FLOER HOMOLOGY AND MIRROR SYMMETRY I

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Abstract. In this survey article, we explain how the Floer homology of Lagrangian submanifold [Fl1],[Oh1] is related to (homological) mirror symmetry [Ko1],[Ko2]. Our discussion is based mainly on [FKO$_3$].

0. Introduction.

This is the first of the two articles, describing a project in progress to study mirror symmetry and D-brane using Floer homology of Lagrangian submanifold. The tentative goal, which we are far away to achiev, is to prove homological mirror symmetry conjecture by M. Kontsevich (see §3.) The final goal, which is yet very very far away from us, is to find a new concept of spaces, which is expected in various branches of mathematics and in theoretical physics.

Together with several joint authors, I wrote several papers on this project [Fu1], [Fu2], [Fu4], [Fu5], [Fu6], [Fu7], [FKO$_3$], [FOh]. The purpose of this article and part II, is to provide an accesible way to see the present stage of our project. The interested readers may find the detail and rigorous proofs of some of the statements, in those papers.

The main purose of Part I is to discribe an outline of our joint paper [FKO$_3$] which is devoted to the obstruction theory to the well-definedness of Floer homology of Lagrangian submanifold. Our emphasis in this article is its relation to mirror symmetry. So we skip most of its application to the geometry of Lagrangian submanifolds.

In §1, we review Floer homology of Lagrangian submanifold in the form introduced by Floer and Oh. They assumed various conditions on Lagrangian submanifold and symplectic manifold, to define Floer homology. These assumptions are not technical one. Understanding the reason why those conditions are imposed, is an essential step toward building the obstruction theory. So, in §1, we explain it a bit.

§2 is devoted to a discussion on the moduli space of Lagrangian submanifolds. This topic recently calls attention of various mathematicians (see [Gs1], [Gs2], [Mc], [SW]). Their interest comes mainly from geometric mirror symmetry conjecture by Strominger-Yau-Zaslow [SYZ]. Our point view is slightly different from those. Namely we study the space of Hamiltonian equivalence classes of Lagrangian submanifolds, rather than studying the moduli space of special Lagrangian submanifolds as in [SYZ]. Our point of view seems to be more natural from the point of

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view of homological mirror symmery. (The idea to restrict oneselves to special Lagrangian submanifolds have also various advantages. Especially it seems more reasonable to do so to study the compactification of the modulis spaces.) The author believe that, for a good understanding of the moduli space of Lagrangian submanifolds, we need to study both of the two approaches and to find a relation between them.

In §3, we join §1 and §2 and will explain the following : Floer homology is not always well-defined : There is an obstruction for it to be well-defined : Even in the case it is well-defined, there is a moduli space $\mathcal{M}(L)$ which parametrize the possible ways to define Floer homology. In other words, our obstruction theory provides a "quantum correction" to the moduli space of Lagrangian submanifolds discussed in §2. Then a version of homological mirror conjecture asserts that the moduli space $\mathcal{M}(L)$ will be equal to the moduli space of branes in the mirror. In other words, the obstruction class will become the Kuranishi map in the mirror.

In part II, which will appear elsewhere, we will discuss the following : An idea to prove homological mirror symmery conjecture by using family of Floer homologies : Converging version of Floer homology and its relation to a family of Floer homology, to Hutchings invariant [Hu] and to genus one homological mirror conjecture. Some explicite example of homological mirror conjecture, in the case of Lagrangian tori. And hopefully we will have more to say.

1. Review of Floer homology of Lagrangian Submanifolds.

We begin with a review of resuts by Floer [Fl] and Oh [Oh1] on Floer homology of Lagrangian submanifolds.

Let (M,ω) be a symplectic manifold with $\dim M = 2n$. An n dimensional closed submanifold L of M is said to be a Lagrangian submanifold if the restriction of ω to L vanishes. Floer homology theory of Lagrangian submanifolds is expected to associate a graded abelian group $HF^*(L_0,L_1)$ to each pair (L_0,L_1) of Lagrangian submanifolds. The properties it is expected to safisfy include :

(P.1) If L_0 is transversal to L_1 then

$$\sum_k \mathrm{rank} HF^k(L_0,L_1) \ge \sharp(L_0\cap L_1), \tag{1.1}$$

$$\sum_k (-1)^k \mathrm{rank} HF^k(L_0,L_1) = L_0\cdot L_1. \tag{1.2}$$

(Here the right hand side of (1.1) is the order of the set $L_0\cap L_1$. The right hand side of (1.2) is the intersection numer, that is the order counted with sign of the set $L_0\cap L_1$.)

