the Demazure character formula ([26, Theorem A], transported to this setting in [32]), we find that the definition of $\mathcal{K}$ is interpreted as

$$\mathbb{R}^\bullet(\pi_i)_* \left(\mathcal{O}(-1) \otimes \mathsf{infl}\left(\mathcal{O}_{\mathbf{Q}_G(s_i)}\right)\right).$$

In particular, its twist by $\mathbb{C}_{\varpi_i}$ acquires the P_i-equivariant structure. In addition, this procedure commutes with the $G(\!(z)\!)$-equivariant line bundle twist of $\mathbf{Q}_G^{\mathrm{rat}}$ (as presented in [32, §6], cf. Theorem 1.19). Therefore, we conclude that

$$\mathbb{C}_{-\varpi_i} \otimes H^m(\mathbf{Q}_G, \mathcal{K}(\lambda)) \qquad \lambda \in P \tag{1.15}$$

admits an action of P_i. Since

$$P_i \times^B \mathbf{Q}_G(s_i) \cong (\mathbf{I} \cdot P_i) \times^{\mathbf{I}} \mathbf{Q}_G(s_i),$$

we deduce that (1.15) admits a $(P_i\mathbf{I}')$-action that prolongs the P_i-action.

By [32, Corollary 4.30], we have an inclusion

$$\Gamma(\mathbf{Q}_G(s_i), \mathcal{O}_{\mathbf{Q}_G(s_i)}(\lambda))^\vee \hookrightarrow \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda))^\vee \qquad \lambda \in P \tag{1.16}$$

as $\mathbf{I}'$-modules, and the RHS has a cyclic vector with $(\mathbb{G}_m \times H)$-weight $-\lambda$. We set

$$K(\lambda) := \mathbb{C}_{\varpi_i} \otimes \Gamma(\mathcal{O}_{\mathbf{Q}_G}, \mathcal{K}(\lambda))^\vee \cong \mathbb{C}_{\varpi_i} \otimes \frac{\Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda))^\vee}{\Gamma(\mathbf{Q}_G(s_i), \mathcal{O}_{\mathbf{Q}_G(s_i)}(\lambda))^\vee}.$$

We have a surjection

$$\theta_\lambda : K(\lambda) \longrightarrow\!\!\!\!\rightarrow \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda - \varpi_i))^\vee$$

and (1.14) is an isomorphism if this is an isomorphism for every $\lambda \in P_+$. Since the action of $SL(2, j)$ ($j \neq i \in \mathtt{I}$) commutes with $\mathbb{C}_{\varpi_i} \otimes \bullet$ and preserves $\mathbf{Q}_G(s_i)$, we find a P_j-action on $K(\lambda)$ from (1.16), that coincide with the P_i-action along the intersection $B = P_i \cap P_j$. This particularly implies that $K(\lambda)$ is invariant under the Demazure functor for each $i \in \mathtt{I}$, and hence it acquires the G-action ([25]). Therefore, $K(\lambda)$ admits a G-action, that is upgraded into a G-integrable $G[z]$-module structure.

Being a quotient of $\mathbb{C}_{\varpi_i} \otimes \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda))^\vee$, the $G[z]$-module $K(\lambda)$ is generated by a cyclic vector with $(\mathbb{G}_m \times H)$-weight $(\varpi_i - \lambda)$ by the action of $\mathbf{I}'$, and if a H-weight $-\mu$ appears in $K(\lambda)$, then we have $\mu \le \lambda - \varpi_i$. It follows that

$$\operatorname{gch} K(\lambda) \in \sum_{\mu \le \lambda - \varpi_i} \mathbb{Z}[\![q^{-1}]\!] \cdot \operatorname{ch} L(\mu)^*. \tag{1.17}$$

By Theorem 1.12 3), the $G[z]$-module $\Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda - \varpi_i))^\vee$ is the largest one generated by a cyclic vector with $(\mathbb{G}_m \times H)$-weight $-(\lambda - \varpi_i)$ that satisfies the same condition as $\operatorname{gch} K(\lambda)$ in (1.17). Therefore, we conclude that θ_λ must be an isomorphism for every $\lambda \in P_+$.

This in turn yields that (1.14) is an isomorphism as desired. □

Corollary 1.23. *For each $i \in \mathtt{I}$, we have an equality*

$$[\mathcal{O}_{\mathbf{Q}_G(s_i)}] = [\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)]$$

inside $\widetilde{K}'(\mathbf{Q}_G)$. In particular, it also holds for $K_H(\mathbf{Q}_G)$.

Proof. Apply Lemma 1.15 to Lemma 1.22 by $[\mathcal{O}_{\mathbf{Q}_G(s_i)}], [\mathcal{O}_{\mathbf{Q}_G(e)}] \in \widetilde{K}'(\mathbf{Q}_G)$. □

Remark 1.24. Corollary 1.23 implies $[\mathcal{O}_{\mathbf{Q}_G}(-\varpi_i)] \in K_H(\mathbf{Q}_G) \subset K_H(\mathbf{Q}_G^{\mathrm{rat}})$. However, this does not imply $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}})$ as $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}] \notin K_H(\mathbf{Q}_G^{\mathrm{rat}})$. By the same reason, $\Xi(\lambda)$ in Theorem 1.21 (= tensor product with $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$) is the multiplication by an element in the ring $K_H(\mathbf{Q}_G)$, but not in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In fact, the definition of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ does not equip it with a ring structure. This stems from the fact that $\mathbf{Q}_G^{\mathrm{rat}}$ is a(n infinite) union of Schubert varieties (whose dimensions are also infinite), but not a Schubert variety by itself.

Motivated by Corollary 1.23, we consider a $\mathbb{C}P$-module endomorphism H_i ($i \in \mathtt{I}$) of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ as:

$$H_i : [\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(w)}(-\varpi_i)] \qquad w \in W_{\mathrm{af}}.$$

Proposition 1.25. *The space $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is topologically generated by $[\mathcal{O}_{\mathbf{Q}_G}(\lambda)]$ ($\lambda \in P$), together with the $\mathbb{C}P$-multiplications and the right $Q^\vee$-actions.*

Proof. Let $K_H(\mathbf{Q}_G)_+$ be the (formal) $\mathbb{C}P$-span of $\{[\mathcal{O}_{\mathbf{Q}_G(wt_\beta)}]\}_{w \in W, 0 \neq \beta \in Q^\vee_+}$ in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In view of (1.2), we have $K_H(\mathbf{Q}_G)/K_H(\mathbf{Q}_G(e))_+ \cong K_H(\mathcal{B})$ as $\mathbb{C}P$-modules that sends $[\mathcal{O}_{\mathbf{Q}_G(w)}]$ to $[\mathcal{O}_{\mathcal{B}(w)}]$ ($w \in W$). By Theorem 1.19 and Theorem 1.4, this intertwines the action of D_i ($i \in \mathtt{I}$). By the Pieri-Chevalley formula [32, Theorem 5.13], we see that

$$[\mathcal{O}_{\mathbf{Q}(w_0)}(\lambda)] \mod K_H(\mathbf{Q}_G)_+ = e^{w_0\lambda}[\mathcal{O}_{\mathcal{B}(w_0)}] = [\mathcal{O}_{\mathcal{B}(w_0)}(\lambda)] \qquad \lambda \in P.$$

By the Demazure character formulas ([26] and [38, VIII]), we conclude that

$$[\mathcal{O}_{\mathbf{Q}(w)}(\lambda)] \mod K_H(\mathbf{Q}_G)_+ = [\mathcal{O}_{\mathcal{B}(w)}(\lambda)] \quad w \in W, \lambda \in P.$$

Therefore, the first two actions generate $K_H(\mathbf{Q}_G)/K_H(\mathbf{Q}_G)_+ \cong K_H(\mathcal{B})$ from $[\mathcal{O}_{\mathbf{Q}_G}]$. Now we use the right $Q^\vee$-action to conclude the result. □

1.6 Graph and map spaces

We refer [36, 17, 20] for the precise definitions of the notions appearing in this subsection.

We have W-equivariant isomorphisms $H^2(\mathcal{B}, \mathbb{Z}) \cong P$ and $H_2(\mathcal{B}, \mathbb{Z}) \cong Q^\vee$. This identifies the (integral points of the) nef cone of $\mathcal{B}$ with $P_+ \subset P$ and the effective cone of $\mathcal{B}$ with $Q^\vee_+$. For each non-negative integer n and $\beta \in Q^\vee_+$, we set $\mathcal{GB}_{n,\beta}$ to be the space of stable maps of genus zero curves with n-marked points to $(\mathbb{P}^1 \times \mathcal{B})$ of bidegree $(1, \beta)$, that is also called the graph space of $\mathcal{B}$. A point of $\mathcal{GB}_{n,\beta}$ is a(n arithmetic) genus zero curve C with n-marked points $\{x_1, \ldots, x_n\}$, together with a map to $\mathbb{P}^1$ of degree one. Hence, we have a unique $\mathbb{P}^1$-component of C that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of C and denote it by C_0. By discarding the map to $\mathbb{P}^1$, we obtain the space of stable maps $\mathcal{B}_{n,\beta}$ of genus zero curves with n-marked points to $\mathcal{B}$ of degree β, together with the natural projection map $\mathtt{f} : \mathcal{GB}_{n,\beta} \longrightarrow \mathcal{B}_{n,\beta}$. The spaces $\mathcal{GB}_{n,\beta}$ and $\mathcal{B}_{n,\beta}$ are normal projective varieties by [17, Theorem 2] that have at worst quotient singularities arising from the automorphism of stable maps. The natural $(\mathbb{G}_m \times H)$-action on $(\mathbb{P}^1 \times \mathcal{B})$ induces a natural $(\mathbb{G}_m \times H)$-action on $\mathcal{GB}_{n,\beta}$.

Let $\widetilde{\mathtt{ev}}_j : \mathcal{GB}_{n,\beta} \to \mathbb{P}^1 \times \mathcal{B}$ $(1 \le j \le n)$ be the evaluation at the j-th marked point, and let $\mathtt{ev}_j : \mathcal{GB}_{n,\beta} \to \mathcal{B}$ be its composition with the second projection. For a $(\mathbb{G}_m \times H)$-equivariant coherent sheaf $\mathcal{F}$ on a projective $(\mathbb{G}_m \times H)$-variety $\mathcal{X}$, let $\chi_q(\mathcal{X}, \mathcal{F}) \in \mathbb{C}[q^{\pm 1}]P$ denote its $(\mathbb{G}_m \times H)$-equivariant Euler-Poincaré characteristic.

With these notation, we define the q-deformed n-point equivariant K-theoretic Gromov-Witten correlation function for $\xi_1, \ldots, \xi_n \in K_H(\mathcal{B})$ as:

$$\langle \xi_1, \ldots, \xi_n \rangle^q_{\mathtt{GW}} := \sum_{\beta \in Q^\vee_+} Q^\beta \chi_q(\mathcal{GB}_{n,\beta}, \bigotimes_{j=1}^n \mathtt{ev}_j^* \xi_j) \in (\mathbb{C}[q^{\pm 1}]P)[\![Q^\vee_+]\!], \tag{1.18}$$

where we regard $\xi_1, \ldots, \xi_n$ as $\mathbb{G}_m$-equivariant objects with trivial $\mathbb{G}_m$-action.

The following result is well-known:

Theorem 1.26. *The $q = 1$ specialization of the q-deformed n-point equivariant K-theoretic Gromov-Witten correlation function is the usual n-point H-equivariant K-theoretic Gromov-Witten correlation function calculated by replacing $\mathcal{GB}_{n,\beta}$ $(0 \neq \beta \in Q^\vee_+)$ in* (1.18) *with $\mathcal{B}_{n,\beta}$.*

Sketch of proof. By adjunction, this comparison follows if we have $\mathbb{R}^\bullet \mathtt{f}_* \mathcal{O}_{\mathcal{GB}_{n,\beta}} \cong \mathcal{B}_{n,\beta}$. The latter fact can be deduced from [34, Theorem 7.1] since the both spaces are normal with at worst rational singularities (cf. Remark 4.1), $\mathtt{f}$ is projective with connected fibers, and the general fiber of $\mathtt{f}$ is $\mathbb{P}^3 = \overline{PGL(2, \mathbb{C})}$. □

1.7 Equivariant quantum K-group of $\mathcal{B}$

Consider the formal power series ring $\mathbb{C}[\![Q_+^\vee]\!]$ with its variables $Q_i = Q^{\alpha_i^\vee}$ $(i \in \mathtt{I})$. We set $Q^\beta := \prod_{i \in \mathtt{I}} Q_i^{\langle \beta, \varpi_i \rangle}$ for each $\beta \in Q^\vee$. We define the H-equivariant (small) quantum K-group of $\mathcal{B}$ as:

$$qK_H(\mathcal{B}) := K_H(\mathcal{B})[\![Q_+^\vee]\!], \tag{1.19}$$

that contains $\mathbb{C}[\![Q_+^\vee]\!][\mathcal{O}_{\mathcal{B}}] \cong \mathbb{C}[\![Q_+^\vee]\!]$. Thanks to (the H-equivariant versions of) [19, 43], it is equipped with the commutative and associative product $\star$ (called the quantum multiplication) characterized as:

$$\langle \xi_1 \star \xi_2, \xi_3 \rangle^q_{\mathtt{GW}}|_{q=1} = \langle \xi_1, \xi_2, \xi_3 \rangle^q_{\mathtt{GW}}|_{q=1} \qquad \xi_1, \xi_2, \xi_3 \in qK_H(\mathcal{B}),$$

where the forms on the both sides are understood to be linear with respect to $\mathbb{C}[\![Q_+^\vee]\!]$. The product $\star$ satisfies the following properties:

1. the element $[\mathcal{O}_{\mathcal{B}}] = [\mathcal{O}_{\mathcal{B}}]Q^0 \in qK_H(\mathcal{B})$ is the identity;
2. the map $Q^\beta \star$ $(\beta \in Q_+^\vee)$ is the multiplication of Q^β in the RHS of (1.19);
3. we have $\xi \star \eta \equiv \xi \cdot \eta \mod (Q_i; i \in \mathtt{I})$ for every $\xi, \eta \in K_H(\mathcal{B}) \otimes 1$.

From the above properties, we can localize $qK_H(\mathcal{B})$ with respect to the multiplicative system $\{Q^\beta\}_{\beta \in Q_+^\vee}$ to obtain a ring $qK_H(\mathcal{B})_{\mathrm{loc}}$.

We set

$$qK_{\mathbb{G}_m \times H}(\mathcal{B}) := \left(K_H(\mathcal{B})(\!(q^{-1})\!)\right)[\![Q_+^\vee]\!].$$

We sometimes identify $K_H(\mathcal{B})$ with the submodule of $qK_H(\mathcal{B})$ or $qK_{\mathbb{G}_m \times H}(\mathcal{B})$ that is constant with respect to Q_i $(i \in \mathtt{I})$ and q. We set $p_i := [\mathcal{O}_{\mathcal{B}}(\varpi_i)]$ for $i \in \mathtt{I}$, and we consider it as an endomorphism of $qK_H(\mathcal{B})$ or $qK_{\mathbb{G}_m \times H}(\mathcal{B})$ through the scalar extension of the product of $K_H(\mathcal{B})$ (i.e. the classical product; we always understand that $p_i^{\pm 1}$ acts via the classical product). For each $i \in \mathtt{I}$, let $q^{Q_i \partial_{Q_i}}$ denote the $(\mathbb{C}P)(\!(q^{-1})\!)$-endomorphism of $qK_{\mathbb{G}_m \times H}(\mathcal{B})$ such that

$$q^{Q_i \partial_{Q_i}}(\xi \otimes Q^\beta) = q^{\langle \beta, \varpi_i \rangle} \xi \otimes Q^\beta \qquad \xi \in K_H(\mathcal{B}), \beta \in Q_+^\vee.$$

Following [24, §2.4], we consider the operator $T \in \mathrm{End}_{(\mathbb{C}P)(\!(q^{-1})\!)}\, qK_{\mathbb{G}_m \times H}(\mathcal{B})$ (obtained from the same named operator in [24] by setting $0 = t \in K(\mathcal{B})$). Then, we have the shift operator (also obtained from an operator $A_i(q,t)$ in [24] by setting $t = 0$) defined by

$$A_i(q) = T^{-1} \circ p_i^{-1} q^{Q_i \partial_{Q_i}} \circ T \in \mathrm{End}\, qK_{\mathbb{G}_m \times H}(\mathcal{B}) \qquad i \in \mathtt{I}. \tag{1.20}$$

An element $J(Q,q) := T([\mathcal{O}_{\mathcal{B}}]) \in qK_{\mathbb{G}_m \times H}(\mathcal{B})$ is called the (equivariant K-theoretic) small quantum J-function, and is computed in [20, 6] (cf. Theorem 3.8).

