The formal model of semi-infinite flag manifolds

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Flag manifolds

The flag manifold X of a (simply conn.) simple group G over $\Bbbk$ is:

- the largest compact G-homogeneous manifold (over $\Bbbk = \mathbb{C}$);
- realizes all finite-dimensional G-modules in a nice way (Borel-Weil, Demazure CF, PRV-Kumar theorem, SMT...);
- a source of concrete realization of the classifying spaces ($\rightsquigarrow$ characteristic classes and equivariant cohomology);
- categorifies the induction process in representation theory (localization theorem, Deligne-Lusztig theory when $\Bbbk = \overline{\mathbb{F}}_q$).

These four items are somehow mutually connected.

Affine flag varieties

In 1980/90s, the extension of the above results to Kac-Moody groups are pursued. This includes:

- construction of affine flag manifolds $(X_{\mathrm{af}}, \mathbf{X}_{\mathrm{af}})$ via various loop groups/Kac-Moody groups, incl. affine Grassmannian Gr_G;
- realization of integrable highest weight modules (Borel-Weil, Demazure CF, PRV theorem, SMT...);
- uniformizes moduli spaces of geometric data on curves;
- Kazhdan-Lusztig algorithm and the Lusztig program for affine Kac-Moody algebras.

As we have

$$\mathrm{Gr}_{GL(2)} \cong \frac{\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{C}(\!(z)\!), \det A \neq 0\}}{\{B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{C}[\![z]\!], \det B \in \mathbb{C}[\![z]\!]^{\times}\}},$$

this is not the field extension of the flag manifold $\mathbb{P}^1_{\mathbb{C}}$ of $GL(2, \mathbb{C})$.

Semi-infinite combinatorics and semi-infinite flag manifolds

In the meantime, objects of the form (when $G = GL(2)$)

$$X^{\frac{\infty}{2}} = \frac{\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{C}(\!(z)\!), \det A \neq 0\}}{\{B = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, b, d \in \mathbb{C}(\!(z)\!), \det B \in \mathbb{C}[\![z]\!]^\times\}}$$

gradually attracted attention:

- Lusztig (1981,1990): generic Bruhat order ($\rightsquigarrow$ orbit closures);
- Feigin-Frenkel (1990): affine Lie algebra at $-h^\vee$ on $X^{\frac{\infty}{2}}$;
- Drinfeld-Finkelberg-Mirković (1999) constructed a variant of $X^{\frac{\infty}{2}}$ as the space of rational maps;
- Arkhipov-Braverman-Bezrukavnikov-Gaitsgory-Mirković (2005) connected them to representation theory (of u_q);
- Givental-Lee (2005) and Braverman-Finkelberg (2014) connected them to quantum K-theory of $X = G/B$.

Semi-infinite flag manifolds

In order to equip semi-infinite flag manifolds with Hausdorff topology, we should consider a variant (when $G = GL(2)$):

$$\mathbf{Q}_G^{\mathrm{rat}} = \frac{\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}((z)), \det A \neq 0\}}{\{B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b \in \mathbb{C}^{\times}, d \in \mathbb{C}((z))\}}$$

- Recorded in Finkelberg-Mirković (1999), but they mainly use quasi-map spaces (= the Drinfeld compactification of the space of maps $\mathbb{P}^1 \to X$) and Zastava spaces ($\subset \mathbf{Q}_G^{\mathrm{rat}}$);
- The union of quasi-map spaces define a "Zariski" dense subset of $\mathbf{Q}_G^{\mathrm{rat}}$ (the ind-model of $\mathbf{Q}_G^{\mathrm{rat}}$);
- Being algebro-geometric object, one should be able to write $\mathbf{Q}_G^{\mathrm{rat}}$ down by its coordinate rings (the formal model of $\mathbf{Q}_G^{\mathrm{rat}}$);
- If one carry this out, then one might pursue loop analogues for the various subjects related to X.

The goal of this talk is to exhibit the status quo of such an idea.

Homogeneous coordinate ring of flag manifolds

G : simply connected simple algebraic group over $\mathbb{C} \supset$ a maximal solvable (Borel) subgroup B and maximal torus $T \subset B$.

$$X = G/B = (G/N)/T, \quad N = [B, B] \;\; (B = T \ltimes N)$$

is the flag variety of G.

$$\mathbb{C}[G/N] = \bigoplus_{\lambda \in P_+} V_\lambda^*, \quad V_\lambda^* \otimes V_\mu^* \to V_{\lambda+\mu}^* \text{ is multi.}$$

where $P_+ \subset P$ is a submonoid of the character group P of T, and V_λ is a fin. dim'l irred. G-module with h.w. $\lambda \in P_+$. We have

$$X = (\operatorname{Spec} \mathbb{C}[G/N] \setminus \{\text{locus of non-free } T\text{-action}\})/T.$$

This is a general recipé to produce an algebraic variety from a tensor-compatible family of representaions of an algebra.

