2nd Day

Bulk deformation and Mirror symmetry.

Operator q. ([TOU] §13.)

(X, ω) symplectic, $L \subset X$ rel spin Lag. submanifold. Filtered A_∞ str is deg $=1$ map:

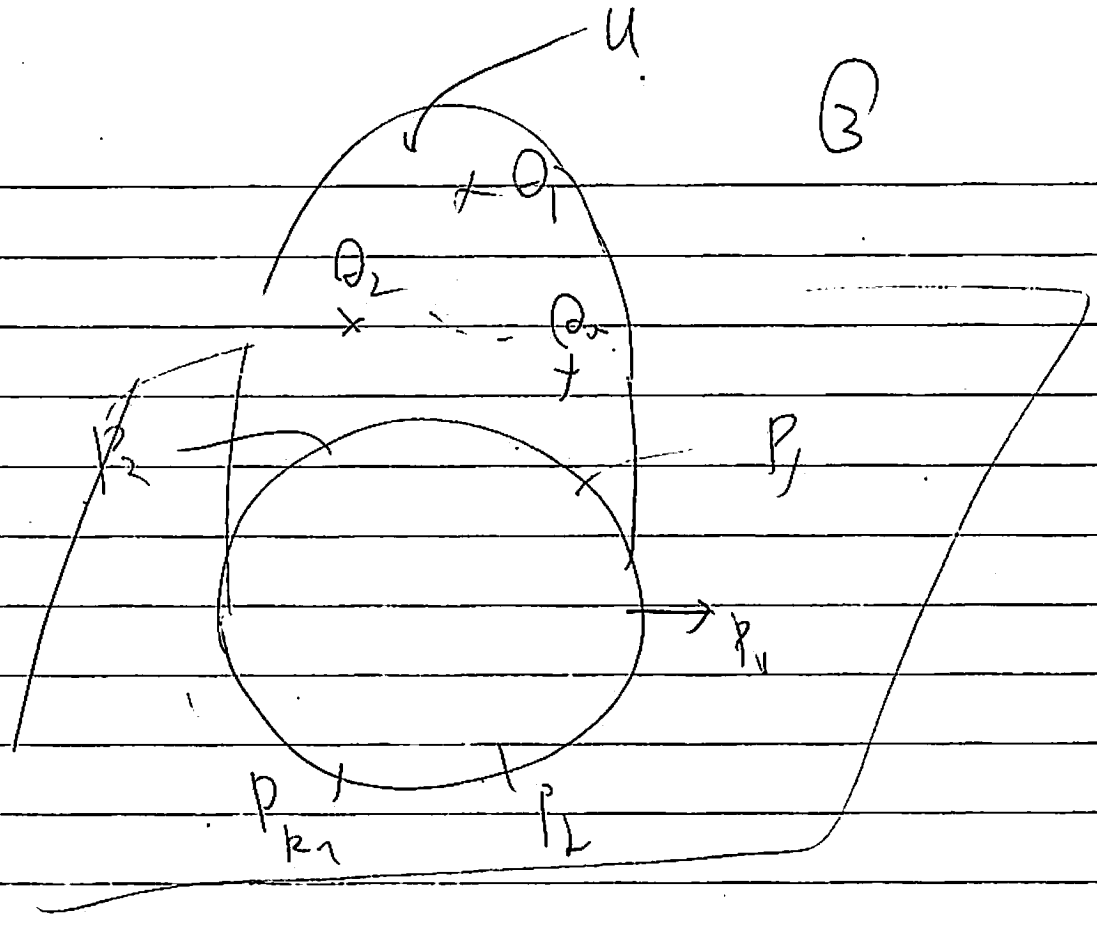
$$m_k : B_k H(L : \Lambda_{0,nov})[2] \longrightarrow H(L : \Lambda_{0,nov})[2]$$

We ext. it to

$$q_{0,k} : E_\ell H(X : \Lambda_{0,nov})[2] \otimes B_k H(L : \Lambda_0)[2]$$
$$\longrightarrow H(L : \Lambda_{0,nov})[2]$$