Theorem 1.27 (Givental and Lee [20, 19, 43]). *For each $\xi, \zeta \in K_H(\mathcal{B})$, we have*

$$\chi(T(\xi) \cdot \overline{T}(\zeta)) = \langle \xi, \zeta \rangle^q_{\mathtt{GW}} \in (\mathbb{C}[q^{\pm 1}]P)[\![Q_+^\vee]\!]. \tag{1.21}$$

Here we warn that the operators $\star$, T, and the q-conjugate $\overline{T}$ of T introduces variables q and Q^β $(\beta \in Q_+^\vee)$ in the calculation of the LHS, and they are understood to be scalars.

Remark 1.28. The operator T intertwines the classical and "q-deformed" quantum inner products if we replace $\overline{T}(\zeta)$ with $\overline{T(\zeta)}$ in (1.21). However, the operator T usually has a pole at $q = 1$, and hence it does not induce an intertwiner between the classical and the quantum inner products naively.

Theorem 1.29 (Reconstruction theorem [24] Proposition 2.20)**.** *For each*

$$f(q, x_1, \ldots, x_r, Q) \in (\mathbb{C}P[q^{\pm 1}, x_1, \ldots, x_r])[\![Q_+^\vee]\!],$$

we have the following equivalence:

$$\begin{aligned} f(q, p_1^{-1} q^{Q_1 \partial_{Q_1}}, \ldots, p_r^{-1} q^{Q_r \partial_{Q_r}}, Q) J(Q, q) = 0 \in qK_{\mathbb{G}_m \times H}(\mathcal{B}) \\ \Leftrightarrow f(q, A_1(q), \ldots, A_r(q), Q)[\mathcal{O}_\mathcal{B}] = 0 \in qK_{\mathbb{G}_m \times H}(\mathcal{B}). \end{aligned}$$

Remark 1.30. The original form of Theorem 1.29 is about big quantum K-group. We have made the specialization $t = 0$ to deduce our form. It should be noted that **1)** this equivariant setting is automatic from the construction, and **2)** we state Theorem 1.29 for unmodified quantum J-function instead of the modified one employed in [24, Proposition 2.20].

For each $i \in \mathtt{I}$, we set $a_i := A(1)$ (thanks to [24, Remark 2.14]).

Theorem 1.31 ([24] Corollary 2.9)**.** *For $i \in \mathtt{I}$, the operator a_i defines the quantum multiplication by $a_i([\mathcal{O}_\mathcal{B}])$ in $qK_H(\mathcal{B})$.*

Proof. By [24, Corollary 2.9], the set $\{a_i\}_{i \in \mathtt{I}}$ defines mutually commutative endomorphisms of $qK_H(\mathcal{B})$ that commutes with the $\star$-multiplication. Since $\mathrm{End}_R R \cong R$ for every ring R, we conclude the assertion. ☐

Corollary 1.32. *For $i \in \mathtt{I}$ and $\xi \in K_H(\mathcal{B})$, we have*

$$a_i(\xi) \equiv [\mathcal{O}_\mathcal{B}(-\varpi_i)] \cdot \xi \equiv [\mathcal{O}_\mathcal{B}(-\varpi_i)] \star \xi \mod (Q_i; i \in \mathtt{I}).$$

Proof. The first equality follows from [24, Proposition 2.10]. The second equality is a property of quantum K-theoretic product listed in the above. ☐

Theorem 1.33 (Anderson-Chen-Tseng [2] Lemma 6, see also [1])**.** *For each $i \in \mathtt{I}$, we have $A_i(q)([\mathcal{O}_\mathcal{B}]) = [\mathcal{O}_\mathcal{B}(-\varpi_i)] \in qK_H(\mathcal{B}) \subset qK_{\mathbb{G}_m \times H}(\mathcal{B})$.*

We give an alternative proof of Theorem 1.33 in §4.5.

2 Relation with affine Grassmannians

We work in the same settings as in the previous section.

2.1 Transporting the $\mathcal{H}$-action to $\mathcal{C}$

Proposition 2.1. *The $\mathcal{H}$-action of $K_H(\mathrm{Gr})$ induces a $\mathcal{H}$-action on $\mathcal{C}$ as:*

$$\begin{aligned} D_0(f \otimes t_\beta) &= \frac{f}{1 - e^{-\vartheta}} \otimes t_\beta - \frac{e^{-\vartheta} s_\vartheta(f)}{1 - e^{-\vartheta}} \otimes t_{s_\vartheta(\beta - \vartheta^\vee)} & f \in \mathbb{C}(P) \\ D_i(f \otimes t_\beta) &= \frac{f}{1 - e^{\alpha_i}} \otimes t_\beta - \frac{e^{\alpha_i} s_i(f)}{1 - e^{\alpha_i}} \otimes t_{s_i \beta} & i \in \mathtt{I}, \beta \in Q^\vee \\ e^\mu(f \otimes t_\beta) &= e^\mu f \otimes t_\beta & \mu \in P. \end{aligned}$$

Proof. For $i \in \mathtt{I}_{\mathrm{af}}$, the action of D_i on $\mathcal{A}$ is the left multiplication of $\frac{1}{1-e^{\alpha_i}} \otimes 1 - \frac{e^{\alpha_i}}{1-e^{\alpha_i}} \otimes s_i$ (if we understand $\alpha_0 = -\vartheta$). Applying to an element $f \otimes t_\beta u \in \mathcal{A}$ ($f \in \mathbb{C}(P), \beta \in Q^\vee, u \in W$), we deduce

$$\begin{aligned} D_i(f \otimes t_\beta u) &= \frac{f}{1-e^{\alpha_i}} \otimes t_\beta u - \frac{e^{\alpha_i} s_i(f)}{1-e^{\alpha_i}} \otimes t_{s_i\beta} s_i u \qquad i \neq 0 \\ D_0(f \otimes t_\beta u) &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes s_0 t_\beta u \\ &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes s_\vartheta t_{-\vartheta^\vee} t_\beta u \\ &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes t_{s_\vartheta(\beta-\vartheta^\vee)} s_\vartheta u. \end{aligned}$$

Hence, applying pr yields the desired formula on D_i for $i \in \mathtt{I}_{\mathrm{af}}$. Together with the left multiplication of $e^\lambda \otimes 1$, these formulae transplant the $\mathcal{H}$-action from $K_H(\mathrm{Gr})$ to $K_H(\mathrm{Gr}) \cap \mathcal{C}$.

Since $K_H(\mathrm{Gr}) = \mathcal{C} \cap K_H(X)$, we have $\mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr}) \subset \mathcal{C}$. By comparing the leading terms of $\{[\mathrm{Gr}_\beta]\}_{\beta \in Q^\vee} \subset \mathcal{C}$ with respect to the Bruhat order (on the second component of $\mathcal{C} \subset \mathcal{A} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}}$), we derive $\mathcal{C} \subset \mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr})$. It follows that $\mathcal{C} = \mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr})$. Hence, the above formulas define the $\mathcal{H}$-action on $\mathcal{C}$ as the scalar extension of that on $K_H(\mathrm{Gr}) \subset \mathcal{C}$ as required (one can also directly check the relations of $\mathcal{H}$). ☐

Below, we may write the action of D_i on $\mathcal{C}$ by $D_i^\#$ to distinguish with the action on $K_H(X)$ or $\mathcal{A}$.

Corollary 2.2. *Let $i \in \mathtt{I}$. Let $\xi \in \mathcal{C}$ be an element such that $D_i^\#(\xi) = \xi$. Then, ξ is a $\mathbb{C}$-linear combination of*

$$f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta} \qquad f \in \mathbb{C}(P), \beta \in Q^\vee.$$

Proof. By Proposition 2.1, the action of $D_i^\#$ preserves $\mathbb{C}(P) \otimes t_\beta + \mathbb{C}(P) \otimes t_{s_i\beta}$ for each $i \in \mathtt{I}$ and $\beta \in Q^\vee$. Hence, it suffices to find a condition that $a \otimes t_\beta + b \otimes t_{s_i\beta}$ ($a, b \in \mathbb{C}(P)$) is stable by the action of $D_i^\#$. It reads as:

$$\begin{aligned} D_i^\#(a \otimes t_\beta + b \otimes t_{s_i\beta}) &= \frac{a - e^{\alpha_i} s_i(b)}{1-e^{\alpha_i}} \otimes t_\beta + \frac{b - e^{\alpha_i} s_i(a)}{1-e^{\alpha_i}} \otimes t_{s_i\beta} \\ &= a \otimes t_\beta + b \otimes t_{s_i\beta}. \end{aligned}$$

This is equivalent to $b = s_i(a)$ (or $s_i(a+b) = a+b$ in the case of $s_i\beta = \beta$) as required. ☐

Corollary 2.3. *Let $i \in \mathtt{I}$. Let $\xi, \xi' \in \mathcal{C}$ be elements such that $D_i^\#(\xi) = \xi$. We have*

$$D_i^\#(\xi\xi') = \xi D_i^\#(\xi').$$

Proof. By Corollary 2.2, it suffices to prove

$$D_i^\#((f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta}) g \otimes t_\gamma) = (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta}) D_i^\#(g \otimes t_\gamma)$$

for every $f, g \in \mathbb{C}(P)$ and $\beta, \gamma \in Q^\vee$. We derive as:

$$
\begin{aligned}
D_i^{\#}((f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})g \otimes t_\gamma) &= D_i^{\#}(fg \otimes t_{\beta+\gamma} + s_i(f)g \otimes t_{s_i\beta+\gamma}) \\
&= \frac{fg}{1-e^{\alpha_i}} \otimes t_{\beta+\gamma} - \frac{e^{\alpha_i}s_i(fg)}{1-e^{\alpha_i}} \otimes t_{s_i\beta+s_i\gamma} \\
&+ \frac{s_i(f)g}{1-e^{\alpha_i}} \otimes t_{s_i\beta+\gamma} - \frac{e^{\alpha_i}fs_i(g)}{1-e^{\alpha_i}} \otimes t_{\beta+s_i\gamma} \\
&= (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})(\frac{g}{1-e^{\alpha_i}} \otimes t_\gamma - \frac{e^{\alpha_i}s_i(g)}{1-e^{\alpha_i}} \otimes t_{s_i\gamma}) \\
&= (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})D_i^{\#}(g \otimes t_\gamma).
\end{aligned}
$$

This completes the proof. □

Lemma 2.4. *Let $\xi, \xi' \in \mathcal{C}$ be elements such that $D_i^{\#}(\xi) = \xi$ for every $i \in \mathtt{I}$. We have*

$$D_0^{\#}(\xi\xi') = \xi D_0^{\#}(\xi').$$

Proof. By Corollary 2.2, we deduce $w\xi w^{-1} = \xi \in \mathcal{A}$ for every $w \in W$. In particular, we have $s_\vartheta \xi s_\vartheta = \xi$.

Therefore, it suffices to prove

$$D_0^{\#}((f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})g \otimes t_\gamma) = (f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})D_0^{\#}(g \otimes t_\gamma)$$

for every $f, g \in \mathbb{C}(P)$ and $\beta, \gamma \in Q^\vee$. We derive as:

$$
\begin{aligned}
D_0^{\#}((f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})g \otimes t_\gamma) &= D_0^{\#}(fg \otimes t_{\beta+\gamma} + s_\vartheta(f)g \otimes t_{s_\vartheta\beta+\gamma}) \\
&= \frac{fg}{1-e^{-\vartheta}} \otimes t_{\beta+\gamma} - \frac{e^{-\vartheta}s_\vartheta(fg)}{1-e^{-\vartheta}} \otimes t_{s_\vartheta\beta+s_\vartheta(\gamma-\vartheta^\vee)} \\
&+ \frac{s_\vartheta(f)g}{1-e^{-\vartheta}} \otimes t_{s_\vartheta\beta+\gamma} - \frac{e^{-\vartheta}fs_\vartheta(g)}{1-e^{-\vartheta}} \otimes t_{\beta+s_\vartheta(\gamma-\vartheta^\vee)} \\
&= (f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})D_0^{\#}(g \otimes t_\gamma).
\end{aligned}
$$

This completes the proof. □

Theorem 2.5. *For each $\beta \in Q^\vee_<$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have*

$$D_i([\mathcal{O}_{\mathrm{Gr}_\beta}] \odot \xi) = [\mathcal{O}_{\mathrm{Gr}_\beta}] \odot D_i(\xi) \qquad \xi \in K_H(\mathrm{Gr}).$$

Proof. By construction, we have

$$\pi^*([\mathcal{O}_{\mathrm{Gr}_\beta}]) = D_{t_\beta}D_{w_0} = D_iD_{t_\beta}D_{w_0} \qquad i \in \mathtt{I},$$

where the second identity follows from $\ell(t_\beta) = \ell(s_it_\beta) + 1$. By Proposition 2.1, we deduce that $r^*([\mathcal{O}_{\mathrm{Gr}_\beta}])$ satisfies the $D_i^{\#}$-invariance for each $i \in \mathtt{I}$. Therefore, Corollaries 2.3 and 2.4 imply the result. □

Corollary 2.6. *For $\beta \in Q^\vee_<$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have $D_i = \mathtt{t}_{-\beta} \circ D_i \circ \mathtt{t}_\beta$. In particular, we have a natural extension of the $\mathcal{H}$-action from $K_H(\mathrm{Gr})$ to $K_H(\mathrm{Gr})_{\mathrm{loc}}$.*

Proof. The first assertion is a direct consequence of Theorem 2.5. As we have $K_H(\mathrm{Gr})_{\mathrm{loc}} = K_H(\mathrm{Gr})[\mathtt{t}_\beta \mid \beta \in Q^\vee_<]$, the latter assertion follows. □

2.2 Inclusion as $\mathcal{H}$-modules

Lemma 2.7. *Let $i \in \mathtt{I}_{\mathrm{af}}$. For each $w \in W_{\mathrm{af}}^{-}$, we have*

$$D_i([\mathcal{O}_{\mathrm{Gr}_w}]) = \begin{cases} [\mathcal{O}_{\mathrm{Gr}_{s_i w}}] & (s_i w >_{\frac{\infty}{2}} w) \\ [\mathcal{O}_{\mathrm{Gr}_w}] & (s_i w <_{\frac{\infty}{2}} w) \end{cases}.$$

Proof. By Theorem 1.7 and Corollary 2.6, we can replace $[\mathcal{O}_{\mathrm{Gr}_w}]$ with $[\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}]$ for $\beta \in Q^\vee$ such that $\langle \beta, \varpi_i \rangle \ll 0$ for all $i \in \mathtt{I}$. Therefore, the assertion is a rephrasement of Theorem 1.4 and Theorem 1.5 as $s_i w >_{\frac{\infty}{2}} w$ is equivalent to $s_i w t_\beta > w t_\beta$ (see (1.1)). □

Lemma 2.8. *The vector space $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is a cyclic module with respect to the action of $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$ with its cyclic vector $[\mathcal{O}_{\mathrm{Gr}_0}]$.*

Proof. By construction, it suffices to find every $\{[\mathcal{O}_{\mathrm{Gr}_\beta}]\}_{\beta \in Q^\vee}$ in the linear span of $\mathcal{H} \cdot \{\mathtt{t}_\gamma \odot [\mathcal{O}_{\mathrm{Gr}_0}]\}_{\gamma \in Q^\vee}$. This follows from a repeated application of the actions of $\{D_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ and Theorem 1.7 (cf. [26, Theorem 4.6]). □

Corollary 2.9. *An endomorphism of $K_H(\mathrm{Gr})_{\mathrm{loc}}$ as a $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-module is completely determined by the image of $[\mathcal{O}_{\mathrm{Gr}_0}]$.* □

Proposition 2.10. *By sending $[\mathcal{O}_{\mathrm{Gr}_0}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(e)}]$, we have a unique injective $\mathcal{H}$-module morphism*

$$K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

such that twisting by $\mathtt{t}_\beta$ corresponds to the right action of $\beta \in Q^\vee$. This map particularly gives

$$[\mathcal{O}_{\mathrm{Gr}_{ut_\beta}}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}] \qquad u \in W, \beta \in Q^\vee_<.$$

Proof. The comparison of the D_i-actions on the basis elements in Lemma 2.7 and Theorem 1.19 implies that we indeed obtain a $\mathcal{H}$-module inclusion, by enhancing the assignment $[\mathcal{O}_{\mathrm{Gr}_{ut_\beta}}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}]$ for $u \in W, \beta \in Q^\vee_<$ into a $\mathbb{C}P$-module homomorphism. We know the actions of $\mathtt{t}_\beta$ and β on the both sides by Theorem 1.7 and (1.12), that coincide on elements that generates $K_H(\mathrm{Gr})_{\mathrm{loc}}$ by the actions of $\mathbb{C}P$ and $\{\mathtt{t}_\beta\}_{\beta \in Q^\vee}$. Hence, we deduce a $\mathcal{H}$-module embedding $K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$ that intertwines the $\mathtt{t}_\beta$-action to the right β-action. Such an embedding must be unique by Corollary 2.9. □

Remark 2.11. Lemma 2.7 is a purely combinatorial statement about the comparison of two orders on W_{af}. As a consequence, we obtain an embedding

$$K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathrm{Gr}_G) \hookrightarrow K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G^{\mathrm{rat}})$$

of nil-DAHA modules (with a parameter q) that sends $[\mathcal{O}_{\mathrm{Gr}_w}]$ to $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}(w)}]$ for each $w \in W_{\mathrm{af}}^{-}$ (cf. [37, 41, 32]; see [28] for its further consequences).