Integrable modules of $\widetilde{\mathfrak{g}}$

$\mathfrak{g} = \mathrm{Lie}\, G$: Lie algebra of G with its untwisted affinization $\widetilde{\mathfrak{g}}$.
Analogue of fin. dim'l reps of $\mathfrak{g}$ (or G) for $\widetilde{\mathfrak{g}}$ are integrable reps.

Simple root of $\widetilde{\mathfrak{g}} \Rightarrow \mathfrak{sl}(2, \mathbb{C}) \subset \widetilde{\mathfrak{g}}$. A $\widetilde{\mathfrak{g}}$-module is integrable if such $\mathfrak{sl}(2, \mathbb{C})$-actions integrate to $\oplus$ fin. dim'l $SL(2, \mathbb{C})$-actions.

Theorem (Chari, 1986, Chari-Pressley 2001)

Indec. integrable $\widetilde{\mathfrak{g}}$-module with finite dim'l weight spaces are:

1. Integrable highest weight modules (c.f. Kac's book);
2. Integrable lowest weight modules (dual of the above);
3. Some level zero modules, including global Weyl modules (a maximal cyclic module with the same T-weight as V_λ).

Integrable h.w./l.w. modules describe the usual (thin, thick) affine flag varieties (Kac-Peterson, Kashiwara, Mokler, K).

Iwahori/current/loop algebras

Set $\widehat{\mathfrak{g}} := [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] \cong \widetilde{\mathfrak{g}}/\mathbb{C}d$. For the level zero modules, the $\widehat{\mathfrak{g}}$-action factors through $\mathfrak{g}[z^{\pm 1}] := \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}] \cong \widehat{\mathfrak{g}}/(K = 0)$.

We define the Iwahori and current algebras of $\widetilde{\mathfrak{g}}$ as:

$$\widetilde{\mathfrak{b}} := \mathrm{Lie}\, B + \mathfrak{g} \otimes \mathbb{C}[z]z \subset \mathfrak{g}[z] := \mathfrak{g} \otimes \mathbb{C}[z] \subset \mathfrak{g}[z^{\pm 1}].$$

They have gradings by d (or setting $\deg z = 1$).

- $\mathfrak{g}[z]$ localizes to $\mathfrak{g}[z^{\pm 1}]$, and share similar rep. theory;
- $\widetilde{\mathfrak{b}}$ "generates" $\widehat{\mathfrak{g}}$ by adding $\mathfrak{sl}(2)$ for each simple root $\rightsquigarrow$ Demazure functors $\mathcal{D}_i$ ($i \in \mathtt{I}_{\mathrm{af}}$: index of simple roots of $\widetilde{\mathfrak{g}}$);
- global Weyl modules $\mathbb{W}_\lambda$ ($\lambda \in P_+$) of $\mathfrak{g}[z]$ must satisfy a recursive formula (for some $i_1, \ldots, i_l \in \mathtt{I}_{\mathrm{af}}$):

$$\mathbf{q}^\bullet \mathbb{W}_\lambda \cong (\mathcal{D}_{i_1} \circ \cdots \circ \mathcal{D}_{i_l})(\mathbb{W}_\lambda) =: \mathcal{D}_w(\mathbb{W}_\lambda) \quad \mathbf{q}^\bullet \text{ is grading shift.}$$

($\mathcal{D}_w$ is a version of Macdonald operator)

Irreps of $\mathfrak{g}[z]$ or $\mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}]$ are classified by Drinfeld polynomials, an enrichment of P_+. The module $\mathbb{W}_\lambda$ cares only their degrees.

Adjoint property and orthogonality relations

Propostion ("adjoint property", Feigin-K-Makedonskyi 2020)

For suitable $\widetilde{\mathfrak{b}}$-modules M, N and $i \in \mathtt{I}_{\mathrm{af}}$, we have:

$$\mathrm{Ext}^{\bullet}_{(\widetilde{\mathfrak{b}},\widetilde{\mathfrak{h}})}(\mathcal{D}_i(M), N^*) \cong \mathrm{Ext}^{\bullet}_{(\widetilde{\mathfrak{b}},\widetilde{\mathfrak{h}})}(M, \mathcal{D}_i(N)^*).$$

Enhancement of Polo (1989), and also a part of affine version of Bezrukavnikov's picture (see Khoroshkhin-K-Makedonskyi 2020+).