Construction $Q_1, \cdots, Q_\ell$ cycle on X

$P_0, P_1, \cdots, P_k$ in L



$$\langle q_{\ell,k}(Q_1 \otimes \cdots \otimes Q_\ell, P_1 \otimes \cdots \otimes P_n), P_v \rangle$$

$$= \sum_{u} T^{wrw} e^{\mu(u)/2} \quad \text{\#} \text{ of above } u$$

Basic relation (gluing - cloud fin)

$\Delta \quad E_{-1} \longrightarrow E_{-1} \otimes E_{-1}$

$\Delta^2 \quad B_{-1} \longrightarrow B_{-1} \otimes B_{-1} \otimes B_{-1}$

$\Delta \vec{\theta} = \sum \vec{\theta}^1_c \otimes \vec{\theta}^2_c$

$\Delta^2 \vec{\beta} = \sum_{c'} \vec{\beta}^1_{c'} \otimes \vec{\beta}^2_{c'} \otimes \vec{\beta}^3_{c'}$

$0 = \sum_{c,c'} \pm q(\vec{\theta}^1_c \otimes (\vec{\beta}^1_{c'} \otimes q(\vec{\theta}^2_c \otimes \vec{\beta}^2_{c'}) \otimes \vec{\beta}^3_{c'})$

For η $\quad b \in H(X, \Lambda_{nw})$ $\quad \deg b = 2$

$m_k^b(\alpha) = \sum_\ell q_\ell(b^{\otimes \ell}, \alpha)$ $\quad$ sub $\cdot$ A_∞ rel

In our case

$$L = \underset{\text{fiber}}{L(u)} \subset \underset{\text{toric}}{X}$$

Q_i : T^n equivariant cycle

$b = \sum w_i Q_i$ $\qquad$ $x = \sum x_i \phi_i \in H^1(L; \Lambda_m)$

Def

$$W(b; x) = \sum_{l,k} q_{l,k} (\underbrace{b \cap \cdots \cap b}_{l}, \underbrace{x \cap \cdots \cap x}_{k}) \cap [L]$$

$$\in \Lambda_{0,m}$$

Note $W(0; x) = \mu(x)$.

W is a function

$$W : \underbrace{H(D : \Lambda_{0,m})}_{\substack{\| \\ \oplus \Lambda_{0,m}}} \times H^1(L(h) : \Lambda_{0,m,v}) \to \Lambda_{0,m,v}$$

identify $H^1_{2n}(X : \Lambda_{0,m}) \cong H^0(X \Lambda_{0,m})$

$D = \pi^{-1}(\partial P)$ total divisor

$$H_{+}(D ; \Lambda_{0,m,v}) \oplus \Lambda_{0,m} \longrightarrow H_{+}(X : \Lambda_{0,m})$$

$\| \; H_n(\ )$ surjective

fix a section Regard

$$W : H^{*}(X : \Lambda_{0,m,v}) \times H^1(L(h), \Lambda_{0,m,v}) \to \Lambda_{0,m,v}$$

$\| \; H_{*}(X : \Lambda_{0,m})$

identify by P.D.)

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$W(w_1 \cdots w_p ; y_1 \cdots y_r)$ $\qquad y_i = e^{x_i}$

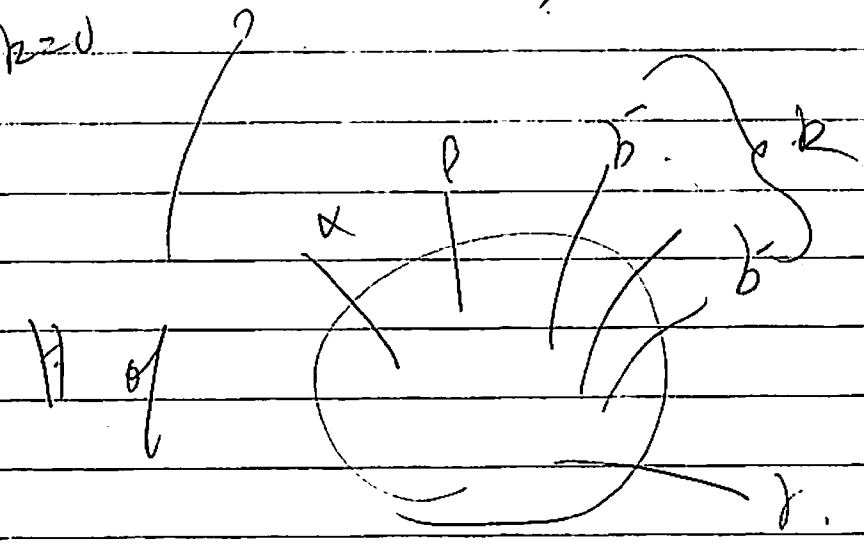
Let $\qquad A_1, \ldots, A_b$ basis of $H(X)$