2.3 The $\mathcal{H}$-module embedding

Theorem 2.12. *We have a $\mathcal{H}$-module embedding*

$$\Phi : K_H(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

such that twisting by $\mathtt{t}_\beta$ corresponds to the right action of $\beta \in Q^\vee$,

$$[\mathcal{O}_{\mathrm{Gr}_{ut_\beta}}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}] \qquad u \in W, \beta \in Q^\vee_<,$$

and sends the Pontryagin product on the LHS to the tensor product on the RHS. More precisely, we have: For each $i \in \mathtt{I}$ and $\xi \in K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$, it holds

$$\Phi(\mathbf{h}_i \odot \xi) = H_i(\Phi(\xi)).$$

In addition, the image of Φ is precisely the set of finite linear combinations of Schubert classes, that forms a dense subset of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$.

Remark 2.13. **1)** It is known that $\{\mathbf{h}_i\}_{i\in\mathtt{I}}$, $\mathbb{C}P$, and $\{\mathtt{t}_\beta\}_{\beta\in Q^\vee}$ generates the ring $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$. One way to prove it is to compare $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ with its original definition in [32, §5]; **2)** We add an extra $\mathbb{G}_m$-action on $K_H(\mathrm{Gr}_G)$ and prove an analogue of Theorem 2.12 in [28] on the basis of the results presented here.

The rest of this subsection is entirely devoted to the proof of Theorem 2.12. The embedding part of Theorem 2.12 as based $\mathcal{H}$-modules is already proved in Proposition 2.10. It also implies that the image of this embedding is the set of finite linear combinations of Schubert classes.

Let $i \in \mathtt{I}$. We have an endomorphism $\Xi(-\varpi_i)$ of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ that commutes with the right $Q^\vee$-action and the left $\mathcal{H}$-action. By Corollary 1.23, the image of $[\mathcal{O}_{\mathbf{Q}_G(e)}]$ under $\Xi(-\varpi_i)$ belongs to the image of $K_H(\mathrm{Gr})_{\mathrm{loc}}$. In particular, $\Xi(-\varpi_i)$ induces a $\mathcal{H}$-module endomorphism of $K_H(\mathrm{Gr})_{\mathrm{loc}}$. In particular, H_i also induces an endomorphism of $K_H(\mathrm{Gr})_{\mathrm{loc}}$. We denote the endomorphisms on $K_H(\mathrm{Gr})_{\mathrm{loc}}$ induced by $\Xi(-\varpi_i)$ and H_i by the same letter.

In order to identify the endomorphisms $\mathbf{h}_i\odot$ and H_i, it suffices to compare some linear combination with the well-understood element, namely id. Therefore, we compare the endomorphisms of $K_H(\mathrm{Gr})_{\mathrm{loc}}$ (as $\mathbb{C}P$-modules) induced by

$$\Theta_i := e^{-\varpi_i}(\mathrm{id} - \mathbf{h}_i\odot)$$

and

$$\Xi(-\varpi_i) = e^{-\varpi_i}(\mathrm{id} - H_i).$$

The both endomorphisms send $[\mathcal{O}_{\mathrm{Gr}_0}]$ to

$$e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \odot [\mathcal{O}_{\mathrm{Gr}_{t_\beta}}]^{-1} \qquad (\beta \in Q^\vee_<)$$

by Proposition 2.10, Corollary 1.8, and Corollary 1.23.

We prove that the both of Θ_i and $\Xi(-\varpi_i)$ commute with the $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-action. It is Theorem 1.21 for $\Xi(-\varpi_i)$. Hence, we concentrate on the action of Θ_i.

The action of Θ_i commutes with $\mathbb{C}P \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$ as $(K_H(\mathrm{Gr})_{\mathrm{loc}}, \odot)$ is a commutative ring. Thus, Corollaries 2.3 and 2.4 (and Theorem 2.5) reduces the problem to

$$D_j(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) = e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \qquad j \in \mathtt{I}, \beta \in Q^\vee_<.$$

If $j \neq i$, then we have $s_j s_i t_\beta < s_i t_\beta$ and $s_j t_\beta < t_\beta$. Moreover, we have $D_j(e^{-\varpi_i} \bullet) = e^{-\varpi_i} D_j(\bullet)$. It follows that

$$\begin{aligned} D_j(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) &= e^{-\varpi_i} D_j([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]). \end{aligned}$$

If $j = i$, then we compute as

$$\begin{aligned} D_i(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) &= e^{-\varpi_i+\alpha_i} D_i([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &\quad + \frac{e^{-\varpi_i} - e^{-\varpi_i+\alpha_i}}{1 - e^{\alpha_i}}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i+\alpha_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{t_\beta}}]) \\ &\quad + e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]). \end{aligned}$$

Hence, Θ_i defines an endomorphism of $K_H(\mathrm{Gr})$ that commutes with the $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-action.

Therefore, Corollary 2.9 guarantees $\Theta_i = \Xi(-\varpi_i) \in \mathrm{End}(K_H(\mathrm{Gr})_{\mathrm{loc}})$. From this, we also deduce $\mathbf{h}_i \odot = H_i \in \mathrm{End}(K_H(\mathrm{Gr})_{\mathrm{loc}})$ as required.

2.4 Example: $SL(2)$-case

Assume that $G = SL(2)$. We make an identification $P_+ = \mathbb{Z}_{\geq 0}\varpi$, $\alpha = 2\varpi$, and $Q^\vee_+ = \mathbb{Z}_{\geq 0}\alpha^\vee$. We have $W = \{e, s\}$. Let $\mathtt{t}$ denote the right $Q^\vee$-action on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ (or $K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G^{\mathrm{rat}})$) corresponding to $\alpha^\vee$, and let q denote the character of $\mathbb{G}_m$ that acts on the variable z (in $G((z))$) by degree one character (so-called the loop rotation action).

The Pieri-Chevalley rule for ϖ ([32, Theorem 5.13]) yields the equations:

$$\begin{aligned} [\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi)] &= \frac{1}{1 - q^{-1}\mathtt{t}}(e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]) \\ [\mathcal{O}_{\mathbf{Q}_G(s)}(\varpi)] &= \frac{1}{1 - q^{-1}\mathtt{t}}(q^{-1}e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]). \end{aligned}$$

Forgetting the extra $\mathbb{G}_m$-action yield:

$$\begin{aligned} [\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi)] &= \frac{1}{1 - \mathtt{t}}(e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]) \\ [\mathcal{O}_{\mathbf{Q}_G(s)}(\varpi)] &= \frac{1}{1 - \mathtt{t}}(e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]). \end{aligned}$$

Inverting this equation yields that

$$\begin{aligned} [\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi)] &= e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}] \\ [\mathcal{O}_{\mathbf{Q}_G(s)}(-\varpi)] &= -e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} [\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi)] &= [\mathcal{O}_{\mathbf{Q}_G(s)}] \\ [\mathcal{O}_{\mathbf{Q}_G(s)}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}(-\varpi)] &= e^{\alpha}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + (1 - e^{\alpha})[\mathcal{O}_{\mathbf{Q}_G(s)}]. \end{aligned}$$

Applying $\mathtt{t}^{-m}$ ($m > 0$) on the both sides, we have

$$[\mathcal{O}_{\mathbf{Q}_G(t_{-m\alpha^\vee})}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(t_{-m\alpha^\vee})}(-\varpi)] = [\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}]$$
$$[\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}(-\varpi)] = e^{\alpha}[\mathcal{O}_{\mathbf{Q}_G(t_{(1-m)\alpha^\vee})}] + (1-e^{\alpha})[\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}].$$

In other words, we have

$$H([\mathcal{O}_{\mathbf{Q}_G(t_{-m\alpha^\vee})}]) = [\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}]$$
$$H([\mathcal{O}_{\mathbf{Q}_G(t_{-m\alpha^\vee})}]) = e^{\alpha}[\mathcal{O}_{\mathbf{Q}_G(t_{(1-m)\alpha^\vee})}] + (1-e^{\alpha})[\mathcal{O}_{\mathbf{Q}_G(st_{-m\alpha^\vee})}]$$

for $H \equiv H_i$ (as we have $|\mathtt{I}| = 1$). By Theorem 2.12, this transplants to

$$\mathbf{h} \odot [\mathcal{O}_{\mathrm{Gr}_{t_{-m\alpha^\vee}}}] = [\mathcal{O}_{\mathrm{Gr}_{st_{-m\alpha^\vee}}}]$$
$$\mathbf{h} \odot [\mathcal{O}_{\mathrm{Gr}_{st_{-m\alpha^\vee}}}] = e^{\alpha}[\mathcal{O}_{\mathrm{Gr}_{t_{(1-m)\alpha^\vee}}}] + (1-e^{\alpha})[\mathcal{O}_{\mathrm{Gr}_{st_{-m\alpha^\vee}}}]$$

for $\mathbf{h} \equiv \mathbf{h}_i$. We have $\mathbf{h} = [\mathcal{O}_{\mathrm{Gr}_{st_{-\alpha^\vee}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{t_{-\alpha^\vee}}}]^{-1}$. By Theorem 1.7, we conclude

$$[\mathcal{O}_{\mathrm{Gr}_{st_{-\alpha^\vee}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{t_{-m\alpha^\vee}}}] = [\mathcal{O}_{\mathrm{Gr}_{st_{-(m+1)\alpha^\vee}}}]$$
$$[\mathcal{O}_{\mathrm{Gr}_{st_{-\alpha^\vee}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{st_{-m\alpha^\vee}}}] = e^{\alpha}[\mathcal{O}_{\mathrm{Gr}_{t_{-m\alpha^\vee}}}] + (1-e^{\alpha})[\mathcal{O}_{\mathrm{Gr}_{st_{-(m+1)\alpha^\vee}}}].$$

for $m > 0$. This coincides with the calculation in [40, (17)].

3 Relation with quantum K-group

We continue to work in the setting of the previous section.

3.1 Quasi-map spaces

Here we recall basics of quasi-map spaces from [16, 13].

A quasi-map (f, D) is a map $f : \mathbb{P}^1 \to \mathcal{B}$ together with a $\Pi^\vee$-colored effective divisor

$$D = \sum_{\alpha^\vee \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee)\alpha^\vee \otimes [x] \in Q^\vee \otimes_{\mathbb{Z}} \mathrm{Div}\, \mathbb{P}^1 \quad \text{with} \quad m_x(\alpha^\vee) \in \mathbb{Z}_{\geq 0}. \tag{3.1}$$

We call D the defect of (f, D), and $\sum_{\alpha \in \Pi^\vee} m_x(\alpha^\vee)\alpha^\vee$ the defect of (f, D) at $x \in \mathbb{P}^1(\mathbb{C})$ (that we denote by $|D|_x$). Here we define the total defect of (f, D) by

$$|D| := \sum_{\alpha^\vee \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee)\alpha^\vee = \sum_{x \in \mathbb{P}^1(\mathbb{C})} |D|_x \in Q^\vee_+.$$

We set $[D] := \{x \in \mathbb{P}^1(\mathbb{C}) \mid \sum_{\alpha^\vee \in \Pi^\vee} m_x(\alpha^\vee) > 0\} \subset \mathbb{P}^1$ and call it the defect locus of (f, D).

For each $\beta \in Q^\vee_+$, we set

$$\mathcal{Q}(\mathcal{B}, \beta) := \{f : \mathbb{P}^1 \to X \mid \text{ quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta\},$$

where $f_*[\mathbb{P}^1]$ is the class of the image of $\mathbb{P}^1$ multiplied by the degree of $\mathbb{P}^1 \to \mathrm{Im}\, f$. We denote $\mathcal{Q}(\mathcal{B}, \beta)$ by $\mathcal{Q}(\beta)$ in case there is no danger of confusion. We understand that $\mathcal{Q}(\beta) = \emptyset$ for $\beta \in Q^\vee \setminus Q^\vee_+$.

Definition 3.1 (Drinfeld-Plücker data)**.** Consider a collection $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_+}$ of inclusions $\psi_\lambda : \mathcal{L}^\lambda \hookrightarrow L(\lambda) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ of line bundles $\mathcal{L}^\lambda$ (as coherent subsheaves) over $\mathbb{P}^1$. The data $\mathcal{L}$ is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of G-modules

$$\eta_{\lambda,\mu} : L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)$$

induces an isomorphism

$$\eta_{\lambda,\mu} \otimes \mathrm{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \xrightarrow{\cong} \psi_\lambda(\mathcal{L}^\lambda) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \psi_\mu(\mathcal{L}^\mu)$$

for every $\lambda, \mu \in P_+$.

Theorem 3.2 (Drinfeld, see Finkelberg-Mirković [16])**.** *The variety* $\mathcal{Q}(\beta)$ *is isomorphic to the variety formed by isomorphism classes of the DP-data* $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_+}$ *such that* $\deg \mathcal{L}^\lambda = -\langle \beta, \lambda \rangle$.

For each $\beta \in Q_+^\vee$ and $w \in W$, we consider two varieties:

$$\mathring{\mathcal{Q}}(\beta) := \{(f, D) \in \mathcal{Q}(\beta) \mid 0 \notin [D]\} \subset \mathcal{Q}(\beta),$$
$$\mathring{\mathcal{Q}}(\beta, w) := \{(f, D) \in \mathring{\mathcal{Q}}(\beta) \mid f(0) \in \mathbb{O}^{\mathcal{B}}(w)\}.$$

In case $\beta, \gamma \in Q_+^\vee$, we have an embedding

$$\imath_\gamma : \mathcal{Q}(\beta) \ni (f, D) \mapsto (f, D + \gamma[0]) \in \mathcal{Q}(\beta + \gamma).$$

We set $\mathcal{Q}(\beta, w) := \overline{\mathring{\mathcal{Q}}(\beta, w)} \subset \mathcal{Q}(\beta)$.

For each $\lambda \in P$, $w \in W$, and $\beta \in Q_+^\vee$, we have a G-equivariant line bundle $\mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)$ obtained by the (tensor product of the) pull-backs $\mathcal{O}_{\mathcal{Q}(\beta,w)}(\varpi_i)$ of the i-th $\mathcal{O}(1)$ via the embedding

$$\mathcal{Q}(\beta, w) \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes_{\mathbb{C}} \mathbb{C}[z]_{\le \langle \beta, \varpi_i \rangle}). \tag{3.2}$$

We have $\mathcal{B} = \mathcal{Q}(0)$ by the Plücker embedding. By expanding the map $\mathbb{P}^1 \to \mathcal{B} \to \mathbb{P}(L(\lambda)^*)$ $(\lambda \in P_+)$ into a collection of formal power series $L(\lambda)^* \otimes \mathbb{C}[\![z]\!]$, we find an embedding $\mathcal{Q}(\beta) \subset \mathbf{Q}_G$ by (1.8). These result in embeddings $\mathcal{B} \subset \mathcal{Q}(\beta) \subset \mathbf{Q}_G$ such that the line bundles $\mathcal{O}(\lambda)$ $(\lambda \in P)$ corresponds to each other by restrictions.

3.2 Factorization structure and its consequences

The contents of this subsection is needed to establish Theorem 4.12 in the next section, and is not used in the rest of this section. Hence this subsection can be safely skipped to understand Theorem 3.11 and Corollary 3.13.

