On quantum K-groups of partial flag manifolds*

Syu KATO†

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Abstract

We show that the equivariant small quantum K-group of a partial flag manifold is a quotient of that of the full flag manifold in a way it respects the Schubert basis. This is a K-theoretic analogue of the parabolic version of Peterson's theorem [Lam-Shimozono, Acta Math. **204** (2010)] that exhibits different shape from the case of quantum cohomology. Our quotient maps send some of the Novikov variables to 1, and its geometric meaning is unclear in quantum K-theory. This paper can be seen as a continuation of [K, arXiv:1805.01718 and arXiv:1810.07106].

Introduction

Let G be a simply connected simple algebraic group over $\mathbb{C}$ with a maximal torus H and a Borel subgroup B that contains H. For each (standard) parabolic subgroup $B \subset P \subset G$, we have a partial flag variety G/P. Let Gr denote the affine Grassmannian of G. In this paper, we describe the H-equivariant small quantum K-group $qK_H(G/P)$ of G/P as a quotient of the H-equivariant small quantum K-group $qK_H(G/B)$ of G/B.

The work of Peterson [34] (on quantum cohomology), whose main results appeared as Lam-Shimozono [31], states that we can recover the structure of the H-equivariant small quantum cohomology $qH_H(G/P)$ of G/P by using the H-equivariant cohomology of Gr. In this context, we have a ring surjection $qH_H(G/B) \to qH_H(G/P)$ as a consequence of detailed study ([33]).

In [21, 20], we shed a light on the K-theoretic version of the above relation (for G/B) by employing the equivariant K-group of a semi-infinite flag manifold ([23]) as a mediator, following an idea by Givental [14]. From this view point, the connection between $qK_H(G/P)$'s for different P's looks simpler as the structure of $K_H(G/P)$ is known to be governed by that of $K_H(G/B)$ through the pullback $K_H(G/P) \to K_H(G/B)$, while functoriality in quantum K-theory is not well-understood.

The goal of this paper is to take this advantage to prove the following:

Theorem A ($\doteq$ Theorem 2.18). *There exists a surjective morphism*

$$qK_H(G/B) \longrightarrow\!\!\!\!\!\rightarrow qK_H(G/P)$$

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†Department of Mathematics, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502 JAPAN E-mail:syuchan@math.kyoto-u.ac.jp

of algebras that sends a Schubert basis to a Schubert basis. Moreover, if $B \subset P' \subset P$ is an intermediate standard parabolic subgroup, then the above algebra map factors through $qK_H(G/P')$.

The same proof also works for its non-commutative variant (Corollary 2.19).

Here we stress that the existence of this map is purely of quantum nature, and it does *not* specialize to give an algebra map $K_H(G/B) \to K_H(G/P)$. In fact, our algebra map specializes some of the Novikov variables to 1, as opposed to 0 employed in the cases of quantum cohomologies [31, 33] (in particular, this procedure makes sense only in the presence of the finiteness of the quantum K-groups [1, 21]). Our algebra map exhibits mixed nature of [31] and [33], whose exact meaning is unclear at the moment. By setting $P = G$, we obtain the ring morphism

$$qK_H(G/B) \to qK_H(\mathrm{pt}) = K_H(\mathrm{pt})$$

presented in Buch-Chung-Li-Mihalcea [7, Corollary 10].

In view of the K-theoretic version of the Peterson isomorphism (conjectured in [29] and proved as [21, Corollary C]), we also conclude a surjective morphism

$$K_H(\mathrm{Gr})_{\mathrm{loc}} \longrightarrow\!\!\!\!\!\rightarrow qK_H(G/P)_{\mathrm{loc}} \tag{0.1}$$

of suitably localized algebras (Theorem 2.22). This also sends a Schubert basis to a Schubert basis (up to a Novikov monomial), and hence enforces the theme developed in [8, 29, 6] and the references therein.

We remark that the explicit nature of Theorem A and (0.1) allows us to transplant various multiplication formulas of $qK_H(G/B)$ (that can be seen in [30, 23] etc...) to the setting of $qK_H(G/P)$.

The organization of this paper is as follows. In §1, we collect preliminary results including those of equivariant quantum K-groups and quasi-map spaces. In §2, we cite results from [21, 20] to establish that certain Schubert varieties of parabolic quasi-map spaces have rational singularities (Theorem 2.11). Also, we introduce variants of equivariant K-groups $K_H(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$ of the semi-infinite (partial) flag manifold $\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$ different from those in [23] and [21] that are more suited for our purpose (Theorem 2.5 and the proof of Theorem 2.14). These reformulations enable us to deduce the equality of structure constants in Theorem 2.18 using key observations made in this paper (Lemma 2.16) and [10, 6]. Other than these, our overall arguments follow those of [21] with necessary modifications, though we tried to exhibit them slightly different in flavor. We also provide example calculations for $G = SL(3)$ in §3.

1 Preliminaries

A vector space is always a $\mathbb{C}$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above. Tensor products are taken over $\mathbb{C}$ unless stated otherwise. We define the graded dimension of a graded vector space as

$$\mathrm{gdim}\, M := \sum_{i \in \mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}(\!(q^{-1})\!).$$

We set $\mathbb{C}_q^0 := \mathbb{C}[q^{-1}]$, $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, and $\mathbf{C}_q := \mathbb{C}(\!(q^{-1})\!)$ for the notational convention. As a rule, we suppress $\emptyset$ and associated parenthesis from notation.

This particularly applies to $\emptyset = \mathtt{J} \subset \mathtt{I}$ frequently used to specify parabolic subgroups.

1.1 Groups, root systems, and Weyl groups

We refer to [9, 28] for precise expositions of general material presented in this subsection.

Let G be a connected, simply connected simple algebraic group of rank r over $\mathbb{C}$, and let B and H be a Borel subgroup and a maximal torus of G such that $H \subset B$. We set N $(= [B, B])$ to be the unipotent radical of B. We denote the Lie algebra of an algebraic group by the corresponding German small letter. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E, we denote its set of $\mathbb{C}[\![z]\!]$-valued points by $E[\![z]\!]$, and its set of $\mathbb{C}(\!(z)\!)$-valued points by $E(\!(z)\!)$ etc... Let $\mathbf{I} \subset G[\![z]\!]$ be the preimage of $B \subset G$ via the evaluation at $z = 0$ (the Iwahori subgroup of $G[\![z]\!]$). By abuse of notation, we might consider $\mathbf{I}$ and $G[\![z]\!]$ as group schemes over $\mathbb{C}$ whose $\mathbb{C}$-valued points are given as these.

Let $P := \mathrm{Hom}_{gr}(H, \mathbb{G}_m)$ be the weight lattice of H, let $\Delta \subset P$ be the set of roots, let $\Delta_+ \subset \Delta$ be the set of roots that yield root subspaces in $\mathfrak{b}$, and let $\Pi \subset \Delta_+$ be the set of simple roots. Each $\alpha \in \Delta_+$ defines a reflection $s_\alpha \in W$. Let $Q^\vee$ be the dual lattice of P with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times P \to \mathbb{Z}$. We define $\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q^\vee_+ \subset Q^\vee$ be the set of non-negative integer span of $\Pi^\vee$. For $\beta, \gamma \in Q^\vee$, we define $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^\vee_+$. We set $P_+ := \{\lambda \in P \mid \langle \alpha^\vee, \lambda \rangle \geq 0, \ \forall \alpha^\vee \in \Pi^\vee\}$ and $P_{++} := \{\lambda \in P \mid \langle \alpha^\vee, \lambda \rangle > 0, \ \forall \alpha^\vee \in \Pi^\vee\}$. Let $\mathtt{I} := \{1, 2, \ldots, r\}$. We fix bijections $\mathtt{I} \cong \Pi \cong \Pi^\vee$ such that $i \in \mathtt{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i = s_{\alpha_i} \in W$. Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset P_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij}$).

For a subset $\mathtt{J} \subset \mathtt{I}$, we define $P(\mathtt{J})$ as the standard parabolic subgroup of G corresponding to $\mathtt{J}$. I.e. we have $\mathfrak{b} \subset \mathfrak{p}(\mathtt{J}) \subset \mathfrak{g}$ and $\mathfrak{p}(\mathtt{J})$ contains the root subspace corresponding to $-\alpha_i$ $(i \in \mathtt{I})$ if and only if $i \in \mathtt{J}$. We set $\mathtt{J}^c := \mathtt{I} \setminus \mathtt{J}$. Then, the set of characters of $P(\mathtt{J})$ is identified with $P_\mathtt{J} := \sum_{i \in \mathtt{J}^c} \mathbb{Z}\varpi_i$. We also set $P_{\mathtt{J},+} := \sum_{i \in \mathtt{J}^c} \mathbb{Z}_{\geq 0}\varpi_i = P_+ \cap P_\mathtt{J}$ and $P_{\mathtt{J},++} := \sum_{i \in \mathtt{J}^c} \mathbb{Z}_{\geq 1}\varpi_i = P_{++} \cap P_\mathtt{J}$. We set $Q^\vee_\mathtt{J} := \sum_{i \in \mathtt{J}^c} \mathbb{Z}\alpha_i^\vee$ and $Q^\vee_{\mathtt{J},+} := \sum_{i \in \mathtt{J}^c} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. We define $W_\mathtt{J} \subset W$ to be the reflection subgroup generated by $\{s_i\}_{i \in \mathtt{J}}$. It is the Weyl group of the semisimple quotient of $P(\mathtt{J})$.

Let $\Delta_{\mathrm{af}} := \Delta \times \mathbb{Z}\delta \cup \{m\delta\}_{m \neq 0}$ be the untwisted affine root system of Δ with its positive part $\Delta_+ \subset \Delta_{\mathrm{af},+}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, and $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$, where ϑ is the highest root of Δ_+. We set $W_{\mathrm{af}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element in $W \subset W_{\mathrm{af}}$. Together with the normalization $t_{-\vartheta^\vee} := s_\vartheta s_0$ (for the coroot $\vartheta^\vee$ of ϑ), we introduce the translation element $t_\beta \in W_{\mathrm{af}}$ for each $\beta \in Q^\vee$. By abuse of notation, we denote by $W/W_\mathtt{J}$ the set of minimal length $W_\mathtt{J}$-coset representatives in W.