$$b = \sum w_i A_i \in H^*(X ; \Lambda_{0,nov})$$

U^b deformed cup product

$\Leftrightarrow \quad \langle \alpha U^b \beta, \gamma \rangle$

$$= \sum_{k=0}^{\infty} GW(\alpha \beta b^{\otimes k} \gamma)$$



(b)

need to def. completion of Z the more precisely

$$\mathrm{Jac}\, W(\gamma; \cdot) \text{ "}=\text{" } \underline{\Lambda_{0,m}[y_1 \cdots y_n^{-1}]}$$

$$\mathrm{Cha}\left(\frac{\partial W}{\partial y_i} \quad i=1 \cdots n\right)$$

<u>Thm</u>

(1) $$T_{\gamma} H(X; \Lambda_{0,m}) \longrightarrow \mathrm{Jac}\, W(\gamma; \ )$$

$$\frac{\partial}{\partial w_i} \longmapsto \left[\frac{\partial W}{\partial w_i}\right]$$

is a ring iso (We put U^{γ} in l.h.s.)

③ P.D. on $T_Y H(X; \mathrm{Arm})$

becomes

Residue pairing on $\mathrm{Jac}\, W$ by

the above iso when

$W(Y_b; -)$ is Morse.

<u>Cor</u>

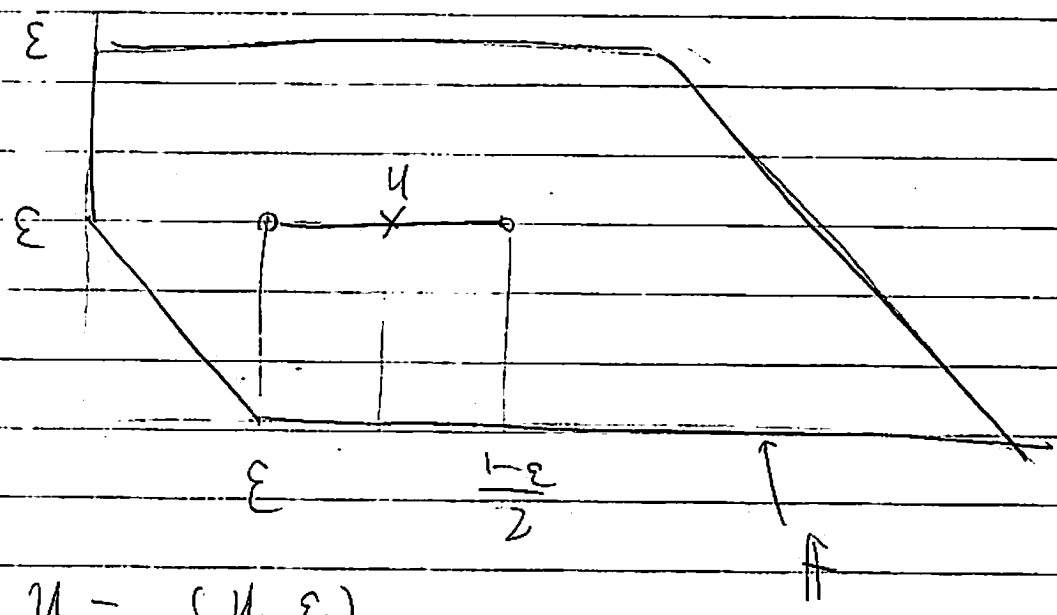
Fix Y_b s.t. $W(Y_b; \cdot)$ is Morse

$\#\{(u, b) \mid HF(L(u, (Y_b, b))) \neq 0\}$

$= \mathrm{rk}\, H(X; \mathbb{Q})$

An example of application to symplectic topology

2 pts blow up of $\mathbb{CP}^2$



$u = (u_1, \varepsilon)$

$$PO(T^{\delta} f, u, y) = T^{\varepsilon}(e^{T^{\delta}} y_2 + y_2^{-1}) + T^{u_1}(y_1 + y_1 y_2)$$