Here we temporarily switch to the complex analytic topology in order to state Theorem 3.3. (Although the whole results are of algebraic nature as recorded in [5, 9], it looks simpler to present them by their analytic counterparts, see Remarks 3.4 3) and 3.5.) For each $\beta \in Q_+^\vee$, we set $\mathcal{Z}(\beta) := \mathcal{Q}(\beta) \cap \mathbb{O}(w_0)$ and call it the zastava space (of degree β). This is an affine open subset of $\mathcal{Q}(\beta, w_0)$ that is stable under the action of $(\mathbb{G}_m \times B)$. We set

$$C^{(\beta)} := \prod_{i \in \mathtt{I}} (C^{m_i} / \mathfrak{S}_{m_i}), \text{ where } \beta = \sum_{i \in \mathtt{I}} m_i \alpha_i^\vee$$

for a Riemann surface C (or a finite set of points), where $\mathfrak{S}_m$ is the symmetric group of order m. We note that the space $C^{(\beta)}$ is the same as the space of $\Pi^\vee$-colored divisors of degree β on C. We set $\mathbb{A}^1_x := \mathbb{P}^1 \setminus \{x\}$ for a point $x \in \mathbb{P}^1$ for the sake of notational simplicity, that we may regard it as an open Riemann surface and an algebraic curve interchangeably.

For each $i \in \mathtt{I}$, a point $(f, D) \in \mathcal{Z}(\beta)$ defines an element of $u_i(f, D) \in L(\varpi_i)^* \otimes_{\mathbb{C}} \mathbb{C}[z]$ through (3.2) for each $i \in \mathtt{I}$ (up to a scalar multiple), that also yields a polynomial $\phi_i(f, D; z) \in \mathbb{C}[z]$ by pairing with the lowest weight vector of $L(\varpi_i)$. By examining the roots of $\phi_i(f, D; z)$ (the multiplicity at ∞ is understood as $\langle \beta, \varpi_i \rangle - \deg \phi_i(f, D; z)$), we obtain the factorization morphism

$$\mathfrak{f}^\beta : \mathcal{Z}(\beta) \longrightarrow (\mathbb{A}^1_0)^{(\beta)}$$

since $0 \in \mathbb{P}^1$ is never a root of such polynomials (see e.g. [16, §5.2.2]). By construction, the point $\mathfrak{f}^\beta(f, D) \in (\mathbb{P}^1)^{(\beta)}$ contains at least $\langle |D|_x, \varpi_i \rangle$-copies of the point $x \in \mathbb{P}^1$ in the i-th configuration for $(f, D) \in \mathcal{Z}(\beta)$. The constant map $c : \mathbb{P}^1 \to B/B \subset \mathcal{B}$ yields a point $(c, \beta[\infty]) \in \mathcal{Z}(\beta)$, that we refer as the origin $\mathbf{0}$ of $\mathcal{Z}(\beta)$. The $\mathbb{G}_m$-action attracts every point of $\mathcal{Z}(\beta)$ to the origin.

Theorem 3.3 (Finkelberg-Mirković [16] §6.3.2)**.** *Let β, β_1, β_2 be elements in $Q^\vee_+$ such that $\beta = \beta_1 + \beta_2$, and let $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{A}^1_0$ be a pair of disjoint complex analytic open subsets. We have an isomorphism of complex analytic sets*

$$(\mathfrak{f}^\beta)^{-1}(\mathcal{U}_1^{(\beta_1)} \times \mathcal{U}_2^{(\beta_2)}) \cong (\mathfrak{f}^{\beta_1})^{-1}(\mathcal{U}_1^{(\beta_1)}) \times (\mathfrak{f}^{\beta_2})^{-1}(\mathcal{U}_2^{(\beta_2)}),$$

where $\mathcal{U}_1^{(\beta_1)} \times \mathcal{U}_2^{(\beta_2)} \subset (\mathbb{P}^1)^{(\beta)}$ is a natural inclusion.

Remark 3.4. **1)** Theorem 3.3 implies that $\mathcal{Z}(\beta)$, and hence also $\mathcal{Q}(\beta)$, must be irreducible; **2)** In [6, 8], it is established that $\mathcal{Z}(\beta)$ is a normal variety and the morphism $\mathfrak{f}^\beta$ is flat (in the course of their proof of Theorem 4.2); **3)** In the algebraic treatments in [5, 9], we work over an open subscheme of $(\mathbb{A}^1_0)^{(\beta_1)} \times (\mathbb{A}^1_0)^{(\beta_2)}$ such that there is no coincidence between the first and second groups of points. This is not a subspace of $(\mathbb{A}^1_0)^{(\beta)}$ in general.

Remark 3.5. In Theorem 3.3, we can fix a point $x = (f, D) \in (\mathfrak{f}^{\beta_1})^{-1}(\mathcal{U}_1^{(\beta_1)})$ and take the completion $\mathcal{O}^\wedge_x$ of the germ of the analytic structure sheaf of $(\mathfrak{f}^{\beta_1})^{-1}(\mathcal{U}_1^{(\beta_1)})$ at a point x. Theorem 3.3 asserts that that $\operatorname{Spec} \mathcal{O}^\wedge_x$ is the formal completion of the normal direction of

$$\{x\} \times (\mathfrak{f}^{\beta_2})^{-1}(\mathcal{U}_2^{(\beta_2)}) \subset \mathcal{Z}(\beta).$$

In this description, $\mathfrak{f}^{\beta_1}(x)$ corresponds to a specific configuration of points in $\mathbb{A}^1_0$, that defines a finite set $[D] \subset S \subset \mathbb{A}^1_0$. Therefore, we can set $\mathcal{U}_2 := (\mathbb{A}^1_0 \setminus S)^{(\beta_2)}$ as the limiting case. In view of the comparison with [5] and Theorem 3.3, the locally ringed spaces and the maps

$$\{x\} \times (\mathfrak{f}^{\beta_2})^{-1}((\mathbb{A}^1_0 \setminus S)^{(\beta_2)}) \subset \operatorname{Spec} \mathcal{O}^\wedge_x \times (\mathfrak{f}^{\beta_2})^{-1}((\mathbb{A}^1_0 \setminus S)^{(\beta_2)}) \longrightarrow \mathscr{Z}(\beta), \quad (3.3)$$

admit structures of schemes and morphisms between them, and now the base space $\mathfrak{f}^{\beta_1}(x) \times (\mathbb{A}^1_0 \setminus S)^{(\beta_2)}$ is a subvariety of $(\mathbb{A}^1_0)^{(\beta)}$. The middle scheme in (3.3) is an algebraic fiber bundle over $(\mathfrak{f}^{\beta_2})^{-1}((\mathbb{A}^1_0 \setminus S)^{(\beta_2)})$, meaning that it is a flat morphism with its fiber isomorphic to $\operatorname{Spec} \mathcal{O}^\wedge_x$ as schemes. This exhibits the reason why we need formal completions in the proof of Theorem 4.9, and why Proposition 3.6 can be useful in its analysis.

Proposition 3.6. *Let $\beta, \gamma \in Q^\vee_+$. We have a local transversal slice $\mathcal{S}_y$ of $y \in \mathring{\mathcal{Q}}(\beta)$ inside $\mathcal{Q}(\beta + \gamma)$ as complex analytic spaces through the embedding $\imath_\gamma$ such that*

$$\eta : \mathcal{S}_y \times U \hookrightarrow \mathcal{Q}(\beta + \gamma),$$

where $\mathbf{0} \in \mathcal{S}_y \subset \mathcal{Z}(\gamma)$ and $y \in U \subset \mathring{\mathcal{Q}}(\beta)$ are (complex analytic) open subsets and $\imath_\gamma(y) = \eta(\mathbf{0}, y)$.

Proof. We make a swap $z \mapsto z^{-1}$ that effects on the coordinate of $\mathbb{P}^1$ (and hence the origin of a zastava space have defect only at 0 instead of ∞). By convention, $\mathcal{Z}(\beta + \gamma)$ now consists of quasi-maps (f, D) of degree $(\beta + \gamma)$ such that $\infty \notin [D]$ and $f(\infty) \in B/B$. Since the former is an open condition and $N^- \times B/B \subset \mathcal{B}$ is open dense, we deduce $N^- \times \mathcal{Z}(\beta + \gamma) \subset \mathcal{Q}(\beta + \gamma)$ is also a dense open subset. Since $y \in \mathring{\mathcal{Q}}(\beta)$, we find that the quasi-map (f_y, D_y) corresponding to $\imath_\gamma(y) \in \mathcal{Q}(\beta + \gamma)$ (that we might also denote by y in the below) satisfies $|D_y|_0 = \gamma$ and $|D_y|_p = 0$ for some $p \in \mathbb{A}^1_\infty$. Let us apply the action of some $\psi \in PSL(2, \mathbb{C})$ on $\mathbb{P}^1$ that fixes 0 and send p to ∞. In addition, we apply some $g \in G$ such that $gf_y(p) \in B/B$. The actions of ψ and G preserves $\mathring{\mathcal{Q}}(\beta)$ (as $0 \in \mathbb{P}^1$ is fixed and the space is G-stable). Thus, we have $\imath_\gamma(y) \in (\psi^{-1} \times g^{-1})(N^- \times \mathcal{Z}(\beta + \gamma)) \cap \imath_\gamma(\mathring{\mathcal{Q}}(\beta)) \subset \mathcal{Q}(\beta + \gamma)$. In particular, it suffices to choose $\imath_\gamma(y) \in \mathcal{Z}(\beta + \gamma) \cap \imath_\gamma(\mathring{\mathcal{Q}}(\beta))$ to construct a local transversal slice.

Now we have $|D_y|_0 = \gamma$ and $|D_y|_\infty = 0$. Since the order of common zero at 0 of the vector valued function $u_i(f, D)$ is exactly $\langle \gamma, \varpi_i \rangle$, we have some $h \in N$ such that $hu_i(f, D)$, when paired with the lowest weight vector of $L(\varpi_i)$, yields exactly zero of order $\langle \gamma, \varpi_i \rangle$ at 0 (since the N-action on $L(\varpi_i)^*$ is cocyclic to the highest weight vector, we can throw in the lowest z-degree part into the coefficient of the highest weight vector by the N-action). The twist by h preserves $\mathcal{Z}(\beta + \gamma)$, and changes the factorization morphism only. We employ this modified factorization morphism (that we denote by $\mathfrak{f}$) and define

$$\mathfrak{f}^+ : N^- \times \mathcal{Z}(\beta + \gamma) \ni (h, (f, D)) \mapsto \mathfrak{f}(f, D) \in (\mathbb{A}_\infty)^{(\beta+\gamma)}.$$

Let us find disjoint open subsets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{A}^1_\infty$ such that $0 \in \mathcal{U}_1$ and $\mathcal{U}_2$ contains the support of the configuration of points $\mathfrak{f}^+(y)$ except for 0. By identifying $\mathfrak{f}$ with the original factorization morphism, Theorem 3.3 yields an isomorphism

$$(\mathfrak{f}^+)^{-1}(\mathcal{U}_1^{(\gamma)} \times \mathcal{U}_2^{(\beta)}) \cong (\mathfrak{f}^\gamma)^{-1}(\mathcal{U}_1^{(\gamma)}) \times \left(N^- \times (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)})\right)$$

that defines an open subset of $\mathcal{Q}(\beta + \gamma)$ such that

$$(\mathfrak{f}^\gamma)^{-1}(\mathcal{U}_1^{(\gamma)}) \times \left(N^- \times (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)})\right) \cap \imath_\gamma(\mathring{\mathcal{Q}}(\beta)) = \{\mathbf{0}\} \times \left(N^- \times (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)})\right)$$

and $y \in (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)}) \subset N^- \times (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)})$. Now we set

$$\mathcal{S}_y := (\mathfrak{f}^\gamma)^{-1}(\mathcal{U}_1^{(\gamma)}) \quad \text{and} \quad U := \left(N^- \times (\mathfrak{f}^\beta)^{-1}(\mathcal{U}_2^{(\beta)})\right)$$

to conclude the assertion. □

Corollary 3.7. *Keep the setting of Proposition 3.6 (with possible rearrangement of $\mathcal{S}_y$ and U). Let $p \in \mathbb{P}^1(\mathbb{C})$. If $y = (f, D)$ satisfies $0 \neq p \notin [D]$, then we have the following commutative diagram for an arbitrary $\delta \in Q^\vee_+$:*

$$\begin{array}{ccccccc}
 & & \mathcal{S}_y \times U & \hookrightarrow & \mathcal{Q}(\beta+\gamma) & \xleftarrow{\imath_\gamma} & \mathring{\mathcal{Q}}(\beta) \\
 & & \downarrow{\scriptstyle \mathrm{id}\times\imath} & & \downarrow{\scriptstyle \imath'} & & \downarrow{\scriptstyle \imath''} \\
\mathcal{S}_y \times U \times U'(\delta) & \xrightarrow{\cong} & \mathcal{S}_y \times U(\delta) & \hookrightarrow & \mathcal{Q}(\beta+\gamma+\delta) & \xleftarrow{\imath_\gamma} & \mathring{\mathcal{Q}}(\beta+\delta)
\end{array}$$

such that

$$\mathcal{S}_y \times U \times \{\mathbf{0}_\delta\} \xrightarrow{\cong} ((\mathcal{S}_y \times U(\delta)) \cap \mathcal{Q}(\beta+\gamma)) \subset \mathcal{Q}(\beta+\gamma+\delta),$$

where the map $\imath' : \mathcal{Q}(\beta+\gamma) \hookrightarrow \mathcal{Q}(\beta+\gamma+\delta)$ is obtained by adding the defect $\delta[p]$ to each point, $U(\delta) \subset \mathring{\mathcal{Q}}(\beta+\delta)$ is an neighbourhood of the image of y in $\mathring{\mathcal{Q}}(\beta+\delta)$, $U'(\delta) \subset \mathcal{Z}(\delta)$ is an open neighborhood of the origin $\mathbf{0}_\delta \in \mathcal{Z}(\delta)$, and $\imath$ and $\imath''$ are the induced maps.

Proof. In the proof of Proposition 3.6, we can modify the factorization morphism $\mathfrak{f}^{(\beta)}$ and $\mathcal{U}_2$ if necessary to assume that $\mathcal{U}_2 = \mathcal{U}_2^{(1)} \sqcup \mathcal{U}_2^{(2)}$, $p \in \mathcal{U}_2^{(2)}$, and $\mathfrak{f}^\beta(y)$ does not contain a point in $\mathcal{U}_2^{(2)}$. Then, Theorem 3.3 separates out the effect of $\delta[p]$ as a product factor $U'(\delta)$ isomorphic to an open neighbourhood of $(c, \delta[p]) \in \mathcal{Z}(\delta)$ (where c is the constant map to $B/B \subset \mathcal{B}$) as required. □

In view of Proposition 3.6 and Corollary 3.7, the structure of local transversal slices of the open subset $\mathring{\mathcal{Q}}(\beta)$ with respect to $\imath_\gamma$ only depends on γ, and not on the choice of β. We need an analogous local transversal slices inside $\mathcal{Q}(\beta+\gamma, w)$ in the course of our proof of Theorem 4.9. The main obstacle there is that it is not clear whether a local transversal slice exists on the neighbourhood of every point of $\mathring{\mathcal{Q}}(\beta)$ in a uniform fashion, that is guaranteed by Theorem 3.3 when $w = e$. This uniformity is resurrected by identifying the situation with the (formal completions of the) transversal slices between $\mathbf{I}$-orbits of $\mathbf{Q}_G^{\mathrm{rat}}$ by the fact that all the points of $\mathring{\mathcal{Q}}(\beta)$ lie on the same $G[\![z]\!]$-orbit (note that two points in $\mathring{\mathcal{Q}}(\beta)$ are not transferred to each other by the action of the smaller group $G[z]$ in general as it preserves the defect at $\mathbb{P}^1 \setminus \{0, \infty\}$).

3.3 Quantum J-functions and generating functions

In this subsection, we reformulate results provided in Givental-Lee [20] and Braverman-Finkelberg [6]. Hence, both of the "theorems" in this subsection are understood as blends of their results, and their "proofs" are just explanations on how they work.

Theorem 3.8. *There exists an element $J'(Q,q) \in (\mathbb{C}P[\![q]\!])[\![Q^\vee_+]\!] \cap \mathbb{C}(P,q)[\![Q^\vee_+]\!]$ with the following properties:*

1. *the composition of maps*

$$(\mathbb{C}P[\![q]\!])[\![Q^\vee_+]\!] \cong (K_G(\mathcal{B})[\![q]\!])[\![Q^\vee_+]\!] \subset (K_H(\mathcal{B})[\![q]\!])[\![Q^\vee_+]\!]$$

sends $J'(Q,q)$ to $J(Q,q)$;

2. for each $\lambda \in P$, we have an identity in $(\mathbb{C}[q^{\pm 1}]P)[\![Q_+^\vee]\!]$:

$$D_{w_0}(J'(Qq^\lambda, q)e^{w_0\lambda}J'(Q, q^{-1})) = \sum_{\beta \in Q_+^\vee} \chi_q(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}}(\lambda))Q^\beta,$$

where we understand that Qq^λ sends Q^β to $Q^\beta q^{-\langle \beta, \lambda \rangle}$ for each $\lambda \in P$.