Let W_{af}^- denote the set of minimal length representatives of W_{af}/W in W_{af}. We set

$$Q^\vee_< := \{\beta \in Q^\vee \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in \mathtt{I}\}.$$

For each $\lambda \in P_+$, we denote by $L(\lambda)$ the corresponding irreducible G-module with a highest B-weight λ. I.e. $L(\lambda)$ has a B-eigenvector with its H-weight λ.

For a semi-simple H-module V, we set

$$\operatorname{ch} V := \sum_{\lambda \in P} e^{\lambda} \cdot \dim \operatorname{Hom}_H(\mathbb{C}_\lambda, V).$$

If V is a $\mathbb{Z}$-graded H-module in addition, then we set

$$\operatorname{gch} V := \sum_{\lambda \in P, n \in \mathbb{Z}} q^n e^{\lambda} \cdot \dim \operatorname{Hom}_H(\mathbb{C}_\lambda, V_n).$$

Let $\mathcal{B}_{\mathtt{J}} := G/P(\mathtt{J})$ and call it the (partial) flag manifold of G. We have the Bruhat decomposition

$$\mathcal{B}_{\mathtt{J}} = \bigsqcup_{u \in W/W_{\mathtt{J}}} \mathbb{O}_{\mathtt{J}}(u) \tag{1.1}$$

into B-orbits such that $\operatorname{codim}_{\mathcal{B}_{\mathtt{J}}} \mathbb{O}_{\mathtt{J}}(u) = \ell(u)$ for each $u \in W/W_{\mathtt{J}} \subset W_{\mathrm{af}}$. We set $\mathcal{B}_{\mathtt{J}}(u) := \overline{\mathbb{O}_{\mathtt{J}}(u)} \subset \mathcal{B}$.

For each $\lambda \in P_{\mathtt{J}}$, we have a line bundle $\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\lambda)$ such that

$$H^0(\mathcal{B}_{\mathtt{J}}, \mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\lambda)) \cong L(-w_0\lambda), \quad \mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}} \mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\mu) \cong \mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\lambda - \mu) \qquad \lambda, \mu \in P_{\mathtt{J},+}.$$

For each $u \in W/W_{\mathtt{J}}$, let $p_u \in \mathbb{O}_{\mathtt{J}}(u)$ be the unique H-fixed point. We normalize p_u (and hence $\mathbb{O}_{\mathtt{J}}(u)$) such that the restriction of $H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\lambda))$ to p_u is isomorphic to $\mathbb{C}_{-uw_0\lambda}$ for every $\lambda \in P_{\mathtt{J},+}$. (Here we warn that the convention differs from [21] by the twist of $-w_0$. This change of convention also applies to $Q^\vee$ in §1.2 in order to keep the degree in Theorem 1.2.)

1.2 Quasi-map spaces

Here we recall basics of quasi-map spaces from [12, 11, 20].

We have isomorphisms $H^2(\mathcal{B}_{\mathtt{J}}, \mathbb{Z}) \cong P_{\mathtt{J}}$ and $H_2(\mathcal{B}_{\mathtt{J}}, \mathbb{Z}) \cong Q^\vee_{\mathtt{J}}$. This identifies the (integral points of the) nef cone of $\mathcal{B}_{\mathtt{J}}$ with $P_{\mathtt{J},+} \subset P_{\mathtt{J}}$ and the effective cone of $\mathcal{B}_{\mathtt{J}}$ with $Q^\vee_{\mathtt{J},+}$. A quasi-map (f, D) is a map $f : \mathbb{P}^1 \to \mathcal{B}_{\mathtt{J}}$ together with a colored effective divisor

$$D = \sum_{x \in \mathbb{P}^1(\mathbb{C})} \beta_x \otimes [x] \in Q^\vee_{\mathtt{J}} \otimes_{\mathbb{Z}} \operatorname{Div} \mathbb{P}^1 \qquad \beta_x \in Q^\vee_{\mathtt{J},+}$$

We call D the defect of (f, D), and we define the total defect of (f, D) by

$$|D| := \sum_{x \in \mathbb{P}^1(\mathbb{C})} \beta_x \in Q^\vee_{\mathtt{J},+}.$$

For each $\beta \in Q^\vee_{\mathtt{J},+}$, we set

$$\mathcal{Q}(\mathcal{B}_{\mathtt{J}}, \beta) := \{ f : \mathbb{P}^1 \to X \mid \text{ quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta \},$$

where $f_*[\mathbb{P}^1]$ is the class of the image of $\mathbb{P}^1$ multiplied by the degree of $\mathbb{P}^1 \to \operatorname{Im} f$. We denote $\mathcal{Q}(\mathcal{B}_{\mathtt{J}}, \beta)$ by $\mathcal{Q}_{\mathtt{J}}(\beta)$ in case there is no danger of confusion.

Definition 1.1 (Drinfeld-Plücker data)**.** Consider a collection $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_{\mathtt{J},+}}$ of inclusions $\psi_\lambda : \mathcal{L}^\lambda \hookrightarrow L(\lambda) \otimes \mathcal{O}_{\mathbb{P}^1}$ of line bundles $\mathcal{L}^\lambda$ over $\mathbb{P}^1$. The data $\mathcal{L}$ is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of G-modules

$$\eta_{\lambda,\mu} : L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)$$

induces an isomorphism

$$\eta_{\lambda,\mu} \otimes \mathrm{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \xrightarrow{\cong} \psi_{\lambda}(\mathcal{L}^{\lambda}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \psi_{\mu}(\mathcal{L}^{\mu})$$

for every $\lambda, \mu \in P_{\mathtt{J},+}$.

Theorem 1.2 (Drinfeld, see [12, 2] and [20])**.** *The variety $\mathcal{Q}_{\mathtt{J}}(\beta)$ is isomorphic to the variety formed by isomorphism classes of the DP-data $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_{\mathtt{J},+}}$ such that* $\deg \mathcal{L}^\lambda = - \langle \beta, \lambda \rangle$.

For each $u \in W/W_{\mathtt{J}}$, let $\mathcal{Q}_{\mathtt{J}}(\beta, u) \subset \mathcal{Q}_{\mathtt{J}}(\beta)$ be the closure of the set formed by quasi-maps that are defined at $z = 0$, and their values at $z = 0$ are contained in $\mathcal{B}_{\mathtt{J}}(u) \subset \mathcal{B}_{\mathtt{J}}$. (Hence, we have $\mathcal{Q}_{\mathtt{J}}(\beta) = \mathcal{Q}_{\mathtt{J}}(\beta, e)$.)

For each $\lambda \in P_{\mathtt{J}}$ and $u \in W$, we have a G-equivariant line bundle $\mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda)$ obtained by the (tensor product of the) pull-backs $\mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\varpi_i)$ of the i-th $\mathcal{O}(1)$ via the embedding

$$\mathcal{Q}_{\mathtt{J}}(\beta, u) \hookrightarrow \prod_{i \in \mathtt{J}^c} \mathbb{P}(L(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[z]_{\leq \langle \beta, \varpi_i \rangle}) \tag{1.2}$$

for each $\beta \in Q^\vee_{\mathtt{J},+}$. Using this, we set

$$\chi(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}}(\lambda)) := \sum_{i \geq 0} (-1)^i \mathrm{gch}\, H^i(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda)) \in \mathbb{C}^0_q P$$

for each $\beta \in Q^\vee_{\mathtt{J}}$ and $\lambda \in P_{\mathtt{J}}$, where the grading q is understood to count the degree of z detected by the $\mathbb{G}_m$-action. Here we understand that

$$\chi(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda)) = 0 \quad\quad \beta \notin Q^\vee_{\mathtt{J},+}.$$

1.3 Graph and map spaces and their line bundles

For each non-negative integer n and $\beta \in Q^\vee_{\mathtt{J},+}$, we set $\mathcal{GB}_{\mathtt{J},n,\beta}$ to be the space of stable maps of genus zero curves with n-marked points to $(\mathbb{P}^1 \times \mathcal{B}_{\mathtt{J}})$ of bidegree $(1, \beta)$, that is also called the graph space of $\mathcal{B}_{\mathtt{J}}$. A point of $\mathcal{GB}_{\mathtt{J},n,\beta}$ is a genus zero quasi-stable curve C with n-marked points, together with a map to $\mathbb{P}^1$ of degree one. Hence, we have a unique $\mathbb{P}^1$-component of C that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of C and denote it by C_0. The space $\mathcal{GB}_{\mathtt{J},n,\beta}$ is a normal projective variety by [13, Theorem 2] that have at worst quotient singularities arising from the automorphism of curves. The natural $(H \times \mathbb{G}_m)$-action on $(\mathbb{P}^1 \times \mathcal{B}_{\mathtt{J}})$ induces a natural $(H \times \mathbb{G}_m)$-action on $\mathcal{GB}_{\mathtt{J},n,\beta}$. Moreover, $\mathcal{GB}_{\mathtt{J},0,\beta}$ has only finitely many isolated $(H \times \mathbb{G}_m)$-fixed points, and thus we can apply the formalism of Atiyah-Bott-Lefschetz localization (cf. [16, p200L26] and [4, Proof of Lemma 5]).

We have a morphism $\pi_{\mathtt{J},n,\beta} : \mathcal{GB}_{\mathtt{J},n,\beta} \to \mathcal{Q}_{\mathtt{J}}(\beta)$ that factors through $\mathcal{GB}_{\mathtt{J},0,\beta}$ (Givental's main lemma [17]; see [11, §8] and [13, §1.3]). Let $\widetilde{\mathtt{ev}}_j : \mathcal{GB}_{\mathtt{J},n,\beta} \to \mathbb{P}^1 \times \mathcal{B}_{\mathtt{J}}$ $(1 \leq j \leq n)$ be the evaluation at the j-th marked point, and let $\mathtt{ev}_j : \mathcal{GB}_{\mathtt{J},n,\beta} \to \mathcal{B}_{\mathtt{J}}$ be its composition with the second projection.