$$+ T^{1-u_1-\varepsilon}(y_1 y_2)^{-1}$$

$$\frac{\partial \mathcal{P}_0}{\partial y_2} = T^{\varepsilon}\left(1 + e^{T^{\delta}} - y_2^{-2} + \text{higher}\right)$$

$$\therefore \quad (\varepsilon + \delta < 1 - u_1 - \varepsilon)$$

$$y_2 = -1 - \frac{1}{2}T^{\delta} + \cdots$$

$$\frac{\partial \mathcal{P}_0}{\partial y_1} = T^{u_1}(1 + y_2)$$

$$- T^{1-u_1-\varepsilon} y_1^{-2} y_2^{-1} + \text{higher}$$

$$= -T^{u_1+\delta}\frac{1}{2} - T^{1-u_1-\varepsilon} y_1^{-2}$$

Put $\quad u_1 + \delta = 1 - u_1 - \varepsilon$

$$\delta = 1 - 2u_1 - \varepsilon \in (0, 1-3\varepsilon)$$

K. Saito theory and Mirror sym.

2 Sources of Frobenius mfd

① Singularity -(K. Saito '80)

② Quantum cohomology ('90)

$W(w, y)$ fmly of holomorphic f.c.n.'s of y

parametrized by $w \in S$

○ W is universal unfolding

$\Leftrightarrow$ $T_w S \cong \mathrm{Jac}\, W(w, \cdot)$

$$\| $$

$$\bigoplus_{p \,:\, \nabla_p W = 0} \mathcal{O}_p \Big/ \left(\frac{\partial W}{\partial y_i}\right)$$

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TS has finite of inner product

—

In many cases it has flat coordinate etc ...

$\Downarrow$

str. of Frob. mfd on S

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(2)

Quantum cohomology

X sym.

$S = H(X : \Lambda_{0,\mathrm{novi}})$

$b \in S \Rightarrow U^b$ deformed Q.

cup product

It gives family of ring str on

$T_b S$

Θ P.D. defines inner product on $T_b S$

$\Rightarrow S$ is a (formal) Frobenius mfd.

Our Th says that these two

are isomorphic in case

W = super potential

Sketch of the proof.

Norm on $\Lambda_{G,mod}[y_1 \quad y_n^{-1}]$

Put $z_i = T^{l_i(u)} y^{\vec{v}_i}$

$\Psi_{u'u} : \Lambda_{mod}[y_1(u), \quad y_n(u)^{-1}] \circlearrowleft$

$$y_i \longmapsto T^{u_i' - u_i} y_i$$

Identify them by this isu. and put

R. Let $y_i(u) \in R$ (depends on u)

Valuation $v_T^u : R \longrightarrow \mathbb{R}$

$$y_i(u) \longmapsto 0$$

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Namely :

$$v_T^u(T^\alpha y_i(u)) = \alpha.$$

$$v_T = \inf v_T^u.$$

$$v_T(z_i) = 0$$

Completion of R, wrt v_T

$= \overline{R}$.

$W \in \overline{R}$

QS Relation

$$\overline{R} = \Lambda_{0,mn}\langle\langle z_1 \cdots z_n \rangle\rangle / \sim$$

closure of polynomial ring

$\parallel$

strongly conv. power series ring

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Def

$$\mathrm{Jac}\, W(w, \cdot) = \frac{\bar{R}}{\mathrm{cls}\left(\frac{\partial W}{\partial y_i}\right)}$$