Proof. For the first assertion, it is actually (the individual Q^β-coefficients of) $J'(Q, q)$ that is calculated as the graded characters of the rings of regular functions of zastava spaces in [6, 8] (cf. §4.2). As explained in [6, §1.3], these graded character counts the push-forward of the (corresponding part of the) denominator of the localization formula for the corresponding Kontsevich graph spaces. This can be understood to be a definition of the (K-theoretic) J-function as in [20, §2.2] (since the push-forward of the structure sheaf does not depend on the choice of a resolution; cf. Remark 4.1).

The second assertion is equivalent to [7, Lemma 5]. □

For $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$, we set $x^{\vec{n}} := x_1^{n_1} \cdots x_r^{n_r}$. For $\lambda \in P$, we set $\lambda[\vec{n}] := \lambda - \sum_{i=1}^r n_i \varpi_i$.

Theorem 3.9. *For each $\sum_{\beta \in Q_+^\vee, \vec{n} \in \mathbb{Z}_{\geq 0}^r} f_{\beta, \vec{n}}(q) x^{\vec{n}} Q^\beta \in (\mathbb{C}P[q^{\pm 1}, x_1, \ldots, x_r])[\![Q_+^\vee]\!]$ such that*

$$\sum_{\beta \in Q_+^\vee, \vec{n} \in \mathbb{Z}_{\geq 0}^r} f_{\beta, \vec{n}}(q) \otimes_{R(G)[q^{\pm 1}]} \left(\prod_{j=1}^r (p_i^{-1} q^{Q_i \partial_{Q_i}})^{n_i} \right) Q^\beta J(Q, q) = 0, \qquad (3.4)$$

we have the following equalities:

$$\sum_{\beta \in Q_+^\vee, \vec{n} \in \mathbb{Z}_{\geq 0}^r} f_{\beta, \vec{n}}(q) q^{-\langle \beta, \lambda[\vec{n}] \rangle} \chi_q(\mathcal{Q}(\gamma - \beta), \mathcal{O}_{\mathcal{Q}}(\lambda[\vec{n}])) = 0 \quad \lambda \in P_+, \gamma \in Q_+^\vee.$$

Proof. The assertion is [20, §4.2] (see also [7, Lemma 5] and [8, §5]), that employs the localization theorem applied to a resolution of $\mathcal{Q}(\beta)$, such as $\mathcal{GB}_{0,\beta}$ (cf. §4.1) or the Laumon spaces (when $G = SL(n, \mathbb{C})$).

Here we demonstrate an alternative proof (it depends on the argument in the previous paragraph through Theorem 3.8, though). We can substitute Q with Qq^λ in (3.4) multiplied with $\prod_{i \in \mathtt{I}} p_i^{-m_i}$ for $\lambda = \sum_{i \in \mathtt{I}} m_i \varpi_i$, that is identified with $e^{w_0 \lambda}$ through the isomorphism $K_G(\mathcal{B}) \cong K_B(\mathrm{pt}) \cong \mathbb{C}P$. By factoring out the effect of additional powers of q coming from $q^{Q_i \partial_{Q_i}}$'s, we derive a formula

$$\sum_{\beta \in Q_+^\vee, \vec{n} \in \mathbb{Z}_{\geq 0}^r} f_{\beta, \vec{n}}(q) \otimes_{R(G)[q^{\pm 1}]} q^{-\langle \beta, \lambda[\vec{n}] \rangle} Q^\beta J'(Qq^{\lambda[\vec{n}]}, q) e^{w_0(\lambda[\vec{n}])} = 0.$$

Applying Theorem 3.8 2), we conclude the desired equation. □

3.4 Identification of defining equations

Proposition 3.10. *For each $\lambda \in P$, we have*

$$\lim_{\beta \to \infty} \chi_q(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)) = \mathrm{gch}\, H^0(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda)). \qquad (3.5)$$

Proof. The limit in (3.5) exists, and it gives the character of the (dual of the) global Weyl module (= character of the RHS of (1.11)) by [7, §4.2] and [8] (here we use Theorem 3.8 2)). □

Theorem 3.11. *We have a well-defined $\mathbb{C}P$-linear isomorphism*

$$\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \longrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends $[\mathcal{O}_{\mathcal{B}}]$ to $[\mathcal{O}_{\mathbf{Q}_G}]$, the quantum multiplication by $a_i([\mathcal{O}_{\mathcal{B}}])$ to the endomorphism $\Xi(-\varpi_i)$ $(i \in \mathtt{I})$, and the multiplication by Q^β to the right $Q^\vee$-action of β for each $\beta \in Q^\vee$.

Remark 3.12. Our proof of Theorem 3.11 says that Ψ is actually the $q = 1$ specialization of an isomorphism

$$\Psi_q : \mathbb{C}[q^{\pm 1}] \otimes_{\mathbb{C}} qK_H(\mathcal{B})_{\mathrm{loc}} \longrightarrow \mathbb{C}Q^\vee \otimes_{\mathbb{C}Q^\vee_+} \widetilde{K}(\mathbf{Q}_G)$$

such that $\Psi_q \circ A_i(q) = \Xi_q(-\varpi_i) \circ \Psi_q$ $(i \in \mathtt{I})$, where $\Xi_q(-\varpi_i)$ denotes the tensor product of $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)$ in the $\mathbb{G}_m$-equivariant setting (see Theorem 1.20).

Proof of Theorem 3.11. We first show that the map Ψ is well-defined. To prove this, it suffices to write down all potential relations of $qK_H(\mathcal{B})$ in $qK_{\mathbb{G}_m \times H}(\mathcal{B})$ and see that the assignment predicted by Ψ yields a relation of $\widetilde{K}'(\mathbf{Q}_G)$. The ring $K_H(\mathcal{B})$ is generated by $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ $(i \in \mathtt{I})$ and $\mathbb{C}P$. In view of Corollary 1.32, the topological linear space $qK_{\mathbb{G}_m \times H}(\mathcal{B})$ contains a dense linear subspace $\mathfrak{K}$ cyclically generated from $[\mathcal{O}_{\mathcal{B}}]$ by the action of $(\mathbb{C}P[q^{\pm 1}, A_1(q), \ldots, A_r(q)])[\![Q^\vee_+]\!]$. The specialization $q = 1$ is possible on $\mathbb{C}P[q^{\pm 1}, A_1(q), \ldots, A_r(q)]$ (as in §1.7). Since the Q^β-coefficient $(\beta \in Q^\vee_+)$ of an element of $(\mathbb{C}P[q^{\pm 1}, A_1(q), \ldots, A_r(q)])[\![Q^\vee_+]\!]$ is a polynomial of the Q^γ-coefficients $(0 \le \gamma \le \beta)$ of $A_i(q)$ $(i \in \mathtt{I})$ with $\mathbb{C}P$-coefficients, we deduce that the $q = 1$ specialization of $\mathfrak{K}$ is possible, and yields $qK_H(\mathcal{B})$. A general element of $\mathfrak{K}$ is presented as:

$$f[\mathcal{O}_{\mathcal{B}}], \qquad \text{where } f = \sum_{\beta \in Q^\vee_+, \vec{n} \in \mathbb{Z}^r_{\ge 0}} f_{\beta,\vec{n}}(q) x^{\vec{n}} Q^\beta \in (\mathbb{C}P[q^{\pm 1}, x_1, \ldots, x_r])[\![Q^\vee_+]\!]$$

(where x_i acts as $A_i(q)$ for each $i \in \mathtt{I}$). By Theorem 3.9, the equations of the form

$$f(q, p_1^{-1} q^{Q_1 \partial_{Q_1}}, \ldots, p_r^{-1} q^{Q_r \partial_{Q_r}}, Q) J(Q, q) = 0,$$

that yield all representatives of $0 \in \mathfrak{K}$ by Theorem 1.29, imply

$$\sum_{\beta \in Q^\vee_+, \vec{n} \in \mathbb{Z}^r_{\ge 0}} f_{\beta,\vec{n}}(q) q^{-\langle \beta, \lambda[\vec{n}] \rangle} \chi_q(\mathcal{Q}(\gamma - \beta), \mathcal{O}_{\mathcal{Q}(\gamma-\beta)}(\lambda[\vec{n}])) = 0 \quad \lambda \in P, \gamma \in Q^\vee_+.$$

By Proposition 3.10 and [32, Proposition D.1], this further implies

$$\sum_{\beta \in Q^\vee_+, \vec{n} \in \mathbb{Z}^r_{\ge 0}} f_{\beta,\vec{n}}(q) \mathrm{gch}\, H^0(\mathbf{Q}(t_\beta), \mathcal{O}_{\mathbf{Q}(t_\beta)}(\lambda[\vec{n}])) = 0 \quad \lambda \in P.$$

Taking [32, Corollary 5.9] into account (and the fact that our K-group intersects with the dense subset of the K-group in [32, §6]), we derive

$$\sum_{\beta \in Q^\vee_+, \vec{n} \in \mathbb{Z}^r_{\ge 0}} f_{\beta,\vec{n}}(q) [\mathcal{O}_{\mathbf{Q}(t_\beta)}(-\sum_{i=1}^r n_i \varpi_i)] = 0 \in \widetilde{K}'(\mathbf{Q}_G),$$

whose $q = 1$ specialization yields

$$\sum_{\beta \in Q^\vee_+, \vec{n} \in \mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(1)[\mathcal{O}_{\mathbf{Q}(t_\beta)}(-\sum_{i=1}^r n_i \varpi_i)] = 0 \in K_H(\mathbf{Q}^{\mathrm{rat}}_G).$$

This induces a $\mathbb{C}P$-linear map $\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \longrightarrow K_H(\mathbf{Q}^{\mathrm{rat}}_G)$ that sends $[\mathcal{O}_{\mathcal{B}}]$ to $[\mathcal{O}_{\mathbf{Q}_G}]$, and the multiplication by Q^β to the right multiplication by β for each $\beta \in Q^\vee$. In Theorem 3.8 2), the multiplication by $p_i^{-1} q^{Q_i \partial_{Q_i}}$ on the first factor $J'(Qq^\lambda, q)$ results in the line bundle twist by $\mathcal{O}_{\mathfrak{Q}(\beta)}(-\varpi_i)$. Hence, it corresponds to the line bundle twist by $\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)$ in $K_H(\mathbf{Q}^{\mathrm{rat}}_G)$. In view of Theorem 1.29 (and the definition of the shift operators), the quantum multiplication by $a_i([\mathcal{O}_{\mathcal{B}}])$ becomes the endomorphism $\Xi(-\varpi_i)$ ($i \in \mathtt{I}$) via Ψ.

By Proposition 1.25, the map Ψ is surjective. It must be injective as the both sides are free modules of rank $|W|$ over the Noetherian rings

$$\mathbb{C}P \otimes (\mathbb{C}Q^\vee \otimes_{\mathbb{C}Q^\vee_+} \mathbb{C}[\![Q^\vee_+]\!]) \xrightarrow{\cong} \mathbb{C}Q^\vee \otimes_{\mathbb{C}Q^\vee_+} \mathcal{R}$$

identified through Ψ (see Lemma 1.18). □

Corollary 3.13. *We have a $\mathbb{C}P$-module isomorphism*

$$\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \xrightarrow{\cong} K_H(\mathbf{Q}^{\mathrm{rat}}_G),$$

that sends $[\mathcal{O}_{\mathcal{B}}]$ to $[\mathcal{O}_{\mathbf{Q}_G}]$, quantum product of a line bundle $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ ($i \in \mathtt{I}$) to the tensor product of $\mathcal{O}_{\mathbf{Q}^{\mathrm{rat}}_G}(-\varpi_i)$, and the multiplication by Q^β to the right $Q^\vee$-action of β for each $\beta \in Q^\vee$.

Proof. Combine Theorem 3.11 and Theorem 1.33 (cf. Theorem 1.31). □

4 Schubert classes under Ψ and consequences

Keep the setting of the previous sections.

4.1 Graph spaces and quasi-map spaces

A variety $\mathfrak{Y}$ is said to have rational singularities if there exists a resolution of singularities $g : \mathfrak{Z} \to \mathfrak{Y}$ that satisfies $g_* \mathcal{O}_{\mathfrak{Z}} \cong \mathcal{O}_{\mathfrak{Y}}$ and $\mathbb{R}^{>0} g_* \mathcal{O}_{\mathfrak{Z}} \cong \{0\}$ ([35, Theorem 5.10]). This is equivalent to assume that all the resolutions of singularities of $\mathfrak{Y}$ have the same property ([*loc. cit.*]).

By the theorem on formal functions ([22, II §11]), the rationality of singularities on a normal variety (i.e. the vanishing of the higher direct images of the structure sheaf from a resolution) is detected by its completion. Therefore, in case $\mathfrak{Y}$ is a normal variety with its (not necessarily closed) point y, we say the completion of $\mathfrak{Y}$ along y has a rational singularity if $\mathfrak{Y}$ has a rational singularity along y (cf. [50, Lemma 15.51.6]).

Remark 4.1. In the below, Theorem 4.2 and other assertions about the rational resolutions of singularities $f : \mathfrak{X} \to \mathfrak{Y}$ are used only to ensure $f_* \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_{\mathfrak{Y}}$ and $\mathbb{R}^{>0} f_* \mathcal{O}_{\mathfrak{X}} \cong \{0\}$. To this end, it suffices to assume that f is a birational projective morphism and $\mathfrak{X}$ has only rational singularities (or quotient singularities by [35, Proposition 5.15]) as we can replace $\mathfrak{X}$ with its resolution of singularities.

We have a morphism $\pi_{n,\beta} : \mathscr{GB}_{n,\beta} \to \mathscr{Q}(\beta)$ ($n \in \mathbb{Z}_{\geq 0}, \beta \in Q_+^\vee$) that factors through $\mathscr{GB}_{0,\beta}$ (Givental's main lemma [21]; see [13, §8] and [17, §1.3]). We define

$$\mathcal{O}_{\mathscr{GB}_{n,\beta}}(\lambda) := \pi_{n,\beta}^* \mathcal{O}_{\mathscr{Q}(\beta)}(\lambda) \qquad \lambda \in P. \tag{4.1}$$

Theorem 4.2 (Braverman-Finkelberg [6, 7, 8]). *The morphism $\pi_{0,\beta}$ is a rational resolution of singularities* (*in an orbifold sense*). □

We note that $\mathscr{GB}_{n,\beta}$ is irreducible ([51, 33]).

Let $\mathscr{X}(\beta)$ denote the subvariety of $\mathscr{GB}_{2,\beta}$ consisting of the stable maps whose first marked point projects to $0 \in \mathbb{P}^1$, and whose second marked point projects to $\infty \in \mathbb{P}^1$ through the projection of a genus zero domain curve C to the main component $C_0 \cong \mathbb{P}^1$. Let us denote the restriction of $\mathtt{ev}_i$ ($i = 1, 2$) to $\mathscr{X}(\beta)$ by the same letter. By Theorem 4.2, $\mathscr{X}(\beta)$ also gives a resolution of singularities $\pi_\beta : \mathscr{X}(\beta) \to \mathscr{Q}(\beta)$. The following is a result of Buch-Chaput-Mihalcea-Perrin [10]:

Theorem 4.3 ([10] Corollary 3.8, cf. [29] Theorem 5.1). *The variety*

$$\mathtt{ev}_1^{-1}(\mathscr{B}(w)) \cap \mathtt{ev}_2^{-1}(\mathscr{B}^{\mathrm{op}}(v)) \subset \mathscr{X}(\beta)$$

is irreducible, normal, and has rational singularities (*that we denote by* $\mathscr{X}(\beta, w, v)$) *for each* $w, v \in W$. □

We remark that $\mathscr{X}(\beta) = \mathscr{X}(\beta, e, w_0)$. We set $\mathscr{X}(\beta, w) := \mathscr{X}(\beta, w, w_0)$ and $\mathscr{Q}(\beta, w, v) := \pi_\beta(\mathscr{X}(\beta, w, v)) \subset \mathscr{Q}(\beta, w)$. Then, the map π_β restricts to a $(\mathbb{G}_m \times H)$-equivariant birational proper map

$$\pi_{\beta,w,v} : \mathscr{X}(\beta, w, v) \to \mathscr{Q}(\beta, w, v)$$

by [29, §5.2]. We denote π_{β,w,w_0} by $\pi_{\beta,w}$ for simplicity. Let $\mathcal{O}_{\mathscr{X}(\beta,w,v)}(\lambda)$ denote the restriction of $\mathcal{O}_{\mathscr{GB}_{2,\beta}}(\lambda)$ to $\mathscr{X}(\beta, w, v)$ for each $\lambda \in P$.