The following result is responsible for the basic case (the case of $\mathtt{J} = \emptyset$) of our computation:

Theorem 1.3 (Braverman-Finkelberg [3, 4, 5])**.** *The morphism $\pi_{0,\beta}$ is a rational resolution of singularities* (*in an orbifold sense*). $\square$

We note that $\mathcal{GB}_{\mathtt{J},n,\beta}$ is irreducible ([24]).

For each $\lambda \in P_{\mathtt{J}}$, we have a line bundle $\mathcal{O}_{\mathcal{GB}_{\mathtt{J},n,\beta}}(\lambda) := \pi^*_{\mathtt{J},n,\beta}\mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta)}(\lambda)$. For a $(H \times \mathbb{G}_m)$-equivariant coherent sheaf on a projective $(H \times \mathbb{G}_m)$-variety $\mathcal{X}$, let $\chi(\mathcal{X}, \mathcal{F}) \in \mathbb{C}_q P$ denote its $(H \times \mathbb{G}_m)$-equivariant Euler-Poincaré characteristic (that enhances the element $\chi(\mathcal{Q}_{\mathtt{J}}(\beta, w), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,w)}(\lambda))$ defined in §1.2).

1.4 Equivariant quantum K-group of $\mathcal{B}_{\mathtt{J}}$

We introduce a polynomial ring $\mathbb{C}Q^{\vee}_{\mathtt{J},+}$ with its variables $Q_i = Q^{\alpha_i^{\vee}}$ ($i \in \mathtt{J}^c$). We set $Q^{\beta} := \prod_{i \in \mathtt{J}^c} Q_i^{\langle \beta, \varpi_i \rangle}$ for each $\beta \in Q^{\vee}_{\mathtt{J}}$. We define the H-equivariant (small) quantum K-group of $\mathcal{B}_{\mathtt{J}}$ as:

$$qK_H(\mathcal{B}_{\mathtt{J}}) := K_H(\mathcal{B}_{\mathtt{J}}) \otimes \mathbb{C}Q^{\vee}_{\mathtt{J},+}, \tag{1.3}$$

where $K_H(\mathcal{B}_{\mathtt{J}})$ is the complexified H-equivariant K-group of $\mathcal{B}_{\mathtt{J}}$.

Thanks to (the H-equivariant versions of) [15, 32] and the finiteness of the quantum multiplication [1], $qK_H(\mathcal{B}_{\mathtt{J}})$ is equipped with the commutative and associative product $\star$ (called the quantum multiplication) such that:

1. the element $[\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}] \otimes 1 \in qK_H(\mathcal{B}_{\mathtt{J}})$ is the identity (with respect to $\cdot$ and $\star$);
2. the map $Q^{\beta}\star$ ($\beta \in Q^{\vee}_{\mathtt{J},+}$) is the multiplication of Q^{β} in the RHS of (1.3);
3. we have $\xi \star \eta \equiv \xi \cdot \eta \mod (Q_i; i \in \mathtt{J}^c)$ for every $\xi, \eta \in K_H(\mathcal{B}_{\mathtt{J}}) \otimes 1$.

We set

$$qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}}) := K_H(\mathcal{B}_{\mathtt{J}}) \otimes \mathbb{C}_q Q^{\vee}_+ \quad \text{and} \quad qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge} := K_H(\mathcal{B}_{\mathtt{J}}) \otimes \mathbf{C}_q[\![Q^{\vee}_+]\!].$$

We can localize $qK_H(\mathcal{B}_{\mathtt{J}})$ (resp. $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})$ and $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}$) in terms of $\{Q^{\beta}\}_{\beta \in Q^{\vee}_{\mathtt{J},+}}$ to obtain a ring $qK_H(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}}$ (resp. vector spaces $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}}$ and $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}_{\mathrm{loc}}$).

We sometimes identify $K_H(\mathcal{B}_{\mathtt{J}})$ with the submodule $K_H(\mathcal{B}_{\mathtt{J}}) \otimes 1$ of $qK_H(\mathcal{B}_{\mathtt{J}})$ or $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})$. We set $p_i := [\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(\varpi_i)]$ for $i \in \mathtt{J}^c$, and we sometimes consider it as an endomorphism of $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})$ through the scalar extension of the product of $K_H(\mathcal{B}_{\mathtt{J}})$ (i.e. the classical product). For each $i \in \mathtt{J}^c$, let $q^{Q_i \partial_{Q_i}}$ denote the $\mathbf{C}_q P$-endomorphism of $qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})$ such that

$$q^{Q_i \partial_{Q_i}}(\xi \otimes Q^{\beta}) = q^{\langle \beta, \varpi_i \rangle} \xi \otimes Q^{\beta} \qquad \xi \in K_H(\mathcal{B}_{\mathtt{J}}), \beta \in Q^{\vee}_{\mathtt{J},+}.$$

Following [18, §2.4], we consider the operator $T \in \mathrm{End}_{\mathbf{C}_q P}\, qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}$ (obtained from the same named operator in [18] by setting $0 = t \in K(\mathcal{B}_{\mathtt{J}})$). Then, we have the shift operator (also obtained from an operator $A_i(q,t)$ in [18] by setting $t = 0$) defined by

$$A_i(q) = T^{-1} \circ p_i^{-1} q^{Q_i \partial_{Q_i}} \circ T \in \mathrm{End}\, qK_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge} \qquad i \in \mathtt{J}^c. \tag{1.4}$$

Theorem 1.4 ([18] and [1]). *For $i \in \mathtt{J}^c$, the operator $A_i(1)$ is well-defined and defines the $\star$-multiplication by $[\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\varpi_i)]$ in $qK_H(\mathcal{B}_{\mathtt{J}})$.*

Proof. The well-definedness of the substitution $q = 1$ is by [18, Remark 2.14]. By [18, Corollary 2.9] and [1, Theorem 8], the set $\{A_i(1)\}_{i \in \mathtt{J}^c}$ defines mutually commutative endomorphisms of $qK_H(\mathcal{B}_{\mathtt{J}})$ that commutes with the $\star$-multiplication. Since $\mathrm{End}_R R \cong R$ for every ring R, we conclude the assertion by $A_i(1)([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}]) = [\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\varpi_i)]$ ([1, Lemma 6]). $\square$

2 A description of the quantum K-groups

We continue to work in the setting of the previous section.

2.1 K-groups of semi-infinite partial flag manifolds

Let $\mathtt{J} \subset \mathtt{I}$ be a subset. The semi-infinite partial flag manifold $\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$ is an ind-scheme whose set of $\mathbb{C}$-valued points is

$$G(\!(z)\!)/H(\mathbb{C}) \cdot ([P(\mathtt{J}), P(\mathtt{J})](\!(z)\!)).$$

This is a pure ind-scheme of ind-infinite type [20]. Note that the group $Q^\vee \subset H(\!(z)\!)/H$ acts on $\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$ from the right, whose action factors through $Q_{\mathtt{J}}^\vee$ via the projection described in the below. The indscheme $\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$ is equipped with a $G[\![z]\!]$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}}(\lambda)$ for each $\lambda \in P_{\mathtt{J}}$. Here we normalized such that $\Gamma(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}, \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}}(\lambda))$ is co-generated by its H-weight $(-\lambda)$-part as a $B^-[\![z]\!]$-module.

The following two results are not recorded in the literature in a strict sense, but they are straight-forward consequences of the set-theoretic consideration that is allowed in view of [20, Theorem A].

Theorem 2.1. *We have an* $\mathbf{I}$*-orbit decomposition*

$$\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}} = \bigsqcup_{u \in W/W_{\mathtt{J}}, \beta \in Q_{\mathtt{J}}^\vee} \mathbb{O}_{\mathtt{J}}(ut_\beta).$$

Corollary 2.2. *The natural quotient map* $\mathbf{Q}^{\mathrm{rat}} \to \mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$ *sends the* $\mathbf{I}$*-orbit* $\mathbb{O}(ut_\beta)$ *to* $\mathbb{O}_{\mathtt{J}}(u' t_{\beta'})$*, where* $u' \in uW_{\mathtt{J}}$ *is characterized by* $u' \in W/W_{\mathtt{J}}$ *and* $\beta' \in Q_{\mathtt{J}}^\vee$ *is defined as the projection:*

$$\beta' := \beta - \sum_{j \in \mathtt{J}} \langle \beta, \varpi_i \rangle \, \alpha_i^\vee.$$

For $u \in W$ and $\beta \in Q^\vee$, we denote the element $u' t_{\beta'} \in W_{\mathrm{af}}$ obtained in Corollary 2.2 by $[ut_\beta]_{\mathtt{J}}$. By abuse of notation, we also write β' by $[\beta]_{\mathtt{J}}$.

For each $u \in W/W_{\mathtt{J}}$ and $\beta \in Q_{\mathtt{J}}^\vee$, we set $\mathbf{Q}_{\mathtt{J}}(ut_\beta) := \overline{\mathbb{O}_{\mathtt{J}}(ut_\beta)} \subset \mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}$. We have embeddings $\mathcal{B}_{\mathtt{J}}(u) \subset \mathcal{Q}_{\mathtt{J}}(\beta, u) \subset \mathbf{Q}_{\mathtt{J}}(u)$ $(u \in W/W_{\mathtt{J}})$ such that the line bundles $\mathcal{O}(\lambda)$ $(\lambda \in P_{\mathtt{J}})$ corresponds to each other by restrictions ([4, 19, 23]).

Theorem 2.3 ([20] Corollary C and Appendix A)**.** *For each* $u \in W/W_{\mathtt{J}}$*, and* $\lambda \in P_{\mathtt{J},+}$*, we have*

$$\lim_{\beta \to \infty} \chi(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda)) = \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(u), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(u)}(\lambda)) \in \mathbf{C}_q P. \tag{2.1}$$

Moreover, we have $H^{>0}(\mathbf{Q}_{\mathtt{J}}(u), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(u)}(\lambda)) = \{0\}$.