(P.2) If $\phi : M \to M$ is a Hamiltonian symplectic diffeomorphism then

$$HF(\phi_1(L_0),\phi_2(L_1)) \simeq HF(L_0,L_1).$$

Here a Hamiltonian symplectic diffeomorphism is a time one map of a (time dependent) Hamiltonian vector field[1].

Floer established the following Theorem 1.1. We need the following technical condition.

[1]See §2 Definition 2.7.

Condition T. Let L_0 and L_1 be Lagrangian submanifolds. One of the following is satisfied.

(1) The images of $H_1(L_0;\mathbb{Z}), H_1(L_1;\mathbb{Z})$ in $H_1(M;\mathbb{Z})$ are finite.
(2) L_1 is homotopic to L_2 in M.

Theorem 1.1. (Floer [Fl1]) *Let L_0, L_1 be Lagrangian submanifolds. We assume $\pi_2(M, L_i) = 0$ and Contition T. Then, there exists a $\mathbb{Z}$-graded Floer homology group $HF^*(L_0, L_1)$ with $\mathbb{Z}_2$ coefficient satisfying* (P.1), (P.2) *and*

(P.3) $\quad HF^*(L, L) \simeq H^*(L;\mathbb{Z}_2),$

where $H^*(L;\mathbb{Z}_2)$ is the cohomology group of L with $\mathbb{Z}_2$ coefficient.

Remark 1.2. The Condition T is imposed in order the degree of Floer homology to be well-defined. In fact, one can define Floer homology without this condition. But, then, the Floer homology group will not be graded. The same remark applies to Theorem 1.7.

Theorem 1.1 was generalized by Y. Oh [Oh1]. To state his result, we first define two homomorphisms, the energy $\mathcal{E} : \pi_2(M, L) \to \mathbb{R}$ and the Maslov index $\mu : \pi_2(M, L) \to \mathbb{Z}$. Let $\beta \in \pi_2(M, L)$ be represened by $\varphi : (D^2, \partial D^2) \to (M, L)$.

Definition 1.3.

$$\mathcal{E}(\beta) = \int_{D^2} \varphi^*\omega.$$

To define μ, we consider the Lagrangian Grassmannian manifold Lag_n consisting of all n-dimensional $\mathbb{R}$ linear subspaces V of $\mathbb{C}^n$ such that the Kähler form ω of $\mathbb{C}^n$ vanishes on V. Let $Gr(n, 2n)$ be the Grassmannian manifold consisting of all n-dimensional $\mathbb{R}$ linear subspaces V of $\mathbb{C}^n$. There is a natural inclusion $Lag_n \to Gr(n, 2n)$.

Lemma 1.4. $\pi_1(Lag_n) = \mathbb{Z}$. *The generator of $\pi_1(Lag_n)$ goes to the first Stiefel-Whitney class (the generator) of $\pi_1(Gr(n, 2n)) \simeq \mathbb{Z}_2$.*

See [AG] for the proof. We now define μ. Let $\beta = [\varphi]$. We consider a vector bundle $\varphi^*TM \to D^2$ together with nondegenerate anti-symmetric 2 form ω on it (that is the pull back of symplectic form). Since D^2 is contractible, we have a trivialization $\varphi^*TM \simeq D^2 \times (\mathbb{C}^n, \omega)$ preserving ω. The trivialization is unique up to homotopy. We restrict it to $S^1 = \partial D^2$. We then have a family of n-dimensional $\mathbb{R}$-linear subspaces $T_{\varphi(t)}(L) \subset T_{\varphi(t)}(M) \simeq (\mathbb{C}^n, \omega)$. Since L is a Lagrangian submanifold, $T_{\varphi(t)}(L)$ is an element of Lag_n. Thus $t \mapsto T_{\varphi(t)}(L)$ defines an element of $\pi_1(Lag_n) = \mathbb{Z}$. We let $\mu(\beta)$ be this element. It is easy to see that it depends only on β and is independent of φ and of the trivialization of $\varphi^*(TM)$. We call $\mu(\beta)$ the *Maslov index* of β.

Maslov index is related to the pseudoholomorphic disk as follows. We consider the moduli space of peudoholomorphic disks $\varphi : (D^2, \partial D^2) \to (M, L)$ whose homology class is β. (We identify φ and φ' if they are transformed by an element of $Aut(D^2) \simeq PSL(2;\mathbb{R})$. See Definition 3.8.) Then its virtual dimension is $n + \mu(\beta) - 2$ (see (3.4)).

Definition 1.5. L is said to be *monotone* if there exists $c > 0$ independent of $\beta \in \pi_2(M, L)$ such that $\mu(\beta) = c\mathcal{E}(\beta)$.