The space $\mathscr{Q}(\beta, w, v)$ is called the Richardson variety of $\mathbf{Q}_G^{\mathrm{rat}}$ in [29] (see also Remark 1.11). In particular, we have

$$\mathscr{Q}(\beta, w, v) = \overline{\mathbf{I} p_w \cap \mathbf{I}^- p_{vt_\beta}} = \mathbf{Q}_G(w) \cap \overline{\mathbf{I}^- p_{vt_\beta}} \subset \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}(\!(z)\!)) \tag{4.2}$$

by comparing the definitions in [29, §4.1] and [29, §5.2]. We have

$$\dim \mathscr{Q}(\beta, w, v) = 2 \langle \beta, \rho \rangle - \ell(w) + \ell(v) \tag{4.3}$$

and $\mathscr{Q}(\beta, w, v) \neq \emptyset$ if and only if $w \geq_{\frac{\infty}{2}} vt_\beta$ by [29, Corollary 5.4].

Let us gather several results on $\mathscr{Q}(\beta, w, v)$ from various places in [29]:

Theorem 4.4 ([29]). *For each $w, v \in W$ and $\beta \in Q_+^\vee$, we have:*

1. (Theorem 5.20) *The variety $\mathscr{Q}(\beta, w, v)$ is normal;*

2. (Theorem 4.33) *For each $\lambda \in P_+$, we have*

$$H^{>0}(\mathscr{Q}(\beta, w, v), \mathcal{O}_{\mathscr{Q}(\beta,w,v)}(\lambda)) = \{0\};$$

3. (Theorem 4.33) *For $\beta' \in Q^\vee_+$ such that $\beta < \beta'$ and $\lambda \in P_+$, the natural restriction map*

$$H^0(\mathcal{Q}(\beta', w), \mathcal{O}_{\mathcal{Q}(\beta',w)}(\lambda)) \longrightarrow H^0(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda))$$

is surjective;

4. (Proposition 4.39 and §5.2) *Let $i \in \mathtt{I}$ be such that $s_i w < w$ and $s_i v < v$. Then, the variety $\mathcal{Q}(\beta, w, v)$ is B_i-stable, and we have an inflation map $\pi_i : SL(2, i) \times^{B_i} \mathcal{Q}(\beta, w, v) \to \mathcal{Q}(\beta, s_i w, v)$. We have*

$$\mathbb{R}^\bullet(\pi_i)_* \mathcal{O}_{SL(2,i)\times^{B_i}\mathcal{Q}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta,s_i w,v)};$$

5. (Proposition 4.39 and Lemma 4.6) *Assume that $s_\vartheta w > w$ and $s_\vartheta v > v$. Then, the variety $\mathcal{Q}(\beta, w, v)$ admits a B_0-action and we have an inflation map*

$$\pi_0 : SL(2, 0) \times^{B_0} \mathcal{Q}(\beta, w, v) \to \mathcal{Q}(\beta - w^{-1}\vartheta^\vee, s_\vartheta w, v).$$

We have $\mathbb{R}^\bullet(\pi_0)_* \mathcal{O}_{SL(2,0)\times^{B_0}\mathcal{Q}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta - w^{-1}\vartheta^\vee, s_\vartheta w, v)}$.

For each $\beta \in Q^\vee_+, w, v \in W$, we consider the open subset $\mathring{\mathcal{Q}}(\beta, w, v) \subset \mathcal{Q}(\beta, w, v)$ consisting of quasi-maps defined at $0 \in \mathbb{P}^1$ (i.e. have no defect at 0) and their values at 0 belong to $\mathbb{O}_{\mathcal{B}}(w)$. The variety $\mathcal{Q}(\beta, w, v)$ is decomposed as

$$\mathcal{Q}(\beta, w, v) = \bigsqcup_{0 \le \beta' \le \beta} \bigsqcup_{\substack{w' \in W \\ w \ge_{\frac{\infty}{2}} w' t_{\beta'}}} \mathring{\mathcal{Q}}(\beta - \beta', w', v),$$

where the inclusion map is given by the restriction of the map $\imath_{\beta'}$ to $\mathcal{Q}(\beta - \beta', w', v) \subset \mathcal{Q}(\beta - \beta')$, that lands on $\mathcal{Q}(\beta, w, v)$ by (4.2). We have $\mathring{\mathcal{Q}}(\beta, w) = \mathring{\mathcal{Q}}(\beta, w, w_0)$, and $\mathring{\mathcal{Q}}(\beta) = G\mathring{\mathcal{Q}}(\beta, w)$ for every $w \in W$.

4.2 Formal neighborhoods of Bruhat cells

For two (affine) schemes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{Y}$ is the spectrum of a complete local ring (and hence has a unique closed point $0 \in \mathfrak{Y}$), we denote by $\mathfrak{X} \widehat{\times} \mathfrak{Y}$ the formal completion of $\mathfrak{X} \times \mathfrak{Y}$ along the point $\mathfrak{X} \times 0$.

Proposition 4.5. *Let $\beta \in Q^\vee_+$. The formal completion $\mathfrak{N}$ of $\mathbf{Q}_G$ along $\mathbb{O}(t_\beta)$ is isomorphic to*

$$\mathfrak{N} \cong \mathbb{O}(t_\beta) \widehat{\times} \operatorname{Spec} \mathbb{C}[\mathcal{Z}(\beta)]^\wedge_{\mathbf{0}}. \tag{4.4}$$

Proof. We have an embedding $\mathbf{Q}_G(t_\beta) \subset \mathbf{Q}_G$ as schemes of infinite types, equipped with the $\mathbf{I}$-action. In particular, we have a sheaf $\mathcal{I}$ of ideals of $\mathbf{Q}_G(t_\beta)$ in $\mathcal{O}_{\mathbf{Q}_G}$. The completion of $\mathcal{O}_{\mathbf{Q}_G}$ with respect to $\mathcal{I}$ yields the infinitesimal neighbourhood $\mathfrak{N}^+$ of $\mathbf{Q}_G(t_\beta)$ in $\mathbf{Q}_G$. Since $\mathbb{O}(t_\beta)$ is an affine scheme, $\mathfrak{N}$ is also affine. Let us impose ($\mathbb{G}_m \times H$-stable) set of equations by lifting the maximal ideal of $\mathbb{C}[\mathbb{O}(t_\beta)]$ that defines p_{t_β} through the pullback:

$$\mathbb{C}[\mathfrak{N}] \supset \sum_{i \in \mathtt{I}} \Gamma(\mathbf{Q}_G, \mathcal{O}(\varpi_i))\phi^{-1}_{i,t_\beta} \longrightarrow \sum_{i \in \mathtt{I}} \Gamma(\mathbf{Q}_G(t_\beta), \mathcal{O}(\varpi_i))\phi^{-1}_{i,t_\beta} \subset \mathbb{C}[\mathbb{O}(t_\beta)].$$

This construction factors through the formal completion of $\mathcal{Q}(\beta+\gamma)$ along $\imath_{\gamma}(\mathring{\mathcal{Q}}(\beta))$ for an arbitrary $\gamma \in Q_{+}^{\vee}$ in view of (4.2). By Proposition 3.6, the resulting ring is the formal completion of $\mathbb{C}[\mathcal{Z}(\beta)]$ along the origin.

The ring $\mathbb{C}[\mathcal{Z}(\beta)]$ is $\mathbb{G}_m$-stable, its grading is concentrated in ≥ 0, and the degree 0-part is $\mathbb{C}$. We consider the ($\mathbb{G}_m \times H$-stable) ideals $\mathbb{C}[\mathcal{Z}(\beta)]_{\geq m} \subset \mathbb{C}[\mathcal{Z}(\beta)]$ consisting of functions of degree $\geq m$ for each $m \in \mathbb{Z}_{\geq 0}$. Then, we have an isomorphism of rings

$$\mathbb{C}[\mathcal{Z}(\beta)]^{\wedge}_{\mathbf{0}} \cong \prod_{m \geq 0} \left(\frac{\mathbb{C}[\mathcal{Z}(\beta)]_{\geq m}}{\mathbb{C}[\mathcal{Z}(\beta)]_{\geq (m+1)}} \right),$$

where the multiplication of the LHS is that of the completion, and the multiplication of the RHS is that of the associated graded of a decreasing filtration $\{\mathbb{C}[\mathcal{Z}(\beta)]_{\geq m}\}_m$. This identification is also B^--equivariant (instead of B, due to the choice of p_{t_β} as the origin).

Here, $\mathbb{O}(t_\beta)$ admits a free homogeneous action by a pro-unipotent subgroup of $\mathbf{I}$ (namely $H[\![z]\!]_1 N[\![z]\!]$, where $H[\![z]\!]_1 = \ker(H[\![z]\!] \to H)$, see e.g. [29, §4.2]). The scheme $\mathfrak{N}$ also admits $\mathbb{G}_m \ltimes \mathbf{I}$-action. Thus, we use the $B[\![z]\!]$-action to construct a map

$$(\mathbb{G}_m \ltimes B[\![z]\!]) \times^{(\mathbb{G}_m \times H)} \operatorname{Spec} \mathbb{C}[\mathcal{Z}(\beta)]^{\wedge}_{\mathbf{0}} \longrightarrow \mathfrak{N} \tag{4.5}$$

such that the zero section maps isomorphically to $\mathbb{O}(t_\beta)$. In the LHS of (4.5), the formal completion and the (suitable) associated graded are still isomorphic as the effect of $(\mathbb{G}_m \ltimes B[\![z]\!]) \times^{(\mathbb{G}_m \times H)} \bullet$ to the coordinate ring is just to take the (completed) tensor product with $\mathbb{C}[H[\![z]\!]_1 N[\![z]\!]]$.

The map (4.5) can be also obtained through the formal completion of the map

$$(\mathbb{G}_m \ltimes B[\![z]\!]) \times^{(\mathbb{G}_m \times H)} \mathcal{S} \to \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes \mathbb{C}[\![z]\!]) \tag{4.6}$$

along $\mathbb{O}(t_\beta)$ (and its preimages) if we take a $(\mathbb{G}_m \times H)$-stable ambient space $p \in \mathcal{Z}(\beta) \subset \mathcal{S}$, though this map itself *cannot* be injective. Nevertheless, Theorem 1.10 asserts that the image $(\mathbb{G}_m \ltimes B[\![z]\!]) \times^{(\mathbb{G}_m \times H)} \mathcal{Z}(\beta)$ under (4.6) contains a dense open subset $\mathbb{O}(e) \subset \mathbf{Q}_G$. Thus, we conclude that the induced map

$$\imath : \mathbb{C}[\mathfrak{N}] \longrightarrow \mathbb{C}[\mathbb{O}(t_\beta)] \mathbin{\widehat{\otimes}} \mathbb{C}[\mathcal{Z}(\beta)]^{\wedge}_{\mathbf{0}} \tag{4.7}$$

is injective. Since $\imath$ is a continuous (injective) map of complete topological rings, it is an isomorphism if the image is dense. The both sides of (4.7) carry actions of $B[\![z]\!]$, and the effect differs as we regard the RHS as the associated graded that neglects the higher order contribution of the action. As higher order modifications preserve the surjectivity of a morphism between complete local rings, we conclude that each $B[\![z]\!]$-translation of the specialization in the first paragraph turns (4.7) into a surjection.

For each $\gamma \in Q_{+}^{\vee}$, the formal completion of $\mathcal{Q}(\beta+\gamma)$ along $\imath_\beta(\mathring{\mathcal{Q}}(\gamma, e))$ yields a closed subscheme $\mathfrak{Q}(\gamma, \beta) \subset \mathfrak{N}$. We have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{N}] & \hookrightarrow & \mathbb{C}[\mathbb{O}(t_\beta)] \mathbin{\widehat{\otimes}} \mathbb{C}[\mathcal{Z}(\beta)]^{\wedge}_{\mathbf{0}} \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathfrak{Q}_{\beta,\gamma}] & \xrightarrow{h_\gamma} & \mathbb{C}[\mathring{\mathcal{Q}}(\gamma, e)] \mathbin{\widehat{\otimes}} \mathbb{C}[\mathcal{Z}(\beta)]^{\wedge}_{\mathbf{0}} \end{array},$$

where h_γ is the map obtained by restricting the pullback under (4.6). In view of the previous paragraph, the map h_γ is an isomorphism when we specialize to each closed point of $\mathring{\mathcal{Q}}(\gamma, e)$ (note that $\mathfrak{Q}_{\beta,\gamma}$ is locally an algebraic fiber bundle over $\mathring{\mathcal{Q}}(\gamma, e)$ whose fiber is $\mathrm{Spec}\,\mathbb{C}[\mathcal{Z}(\beta)]^\wedge_{\mathbf{0}}$). Therefore, Nakayama's lemma, applied to an arbitrary finite-degree truncation of $\mathbb{C}[\mathring{\mathcal{Q}}(\gamma, e)] \widehat{\otimes}\, \mathbb{C}[\mathcal{Z}(\beta)]^\wedge_{\mathbf{0}}$ with respect to the second factor, yields that h_γ is an isomorphism.

In particular, we find a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{N}] & \hookrightarrow & \mathbb{C}[\mathbb{O}(t_\beta)] \,\widehat{\otimes}\, \mathbb{C}[\mathcal{Z}(\beta)]^\wedge_{\mathbf{0}} \\ & \searrow \quad \swarrow & \\ & \mathbb{C}[\mathfrak{Q}_{\beta,\gamma}] & \end{array}.$$

By taking the projective limit (cf. Corollary 3.7), we deduce that the both of

$$\mathbb{C}[\mathfrak{N}] \subset \varprojlim_{\gamma} \mathbb{C}[\mathfrak{Q}_{\beta,\gamma}] \supset \mathbb{C}[\mathbb{O}(t_\beta)] \,\widehat{\otimes}\, \mathbb{C}[\mathcal{Z}(\beta)]^\wedge_{\mathbf{0}}$$

are dense subsets. Therefore, we conclude that $\imath$ is an isomorphism as required. $\square$

Corollary 4.6. *Let $\beta, \beta' \in Q^\vee_+$, and let $w, w', v \in W$ such that $\mathcal{Q}(\beta - \beta', w', v) \subset \mathcal{Q}(\beta, w, v)$. Then, the formal completion of $\mathcal{Q}(\beta, w, v)$ along $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$ yields a locally trivial fiber bundle over $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$ whose fiber is the spectrum of a normal Noetherian ring completed at a maximal ideal. In addition, the structure of the fiber (that we denote by $\mathfrak{X}_{\beta', w, w'}$) is the same as that of the formal completion of $\mathbf{Q}_G(w)$ along $\mathbb{O}(w' t_{\beta'})$, and does not depend on β and v.*

Proof. Consider the formal completion $\mathfrak{Q}$ of $\mathcal{Q}(\beta, e, v)$ along $\mathring{\mathcal{Q}}(\beta - \beta', e, v)$. We have a map $\eta : \mathfrak{Q} \hookrightarrow \mathfrak{N}$ by the natural inclusions $\mathring{\mathcal{Q}}(\beta - \beta', e, v) \subset \mathbb{O}(t_{\beta'})$ and $\mathcal{Q}(\beta, e, v) \subset \mathbf{Q}_G$ induced from (4.2). We have a set-theoretic defining equations of $\mathcal{Q}(\beta, e, v) \subset \mathbf{Q}_G$ consisting of some of $\phi_{i,u}$ for $u <_{\frac{\infty}{2}} t_{\beta'}$ and $i \in \mathtt{I}$ (see the last paragraphs of §1.4). Since they are obtained as the (uniquely determined) preimages of of the middle map

$$\mathbb{C}[\mathfrak{N}] \supset \Gamma(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\varpi_i)) \twoheadrightarrow \Gamma(\mathbf{Q}_G(t_{\beta'}), \mathcal{O}_{\mathbf{Q}_G(t_{\beta'})}(\varpi_i)) \subset \mathbb{C}[\mathbb{O}(t_{\beta'})],$$

we can regard them as functions on $\mathbb{C}[\mathbb{O}(t_{\beta'})] \subset \mathbb{C}[\mathfrak{N}]$. Thus, the trivial fiber bundle structure of $\mathfrak{N}$ in Proposition 4.5 restricts to a trivial algebraic fiber bundle structure on $\mathfrak{Q}$ along $\mathring{\mathcal{Q}}(\beta - \beta', e, v)$ whose fiber is isomorphic to $\mathrm{Spec}\,\mathbb{C}[\mathcal{Z}(\beta')]^\wedge_{\mathbf{0}}$ (as $\mathfrak{Q}$ is reduced, each fiber must be reduced). In other words, we can understand this fiber bundle as the restriction of (4.4) from $\mathbb{O}(t_{\beta'})$ to $\mathring{\mathcal{Q}}(\beta - \beta', e, v)$ by intersecting the base with the opposite Schubert variety corresponding to $v t_\beta$.