We define a(n uncompleted version of the) $\mathbb{C}_q^0 P$-module $K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$ as:

$$K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) := \{ \sum_{u \in W/W_{\mathtt{J}}, \beta \in Q_{\mathtt{J}}^\vee} a_{u,\beta} [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}] \mid a_{u,\beta} \in \mathbb{C}_q^0 P \}.$$

Here we remark that the sum in the definition of $K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$ is a finite sum. We set $K'_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) := \mathbb{C}_q \otimes_{\mathbb{C}_q^0} K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$. For each $\gamma \in Q_{\mathtt{J}}^\vee$, we also define

$$K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}(t_\gamma)) := \{ \sum_{u \in W/W_{\mathtt{J}}, \beta - \gamma \in Q_{\mathtt{J},+}^\vee} a_{u,\beta} [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}] \in K''_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) \}.$$

We sometimes also consider its completion

$$K_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})^\wedge := \mathbb{C}_q \otimes_{\mathbb{C}_q^0} \varprojlim_{\gamma} K''_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})/K''_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}(t_\gamma))$$

and its subset

$$K^+_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) := \{ \sum_{u\in W/W_{\mathtt{J}},\beta\in Q^\vee_{\mathtt{J}}} a_{u,\beta}[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}] \in K_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})^\wedge \mid \sum_{u,\beta} |a_{u,\beta}| \in \mathbf{C}_q P\},$$

where the absolute value is taken for each coefficient of monomials.

We have a $\mathbb{C}_q P$-linear surjective morphism

$$\phi_{\mathtt{J}} : K'_{H\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \ni [\mathcal{O}_{\mathbf{Q}(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}([w]_{\mathtt{J}})}] \in K'_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) \qquad w \in W_{\mathrm{af}}.$$

Theorem 2.4 ([23] Corollary 4.31 and [20] Appendix A)**.** *For $w \in W_{\mathrm{af}}$ and $\lambda \in P_{\mathtt{J}}$, we have*

$$\mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}([w]_{\mathtt{J}}), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}([w]_{\mathtt{J}})}(\lambda)) = \mathrm{gch}\, H^0(\mathbf{Q}(w), \mathcal{O}_{\mathbf{Q}(w)}(\lambda)) \in \mathbb{Z}_{\ge 0}[\![q^{-1}]\!]P.$$

They yields zero if $\lambda \notin P_{\mathtt{J},+}$. Moreover, their higher cohomologies vanish. □

Let $\mathrm{Fun}_{P_{\mathtt{J}}}(\mathbf{C}_q P)$ denote the set of functionals on $P_{\mathtt{J}}$ whose value is in $\mathbf{C}_q P$. We set

$$\mathrm{Fun}^{\mathrm{neg}}_{P_{\mathtt{J}}}(\mathbf{C}_q P) := \{f \in \mathrm{Fun}_{P_{\mathtt{J}}}(\mathbf{C}_q P) \mid \exists \gamma \in P_{\mathtt{J}} \text{ s.t. } f(\lambda) = 0 \text{ for each } \lambda \in \gamma + P_{\mathtt{J},+}\}$$

and $\mathrm{Fun}^{\mathrm{ess}}_{P_{\mathtt{J}}}(\mathbf{C}_q P) := \mathrm{Fun}_{P_{\mathtt{J}}}(\mathbf{C}_q P)/\mathrm{Fun}^{\mathrm{neg}}_{P_{\mathtt{J}}}(\mathbf{C}_q P)$.

Theorem 2.5. *The assignment*

$$K'_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) \ni \sum_{u\in W/W_{\mathtt{J}},\beta\in Q^\vee_{\mathtt{J}}} a_{u,\beta}[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}]$$
$$\mapsto \left(\lambda \mapsto \sum_{u,\beta} a_{u,\beta} \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}, \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda)) \right) \in \mathrm{Fun}^{\mathrm{ess}}_{P_{\mathtt{J}}}(\mathbf{C}_q P)$$

is an injective $\mathbb{C}_q P$-linear map. This prolongs to a $\mathbb{C}_q P$-linear map

$$K^+_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) \longrightarrow \mathrm{Fun}^{\mathrm{ess}}_{P_{\mathtt{J}}}(\mathbf{C}_q P).$$

Proof. The first assertion reduces to the $\mathbb{C}_q P$-linear independence of the functionals

$$P_{\mathtt{J},+} \ni \lambda \mapsto \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(ut_\beta), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda)) \qquad u \in W/W_{\mathtt{J}}, \beta \in Q^\vee_{\mathtt{J}}.$$

In view of Theorem 2.4, this follows as in [23, Proof of Proposition 5.11].

We prove the second assertion. The $\mathbb{C}_q P$-coefficients $\{a_{u,\beta}\}$ of an element of $K^+_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$ satisfies $a_{u,\beta} = 0$ for $\beta \not\ge \beta_0$ for some $\beta_0 \in Q^\vee_{\mathtt{J},+}$, each of them are Laurant polynomials with a uniform upper bound on its q-degree, and $\sum_{u,\beta} |a_{u,\beta}| \in \mathbf{C}_q P$.

In view of [20, Theorem 2.32] and Theorem 2.4, we have

$$\mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(ut_\beta), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda)) \le \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(t_{\beta_0}), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(t_{\beta_0})}(\lambda)) \qquad (2.2)$$

for each $\lambda \in P_{\mathtt{J}}$, $u \in W$, and $\beta_0 \le \beta \in Q^{\vee}_{\mathtt{J},+}$, where the inequality is understood to be coefficient-wise (in $\mathbb{Z}_{\ge 0}$). The RHS of (2.2) belongs to $\mathbf{C}_q P$ (cf. [4]).

We set $a := \sum_{u,\beta} |a_{u,\beta}| \in \mathbf{C}_q P$. From the above, we deduce

$$\sum_{u,\beta} |a_{u,\beta}| \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(ut_\beta), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda)) \le a \cdot \mathrm{gch}\, H^0(\mathbf{Q}_{\mathtt{J}}(t_{\beta_0}), \mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(t_{\beta_0})}(\lambda)),$$

that implies the convergence of our functional for each $\lambda \in P_{\mathtt{J}}$. $\square$

We define $K_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ as the image of $K^{+}_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ in $\mathrm{Fun}^{\mathrm{ess}}_{P_{\mathtt{J}}}(\mathbf{C}_q P)$.

We have $q = 1$ specializations of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ and $K_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ are possible as each coefficient of $[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}([w]_{\mathtt{J}})}]$ ($w \in W_{\mathrm{af}}$) belongs to $\mathbb{C}_q P$. They are denoted by $K'_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ and $K_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$, respectively.

Theorem 2.6 ([23] Theorem 6.5 for the case $\mathtt{J} = \emptyset$)**.** *For each $\lambda \in P_{\mathtt{J}}$, there exists a $\mathbb{C}_q P$-linear endomorphism*

$$[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}] \mapsto [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda)] \in K_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}}) \quad u \in W/W_{\mathtt{J}}, \beta \in Q^{\vee}_{\mathtt{J}}$$

which is an automorphism of $K_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ (that we call $\Xi(\lambda)$ in the below).

Proof. The reasoning we need is the same as those provided in [23, Proof of Theorem 6.5] and [21, Proof of Theorem 1.13] in view of the definition of $K_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ and Theorem 2.4. $\square$

Remark 2.7. In view of [21, Lemma 1.14] and [22, Corollary 3.3] (or combine [21, Corollary 3.3, Theorem 4.2, and their proofs]; cf. [21, Remark 2.13 and Remark 3.12]), we deduce that $\Xi(-\varpi_i)$ ($i \in \mathtt{I}$) defines an automorphism of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$. However, an explicit formula [23, Theorem 5.13] tells that $\Xi(\varpi_i)$ ($i \in \mathtt{I}$) never defines an automorphism of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$.

Theorem 2.8. *For each $i \in \mathtt{J}^c$, the endomorphism $\Xi(-\varpi_i)$ descends to an endomorphism $\Xi_{\mathtt{J}}(-\varpi_i)$ of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ through $\phi_{\mathtt{J}}$. In addition, the map $\phi_{\mathtt{J}}$ induces a surjective $\mathbb{C}P$-module map $K'_H(\mathbf{Q}^{\mathrm{rat}}) \to K'_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ such that $\Xi(-\varpi_i)$ induces an endomorphism of $K'_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$.*

Proof. Consider the $\mathbb{C}_q P$-linear map generated by

$$K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}) \ni \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}(w)}]$$
$$\mapsto \left(\lambda \mapsto \sum_{w} a_w \mathrm{gch}\, H^0(\mathbf{Q}^{\mathrm{rat}}, \mathcal{O}_{\mathbf{Q}(w)}(\lambda)) \right) \in \mathrm{Fun}_{P_{\mathtt{J}}}(\mathbf{C}_q P).$$

By Theorem 2.4, this map factors through $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ as $[\mathcal{O}_{\mathbf{Q}(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}([w]_{\mathtt{J}})}]$ for $w \in W_{\mathrm{af}}$. By Remark 2.7, we know that $\Xi(-\varpi_i)$ ($i \in \mathtt{I}$) is an endomorphism of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$. In view of Theorem 2.5, the endomorphism $\Xi(-\varpi_i)$ on $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ descends to an endomorphism of $K'_{H \times \mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ for each $i \in \mathtt{J}^c$ via the map $\phi_{\mathtt{J}}$. By specializing $q = 1$, we conclude that $\phi_{\mathtt{J}}$ induces a $\mathbb{C}P$-module surjection $K'_H(\mathbf{Q}^{\mathrm{rat}}) \to K'_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ on which $\Xi(-\varpi_i)$ descends to an endomorphism. $\square$

By abuse of notation, we denote the surjective map $K'_H(\mathbf{Q}^{\mathrm{rat}}) \to K'_H(\mathbf{Q}^{\mathrm{rat}}_{\mathtt{J}})$ in Theorem 2.8 by $\phi_{\mathtt{J}}$. We also denote the $q = 1$ specializations of the automorphisms $\Xi(-\varpi_i)$ and $\Xi_{\mathtt{J}}(-\varpi_i)$ in Theorem 2.8 by the same symbols.