☆

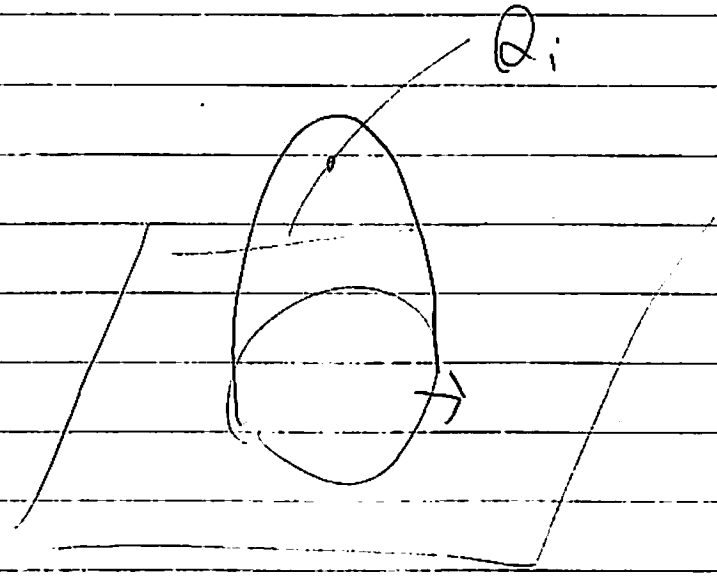
$$T_w H(X : \Lambda_{0,m}) \longrightarrow \mathrm{Jac}\, W(w, \cdot)$$

$$\frac{\partial}{\partial w_i} \longmapsto \frac{\partial W}{\partial w_i}$$

Step 1

This is ring homomorphism

$W=0$ for simplicity.



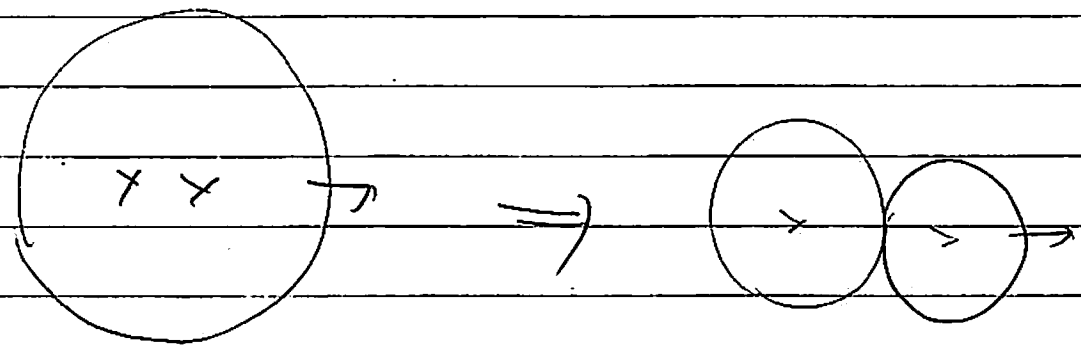
w_i

coordinate wrt the basis Q_i

$Q_i \in H(X)$

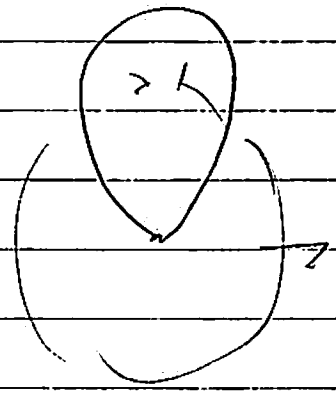
$$\frac{\partial W}{\partial w_i} = \sum_{\beta} \left[\mathcal{M}_{1,1}(\beta) \times_M Q_i \right] \cap [L].$$

moduli of disk with $\begin{cases} \text{one interior mkt point} \\ \text{one boundary mkt point} \end{cases}$



product is Jue

$\Downarrow$



product

or Q1-1

Step 2

This is surjective

Consider $D_1, \cdots, D_m \in H^2(X)$

$$D_i = \pi^{-1}(\partial_i P)$$

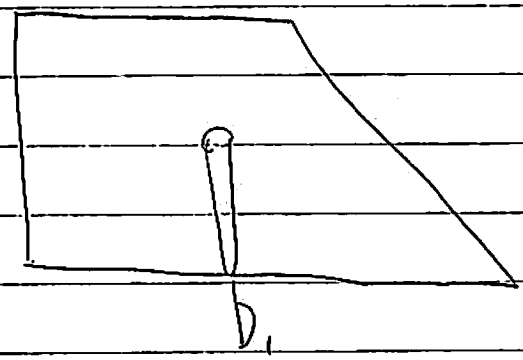
w_i : coordinate corresponding to D_i

then

$$\frac{\partial W}{\partial w_i} = (\text{nonzero const}) \times z_i$$

$\Uparrow$

follows from Chi-Oh



count such discs

<u>Step 3</u> Injective

$T_w H(X; \Lambda_{0,nov})$ is free $\Lambda_{0,nov}$ module

It suffices to show

$$\mathrm{rank}_{\Lambda_{nov}} \mathrm{Jac}(W) \otimes \Lambda_{nov} = \sum \mathrm{rank}\, H(X; D)$$