L is said to be *semipositive* if there exit no $\beta \in \pi_2(M, L)$ such that $0 > \mu(\beta) \geq 3 - n$ and $\mathcal{E}(\beta) > 0$.

The *minimal Maslov number* of L is $\min\{\mu(\beta)|\beta \in \pi_2(M, L), \mu(\beta) > 0\}$.

Examples 1.6. Let $L = S^1 \subset \mathbb{C}$. Then $\pi_2(\mathbb{C}, S^1) = \mathbb{Z}$, $\mu(1) = 2$ and $\mathcal{E}(1) = \pi$. Hence L is monotone with minimal Maslov number 2.

Let $L \simeq S^1 \subset S^2$ be a small circle. Let A and B be areas of the components of $S^2 - L$. We have $\pi_2(S^2, L) = \mathbb{Z}^2$. We can choose generator x, y of $\pi_2(S^2, L)$ such that $\mu(kx + \ell y) = 2k + 4\ell$, $\mathcal{E}(k, \ell) = kA + \ell(A + B)$. ($y$ is in the image of $\pi_2(S^2) \to \pi_2(S^2, L)$.) Hence L is monotone if and only if $A = B$, that is L is the equator.

Let M be a symplectic manifold. Define a symplectic structure on $M \times M$ by $\pi_1^*\omega - \pi_2^*\omega$. Then the diagonal $M \sim \Delta_M \subset M \times M$ is a Lagrangian submanifold. In this case, Δ_M is a monotone Lagrangian submanifold (resp. semipositive Lagransian submanifold) if and only if M is a monotone symplectic manifold (resp. semipositive symplectic manifold.) See [MS1] the definition of monotonicity and semipositivity of symplectic manifolds.

Theorem 1.7. (Oh [Oh1]) *Let L_1, L_2 be monotone Lagrangian submanifolds with minimal Maslov numbers N_1,N_2 respectively. Suppose $N_1, N_2 \geq 3$. We assume either the image of $H_1(L_1; \mathbb{Z}), H_1(L_2; \mathbb{Z})$ in $H_1(M; \mathbb{Z})$ is finite or L_1 is homotopic to L_2 in M. Then there exists $\mathbb{Z}_N$-graded Floer homology with $\mathbb{Z}_2$ coefficient, which satisfies* (P.1), (P.2). *(Here N is a greatest common divisor of N_1 and N_2.)*

Oh [Oh2] observed that (P.3) may not hold in general. (We will discuss an example in [FKO$_3$] and Part II of this article.) In general, there exists a spectral sequence describing a relation between $HF(L, L)$ and $H(L; \mathbb{Z}_2)$. (See §3.)

Theorems 1.1 and 1.7 above have various applications to the geometry of Lagrangian submanifolds. Since, in this article, we focus relation of Floer homology to mirror symmetry, we do not discuss many of its applications to the geometry of Lagrangian submanifolds. However, describing a few of them is useful for our purpose also, since it shows the main troubles in the definition of the Floer homology of Lagrangian submanifold.

The reader may feel the assumptions of Theorems 1.1 and 1.7 to be rather restrictive, compared to the corresponding result for Floer homology of periodic Hamiltonian system ([FOn1], [LT], [R]), where no assumption is imposed on the symplectic manifold. However properties (P.1), (P.2), (P.3) are too much to be expected to hold for general Lanrangian submanifold. To see this, we first remark :

Lemma 1.8. *If there exists a Floer homology $HF(L, \phi(L))$ satisfying* (P.1), (P.2), (P.3)*, for a Lagrangian submanifold L and a Hamiltonian symplectic diffeomorphism ϕ, then $L \cap \phi(L) \neq \emptyset$.*

Proof. If $L \cap \phi(L) = \emptyset$ then $HF(L, \phi(L))$ is 0 by (P.1). On the other hand, (P.2), (P.3) imply $HF(L, \phi(L)) \simeq H(L) \neq 0$. This is a contradiction.

This implies the following result due to Gromov [Gr].

Theorem 1.9. *Let $L \subset \mathbb{C}^n$ be a compact Lagrangian submanifold. Then $\mathcal{E}$: $\pi_2(\mathbb{C}^n, L) \to \mathbb{Z}$ is nonzero.*

Proof. We prove only a weaker statement $\pi_2(\mathbb{C}^n, L) \neq 0$. Assume $\pi_2(\mathbb{C}^n, L) = 0$. Then the assumption of Theorem 1.1 is satisfied. Hence, by Lemma 1.8, there exists no exact symplectic diffeomorphism ϕ such that $\phi(L) \cap L = \emptyset$. However it is easy to construct such ϕ. This is a constradiction.

Remark 1.10. The author learned this proof from Oh.