2.2 $\mathcal{Q}_{\mathtt{J}}(\beta, w)$ has at worst rational singularities

Let $\mathcal{X}_{\mathtt{J}}(\beta)$ denote the subvariety of $\mathcal{GB}_{\mathtt{J},2,\beta}$ such that the first marked point projects to $0 \in \mathbb{P}^1$, and the second marked point projects to $\infty \in \mathbb{P}^1$ through the projection of quasi-stable curves C to the main component $C_0 \cong \mathbb{P}^1$. Let us denote the restriction of $\mathtt{ev}_i$ $(i = 1, 2)$ to $\mathcal{X}_{\mathtt{J}}(\beta)$ by the same letter. Since $\mathcal{X}_{\mathtt{J}}(\beta)$ is a normal scheme that have at worst quotient singularity, we might regard it as a smooth stack ([13]). As we know that $\mathcal{Q}_{\mathtt{J}}(\beta)$ is normal ([20]), we conclude that $\pi_{\mathtt{J},2,\beta}$ restricted to $\mathcal{X}_{\mathtt{J}}(\beta)$ also gives a resolution of singularities of $\mathcal{Q}_{\mathtt{J}}(\beta)$.

For each $\beta \in Q^{\vee}_{\mathtt{J},+}$ and $u \in W/W_{\mathtt{J}}$, we set $\mathcal{X}_{\mathtt{J}}(\beta, u) := \mathtt{ev}_1^{-1}(\mathcal{B}_{\mathtt{J}}(u))$.

Lemma 2.9. *For each $\beta \in Q^{\vee}_{\mathtt{J},+}$ and $u \in W/W_{\mathtt{J}}$, the variety $\mathcal{X}_{\mathtt{J}}(\beta, u)$ is projective, normal, and has at worst rational singularities.*

Proof. Being a closed subset of a projective variety $\mathcal{GB}_{\mathtt{J},2,\beta}$, we find that $\mathcal{X}_{\mathtt{J}}(\beta, u)$ is projective. The evaluation map $\mathtt{ev}_1 : \mathcal{X}_{\mathtt{J}}(\beta) \to \mathcal{B}_{\mathtt{J}}$ is homogeneous with respect to the G-action. Let $N_{\mathtt{J}} \subset N$ be the opposite unipotent radical of the conjugation of $P(\mathtt{J})$ by a lift of $w_0 \in W$ in $N_G(H)$. By restricting to the open N-orbit $N_{\mathtt{J}} \times \{p_e\} \cong \mathbb{O}_{\mathtt{J}}(e) \subset \mathcal{B}_{\mathtt{J}}$, we deduce that $\mathtt{ev}_1^{-1}(\mathbb{O}_{\mathtt{J}}(e)) \cong N_{\mathtt{J}} \times \mathtt{ev}_1^{-1}(p_e)$. By translating using the G-action, we conclude that $\mathtt{ev}_1$ is a locally trivial fibration. We know that $\mathcal{B}_{\mathtt{J}}(u)$ $(u \in W/W_{\mathtt{J}})$ is normal and has at worst rational singularities (see [25]). Thus, the singularity of $\mathcal{X}_{\mathtt{J}}(\beta, u)$ is locally a product of two rational singularities. From basic properties of rational singularities [27, §5.1], we deduce that being rational singularity is a local condition and it is preserved by taking products. Therefore, we conclude that $\mathcal{X}_{\mathtt{J}}(\beta, u)$ has at worst rational singularities (and the normality is its consequence). □

We have $\mathcal{X}_{\mathtt{J}}(\beta) = \mathcal{X}_{\mathtt{J}}(\beta, e)$. The map $\pi_{\mathtt{J},2,\beta}$ restricts to a $(B \times \mathbb{G}_m)$-equivariant birational proper map

$$\pi_{\mathtt{J},\beta,u} : \mathcal{X}_{\mathtt{J}}(\beta, u) \to \mathcal{Q}_{\mathtt{J}}(\beta, u)$$

by inspection. Let $\mathcal{O}_{\mathcal{X}_{\mathtt{J}}(\beta,u)}(\lambda)$ denote the restriction of $\mathcal{O}_{\mathcal{GB}_{\mathtt{J},2,\beta}}(\lambda)$ to $\mathcal{X}_{\mathtt{J}}(\beta, u)$ for each $\lambda \in P_{\mathtt{J}}$ and $u \in W/W_{\mathtt{J}}$.

Theorem 2.10 (Kollár [26] Theorem 7.1)**.** *Let $f : X \to Z$ be a surjective map between projective varieties, X smooth, and Z normal. Let F be the geometric generic fiber of f and assume that F is connected. The following two statements are equivalent:*

1. *$\mathbb{R}^i f_* \mathcal{O}_X = 0$ for all $i > 0$;*
2. *Z has rational singularities and $H^i(F, \mathcal{O}_F) = 0$ for all $i > 0$.*

Theorem 2.11. *For each $\beta \in Q^{\vee}_{\mathtt{J},+}$ and $u \in W/W_{\mathtt{J}}$, the variety $\mathcal{Q}_{\mathtt{J}}(\beta, u)$ has at worst rational singularities. In addition, we have*

$$(\pi_{\mathtt{J},\beta,u})_* \mathcal{O}_{\mathcal{X}_{\mathtt{J}}(\beta,u)} \cong \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}, \qquad \mathbb{R}^{>0}(\pi_{\mathtt{J},\beta,u})_* \mathcal{O}_{\mathcal{X}_{\mathtt{J}}(\beta,u)} \cong \{0\}.$$

Proof. By [20, Corollary 4.20], the variety $\mathcal{Q}_{\mathtt{J}}(\beta, u)$ is normal. By Lemma 2.9, we know that $\mathcal{X}_{\mathtt{J}}(\beta, u)$ has at worst rational singularities. The same is true for $\mathtt{J} = \emptyset$ by [3, 13]. The coarse moduli property of $\mathcal{X}(\beta)$ yields a morphism $\mathcal{X}(\beta^+) \longrightarrow \mathcal{X}_{\mathtt{J}}(\beta)$ for every $\beta^+ \in Q^{\vee}_+$ such that $\beta = [\beta^+]_{\mathtt{J}}$. In view of [20, Remark 3.36] (cf. Woodward [35]), we can choose β^+ such that $\mathcal{Q}(\beta^+, u) \longrightarrow \mathcal{Q}_{\mathtt{J}}(\beta, u)$ is surjective.

We have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{X}(\beta^+, u) \arrow[r, "\widetilde{\eta}"] \arrow[d, "\pi_{\beta^+,u}"'] & \mathfrak{X}_{\mathtt{J}}(\beta, u) \arrow[d, "\pi_{\mathtt{J},\beta,u}"] \\
\mathcal{Q}(\beta^+, u) \arrow[r, "\eta"] & \mathcal{Q}_{\mathtt{J}}(\beta, u).
\end{tikzcd}$$

Here the maps $\pi_{\beta^+,u}$ and $\pi_{\mathtt{J},\beta,u}$ are birational. Thus, the map $\widetilde{\eta}$ is also surjective. Moreover, we have $\mathbb{R}^\bullet \eta_* \mathcal{O}_{\mathcal{Q}(\beta^+,u)} = \mathcal{O}_{\mathcal{Q}(\beta,u)}$ by [20, Corollary 3.35]. We find $\mathbb{R}^\bullet (\pi_{\beta^+,u})_* \mathcal{O}_{\mathfrak{X}(\beta^+,u)} = \mathcal{O}_{\mathcal{Q}(\beta^+,u)}$ by [21, Theorem 4.9]. By the Leray spectral sequence applied to the composition map $\eta \circ \pi_{\beta^+,u}$, we find that

$$\mathbb{R}^\bullet (\eta \circ \pi_{\beta^+,u})_* \mathcal{O}_{\mathfrak{X}(\beta^+,u)} = \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}.$$

This implies that the geometric generic fiber of the composition map $(\eta \circ \pi_{\beta^+,u})$ has trivial higher cohomology. Since $\pi_{\mathtt{J},\beta,u}$ is birational, the geometric generic fiber of $(\eta \circ \pi_{\beta^+,u})$ is the same as $\widetilde{\eta}$. Therefore, we conclude

$$\mathbb{R}^\bullet \widetilde{\eta}_* \mathcal{O}_{\mathfrak{X}(\beta^+,u)} = \mathcal{O}_{\mathfrak{X}_{\mathtt{J}}(\beta,u)} \tag{2.3}$$

by Theorem 2.10 (by replacing $\mathfrak{X}(\beta^+, u)$ with its resolution of singularity if necessary, cf. [27, Theorem 5.10]). By the above commutative diagram, the Leray spectral sequence applied to the composition map $\pi_{\mathtt{J},\beta,u} \circ \widetilde{\eta} = \eta \circ \pi_{\beta^+,u}$ implies

$$\mathbb{R}^\bullet (\pi_{\mathtt{J},\beta,u})_* \mathcal{O}_{\mathfrak{X}_{\mathtt{J}}(\beta,u)} \cong \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}$$

from (2.3). This shows that $\mathcal{Q}_{\mathtt{J}}(\beta, u)$ has at worst rational singularities by [27, Theorem 5.10]. $\square$

Corollary 2.12. *For each $\beta \in Q^\vee_{\mathtt{J},+}$, $u \in W/W_{\mathtt{J}}$, and $\lambda \in P_{\mathtt{J}}$, we have*

$$\chi(\mathfrak{X}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathfrak{X}_{\mathtt{J}}(\beta,u)}(\lambda)) = \chi(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda)) \in \mathbb{C}^0_q P.$$

Proof. Apply the projection formula to Theorem 2.11. $\square$

For $\vec{n} = \{n_i\}_{i \in \mathtt{J}^c} \in \mathbb{Z}^{\mathtt{J}^c}_{\geq 0}$, we set $x^{\vec{n}} := \prod_{i \in \mathtt{J}^c} x_i^{n_i}$. For $\lambda \in P$, we set $\lambda[\vec{n}] := \lambda - \sum_{i \in \mathtt{J}^c} n_i \varpi_i$.