$$\mathrm{Jac}(W) \otimes \Lambda_{nov} = \frac{\Lambda_{nov}\langle\langle y_1, \cdots, y_n^{-1} \rangle\rangle}{\mathrm{clos}\left(\frac{\partial W}{\partial y_i}\right)}$$

$$\mathrm{Rel}_i \in \Lambda_{nov}\langle\langle y_1 \cdots y_n^{-1}\rangle\rangle$$

$$\mathrm{Rel}_i = \frac{\partial W}{\partial y_i}$$

<u>Lemma 1</u>

$\exists\ \mathrm{Rel}'_i$

☆ $\mathrm{Rel}'_i \equiv \mathrm{Rel}_i$ modulo higher order

s.t. $\mathrm{rk}\dfrac{\Lambda_{nov}\langle\langle y_1 \cdots y_n^{-1}\rangle\rangle}{\mathrm{cls}(\mathrm{Rel}'_i)} = \sum \mathrm{rk}\, H(X; \mathbb{Q})$

<u>Lem 2</u>

If ☆ then

$$\mathrm{rk}\frac{\Lambda_{nov}\langle\langle y_1 \cdots y_n^{-1}\rangle\rangle}{\mathrm{cls}(\mathrm{Rel}'_i)} = \mathrm{rk}\frac{\Lambda_{nov}\langle\langle y_1 \cdots y_n^{-1}\rangle\rangle}{\mathrm{cls}(\mathrm{Rel}_i)}$$

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○ Lemma 1

Seidel + McDuff–Tolman

$\exists\, z_i' \in HQ(X : \Lambda_{0,nov})$

st $\begin{cases} z_i' \text{ satisfies QSR} & a) \\ z_i' \equiv D_i \mod \Lambda_+ & b) \end{cases}$

a) $\Rightarrow$ $\exists$ ring homo

$$\Lambda_{nov}\langle\langle y_1 \cdots y_n^{-1} \rangle\rangle \longrightarrow QH(X, \Lambda_{nov})$$

st $\prod y_i^{D_i(A_i)} y^{v_i} \longmapsto z_i'$

b) $\Rightarrow$ Kernel is gen by R_i'

○ Lemma 2

Semi continuity + properness

$$\mathrm{Rel}_i(s) = s\,\mathrm{Rel}'_i + (1-s)\,\mathrm{Res}_i$$

$$R_s = \frac{\Lambda_{\mathrm{mon}}\langle\langle y_1, \cdots, y_n \rangle\rangle}{\mathrm{clos}(\mathrm{Rel}_i(s))}$$

$$\bigcup_{s \in \mathbb{C}} \mathrm{Spec}\, R_s \times \{s\} \xrightarrow{\pi} \mathbb{C}$$

$$\rotatebox{90}{$\in$}$$

$$s$$

○ fiber is 0-dimentional

rank $\mathrm{Jac} = H^0(\mathrm{fiber})$

locally constant if π is proper.

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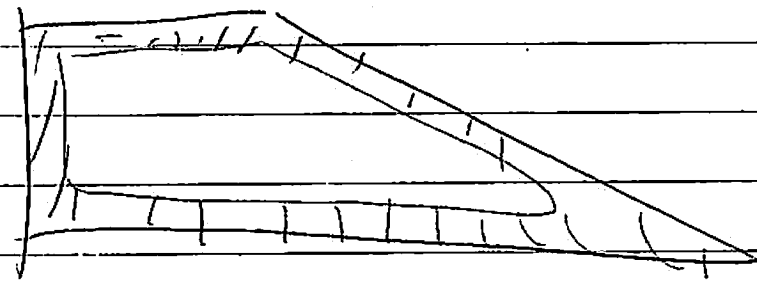
π is proper

$\Leftarrow$



there exists no critical point in a nbd of ∂P.

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