Remark 1.11. In fact, compactness of M is assumed in Theorems 1.1 and 1.7. So the above argument does not directly apply. But one can replace $\mathbb{C}^n$ by T^n (dividing $\mathbb{C}^n$ by $C\mathbb{Z}^n$ for sufficiently large C). (Alternatively, we can use convexity of $\mathbb{C}^n$.)

In fact, Floer homology satisfying (P.1),(P.2),(P.3) *never* exist for a Lagrangian submanifold in $\mathbb{C}^n$. Thus, to find a condition for Floer homology to exist and to clarify how the property (P.3) to be modified, is an important part of the study of Floer homology of Lagrangian submanifold.

We now review the basic idea to construct Floer homology. Let L_0, L_1 be Lagrangian submanifolds of M. We put

$$\Omega(L_0, L_1) = \{\ell : [0,1] \to M | \ell(0) \in L_0, \ell(1) \in L_1\}.$$

We are going to study Morse theory on $\Omega(L_0, L_1)$ to define $\infty/2$-dimensional homology group, the Floer homology. We are going to define the Morse "function" $\mathcal{A}$. In fact, $\mathcal{A}$ is a multivalued function or a function on a covering space of $\Omega(L_0, L_1)$. Choose a base point $\ell_a \in \Omega(L_0, L_1)$ for each connected component of $\Omega(L_0, L_1)$. We define a covering space of $\Omega(L_0, L_1)$ as follows :

$$\hat{\Omega}(L_0, L_1) = \left\{ (\ell, u) \,\middle|\, \begin{array}{l} \ell \in \Omega(L_0, L_1), \quad u : [0,1]^2 \to M, \\ u(0,t) = \ell_a(t), \text{ where } \ell_a \text{ is in the same component as } \ell, \\ u(1,t) = \ell(t), \quad u(s, \cdot) \in \Omega(L_0, L_1). \end{array} \right\}$$

$$\tilde{\Omega}(L_0, L_1) = \hat{\Omega}(L_0, L_1)/\sim$$

where $(\ell, u) \sim (\ell', u')$ if and only if $\ell = \ell'$ and u is homotopic to u' relative to the boundary. It is easy to see that $[\ell, u] \mapsto \ell$, $\tilde{\Omega}(L_0, L_1) \to \Omega(L_0, L_1)$ is a covering map.

Definition 1.12.

$$\mathcal{A}([\ell, u]) = \int u^*\omega,$$

Stokes's theorem implies that the right hand side is independent of $\sim$ equivalence class.

Lemma 1.13. *There exists a closed* 1 *form on $\Omega(L_0, L_1)$ which pulls back to the* 1 *form $d\mathcal{A}$ on $\tilde{\Omega}(L_0, L_1)$.*

See [Fl1] for the proof. Hereafter the closed 1 form on $\Omega(L_0, L_1)$ obtained from Lemma 1.13 is denoted by $d\mathcal{A}$ by abuse of notation. We remark however that this form is not exact in general.

Lemma 1.14. *$d\mathcal{A}(\ell) = 0$ if and only if $\ell(t) = const = L_0 \cap L_1$.*

The proof is easy.

It is easy to see that $\tilde\Omega(L_0, L_1) = \Omega(L_0, L_1)$ if L_0,L_1 satisfies assumptions of Theorem 1.1. Hence $\mathcal{A}$ is single valued on $\Omega(L_0, L_1)$ in that case. Single-valuedness of $\mathcal{A}$ is relatetd to the compactness of the moduli space of gradient trajectories as we will see below.

Thus Floer homology is an infinite dimensional analogue of a Morse theory of closed one form. Morse theory of closed one form was developed by Novikov [No] in the finite dimensional case. Our infinite dimensional case is different from finite dimensional case at various points. We will discuss them later and first review Novikov's theory.

We consider the following situation. Let X be a compact finite dimensional Riemannnian manifold and θ be a closed one form on it. (Later we will put $X = \Omega(L_0, L_1)$, and $\theta = d\mathcal{A}$.) Using Riemannian metric, we identify $TX \simeq \Lambda^1 X$. Hence θ is identified to a vector field, which we denote by gradθ.

Definition 1.15 θ is said to be a *Morse form*, if for each point $p \in X$, there exists a Morse function f_p such that $df_p = \theta$ in a neighborhood of p.

We put $Cr(\theta) = \{p \in X | \theta(p) = 0\}$.

Assume that θ is a Morse form. Let $p \in Cr(\theta)$. Then p is a critical point of the Morse function f_p. The *Morse index* $\mu(p)$ of θ at p is, by definition, the Morse index of f_p at p.