Theorem 2.13 (Iritani-Milanov-Tonita [18], cf. Givental-Lee [16])**.** *For each*

$$\sum_{\beta \in Q^\vee_{\mathtt{J},+}, u \in W/W_{\mathtt{J}}, \vec{n} \in \mathbb{Z}^{\mathtt{J}^c}_{\geq 0}} f_{\beta,u,\vec{n}}(q) x^{\vec{n}} Q^\beta \in (\mathbb{C}^0_q P)[\{x_i\}_{i \in \mathtt{J}^c}][\![Q^\vee_+]\!]$$

such that

$$\sum_{\beta \in Q^\vee_{+}, u \in W/W_{\mathtt{J}}, \vec{n} \in \mathbb{Z}^{\mathtt{J}^c}_{\geq 0}} f_{\beta,u,\vec{n}}(q) \left(\prod_{i \in \mathtt{J}^c}^r A_i^{n_i} \right) Q^\beta [\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}] = 0 \in qK_{G \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^\wedge, \tag{2.4}$$

we have the following equalities:

$$\sum_{\beta \in Q^\vee_{\mathtt{J},+}, u \in W/W_{\mathtt{J}}, \vec{n} \in \mathbb{Z}^{\mathtt{J}^c}_{\geq 0}} f_{\beta,u,\vec{n}}(q) q^{-\langle \beta, \lambda[\vec{n}] \rangle} \chi(\mathfrak{X}_{\mathtt{J}}(\gamma - \beta, u), \mathcal{O}_{\mathfrak{X}_{\mathtt{J}}(\gamma-\beta,u)}(\lambda[\vec{n}])) = 0$$

for each $\lambda \in P_{\mathtt{J},+}$ and $\gamma \in Q^\vee_{\mathtt{J},+}$.

Proof. The assertion follows by plugging (2.4) into [18, Proposition 2.20] and observe that A_i becomes the line bundle twist by $\mathcal{O}(-\varpi_i)$ up to $q^{\langle \beta, \varpi_i \rangle}$, Q_i twists the Novikov variable (and hence the degree of the stable map spaces), and the effect of $\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}$ is to restrict the whole variety to $\mathcal{X}_{\mathtt{J}}(\bullet, u)$ via $\mathtt{ev}_1^*$. It can be also seen as a variant of [21, Theorem 3.8 and Theorem 3.9]. $\square$

2.3 Comparison of equivariant K-groups

Theorem 2.14. *We have a $\mathbb{C}_q P$-module isomorphism*

$$\Psi_{\mathtt{J},q} : qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}} \xrightarrow{\cong} K'_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$$

such that

1. $\Psi_{\mathtt{J},q}([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}] Q^{\beta}) = [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_{\beta})}]$ *for each* $u \in W/W_{\mathtt{J}}$ *and* $\beta \in Q_{\mathtt{J}}^{\vee}$*;*
2. $\Psi_{\mathtt{J},q}(A_i(\bullet)) = \Xi_{\mathtt{J}}(-\varpi_i)(\Psi_{\mathtt{J},q}(\bullet))$ *for each* $i \in \mathtt{J}^c$*.*

Corollary 2.15. *As the $q = 1$ specialization of Theorem 2.14, we obtain a $\mathbb{C}P$-module isomorphism*

$$\Psi_{\mathtt{J}} : qK_{H}(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}} \xrightarrow{\cong} K'_{H}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$$

such that

1. $\Psi_{\mathtt{J}}([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}] Q^{\beta}) = [\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_{\beta})}]$ *for each* $u \in W/W_{\mathtt{J}}$ *and* $\beta \in Q_{\mathtt{J}}^{\vee}$*;*
2. $\Psi_{\mathtt{J}}([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\varpi_i)] \star \bullet) = \Xi_{\mathtt{J}}(-\varpi_i)(\Psi_{\mathtt{J}}(\bullet))$ *for each* $i \in \mathtt{J}^c$*.*

Proof of Corollary 2.15. Taking Theorem 2.14 into account, it remains to observe that $A_i(\bullet)$ specializes to $[\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\varpi_i)]\star$ by Theorem 1.4. $\square$

Proof of Theorem 2.14. By the definitions of $qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}}$ and $K'_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$, we find that $\Psi_{\mathtt{J},q}$ is a $\mathbb{C}_q P$-linear isomorphism. The map $\Psi_{\mathtt{J},q}$ prolongs to an isomorphism

$$qK'_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}_{\mathrm{loc}} \xrightarrow{\cong} K_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}),$$

where $qK'_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}_{\mathrm{loc}}$ is the quotient of some subset of $qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}_{\mathrm{loc}}$ subject to the analogous convergence condition as in $K^{+}_{H \times \mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$ (such that we have $qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}} \subset qK'_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^{\wedge}_{\mathrm{loc}}$).

For each $u \in W/W_{\mathtt{J}}$, we expand $A_i([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}])$ as a formal linear combination

$$A_i([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}]) = \sum_{v \in W/W_{\mathtt{J}}, \gamma \in Q^{\vee}_{\mathtt{J},+}} a_{i,u}^{v,\gamma} Q^{\gamma} [\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(v)}] \qquad a_{i,u}^{v,\gamma} \in \mathbb{C}_q P$$

by [18, Remark 2.14].

Applying Theorem 2.13 and Corollary 2.12, we have

$$\chi(\mathcal{Q}_{\mathtt{J}}(\beta, u), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda - \varpi_i)) = \sum_{v \in W/W_{\mathtt{J}}, \gamma \in Q^{\vee}_{\mathtt{J},+}} a_{i,u}^{v,\gamma} q^{-\langle \gamma, \lambda \rangle} \chi(\mathcal{Q}_{\mathtt{J}}(\beta - \gamma, v), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta-\gamma,v)}(\lambda)) \tag{2.5}$$

for each $\beta \in Q^{\vee}_{\mathtt{J},+}$ and $\lambda \in P_{\mathtt{J}}$. We have $\chi(\mathcal{Q}_{\mathtt{J}}(\beta, v), \mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,v)}(\lambda)) \in \mathbb{C}_q^0 P$ for every $u \in W/W_{\mathtt{J}}$, $\beta \in Q^{\vee}_{\mathtt{J},+}$, and $\lambda \in P_{\mathtt{J},+}$. By [20, Theorem 3.33], the $\mathbb{C}$-coefficients of the series $\{\chi(\mathcal{Q}_{\mathtt{J}}(\beta,u),\mathcal{O}_{\mathcal{Q}_{\mathtt{J}}(\beta,u)}(\lambda))\}_\beta \subset \mathbb{C}_q P$ belong to $\mathbb{Z}_{\geq 0}[q^{-1}]P$ and are monotonically non-decreasing with respect to β. By examining the cases $\beta=\gamma$, we deduce $a_{i,u}^{v,\gamma} \in \mathbb{Z}[q^{-1}]P$ by induction (from the case $\beta=\gamma=0$). Moreover, the limit $\beta \to \infty$ (in $Q^\vee_{\mathtt{J},+}$) of the LHS of (2.5) is convergent ([20, Theorem 3.33]). Hence, in order that the RHS of (2.5) to be equal to the LHS, we further need $\sum_{v,\gamma}|a_{i,u}^{v,\gamma}| \in \mathbf{C}_q P$. Therefore, we conclude $A_i([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}]) \in qK'_{H\times\mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})^\wedge$.

By taking the limit $\beta \to \infty$ (cf. [23, Proposition D.1]), we obtain

$$\chi(\mathbf{Q}_{\mathtt{J}}(ut_\beta),\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(\lambda-\varpi_i)) = \sum_{v\in W/W_{\mathtt{J}},\gamma\in Q^\vee_{\mathtt{J},+}} a_{i,u}^{v,\gamma}\chi(\mathbf{Q}_{\mathtt{J}}(vt_\gamma),\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(vt_\gamma)}(\lambda))$$

for each $\lambda \in P_{\mathtt{J},+}$ by Theorem 2.1. This implies

$$[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(ut_\beta)}(-\varpi_i)] = \sum_{v\in W/W_{\mathtt{J}},\gamma\in Q^\vee_{\mathtt{J},+}} a_{i,u}^{v,\gamma}[\mathcal{O}_{\mathbf{Q}_{\mathtt{J}}(vt_\gamma)}]$$

in view of Theorem 2.5. Hence, we conclude

$$\Psi_{\mathtt{J},q}(A_i([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}])) = \Xi_{\mathtt{J}}(-\varpi_i)(\Psi_{\mathtt{J},q}([\mathcal{O}_{\mathcal{B}_{\mathtt{J}}(u)}])) \qquad u\in W/W_{\mathtt{J}}, \tag{2.6}$$

where the equality is in $K_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$. We can rewrite the operator $\Xi(-\varpi_i)$ into a linear combination of the identity and an operator H_i as in [21, §1.4]. In view of [22, Corollary 3.3] (or [21, Remark 3.12 and Remark 3.13]), we find that the operator H_i is finite as it transplants to the Pontryagin multiplication of a (localized) Schubert classe (cf. [21, Corollary 3.3]). Thus, the operator $\Xi_{\mathtt{J}}(-\varpi_i)$ also exhibits finiteness by Theorem 2.8 (we can also apply [1] directly to obtain this). Consequently, (2.6) is in fact an equality in $K'_{H\times\mathbb{G}_m}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}})$. Since $\Psi_{\mathtt{J},q}$, A_i, and $\Xi_{\mathtt{J}}(-\varpi_i)$ $(i\in\mathtt{J}^c)$ are $\mathbb{C}_qP$-linear, we conclude the result. $\Box$

We consider the subring of $qK_H(\mathcal{B}_{\mathtt{J}})_{\geq 0} \subset qK_H(\mathcal{B}_{\mathtt{J}})$ generated by $\mathbb{C}P$, $\mathbb{C}Q^\vee_{\mathtt{J},+}$, and $\{[\mathcal{O}_{\mathcal{B}_{\mathtt{J}}}(-\varpi_i)]\star\}_{i\in\mathtt{J}^c}$.