For each $u \in W$, the orbit $\mathbb{O}(u)$ is the preimage of $\mathbb{O}^{\mathcal{B}}(u)$ under the (uncountable dimensional) affine fibration $G\mathbb{O}(e) \to \mathcal{B}$ obtained by setting $z = 0$. As $\mathbb{O}^{\mathcal{B}}(u) \subset \dot{u}\mathbb{O}^{\mathcal{B}}(e)$, we deduce that $\mathbb{O}(u) \subset \dot{u}\mathbb{O}(e)$. In addition, the normal direction of $\mathbb{O}(u) \subset \dot{u}\mathbb{O}(e)$ is given by the free action of $(N^- \cap \dot{u} N \dot{u}^{-1})$. The same is true if we multiply u, e in $\mathbb{O}(\bullet)$ by $t_{\beta'}$ (cf. Theorem 1.10). Let $\mathfrak{N}_{w'}$

denote the formal completion of $\mathbf{Q}_G$ along $\mathbb{O}(w't_{\beta'})$. In view of Proposition 4.5, we deduce

$$\mathfrak{N}_{w'} \cong \mathbb{O}(w't_{\beta'}) \mathbin{\widehat{\times}} \mathfrak{D}', \tag{4.8}$$

where $\mathfrak{D}'$ is the completion of $\operatorname{Spec}\mathbb{C}[\mathcal{Z}(\beta')]^{\wedge}_{\mathbf{0}} \times (N^- \cap \dot{u}N\dot{u}^{-1})$ at $(\mathbf{0}, 1)$. If we further replace $\mathbf{Q}_G = \mathbf{Q}_G(e)$ with $\mathbf{Q}_G(w)$ for $w \in W$, then we take the closed subscheme of $\mathfrak{N}_{w'}$ by imposing the defining equations of $\mathbf{Q}_G(w) \subset \mathbf{Q}_G$ near $\mathbb{O}(w't_{\beta'})$. We set $E := \mathbb{G}_m \ltimes ((\dot{w}'B(\dot{w}')^{-1})[\![z]\!] \cap \mathbf{I}) \subset \mathbb{G}_m \ltimes \mathbf{I}$. The E-stabilizer of $p_{w't_{\beta'}}$ is $(\mathbb{G}_m \times H)$. Since $\mathbb{O}(w't_{\beta'})$ is homogeneous under the E-action, we obtain a scheme $\mathfrak{D}''$ such that

$$(\mathfrak{N}_{w'} \cap \mathbf{Q}_G(w)) \cong \mathbb{O}(w't_{\beta'}) \mathbin{\widehat{\times}} \mathfrak{D}''$$

from (4.8). The scheme $\mathfrak{D}''$ is identified with the fiber of the fiber bundle structure on the formal completion of $\mathcal{Q}(\beta, w, v)$ along $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$ as we can interpret the construction of $\mathfrak{D}''$ to be imposing the local defining equations of $\mathcal{Q}(\beta, w, v) \subset \mathcal{Q}(\beta, e, v)$ near $\imath_{\beta'}(\mathring{\mathcal{Q}}(\beta - \beta', w', v))$ to $\mathfrak{D}'$ by (4.2). This construction must yield a locally trivial family as being the restriction of an E-equivariant closed subfamily of the case $w = e$. The scheme $\mathfrak{D}''$ is the completion of a Noetherian ring since $\operatorname{Spec}\mathbb{C}[\mathcal{Z}(\beta')]^{\wedge}_{\mathbf{0}}$, and hence $\mathfrak{D}'$ is so. Thus, it is normal as $\mathcal{Q}(\beta, w)$, and hence the neighbourhood of $\imath_{\beta-\beta'}(\mathring{\mathcal{Q}}(\beta', w)) \subset \mathcal{Q}(\beta, w)$ is normal by [47, Theorem 32.2].

These complete the construction of (the family of) the required locally trivial fibrations. ☐

4.3 Cohomology calculation for $\mathcal{X}(\beta, w)$

Lemma 4.7. *For each $\beta \in Q^\vee_+, w, v \in W$, we have*

$$(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta,w,v)}.$$

Proof. This follows from the normality of $\mathcal{Q}(\beta, w, v)$ and the fact that all the fibers of $\pi_{\beta,w,v}$ are connected ([29, Corollary 5.19]). ☐

Theorem 4.8. *Let $\beta' \in Q^\vee_+$, and let $w, w' \in W$ such that $\mathbb{O}(w't_{\beta'}) \subset \mathbf{Q}_G(w)$. The scheme $\mathfrak{X}_{\beta',w,w'}$ in Corollary 4.6 have rational singularities.*

Proof. We set $\mathfrak{X} := \mathfrak{X}_{\beta',w,w'}$. We assume to the contrary to deduce contradiction. A product of spectrums of local rings have rational singularities if and only if each of them admits rational singularities. By Corollary 4.6, if $\mathfrak{X}$ have singularities worse than rational singularities, then we have

$$\mathcal{Q}(\beta - \beta', w', v) \subset \operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}. \tag{4.9}$$

This containment is independent of the choice of v, though the LHS maybe an empty set. We enlarge β if necessary to guarantee $\mathcal{Q}(\beta - \beta', w', v) \neq \emptyset$ for every $v \in W$, and some point (and hence general points) of $\mathcal{Q}(\beta - \beta', w', v)$ has no defect at ∞ and its value belongs to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$ by [16, Lemma 8.5.1]. Let $Z(v)$ be an irreducible component of $\operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ that contains $\mathcal{Q}(\beta - \beta', w', v)$. General points of $Z(v)$ have no defect at ∞, and their values at ∞ belong to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$. We set $U := N \cap \dot{v}N\dot{v}^{-1}$. Then, the multiplication map $U \times \mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v) \subset \mathcal{B}$ defines an embedding of an open dense subset. Thus, $\overline{U \times Z(v)}$ is an irreducible component of the support of (4.9) for $v = w_0$ that contains $\mathcal{Q}(\beta - \beta', w', w_0)$. Therefore, we obtain a family $\{Z(v)\}_{v \in W}$ of irreducible components of $\mathrm{Supp}\, \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ such that $\mathcal{Q}(\beta - \beta', w', v) \subset Z(v)$, $Z(v) \subset Z(u)$ if $v \le u$, and $\overline{BZ(v)}$ is independent of v.

Let $i \in \mathtt{I}_{\mathrm{af}}$ be such that $SL(2,i)Z(v) \not\subset Z(v)$ for $i \neq 0$ and $s_i v < v$, or $i = 0$ and $s_\vartheta v > v$, inside $\mathcal{Q}(\beta)$ $(i \neq 0)$ or $\mathcal{Q}(\beta - w^{-1}\vartheta)$ ($i = 0$, see Theorem 4.4 5)). Since B_i acts on $\mathcal{Q}(\beta, w, v)$ (by Theorem 4.4), it follows that $Z(v)$ is B_i-stable. We have $s_i w < w$ $(i \neq 0)$ or $s_\vartheta w > w$ $(i = 0)$ as otherwise $SL(2,i)$ acts on $\mathcal{Q}(\beta, w, v)$ and hence on $Z(v)$, that is a contradiction.

The map π_i restricted to $SL(2,i) \times^{B_i} Z(v)$ is birational onto its image. In particular, there exists a Zariski open subset $V \subset SL(2,i)Z(v)$ such that $\pi_i^{-1}(V) \cap SL(2,i) \times^{B_i} Z(v)$ forms an irreducible component of

$$\pi_i^{-1}(V) \cap \left(SL(2,i) \times^{B_i} \mathrm{Supp}\, \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}\right) \subset SL(2,i) \times^{B_i} \mathcal{Q}(\beta, w, v).$$

Hence, $(\pi_i)_*$ sends the inflation of $\mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ to a non-zero sheaf whose support contains $SL(2,i)Z(v)$.

If the variety $\mathcal{Q}(\beta, w, v)$ is B_i-stable for $i \in \mathtt{I}$, then the G-action on $\mathcal{X}(\beta)$ restricts to the B_i-action on $\mathcal{X}(\beta, w, v)$. If the variety $\mathcal{Q}(\beta, w, v)$ is B_0-stable, then there exists a smooth projective variety $\mathcal{X}'(\beta, w, v)$ with the B_0-action that yields a B_0-equivariant resolution of singularities of $\mathcal{Q}(\beta, w, v)$ (see e.g. [52, Corollary 7.6.3]). We can replace $\pi_{\beta,w,v} : \mathcal{X}(\beta, w, v) \to \mathcal{Q}(\beta, w, v)$ with $\mathcal{X}'(\beta, w, v) \to \mathcal{Q}(\beta, w, v)$ in this case since both of $\mathcal{X}(\beta, w, v)$ and $\mathcal{X}'(\beta, w, v)$ have rational singularities and there exists yet another resolution of singularities of $\mathcal{Q}(\beta, w, v)$ that dominates the both. Let us consider the map

$$SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v) \to SL(2,i) \times^{B_i} \mathcal{Q}(\beta, w, v) \xrightarrow{\pi_i} \mathcal{Q}(\gamma, u, v), \tag{4.10}$$

where the first map is the inflation of $\pi_{\beta,w,v}$, $\gamma = \beta$ and $u = s_i w$ $(i \neq 0)$, or $\gamma = \beta - w^{-1}\vartheta^\vee$ and $u = s_\vartheta w$ $(i = 0)$. Since $\mathcal{X}(\beta, w, v)$ has rational singularities, so is $SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v)$. In addition, the composition map (4.10) is birational and projective. Thus, it is another resolution of $\mathcal{Q}(\gamma, u, v)$ by a variety that has rational singularities. Therefore, we can replace $\mathcal{X}(\gamma, u, v)$ with $SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v)$ to compute $\mathbb{R}^\bullet(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)}$. Applying the Leray spectral sequence to (4.10) using Theorem 4.4 4), 5), we have

$$\mathbb{R}^0(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)} \cong \mathcal{O}_{\mathcal{Q}(\gamma,u,v)} \quad \text{and}$$
$$\mathbb{R}^{>0}(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)} \neq \{0\}. \tag{4.11}$$

Moreover, the support of (4.11) contains $SL(2,i)Z(v)$. By construction, general points of $SL(2,i)Z(v)$ have no defect at ∞, and their values belong to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$. Therefore, an irreducible component $Z'(v)$ of the support of (4.11) that contains $SL(2,i)Z(v)$ again comes as a family $\{Z'(v)\}_{v \in W}$ such that $Z'(v) \subset Z'(u)$ if $v \le u$, and $\overline{BZ'(v)}$ is independent of v (in particular, we have $Z'(v)$ even if $i \neq 0$ and $s_i v > v$, or $i = 0$ and $s_\vartheta v < v$). Thus, we can repeat the above procedure by replacing $\mathcal{Q}(\beta, w, v)$ with $\mathcal{Q}(\gamma, u, v)$ and $\{Z(v)\}_{v \in W}$ with $\{Z'(v)\}_{v \in W}$. Note that we eventually attain $Z'(v) = SL(2,i)Z(v)$ for any application of the above procedures as the strict inclusion forces

$$(0 <) \quad \mathrm{codim}_{\mathcal{Q}(\gamma,u,v)} Z'(v) < \mathrm{codim}_{\mathcal{Q}(\gamma,u,v)} SL(2,i)Z(v) = \mathrm{codim}_{\mathcal{Q}(\beta,w,v)} Z(v),$$

that cannot be repeated infinitely many times.

Consider the minimal $\mathcal{Q}(\theta, t, v)$ ($\theta \in Q_+^\vee$ and $t \in W$) that contains $Z(v)$. Here $\mathring{\mathcal{Q}}(\theta, t, v)$ contains a point of $Z(v)$ as $Z(v)$ is irreducible and the inclusion relations among $\mathcal{Q}(\beta - \bullet, \bullet, v)$ obey the closure relation of $\mathbf{I}$-orbits of $\mathbf{Q}_G^{\mathrm{rat}}$ described in Theorem 1.10 by their irreducibility and (4.2). Hence, the condition $SL(2, i)Z(v) \not\subset Z(v)$ is achieved if it holds for $\mathcal{Q}(\theta, t, v)$. Since $\overline{BZ(v)}$ is common for every $v \in W$, we deduce that θ and u are independent of v. In addition, $\mathcal{Q}(\theta, t, v)$ is transformed to $\mathcal{Q}(\theta, s_i t, v)$ ($i \neq 0$) or $\mathcal{Q}(\theta - t^{-1}\vartheta, s_\vartheta t, v)$ ($i = 0$) by an application of the above procedure.

In view of [26, Theorem 4.6] (cf. arguments around there), we repeat these procedures if necessary to assume $t = w_0$. The condition $SL(2, i)Z(v) \not\subset Z(v)$ implies $SL(2, i)\mathcal{Q}(\beta, w, v) \not\subset \mathcal{Q}(\beta, w, v)$ (otherwise $Z(v)$ is $SL(2, i)$-stable) asserts that $s_i t < t$ implies $s_i w < w$ ($i \in \mathtt{I}$). In case $t = w_0$, this implies $w = w_0$. Again by repeating the above procedures, we can rearrange the situation to assume $w = t = e$ and $v = w_0$. In this case, we have $\mathcal{Q}(\beta) = \mathcal{Q}(\beta, e, w_0)$, that has rational singularities by Theorem 4.2. From this, we find a contradiction on the existence of $Z(v)$. This in turn implies a contradiction to the existence of (β', w'). Therefore, we conclude that our $\mathfrak{X}$ have rational singularities. □

The following is our main geometric result in this paper, that is an extension of Theorem 4.2 due to Braverman-Finkelberg [6, 7, 8] from $\mathcal{Q}(\beta)$ to an arbitrary Richardson variety of $\mathbf{Q}_G^{\mathrm{rat}}$:

Theorem 4.9. *For each $w, v \in W$ and $\beta \in Q_+^\vee$, the variety $\mathcal{Q}(\beta, w, v)$ has rational singularities. In particular, $\mathcal{Q}(\beta, w, v)$ is Cohen-Macaulay.*

For each $\beta \in Q_+^\vee, w, v \in W$, we set

$$\mathring{\mathring{\mathcal{Q}}}(\beta, w, v) := \left\{ f \in \mathcal{Q}(\beta) \middle| 0, \infty \not\in [D], f(0) \in \mathbb{O}_{\mathcal{B}}(w), f(\infty) \in \mathbb{O}_{\mathcal{B}}^{\mathrm{op}}(v) \right\}.$$

In view of [29, Corollary 5.4], the inclusion $\mathring{\mathring{\mathcal{Q}}}(\beta, w, v) \subset \mathcal{Q}(\beta, w, v)$ is open dense.

Proof of Theorem 4.9. We assume to the contrary to deduce contradiction. Namely, we assume that $\mathcal{Q}(\beta, w, v)$ has singularities worse than rational singularities. In view of Corollary 4.6 and Theorem 4.8, if the worse than rational singularity locus of $\mathcal{Q}(\beta, w, v)$ is contained in $\mathcal{Q}(\gamma, u, v)$, then the variety $\mathcal{Q}(\gamma, u, v)$ itself have singularities worse than rational singularities. Therefore, by rearranging (β, w) if necessary, we can assume

$$\mathring{\mathcal{Q}}(\beta, w, v) \cap \operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)} \neq \emptyset. \tag{4.12}$$

For (4.12), we can also rearrange (β, v) by applying above arguments by swapping z with z^{-1}, and B with B^-. In particular, we can assume

$$\mathring{\mathring{\mathcal{Q}}}(\beta, w, v) \cap \operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)} \neq \emptyset. \tag{4.13}$$

The variety $\mathring{\mathring{\mathcal{Q}}}(\beta, w, v)$ admits only H-action, but its ambient space $\mathcal{Q}(\beta)$ admits a G-action. The action of $(N^- \cap \dot{w}N\dot{w}^{-1})$ applied to the subspace $\mathbb{O}_{\mathcal{B}}(w) \subset \mathcal{B}$, as well as the action of $(N \cap \dot{v}N\dot{v}^{-1})$ applied to the subspace $\mathbb{O}_{\mathcal{B}}^{\mathrm{op}}(v) \subset \mathcal{B}$ have trivial stabilizers. As a consequence, the action of $(N^- \cap \dot{w}N\dot{w}^{-1})$ applied to the subspace $\mathring{\mathring{\mathcal{Q}}}(\beta, w, u) \subset \mathcal{Q}(\beta)$, and the action of $(N \cap \dot{v}N\dot{v}^{-1})$ applied to the subspace $\mathring{\mathring{\mathcal{Q}}}(\beta, u, v) \subset \mathcal{Q}(\beta)$ also have trivial stabilizers for each $u \in W$. In view of the fact that $\mathbb{O}_{\mathcal{B}}(w)$ is N-stable, the action of $(N \cap \dot{v} N \dot{v}^{-1})$ on $\mathcal{Q}(\beta)$ preserves $\sqcup_{u \in W} \mathring{\mathring{\mathcal{Q}}}(\beta, w, u)$. Therefore, we deduce an embedding

$$(N^- \cap \dot{w} N \dot{w}^{-1}) \times (N \cap \dot{v} N \dot{v}^{-1}) \times \mathring{\mathring{\mathcal{Q}}}(\beta, w, v) \ni (n_1, n_2, x) \mapsto n_1 n_2 x \in \mathcal{Q}(\beta).$$

By the dimension comparison using (4.3), we deduce that this embedding must be open dense in $\mathcal{Q}(\beta) = \mathcal{Q}(\beta, e, w_0)$. The locus Y on which the singularity of $\mathring{\mathring{\mathcal{Q}}}(\beta, w, v)$ is worse than rational singularities gives rise to the locus

$$(N^- \cap \dot{w} N \dot{w}^{-1}) \times (N \cap \dot{v} N \dot{v}^{-1}) \times Y \subset \mathcal{Q}(\beta)$$

on which the singularity of $\mathcal{Q}(\beta)$ is worse than rational singularities.