In the usual study of Morse-Witten complex ([Fl2], [W]), one counts the number of gradient trajectories joining two critical points of a Morse function. However, in our situation, if we consider the set of all trajectories of gradθ joining two $p, q \in Cr(\theta)$ with $\mu(p) + 1 = \mu(q)$, then there are infinitely many of them. Novikov's idea is to use a kind of formal power series ring, which is now called Novikov ring, to "count" it. Let us describe it more precisely.

Let $p, q \in Cr(\theta)$. We put :

Definition 1.16.

$$\tilde{\mathcal{M}}(p,q) = \left\{ \gamma : \mathbb{R} \to X \,\middle|\, \frac{d\gamma}{dt} = \text{grad}\theta, \lim_{\tau\to-\infty} \gamma(\tau) = p, \lim_{\tau\to\infty} \gamma(\tau) = q \right\}.$$

$\mathbb{R}$ acts on $\tilde{\mathcal{M}}(p,q)$ by $s \cdot \gamma(\tau) = \gamma(s + \tau)$. Let $\mathcal{M}(p,q)$ be the quotient space.

The main techinical result one needs to define Novikov homology is the following Theorem 1.17. We put

$$\mathcal{M}(p,q;E) = \left\{ [\gamma] \in \mathcal{M}(p,q) \,\middle|\, \int \gamma^*\theta = E \right\}. \tag{1.3}$$

Theorem 1.17.

(1) *For generic θ, $\mathcal{M}(p,q)$ is a smooth manifold of dimension $\mu(q) - \mu(p) - 1$. ($p \neq q$.)*

(2) *If $\mu(q) - \mu(p) - 1 = 0$. Then, for generic θ and each E, the set $\mathcal{M}(p,q;E)$ is finite.*

(3) *If* $\mu(q)-\mu(p)-1=1$. *Then, for generic* θ *and each* E, *the set* $\mathcal{M}(p,q;E)$ *can be compactified to a one dimensional manifold whose boundary is*

$$\bigcup_{\substack{r\in Cr(\theta)\\ \mu(r)-\mu(p)=1}} \bigcup_{E_1+E_2=E} \mathcal{M}(p,r;E_1)\times\mathcal{M}(r,q:E_2).$$

We remark that

$$\int \gamma^*\theta = f(p)-f(q)$$

if $\theta=df$ and $[\gamma]\in\mathcal{M}(p,q)$. Hence, in this case, the "enery" $\int\gamma^*\theta$ is uniformly bounded on $\mathcal{M}(p,q)$. This is the reason why we can show the finiteness of the order of $\mathcal{M}(p,q)$ in the case when θ is exact (and $\mu(q)-\mu(p)-1=0$). In our situation, the energy $\int\gamma^*\theta$ is unbounded because of the "multivaluedness" of the functional (or equivalently the nonexactness of θ.)

We omit the proof of Theorem 1.17. (See [No].)

We use universal Novikov ring introduced in [FKO$_3$]. Let T be a formal parameter.

Definition 1.18. We consider the formal (countable) sum $\sum c_iT^{\lambda_i}$ such that

$$c_i\in\mathbb{R},\quad \lambda_i\in\mathbb{R},\quad \lim_{i\to\infty}\lambda_i=\infty.$$

The totality of such formal sums becomes a ring in an obvious way. We denote this ring by $\Lambda'_{nov,+}$.

We consider $\sum c_iT^{\lambda_i}$ which satisfies $\lambda_i\ge 0$ in addition and denote it by Λ'_{nov}.

Now let θ be a Morse one form on X.

Definition 1.19. $CF(X;\theta)$ stands for the free Λ'_{nov} module whose generator is identified with the set $Cr(\theta)$. $CF(X;\theta)$ becomes a graded module by putting $\deg[p]=\mu(p)$.

We define coboundary operator $\partial : CF^k(X;\theta)\to CF^{k+1}(X;\theta)$ by

$$\partial[p]=\sum_{\mu(q)=\mu(p)+1}\ \sum_{[\gamma]\in\mathcal{M}(p,q)}\epsilon(\gamma)\,T^{\int\gamma^*\theta}[q] \tag{1.4}$$

Theorem 1.17(1) implies that the sum $\sum_{[\gamma]\in\mathcal{M}(p,q)}\epsilon(\gamma)\,T^{\int\gamma^*\theta}$ is an element of Λ_{nov} if θ is generic and if $\mu(q)=\mu(p)+1$. Hence ∂ is well-defined for generic θ.

Theorem 1.20. (Novikov) $\partial\partial=0$. *The cohomology* $\mathrm{Ker}\partial/\mathrm{Im}\partial$ *depends only on* X *and the De-Rham cohomology class of* θ.

$\partial\partial=0$ is a cosequence of Theorem 1.16(2). The proof of the second half of the statement is omitted. (See [No].)