Theorem ("orthogonality", BBCIKLM 2012–2015)

For each $\lambda, \mu \in P_+$, we have

$$\mathrm{Ext}^{i}_{(\mathfrak{g}[z],\mathfrak{h})}(\mathbb{W}_\lambda, W^*_\mu) = \begin{cases} \mathbb{C}^{\delta_{i,0}} & (V_\lambda \cong V^*_\mu) \\ 0 & (else) \end{cases}. \tag{1}$$

The vanishing part of (1) follows by comparing "eigenvalues" of $\mathcal{D}_w$ transferred by the adjoint property (valid also for char > 0).

The ring R

Universal property of $\mathbb{W}_\lambda$ induces a (unique degree 0) map

$$\mathbb{W}_{\lambda+\mu} \longrightarrow \mathbb{W}_\lambda \otimes \mathbb{W}_\mu \qquad \lambda, \mu \in P_+.$$

This yields a (non-Noetherian) commutative algebra

$$R := \bigoplus_{\lambda \in P_+} \mathbb{W}_\lambda^\vee \qquad \bullet^\vee \text{ is the restricted dual.}$$

We have $G[\![z]\!]/N[\![z]\!] \subset \mathrm{Spec}\, R$, where

$$E[\![z]\!] := E(\mathbb{C}[\![z]\!]) \;\; \text{for an algebraic group } E \text{ over } \mathbb{C}.$$

The global Weyl module $\mathbb{X}_\lambda$ of $\mathfrak{g}[z^{\pm 1}]$ contains $\mathbb{W}_\lambda$.

$$\rightsquigarrow \;\; \mathbb{W}_\lambda \subsetneq \mathbf{q}^\bullet \mathbb{W}_\lambda = \mathcal{D}_w(\mathbb{W}_\lambda) \subsetneq \mathcal{D}_w(\mathbb{X}_\lambda) = \mathbb{X}_\lambda, \qquad \bigcup_w \mathcal{D}_w(\mathbb{W}_\lambda) = \mathbb{X}_\lambda$$

by $\mathcal{D}_i(\mathbb{X}_\lambda) = \mathbb{X}_\lambda$ for $i \in \mathtt{I}_{\mathrm{af}}$.

The formal model of semi-infinite flag manifolds

Dualizing, we obtain a projective system of commutative rings:

$$R \twoheadleftarrow \mathcal{D}^{\dagger}_{w}(R) \twoheadleftarrow \cdots$$

We define

$$\mathbf{Q} := (\mathrm{Spec}\, R \setminus \{\text{non-free } T\text{-action}\})/T \subset \bigcup_{w} (\mathcal{D}^{\dagger}_{w})^{*}(\mathbf{Q}) = \mathbf{Q}^{\mathrm{rat}}.$$

Theorem (K 2021)

The ind-scheme $\mathbf{Q}^{\mathrm{rat}}$ coarsely (ind-)represents the functor

$$\mathrm{CommAlg}_{\mathbb{C}} \ni R \mapsto G(R(\!(z)\!))/(T(R) \cdot N(R(\!(z)\!))) \in \mathrm{Sets}.$$

In case $G = SL(2)$, we have

$$\mathbf{Q} = \mathbb{P}(\mathbb{C}^2[\![z]\!]) \subset \mathbb{P}(\mathbb{C}^2(\!(z)\!)) = \mathbf{Q}^{\mathrm{rat}},$$

where $\mathbb{P}$ is over $\mathbb{C}$. I.e. we need to treat z and z^{-1} differently.

Borel-Weil theorem and equivariant K-group

Theorem (Naito-K-Sagaki 2020)

For each $\lambda \in P$, we have a line bundle $\mathcal{O}(\lambda)$ on $\mathbf{Q}$ such that

$$H^i(\mathbf{Q}, \mathcal{O}(\lambda))^\vee \cong \begin{cases} \mathbb{W}_\lambda & (i = 0, \lambda \in P_+) \\ 0 & (else) \end{cases}.$$

Ind-model analogue of this result is due to Braverman-Finkelberg.