Lemma 2.16. *For each $i\in\mathtt{I}$, the $\mathbb{C}_qP$-subspace $K_i^q \subset qK_{H\times\mathbb{G}_m}(\mathcal{B})$ spanned by the set*

$$\{[\mathcal{O}_{\mathcal{B}(u)}]Q^\beta - [\mathcal{O}_{\mathcal{B}(us_i)}]Q^{\beta'} \mid u\in W, \beta,\beta'\in Q^\vee_+,\ s.t.\ \beta-\beta'\in\mathbb{Z}\alpha_i^\vee\}$$

is stable by the action of $A_j(q)$ $(j\in\mathtt{I})$. In particular, its specialization $q=1$ yields a $\mathbb{C}P$-subspace $K_i\subset qK_H(\mathcal{B})$ that is stable by the $qK_H(\mathcal{B})_{\geq 0}$-action.

Remark 2.17. Lemma 2.16 does not hold if we replace $qK_H(\mathcal{B})$ with $K_H(\mathcal{B})$. We set $G=SL(2)$ (and hence $\mathcal{B}=\mathbb{P}^1$ and $\mathtt{I}=\{1\}$). We have an equality $[\mathcal{O}_{\mathcal{B}(s_1)}(-\varpi_1)] = e^{\varpi_1}[\mathcal{O}_{\mathcal{B}(s_1)}] \in K_H(\mathcal{B})$, that implies

$$[\mathcal{O}_{\mathcal{B}}(-\varpi_1)]-[\mathcal{O}_{\mathcal{B}(s_1)}(-\varpi_1)] = e^{-\varpi_1}[\mathcal{O}_{\mathcal{B}}]-(e^{\varpi_1}+e^{-\varpi_1})[\mathcal{O}_{\mathcal{B}(s_1)}]) \notin \mathbb{C}P([\mathcal{O}_{\mathcal{B}}]-[\mathcal{O}_{\mathcal{B}(s_1)}]).$$

In other words, the vanishing part of Theorem 2.4 is crucial in our consideration.

Proof of Lemma 2.16. Let F be the functional (on P) in Theorem 2.5 (for $\mathtt{J}=\emptyset$). By Theorem 2.4, elements in $\Psi_q^{-1}(K_i^q)$ vanishes via the functional F restricted to $\lambda\in(P_{\{i\}}+\mathbb{Z}_{\leq 0}\varpi_i)$. Conversely, if $a\in\Psi_q^{-1}(qK_{H\times\mathbb{G}_m}(\mathcal{B}))$ vanishes via the functional F restricted to $\lambda\in(P_{\{i\}}+\mathbb{Z}_{\leq 0}\varpi_i)$, then $F(a)$ must also vanish on $\lambda\in P_{\{i\}}$. In view of the injectivity assertion in Theorem 2.5, we conclude that $a \in \Psi_q^{-1}(K_i^q)$. Therefore, elements in $\Psi_q^{-1}(K_i^q)$ is precisely the set of elements of $\Psi_q^{-1}(qK_{H\times\mathbb{G}_m}(\mathcal{B}))$ that vanish via the functional F restricted to $\lambda \in (P_{\{i\}} + \mathbb{Z}_{\le 0}\varpi_i)$. Hence, $\Psi_q^{-1}(K_i^q)$ is stable under the action of $\{\Xi(-\varpi_j)\}_{j\in\mathtt{I}}$. It follows that the set $\Psi^{-1}(K_i)$ is stable by the multiplication by $qK_H(\mathcal{B})_{\ge 0}$. $\square$

2.4 Comparison between equivariant quantum K-groups

The following crucial observation is due to Buch-Chaput-Mihalcea-Perrin [6, §5] (see also [1, §1.2], cf. [10, Lemma 4.1.3]):

- The multiplication rule of $qK_H(\mathcal{B}_\mathtt{J})$ as a $\mathbb{C}P\otimes\mathbb{C}Q^\vee_{\mathtt{J},+}$-algebra is completely determined by the $\star$-multiplication table of $\mathcal{O}_{\mathcal{B}_\mathtt{J}(s_i)}$ for $i \in \mathtt{J}^c$.

In view of the equality (cf. [21, Theorem 1.1])

$$[\mathcal{O}_{\mathcal{B}_\mathtt{J}}(-\varpi_i)] = e^{w_0\varpi_i}([\mathcal{O}_{\mathcal{B}_\mathtt{J}}] - [\mathcal{O}_{\mathcal{B}_\mathtt{J}(s_i)}]) \in K_H(\mathcal{B}_\mathtt{J}) \qquad i \in \mathtt{J}^c,$$

we can rephrase this as:

- The multiplication rule of $qK_H(\mathcal{B}_\mathtt{J})$ as a $\mathbb{C}P\otimes\mathbb{C}Q^\vee_{\mathtt{J},+}$-algebra is completely determined by the $\star$-multiplication table of $\mathcal{O}_{\mathcal{B}_\mathtt{J}}(-\varpi_i)$ for $i \in \mathtt{J}^c$.

These fact holds as $qK_H(\mathcal{B}_\mathtt{J})$ is generated by $\{[\mathcal{O}_{\mathcal{B}_\mathtt{J}}(-\varpi_i)]\star\}_{i\in\mathtt{J}^c}$ after localization to $\mathbb{C}(P\oplus Q^\vee_\mathtt{J})$ [6, Remark 5.10]. In other words, we have

$$\mathbb{C}(P\oplus Q^\vee_\mathtt{J}) \otimes_{\mathbb{C}P\otimes\mathbb{C}Q^\vee_{\mathtt{J},+}} qK_H(\mathcal{B}_\mathtt{J})_{\ge 0} = \mathbb{C}(P\oplus Q^\vee_\mathtt{J}) \otimes_{\mathbb{C}P\otimes\mathbb{C}Q^\vee_{\mathtt{J},+}} qK_H(\mathcal{B}_\mathtt{J})$$

and the multiplication rule of $\{[\mathcal{O}_{\mathcal{B}_\mathtt{J}}(-\varpi_i)]\star\}_{i\in\mathtt{J}^c}$ on some $\mathbb{C}(P\oplus Q^\vee_\mathtt{J})$-basis of $\mathbb{C}(P\oplus Q^\vee_\mathtt{J}) \otimes_{\mathbb{C}P\otimes\mathbb{C}Q^\vee_{\mathtt{J},+}} qK_H(\mathcal{B}_\mathtt{J})$ determines the product structure of $qK_H(\mathcal{B}_\mathtt{J})$.

Theorem 2.18. *We have a surjective morphism*

$$qK_H(\mathcal{B}) \twoheadrightarrow qK_H(\mathcal{B}_\mathtt{J})$$

of commutative algebras such that the image of $[\mathcal{O}_{\mathcal{B}(w)}]$ *is* $[\mathcal{O}_{\mathcal{B}_\mathtt{J}([w]_\mathtt{J})}]$ *for each* $w \in W$, *and the image of* Q^β *is* $Q^{[\beta]_\mathtt{J}}$ *for each* $\beta \in Q^\vee_+$.

Proof. We have a diagram (represented by real arrows) of $\mathbb{C}P\otimes\mathbb{C}Q^\vee_+$-modules

$$\begin{tikzcd} K'_H(\mathbf{Q}_G(e)) \arrow[r, "\Psi"] \arrow[d, "\phi_\mathtt{J}"'] & qK_H(\mathcal{B}) \arrow[d, dashed] \\ K'_H(\mathbf{Q}_\mathtt{J}(e)) \arrow[r, "\Psi_\mathtt{J}"] & qK_H(\mathcal{B}_\mathtt{J}) \end{tikzcd}$$

such that their bases correspond as $\phi_\mathtt{J}([\mathcal{O}_{\mathbf{Q}(w)}]) = [\mathcal{O}_{\mathbf{Q}_\mathtt{J}([w]_\mathtt{J})}]$ ($w \in W \times Q^\vee_+ \subset W_{\mathrm{af}}$). The kernel of the map $\phi_\mathtt{J}$ is the preimage of the sum of K_i (borrowed from Lemma 2.16) for $i \in \mathtt{J}$. This defines an ideal of $\Psi^{-1}(qK_H(\mathcal{B})_{\ge 0})$. Therefore, the map $\phi_\mathtt{J}$ induces *some* $\mathbb{C}P$-algebra structure on

$$\phi_\mathtt{J}(\Psi^{-1}(qK_H(\mathcal{B})_{\ge 0})) \subset K'_H(\mathbf{Q}_\mathtt{J}(e)).$$

(If $\mathtt{J} = \mathtt{I}$, then we have $\phi_{\mathtt{J}}([\mathcal{O}_{\mathbf{Q}(w)}]) \equiv 1$ and $\mathrm{Im}\,\phi_{\mathtt{J}} = K_H(\mathrm{pt}) = \mathbb{C}P$. Hence this algebra structure must be the correct one and the result follows in this case.) In view of Theorem 2.8, we find that

$$\Xi_{\mathtt{J}}(-\varpi_i) \circ \phi_{\mathtt{J}} = \phi_{\mathtt{J}} \circ \Xi(-\varpi_i) \qquad i \in \mathtt{J}^c.$$

Thus, the above observation and Corollary 2.15 imply that the above module map induces an algebra map

$$qK_H(\mathcal{B}) \longrightarrow qK_H(\mathcal{B}_{\mathtt{J}})$$

with the desired properties (here we used that the both sides are algebras also by the $\star$-products). $\square$