Now we go back to our infinite dimensional situation where $X=\Omega(L_0,L_1)$, and $\theta=d\mathcal{A}$. Floer observed that the gradient trajectory of $d\mathcal{A}$ is pseudoholomorphic strip. Namely :

"Observation 1.21".

$$\mathrm{grad}_\ell \mathcal{A} = J_M \left(\frac{d\ell}{dt}\right).$$

We did not mention which Sobolev space we use to define $\Omega(L_0, L_1)$. So we are unable to define its tangent space rigorously. However, natively speaking, it is natural to regard the tangent space $T_\ell\Omega(L_0, L_1)$ as the set of the sections s of ℓ^*TM with the boundary condition $s(0) \in T_{\ell(0)}L_0$, $s(1) \in T_{\ell(1)}L_1$. The right hand side $J_M\left(\frac{d\ell}{dt}\right)$ of "Observation" 1.21 is a section of ℓ^*TM. However the boundary condition may not be satisfied. This is the reason why we put "Observatin" 1.21 in the quote. In fact, we use it only to motivate the definition and define directly "the moduli space of gradient trajectories", as follows. We first assume that L_0 is transversal to L_1. We put

$$Cr(\mathcal{A}) = L_0 \cap L_1.$$

For $p, q \in Cr(\mathcal{A})$, we define :

Definition 1.22.

$$\tilde{\mathcal{M}}(p,q) = \left\{ \varphi : \mathbb{R}\times[0,1] \to M \;\middle|\; \begin{array}{l} \dfrac{d\varphi}{dt} = J_M\dfrac{d\varphi}{d\tau}, \varphi(\tau,0)\in L_0,\ \varphi(\tau,1)\in L_1, \\ \lim\limits_{\tau\to-\infty}\varphi(\tau,t) = p,\ \lim\limits_{\tau\to\infty}\varphi(\tau,t) = q. \end{array} \right\}$$

Here τ is the parameter of $\mathbb{R}$ and t is the parameter of $[0,1]$.

$\mathbb{R}$ acts on $\tilde{\mathcal{M}}(p,q)$ by $s\cdot\varphi(\tau,t) = \varphi(s+\tau,0)$. Let $\mathcal{M}(p,q)$ be the quotient space.

The rough idea is to use Definition 1.22 in place of Definition 1.16 to define Floer homology. However there are various points where our infinite dimensional situation is different from finite dimensional case. Namely :

(a) The (virtual) dimension of $\mathcal{M}(p,q)$ is not well-defined. In other words, it depends on the component.

(b) The compactness or the compactification of $\mathcal{M}(p,q)$ does not go in the same way as the finite dimensional case. Namely the analogy of Theorem 1.17 does not hold in general. The reason is bubbling off of pseudoholomorphic S^2 and D^2.

Let us first explain (a). To define virtual dimension, we consider the linearlized equation of $\frac{d\varphi}{dt} = J_M\frac{d\varphi}{d\tau}$.

Let $\varphi \in \tilde{\mathcal{M}}(p,q)$. Let $L^p_1(\mathbb{R}\times[0,1];\varphi^*TM)$ be the Banach space of the section of φ^*TM on $\mathbb{R}\times[0,1]$ of L^p_1 class. (Namely the section of L^p class whose first derivative is also of L^p class.) Let $L^p(\mathbb{R}\times[0,1];\varphi^*TM\otimes\Lambda^{0,1})$ be the Banach space of L^p section of the bundle $\varphi^*TM\otimes\Lambda^{0,1}$ on $\mathbb{R}\times[0,1]$. We define

$$\overline{\partial}_\varphi : L^p_1(\mathbb{R}\times[0,1];\varphi^*TM) \to L^p(\mathbb{R}\times[0,1];\varphi^*TM\otimes\Lambda^{0,1}) \tag{1.5}$$

as follows. We have a connection ∇ of φ^*TM induced from the Levi-Civita connection on M. Using it, we have $d : L^p_1(\mathbb{R}\times[0,1];\varphi^*TM) \to L^p(\mathbb{R}\times[0,1];\varphi^*TM\otimes\Lambda^1)$. Compsing the projection $\Lambda^1 \to \Lambda^{0,1}$ we obtain (1.5).

Lemma 1.23. *$\overline{\partial}_\varphi$ is a Fredholm operator.*

The proof is straignt forward. (See [Fl1],[Oh1].)