However, the variety $\mathcal{Q}(\beta)$ has only rational singularities (Theorem 4.2). Thus, the locus of $Y \subset \mathcal{Q}(\beta, w, v)$ on which $\mathcal{Q}(\beta, w, v)$ has worse than rational singularity must be empty. Hence $\mathcal{Q}(\beta, w, v)$ must have rational singularities. The latter assertion follows from [35, Theorem 5.10]. □

Corollary 4.10. *Let $\lambda \in P_+$. For each $w \in W$ and $\beta \in Q^\vee$, we have*

$$H^{>0}(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \{0\}.$$

Proof. In view of Theorem 4.9, we apply [35, Theorem 5.10] and the Leray spectral sequence to reduce the assertion to $H^{>0}(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) = \{0\}$. This is Theorem 4.4 2). □

Proposition 4.11. *Let $w \in W$ and $\lambda \in P_+$. We have*

$$\lim_{\beta \to \infty} \chi_q(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)).$$

Proof. By Corollary 4.10, we have

$$\chi_q(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda))$$

for every $\beta \in Q^\vee_+$.

By Theorem 4.9, we deduce

$$H^0(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = H^0(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda))$$

for every $\lambda \in P_+$ and $\beta \in Q^\vee_+$.

By Theorem 4.4 2), we have

$$\lim_{\beta \to \infty} \chi_q(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \lim_{\beta \to \infty} \chi_q(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) \qquad \lambda \in P_+$$

and it is uniquely determined by Theorem 4.4 3). In addition, the comparison of Theorem 4.4 3) with [26, Theorem 4.12] implies

$$\lim_{\beta \to \infty} \chi_q(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \qquad \lambda \in P_+.$$

Combining these implies the desired equality. □

4.4 The image of Schubert classes under Ψ

Theorem 4.12. *The map Ψ constructed in Theorem 3.11 restricts to an isomorphism $qK_H(\mathcal{B}) \cong K_H(\mathbf{Q}_G)$ of $\mathbb{C}P \otimes \mathbb{C}Q^\vee_+$-modules, and we have*

$$\Psi([\mathcal{O}_{\mathcal{B}(w)}]) = [\mathcal{O}_{\mathbf{Q}(w)}] \qquad w \in W.$$

The rest of this subsection is devoted to the proof of Theorem 4.12. In this subsection, $\otimes$ is understood to be $\otimes_{\mathcal{O}_Z}$, where Z is the variety we are considering.

By the proof of Theorem 3.11 and the properties of $\star$-products of H-equivariant quantum K-groups (see §1.7), $qK_H(\mathcal{B})$ is the subspace of $qK_H(\mathcal{B})_{\mathrm{loc}}$ (topologically) generated by $\mathbb{C}P$, $Q^\vee_+$, and a_i ($i \in \mathtt{I}$). (Here we can replace a_i with $a_i([\mathcal{O}_\mathcal{B}])\star$ or $[\mathcal{O}_\mathcal{B}(-\varpi_i)]\star$ freely by Theorem 1.31 and Corollary 1.32, although we do not utilize this fact in the below.) As each of them (transferred by Ψ) preserves $K_H(\mathbf{Q}_G)$ and $\Psi([\mathcal{O}_\mathcal{B}]) = [\mathcal{O}_{\mathbf{Q}_G}]$, we deduce that Ψ embeds $qK_H(\mathcal{B})$ into $K_H(\mathbf{Q}_G)$.

We consider the $\mathbb{C}[q^{\pm 1}]P$-valued functional $F^\lambda_\beta(\bullet)$ on $K_H(\mathcal{B}) \otimes \mathbb{C}[q^{\pm 1}]Q^\vee_+$ with parameters $\beta \in Q^\vee_+$ and $\lambda \in P_+$:

$$\begin{aligned}\sum_{\beta \in Q^\vee_+} F^\lambda_\beta(\bullet) Q^\beta &:= \sum_{\gamma \in Q^\vee_+} \chi_q(\mathfrak{X}(\gamma), \mathcal{O}_{\mathfrak{X}(\gamma)}(\lambda) \otimes \mathtt{ev}_1^*(\bullet) \otimes \mathtt{ev}_2^*(\mathcal{O}_\mathcal{B})) Q^\gamma \\ &= \chi(T(\prod_{i \in \mathtt{I}} A_i(q)^{-\langle \alpha_i^\vee, \lambda\rangle}(\bullet)) \cdot \overline{T}([\mathcal{O}_\mathcal{B}])),\end{aligned}$$

where the second equality is a reformulation of [24, Proposition 2.13] and the last term is connected to the quantum K-theoretic product by Theorem 1.27 and Theorem 1.26. Note that this collection of functionals $\{F^\lambda_\beta(\bullet)\}_{\beta,\lambda}$ is uniquely determined by the calculations from §4.3 and Theorem 3.8

$$\begin{aligned}\sum_{\beta \in Q^\vee_+} F^\lambda_\beta([\mathcal{O}_\mathcal{B}]) Q^\beta &= \sum_{\beta \in Q^\vee_+} \chi_q(\mathfrak{X}(\beta), \mathcal{O}_{\mathfrak{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_\mathcal{B}) \otimes \mathtt{ev}_2^*(\mathcal{O}_\mathcal{B})) Q^\beta \\ &= \sum_{\beta \in Q^\vee_+} \chi_q(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)) Q^\beta = D_{w_0}(J'(Qq^\lambda, q) e^{w_0\lambda} J'(Q, q^{-1})),\end{aligned}$$

as $\sum_\beta F^\lambda_\beta(\bullet) Q^\beta$ commutes with the $\mathbb{C}[q^{\pm 1}]P$-action and the right $\mathbb{C}Q^\vee$-action, and intertwines the shift operator $A_i(q)$ with the line bundle twist by $\mathcal{O}_{\mathfrak{X}(\beta)}(-\varpi_i)$ for each $i \in \mathtt{I}$.

For each $a \in (\mathbb{C}[q^{\pm 1}] \otimes_\mathbb{C} K_H(\mathcal{B}))[\![Q^\vee_+]\!]$, we have the class $\Psi(a|_{q=1}) \in K_H(\mathbf{Q}_G)$ written as

$$\Psi(a|_{q=1}) = \sum_{w \in W_{\mathrm{af}}} c^w(a) [\mathcal{O}_{\mathbf{Q}_G(w)}] \qquad w \in W_{\mathrm{af}}, c^w(a) \in \mathbb{C}P.$$

In view of the definition of our shift operators (1.20) and the proof of Theorem 3.11, the coefficients $c^w(a) \in \mathbb{C}P$ are characterized as

$$c^w(a) = c^w_q(a)|_{q=1}$$

if we have elements $c^w_q(a) \in \mathbb{C}[q^{\pm 1}]P$ ($w \in W_{\mathrm{af}}$) determined by

$$\lim_{\beta \to \infty} F^\lambda_\beta(a) = \sum_{w \in W_{\mathrm{af}}} c^w_q(a) \operatorname{gch} \Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \qquad \lambda \in P_+.$$

We have

$$\lim_{\beta\to\infty} \chi_q(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \mathrm{gch}\, H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(\beta,w)}(\lambda))$$

for each $\lambda \in P_+$ by Proposition 4.11.

Thus, we have

$$\begin{aligned}\lim_{\beta\to\infty} F^\lambda_\beta([\mathcal{O}_{\mathcal{B}(w)}]) &= \lim_{\beta\to\infty} \chi_q(\mathcal{X}(\beta), \mathcal{O}_{\mathcal{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_{\mathcal{B}(w)}) \otimes \mathtt{ev}_2^*(\mathcal{O}_{\mathcal{B}}))\\ &= \lim_{\beta\to\infty} \chi_q(\mathcal{X}(\beta), \mathcal{O}_{\mathcal{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_{\mathcal{B}(w)})) \qquad (4.14)\\ &= \lim_{\beta\to\infty} \chi_q(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \mathrm{gch}\, H^0(\mathbf{Q}(w), \mathcal{O}_{\mathbf{Q}(w)}(\lambda))\end{aligned}$$

for each $\lambda \in P_+$ and $w \in W$.

Therefore, we conclude

$$\Psi([\mathcal{O}_{\mathcal{B}(w)}]) = [\mathcal{O}_{\mathbf{Q}_G(w)}] \qquad w \in W.$$

This proves the second assertion. By examining the $\mathbb{C}P$-bases of $qK_H(\mathcal{B})$ and $K_H(\mathbf{Q}_G)$, we also deduce $\mathrm{Im}\,\Psi = K_H(\mathbf{Q}_G)$. This is the first assertion. These complete the proofs of all the assertions.

4.5 Consequences

Since Theorem 1.33 is used only when we reformulate Theorem 3.11 into Corollary 3.13 (the last result in §3.4), we obtain an alternative proof of the following:

Theorem 4.13 (= Theorem 1.33 due to Anderson-Chen-Tseng)**.** *For each $i \in \mathtt{I}$, we have $A_i(q)([\mathcal{O}_{\mathcal{B}}]) = [\mathcal{O}_{\mathcal{B}}(-\varpi_i)]$.*

Proof. By Theorem 3.11 and Remark 3.12, we know that $\Psi_q(A_i(q)([\mathcal{O}_{\mathcal{B}}])) = [\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)]$. Now we argue as:

$$\begin{aligned}A_i(q)([\mathcal{O}_{\mathcal{B}}]) &= \Psi_q^{-1}([\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)])\\ &= e^{-\varpi_i}\Psi_q^{-1}([\mathcal{O}_{\mathbf{Q}_G(e)}] - [\mathcal{O}_{\mathbf{Q}_G(s_i)}]) && \text{by Corollary 1.23}\\ &= e^{-\varpi_i}([\mathcal{O}_{\mathcal{B}}(e)] - [\mathcal{O}_{\mathcal{B}(s_i)}]) && \text{by Theorem 4.12}\\ &= [\mathcal{O}_{\mathcal{B}}(-\varpi_i)] && \text{by (1.3).}\end{aligned}$$

These imply the result. □

Corollary 4.14. *For each $i \in \mathtt{I}$ and $w \in W$, the element $A_i(q)([\mathcal{O}_{\mathcal{B}(w)}])$ is a finite $\mathbb{C}[q^{\pm 1}]P$-linear combination of $\{[\mathcal{O}_{\mathcal{B}(w)}]Q^\beta\}_{w\in W,\beta\in Q^\vee_+}$.*

Proof. By Remark 3.12 and Theorem 4.12, the problem reduces to the corresponding problem in $K_{\mathbb{G}_m \ltimes \mathbf{I}}(\mathbf{Q}_G^{\mathrm{rat}})$. The latter is explained in either [28, Theorem 3.7] (as in Corollary 4.16) or [48, Theorem 1] (as an explicit formula), though author's original reasoning is by the finiteness of the (global version of the) decomposition procedure in [15] (as their global generalized Weyl modules are exactly $\Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda))^*$; see e.g. [26, §5]). □

Corollary 4.15 (Conjectured by Lam-Li-Mihalcea-Shimozono [40]). *We have a natural $\mathbb{C}P$-algebra dense embedding*

$$\Psi^{-1} \circ \Phi : K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow qK_H(\mathcal{B})_{\mathrm{loc}},$$

such that

$$\Psi^{-1} \circ \Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{-1}) = [\mathcal{O}_{\mathcal{B}(w)}]Q^{\beta-\gamma} \qquad w \in W \tag{4.15}$$

holds for every $\beta, \gamma \in Q^\vee_<$.

Proof. For the first assertion, combine Theorem 2.12 and Corollary 3.13 to obtain the map $\Psi^{-1} \circ \Phi$, that have dense image. Note that the both sides are rings and the identity $[\mathcal{O}_{\mathrm{Gr}_0}]$ goes to the identity $[\mathcal{O}_\mathcal{B}]$. The map $\Psi^{-1} \circ \Phi$ commutes with the natural $Q^\vee$-actions given by $\mathtt{t}_\gamma$ and Q^γ for each $\gamma \in Q^\vee$ by Theorem 2.5 and Corollary 3.13. Moreover, the action of Θ_i (see §2.3) and the quantum multiplication by $[\mathcal{O}_\mathcal{B}(-\varpi_i)]$ corresponds for each $i \in \mathtt{I}$ (by Theorem 2.12 and Corollary 3.13). Therefore, the $\odot$-multiplication by the element $\mathbf{h}_i$ and $\star$-multiplication by $[\mathcal{O}_{\mathcal{B}(s_i)}] = ([\mathcal{O}_\mathcal{B}] - e^{\varpi_i}[\mathcal{O}_\mathcal{B}(-\varpi_i)])$ coincide for each $i \in \mathtt{I}$. Since the ring $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is generated by $\{\mathbf{h}_i\}_{i\in\mathtt{I}}$ up to the $\mathbb{C}P$-action and $\{\mathtt{t}_\gamma\}_\gamma$-action (Remark 2.13), we conclude that $\Psi^{-1} \circ \Phi$ is a ring embedding.

For the second assertion, note that Theorem 2.5 asserts that

$$\Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{\pm 1}) = [\mathcal{O}_{\mathbf{Q}_G(wt_{\beta\pm\gamma})}] \qquad w \in W$$

for each $\beta, \gamma \in Q^\vee_<$ (cf. Lemma 1.9). From this, we derive

$$\begin{aligned}\Psi^{-1} \circ \Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{\pm 1}) &= \Psi^{-1}([\mathcal{O}_{\mathbf{Q}_G(w)}])Q^{\beta\pm\gamma} && \text{by Corollary 3.13}\\ &= [\mathcal{O}_{\mathcal{B}(w)}]Q^{\beta\pm\gamma} && \text{by Theorem 4.12.}\end{aligned}$$

This yields the desired equality. □

In view of [40], we obtain another proof of the finiteness of quantum K-theory of $\mathcal{B}$ originally proved in Anderson-Chen-Tseng [1, 2]. We reproduce the reasoning here for the sake of reference:

Corollary 4.16 (Anderson-Chen-Tseng [1, 2]). *For each $w, v \in W$, we have*

$$[\mathcal{O}_{\mathcal{B}(w)}] \star [\mathcal{O}_{\mathcal{B}(v)}] \in \bigoplus_{\beta \in Q^\vee_+, u \in W} \mathbb{C}P[\mathcal{O}_{\mathcal{B}(u)}]Q^\beta.$$

In other words, the multiplication rule of $qK_H(\mathcal{B})$ is finite.

Proof of Corollary 4.16 due to Lam-Li-Mihalcea-Shimozono [40]. By Corollary 4.15 (cf. Theorem 1.7), the assertion follows from

$$[\mathcal{O}_{\mathrm{Gr}_\beta}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}] \in \bigoplus_{\kappa \in Q^\vee} \mathbb{C}P[\mathcal{O}_{\mathrm{Gr}_\kappa}] \qquad \forall \beta, \gamma \in Q^\vee. \tag{4.16}$$

By definition, the LHS of (4.16) is a product inside the ring $\mathcal{C}$ that has $\{[\mathcal{O}_{\mathrm{Gr}_\kappa}]\}_\kappa$ as its $\mathbb{C}P$-basis (Theorem 1.6). Hence, the assertion follows. □

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