The $\mathbb{C}P$-action commutes with the actions of $A_i(q)$ ($i \in \mathtt{J}^c$), while we have $A_i(q)Q^\beta = q^{-\langle \beta, \varpi_i \rangle} Q^\beta A_i(q)$ for each $i \in \mathtt{J}^c$ and $\beta \in Q^\vee_{\mathtt{J},+}$ by [19, Theorem A]. In particular, we can localize $qK_{H \times \mathbb{G}_m}(\mathcal{B})_{\geq 0}$ and $qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})_{\geq 0}$ to the field $\mathbb{C}(q, P)$ from the (left) $\mathbb{C}_q P$-action, and we can extend the (right) $\mathbb{C}Q^\vee_{\mathtt{J},+}$-action to the $\mathbb{C}[\![Q^\vee_{\mathtt{J},+}]\!]$-action. Since the proof of Theorem 2.18 rely on the comparison of the basis and the actions of $A_i(q)$'s, the same reasoning yields the following:

Corollary 2.19. *We have a surjective $\mathbb{C}_q P$-module morphism*

$$qK_{H \times \mathbb{G}_m}(\mathcal{B}) \longrightarrow\!\!\!\!\!\rightarrow qK_{H \times \mathbb{G}_m}(\mathcal{B}_{\mathtt{J}})$$

that intertwines the actions of $A_i(q)$ ($i \in \mathtt{J}$), and the image of $[\mathcal{O}_{\mathcal{B}(w)}]Q^\beta$ is $[\mathcal{O}_{\mathcal{B}_{\mathtt{J}}([w]_{\mathtt{J}})}]Q^{[\beta]_{\mathtt{J}}}$ for each $w \in W$ and $\beta \in Q^\vee_+$. $\square$

2.5 Comparison with affine Grassmannians

In this subsection, we deal with an algebra $K_H(\mathrm{Gr})$ that can be seen as the H-equivariant K-group of the affine Grassmannian of G whose product structure is given by the Pontryagin product. For background materials, see [30, 21].

For $w \in W^-_{\mathrm{af}}$, we consider a formal symbol Gr_w and set

$$K_H(\mathrm{Gr}) := \bigoplus_{w \in W^-_{\mathrm{af}}} \mathbb{C}P\,[\mathcal{O}_{\mathrm{Gr}_w}].$$

Theorem 2.20 (Lam-Schilling-Shimozono, see [21] §1.3)**.** *There exists a commutative algebra structure (whose multiplication is denoted by $\odot$) on $K_H(\mathrm{Gr})$ such that*

$$[\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}]$$

for each $w \in W^-_{\mathrm{af}}$ and let $\beta \in Q^\vee_<$.

We call the multiplication $\odot$ of $K_H(\mathrm{Gr})$ the *Pontryagin product*. Theorem 2.20 implies that the set

$$\{[\mathcal{O}_{\mathrm{Gr}_\beta}] \mid \beta \in Q^\vee_<\} \subset (K_H(\mathrm{Gr})_{\mathrm{loc}}, \odot)$$

forms a multiplicative system. We denote by $K_H(\mathrm{Gr})_{\mathrm{loc}}$ its localization. The action of an element $[\mathcal{O}_{\mathrm{Gr}_\beta}]$ on $K_H(\mathrm{Gr})$ in Theorem 2.20 is torsion-free, and hence we have an embedding $K_H(\mathrm{Gr}) \hookrightarrow K_H(\mathrm{Gr})_{\mathrm{loc}}$.

Theorem 2.21 ([21] Corollary C). *There exists an isomorphism*

$$\Phi : (K_H(\mathrm{Gr})_{\mathrm{loc}}, \odot) \longrightarrow (qK_H(\mathcal{B})_{\mathrm{loc}}, \star)$$

of algebras such that

$$\Phi([\mathcal{O}_{\mathrm{Gr}_{ut_{\beta_1}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{t_{\beta_2}}}]^{-1}) = [\mathcal{O}_{\mathcal{B}(u)}]Q^{\beta_1-\beta_2} \qquad u \in W, \beta_1, \beta_2 \in Q^\vee_<.$$

Theorem 2.22. *There exist a surjective algebra map*

$$\eta_{\mathtt{J}} : (K_H(\mathrm{Gr})_{\mathrm{loc}}, \odot) \longrightarrow (qK_H(\mathcal{B}_{\mathtt{J}})_{\mathrm{loc}}, \star)$$

such that

$$\eta_{\mathtt{J}}([\mathcal{O}_{\mathrm{Gr}_{ut_{\beta_1}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{t_{\beta_2}}}]^{-1}) = [\mathcal{O}_{\mathcal{B}_{\mathtt{J}}([u]_{\mathtt{J}})}]Q^{[\beta_1-\beta_2]_{\mathtt{J}}} \qquad u \in W, \beta_1, \beta_2 \in Q^\vee_<.$$

Proof. Combine Theorem 2.21 with Theorem 2.18. □

3 Examples: $G = SL(3)$

Keep the setting of the previous section with $G = SL(3)$. We have $W = \langle s_1, s_2\rangle \cong \mathfrak{S}_3$, $P = \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$, and $Q^\vee = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee$. Recall that $\vartheta := \alpha_1 + \alpha_2$ and $\vartheta^\vee := \alpha_1^\vee + \alpha_2^\vee$. We have $w_0 = s_1s_2s_1 = s_2s_1s_2$. In this case, we have three possible choices of $\emptyset \neq \mathtt{J} \subset \mathtt{I} = \{1, 2\}$. In view of [21, Corollary 3.2 or Proposition 2.12], we may consult [29, §4.2] (with the convention of H-characters twisted by w_0) to justify the first equality in each item. The other equalities are consistent with [8, §5.5].

- We have

$$[\mathcal{O}_{\mathcal{B}(s_1)}]\star[\mathcal{O}_{\mathcal{B}(s_1)}] = (1-e^{\alpha_2})[\mathcal{O}_{\mathcal{B}(s_1)}]+e^{\alpha_2}[\mathcal{O}_{\mathcal{B}}]Q^{\alpha_1^\vee}+e^{\alpha_2}[\mathcal{O}_{\mathcal{B}(s_2s_1)}]-e^{\alpha_2}[\mathcal{O}_{\mathcal{B}(s_2)}]Q^{\alpha_1^\vee}.$$

  Applying Theorem 2.18, we deduce

$$\begin{aligned}[\mathcal{O}_{\mathcal{B}_{\{1\}}}] \star [\mathcal{O}_{\mathcal{B}_{\{1\}}}] &= [\mathcal{O}_{\mathcal{B}_{\{1\}}}],\\ [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] \star [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] &= (1-e^{\alpha_2})[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] + e^{\alpha_2}[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2s_1)}].\end{aligned}$$

- We have $[\mathcal{O}_{\mathcal{B}(s_1)}]\star[\mathcal{O}_{\mathcal{B}(s_2)}] = [\mathcal{O}_{\mathcal{B}(s_1s_2)}]+[\mathcal{O}_{\mathcal{B}(s_2s_1)}]-[\mathcal{O}_{\mathcal{B}(w_0)}]$. From this, we deduce

$$\begin{aligned}[\mathcal{O}_{\mathcal{B}_{\{1\}}}] \star [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_2)}] &= [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_2)}],\\ [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] \star [\mathcal{O}_{\mathcal{B}_{\{2\}}}] &= [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}].\end{aligned}$$

- We have $[\mathcal{O}_{\mathcal{B}(s_1)}] \star [\mathcal{O}_{\mathcal{B}(s_1s_2)}] = (1-e^{\alpha_2})[\mathcal{O}_{\mathcal{B}(s_1s_2)}] + e^{\alpha_2}[\mathcal{O}_{\mathcal{B}(w_0)}]$. From this, we deduce

$$\begin{aligned}[\mathcal{O}_{\mathcal{B}_{\{1\}}}] \star [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_1s_2)}] &= [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_1s_2)}],\\ [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] \star [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] &= (1-e^{\alpha_2})[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] + e^{\alpha_2}[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2s_1)}].\end{aligned}$$

- We have $[\mathcal{O}_{\mathcal{B}(s_1)}] \star [\mathcal{O}_{\mathcal{B}(s_2 s_1)}] = (1-e^{\vartheta})[\mathcal{O}_{\mathcal{B}(s_2 s_1)}] + e^{\vartheta}[\mathcal{O}_{\mathcal{B}(s_2)}]Q^{\alpha_1^\vee}$. From this, we deduce

$$[\mathcal{O}_{\mathcal{B}_{\{1\}}}] \star [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_2)}] = [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_2)}],$$
$$[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] \star [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2 s_1)}] = (1-e^{\vartheta})[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2 s_1)}] + e^{\vartheta}[\mathcal{O}_{\mathcal{B}_{\{2\}}}]Q^{\alpha_1^\vee}.$$

- We have

$$[\mathcal{O}_{\mathcal{B}(s_1)}] \star [\mathcal{O}_{\mathcal{B}(w_0)}] = (1-e^{\vartheta})[\mathcal{O}_{\mathcal{B}(w_0)}] + e^{\vartheta}([\mathcal{O}_{\mathcal{B}}]Q^{\vartheta^\vee} + [\mathcal{O}_{\mathcal{B}(s_1 s_2)}]Q^{\alpha_1^\vee} - [\mathcal{O}_{\mathcal{B}(s_1)}]Q^{\vartheta^\vee}).$$

From this, we deduce

$$[\mathcal{O}_{\mathcal{B}_{\{1\}}}] \star [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_1 s_2)}] = [\mathcal{O}_{\mathcal{B}_{\{1\}}(s_1 s_2)}],$$
$$[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_1)}] \star [\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2 s_1)}] = (1-e^{\vartheta})[\mathcal{O}_{\mathcal{B}_{\{2\}}(s_2 s_1)}] + e^{\vartheta}[\mathcal{O}_{\mathcal{B}_{\{2\}}}]Q^{\alpha_1^\vee}.$$

In all cases, the above calculations recover [7, Corollary 10] as:

$$1 \star 1 = 1 \in qK_H(\mathcal{B}_{\{1,2\}}) \equiv qK_H(G/G) = K_H(G/G) = \mathbb{C}P$$

by setting $[\mathcal{O}_{\mathcal{B}(w)}] \equiv 1 \equiv Q^{\alpha_i^\vee}$ $(w \in W, i = 1, 2)$.

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