Let $p, q \in L_0 \cap L_1$. We may regard p, q as elements (constant path) of $\Omega(L_0, L_1)$. Let $\pi_1(\Omega(L_0,L_1);p,q)$ be the set of homotopy classes of the pathes joining p and q in $\Omega(L_0,L_1)$. For each $\varphi \in \tilde{\mathcal{M}}(p,q)$, homotopy class of $\tau \mapsto \varphi(\tau,\cdot)$ defines an element of $\pi_1(\Omega(L_0,L_1);p,q)$.

Definition 1.24. Let $\beta \in \pi_1(\Omega(L_0, L_1); p, q)$. We put :

$$\mathcal{M}(p,q;\beta) = \{[\varphi] \in \mathcal{M}(p,q) | [\tau \mapsto \varphi(\tau,\cdot)] = \beta\}.$$

The following is a consequence of homotopy invariance of the index of Fredholm operators.

Lemma 1.25. *The index of $\overline{\partial}_\varphi$ for $[\varphi] \in \mathcal{M}(p,q;\beta)$ depends only on β.*

Let $\eta(\beta)$ be the index in Lemma 1.25. To state the next lemma, we need a notation. We define $+ : \pi_1(\Omega(L_0, L_1); p, r) \times \pi_1(\Omega(L_0, L_1); r, q) \to \pi_1(\Omega(L_0, L_1); p, q)$ by joining the path. And $- : \pi_1(\Omega(L_0, L_1); p, q) \to \pi_1(\Omega(L_0, L_1); q, p)$ by changing the orientation of the path.

Lemma 1.26. $\eta(\beta + \beta') = \eta(\beta) + \eta(\beta')$. $\eta(-\beta) = -\eta(\beta)$.

The proof follows from sum formula of index. (See for example [Fl].) We put :

$$\mathcal{M}(p,q;k) = \bigcup_{\mu(\beta)=k} \mathcal{M}(p,q;\beta).$$

Now the main technical result we need to prove Theorem 1.7 is the following.

Proposition 1.27. (Oh [Oh1]) *Suppose $L_i \subset M$ ($i = 0, 1$) are monotone with minimal Maslov number ≥ 3. We also assume Condition T. Then we can "perturb" and compactify $\mathcal{M}(p,q;k)$ to obtain $\mathcal{CM}(p,q;k)$ such that the following holds.*
(1) $\mathcal{CM}(p,q;0)$ *is a compact 0 dimensional manifold.*
(2) $\mathcal{CM}(p,q;1)$ *is a comapct 1 dimensional manifold and*

$$\partial\mathcal{CM}(p,q;1) = \bigcup_r \mathcal{CM}(p,r;0) \times \mathcal{CM}(r,q;0).$$

Before explaining the idea of the proof of Proposition 1.27, we show how to use Proposition 1.27 to construct Floer homology. We put

$$\partial[p] = \sum_q \sharp\mathcal{M}(p,q;0)[q].$$

Here $\sharp\mathcal{M}(p,q;0)$ is order modulo 2 of the set $\mathcal{M}(p,q;0)$ which is well-defined because of (1). Note that we work here over $\mathbb{Z}_2$ coeficcient. (We do not need Novikov ring since $\mathcal{A}$ is single valued in this case.)

We remark that the coefficient of $[q]$ in $\partial\partial[p]$ is

$$\sum_r \sharp\mathcal{CM}(p,r;0) \times \sharp\mathcal{CM}(r,q;0).$$

Hence (2) implies $\partial\partial = 0$.

Let us now explain the reason we need to assume monotonicity and minimal Maslov number ≥ 3 in order to prove Proposition 1.27.

Let us first describe an example where (2) does not hold in the case when minimal Maslov number is 2. This is related to point (b) in the last page.

We take $M = \mathbb{C}$, $L_0 = \mathbb{R}$, and $L_1 = S^1 = \{z \in \mathbb{C} | |z| = 1\}$. It is easy to see that $\mu(\beta) = 2$, where $\beta \in \pi_2(\mathbb{C}, S^1) \simeq \mathbb{Z}$ is the generator. We put $p = -1$, $q = 1$. It is also easy to see that $\mathcal{M}(p,q;0)$, $\mathcal{M}(q,p;0)$ both consist of one point. Hence $\partial[p] = [q]$, $\partial[q] = [p]$. Thus $\partial\partial[p] = [p] \neq 0$.

We consider $\mathcal{CM}(p,p;0)$. Let $\varphi \in \mathcal{M}(p,p;1)$. φ is a (pseudo)holomorphic map: $\mathbb{R} \times [0,1] \to \mathbb{C}$ such that $\varphi(\tau,0) \in L_0$. We find that there exists $z \in (-1,1)$ such that the image $\varphi(\mathbb{R} \times \{0\})$ is an open interval $(-1,z)$. By Riemann's mapping theorem the map, $\varphi \mapsto z$ gives a diffeomorphism $\mathcal{M}(p,p;0) \simeq (-1,1)$. Thus the compactification $\mathcal{CM}(p,p;0)$ of $\mathcal{M}(p,p;0)$ is identified with $[-1,1]$.