We define $(T \times \mathbb{G}_m)$-equivariant K-group of $\mathbf{Q}$ via numerics:

$$\mathcal{F} \mapsto \sum_{i \geq 0} (-1)^i \mathrm{gch}\, H^i(\mathbf{Q}, \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(\lambda)) \equiv \chi(\mathbf{Q}, \mathcal{F}(\lambda)) \quad \lambda \in P$$

by imposing suitable restrictions on $(T \times \mathbb{G}_m)$-equivariant sheaf $\mathcal{F}$.

$\rightsquigarrow$ One can define $K_T(\mathbf{Q}^{\mathrm{rat}})$ suitably (not automatically a ring).

Richardson varieties of X

The Chevalley involution of (G, T) sends B to its opposite B^-.

Theorem (Bruhat decomposition and Richardson's theorem)

1. T-fixed points of $X \Leftrightarrow$ the finite Weyl group W of G;
2. B-orbits (resp. B^--orbits) of $X \Leftrightarrow W$ (through X^T)
   $\rightsquigarrow X(w, v)$: intersection of B and B^-orbit for $w, v \in W$.
3. $X(w, v)$ is either empty or an irreducible smooth variety.

B-orbit closure X_w corresponding to $w \in W$ is our Schubert varieties, and $X(w, v)$ are our open Richardson varieties.

Richardson varieties knows the "intersection theory" of X.

$X(w, v)$ is defined over $\mathbb{Z}$, and its structure and topology over $\mathbb{R}$ (often projected) attracted attention (see e.g. Williams' talk).

Schubert and Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$

In general, we have

$$\mathbf{Q}^{\mathrm{rat}} \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(V_{\varpi_i} \otimes_{\mathbb{C}} \mathbb{C}(\!(z)\!)) \subset \prod_{i \in \mathtt{I}} \mathbb{P}(V_{\varpi_i} \otimes_{\mathbb{C}} \mathbb{C}[\![z, z^{-1}]\!]),$$

where $\{\varpi_i\}_{i \in \mathtt{I}} \subset P_+$: fundamental weights. The Chevalley involution θ of $\widetilde{\mathfrak{g}}$ should "swap" $G[z]$ and $G[z^{-1}]$ in $G(\!(z)\!)$.

$\tilde{B} := \mathtt{ev}_0^{-1}(B) \subset G[\![z]\!]$, where $\mathtt{ev}_0 : G[\![z]\!] \to G$ is evaluation.

Theorem (essentially due to Iwahori-Matsumoto 1965)

$\tilde{B} \backslash \mathbf{Q}^{\mathrm{rat}}$ is in bijection with W_{af}, the affine Weyl group of G. Let $\mathbb{O}_w \subset \mathbf{Q}^{\mathrm{rat}}$ be the $\mathbf{I}$-orbit corr. to $w \in W_{\mathrm{af}}$ and set $\mathbf{Q}_w := \overline{\mathbb{O}_w}$.

For $w, v \in W_{\mathrm{af}}$, we define the Richardson variety of $\mathbf{Q}^{\mathrm{rat}}$ as:

$$\mathcal{Q}(w, v) := \mathbf{Q}_w \cap \theta(\mathbf{Q}_{vw_0}) \subset \prod_{i \in \mathtt{I}} \mathbb{P}(V_{\varpi_i} \otimes_{\mathbb{C}} \mathbb{C}[\![z, z^{-1}]\!]),$$

where $\cap$ is understood scheme-theoretically and $w_0 \in W \subset W_{\mathrm{af}}$.

Connection between $\mathbf{Q}^{\mathrm{rat}}$ and the space of rational maps

Theorem (K 2021)

For all $w, v \in W_{\mathrm{af}}$, the variety $\mathcal{Q}(w, v)$ is irreducible and reduced. It is normal and have an explicit dimension formula.

Proof uses the Frobenius splitting (FS) of $\mathbf{Q}^{\mathrm{rat}}$, transplanted as:

$$X_{\mathrm{af}} \rightsquigarrow \mathbf{X}_{\mathrm{af}} \rightsquigarrow \mathbf{Q} \rightsquigarrow \mathbf{Q}^{\mathrm{rat}}.$$

X_{af} has FS by Kumar and Mathieu, $\mathbf{X}_{\mathrm{af}}$ has FS by K (2020), and next two are by the relation between integrable l.w. $\widetilde{\mathfrak{g}}$-modules, $\mathbb{W}_\lambda$s, and $\mathbb{X}_\lambda$s. Then, we lift its consequences from $\{\overline{\mathbb{F}}_p\}_p$ to $\mathbb{C}$.

Corollary (K 2021) For β varying in $H_2(X, \mathbb{Z})$, we have

$$\{G\text{-stable Richardson} \subset \mathbf{Q}^{\mathrm{rat}}\} = \{\overline{\{f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = \beta\}}^{Drin}\}_\beta,$$

where *Drin* denotes the Drinfeld compactification.

(Small) quantum K-groups of flag manifolds

Quantum K-group $qK(X)$ on $X = G/B$ is defined by evaluation on

$$\overline{\{f : (\mathbb{P}^1, \{x_m\}_{m=1}^t) \to X \mid \deg f = \beta\}}^{Kont} \quad t = 2, 3$$

by its original definition (Givental 2000, Lee 2005).

Here, *Kont* denotes the Kontevich compacitification (stable maps).

Reconstruction theorem (Iritani-Milanov-Tonita 2015) reduces calculations to $t = 1, 2$.

Above moduli spaces with $t = 1, 2$ with Schubert class insertions yields "resolutions" of Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$.

Therefore, K-theoretic counts on $\mathbf{Q}^{\mathrm{rat}}$ knows $qK(X)$ of X if the singularities involved are all rational.
$\Leftarrow$ [K, 2018+] (reduced to [Braverman-Finkelberg, 2014, 2017])

Pontryagin product on affine Grassmannians

We have an identification

$$\mathrm{Gr}_G = \frac{G[z^{\pm 1}]}{G[z]} \xleftarrow{\cong} \Omega K = \{f : S^1 \to K \mid f \text{ is poly s.t. } f(1) = e\},$$

where $K \subset G$ is the maximal compact subgroup. The RHS has a product induced by the mult on K. This induces a product $\odot$ on $K^{\mathrm{top}}(\Omega K)$. The Birkhoff factorization induces an identification

$$(K_G(\mathrm{Gr}_G), \mathsf{conv}) \cong (K_K^{\mathrm{top}}(\Omega K), \odot),$$

where the product on the LHS is the convolution product. Its scalar extension to

$$K_T(\mathrm{pt}) = K_{T\cap K}^{\mathrm{top}}(\mathrm{pt}) \supset K_K^{\mathrm{top}}(\mathrm{pt}) = K_G(\mathrm{pt})$$

yields the $(T \cap K)$-equivariant Pontryagin product.
([Lam-Schilling-Shimozono 2010] and [LLMS 2018])

The Peterson isomorphism in K-theory

We have some natural multiplicative systems on $K_T(\mathrm{Gr}_G)$ and $qK_T(X)$ (in an "oppisotite" direction to each other).

Theorem (conjectured by Lam-Li-Mihalcea-Shimozono, K 2018+)

We have a commutative diagram

$$\begin{array}{ccc} & K_T(\mathbf{Q}^{\mathrm{rat}}) & \\ {}^{\Phi}\nearrow & & \nwarrow^{\Psi} \\ K_T(\mathrm{Gr}_G)_{\mathrm{loc}} & \longrightarrow & qK_T(X)_{\mathrm{loc}} \end{array},$$

that respects the Schubert bases and operations in each objects.

We have its parabolic & non-commutative versions. $H_\bullet$-version of the bottom arrow (due to Peterson-Lam-Mihalcea-Shimozono) can be proven via $\mathcal{Q}$, but it does *not* yield a triangle as above (yet?).

Some perspectives

1. We expect virtually *all* results on the geometry of X and their descriptions (including some combinatorics) have reasonable analogues in the setting of $\mathbf{Q}^{\mathrm{rat}}$;
2. Above triangle tells finer structure of $qK_T(X)$ via $K_T(\mathbf{Q}^{\mathrm{rat}})$;
3. Our $\mathbf{Q}^{\mathrm{rat}}$ is an algebraic version of the loop space of X. Hence, the morphism Ψ, that is an instance of the loop space formalism to describe $qH(X)$, would exist in greater generality;
4. We have "global counter-parts" in each piece of the above triangle. $\rightsquigarrow$ global version of the above triangle?
5. The study of tensor products should be replaced with the calculation of a kind of conformal blocks?
6. We now understand some vector bundles, including sheaf of differential operators of $\mathbf{Q}^{\mathrm{rat}}$. $\rightsquigarrow$ some representations of $\widetilde{\mathfrak{g}}$.