Figure 1

The point $1 \in [0,1]$ in the boundary $\partial\mathcal{CM}(p,p;0)$ corresponds to the (unique) element of $\mathcal{M}(p,q;0) \times \mathcal{M}(q,p;0)$, which is one described by (2). In fact, as z approaches to 1, the corresponding element $\varphi_z \in \mathcal{M}(p,p;1)$ will splits into the union of two maps. These two maps represent elements of $\mathcal{M}(p,q;0)$ and $\mathcal{M}(q,p;0)$, respectively.

Let us consider the other point -1 of the boundary $\partial\mathcal{CM}(p,p;0)$. As z approaches -1 then the element φ_z will split also. This time, the "limit" will be a connected sum of two maps $\varphi^{(1)}_{-1} : \mathbb{R} \times [0,1] \to \mathbb{C}$ and $\varphi^{(2)}_{-1} : D^2 \to \mathbb{C}$. Here $\varphi^{(1)}_{-1}(\tau,t) \equiv -1$ and $\varphi^{(2)}_{-1} : D^2 \to \mathbb{C}$ is the unique (psudo)holomorphic map representing $\beta \in \pi_2(\mathbb{C}, S^1)$. In other words, the bubbling off of (pseudo)holomorphic disk gives another element of the boundary.

Oh's argument [Oh1] based on dimension counting, shows that bubbling off of pseudoholomorphic disk occurs in a moduli space of virtual dimension one, only in the case when there exists a nonconstant pseudoholomorphic disks of Maslov index ≤ 2.

The monotonicity of the Lagrangian submanifold implies that there is no nonconstant pseudoholomorphic disk with Maslov index ≤ 0. In fact, a pseudoholomorphic disk φ with Maslov index ≤ 0 would satisfy

$$\int_{D^2} \varphi^*\omega \leq 0,$$

by monotonicity, and must be constant.

Hence, the other assumption (minimal Maslov number ≥ 3) implies that there is no nonconstant pseudoholomorphic disk Maslov index ≤ 2. This is an outline of the proof of Proposition 1.27.

Actually, we use the monotonicity, also to show the precompactness of $\mathcal{M}(p,q;k)$. Namely monotonicity and Condition T imply that, if $\varphi \in \mathcal{CM}(p,q;k)$, then

$$\int_{D^2} \varphi^*\omega = c(p,q)k,$$

which depends only on k,p,q. This fact is essential to show the precompactness of $\mathcal{M}(p,q;k)$. However, as far as this point concerns, we can go around the trouble of noncompactness of $\mathcal{M}(p,q;k)$ by introducing Novikov ring in the same way as the finite dimensional case we already discussed. (Namely we divide $\mathcal{CM}(p,q;k)$ into $\mathcal{CM}(p,q;k;E)$ according to the energy $E = \int_{D^2} \varphi^*\omega$ and can show the compactness of $\mathcal{CM}(p,q;k;E)$.) So this is not the point we need these assumptions in the most serious way.

2. Moduli space of Lagrangian submanifolds.

Moduli space of Lagrangian submanifolds entered to the story of mirror symmery through two different routes. One is homological mirror symmetry conjecture by Kontsevich [Ko1],[Ko2], the other is geometric mirror symmetry conjecture by Strominger-Yau-Zaslow [SYZ].

We first describe a part of homological mirror symmetry conjecture. We describe it (the "Conjecture 2.1") in rather naive and imprecise way. In fact, "Conjecture 2.1" does not hold true as it stands, and we need some modification. The precise story is more delicate and we will explain a part of it later.

"Homological mirror symmetry conjecture 2.1".
(1) *To some symplectic manifold* (M, ω)*, mirror symmetry associates a complex manifold* $(M^\dagger, J_{M^\dagger})$*, its mirror.*
(2) *Let* $(L, \mathcal{L})$ *be a pair of Lagrangian submanifold* L *of* M *and a flat complex line bundle* $\mathcal{L}$ *on* L*. Then, we have its mirror* $\mathcal{E}(L, \mathcal{L})$*, which is an object of the derived category of coherent sheaves on* $(M^\dagger, J_{M^\dagger})$*.*
(3) *We have the following isomorphism*

$$HF((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1)) \simeq Ext(\mathcal{E}(L_0, \mathcal{L}_0), \mathcal{E}(L_1, \mathcal{L}_1)).$$

(*We will explain the way to include flat bundles to Floer homology later in* §*3.*)
(4) *The Yoneda product*

$$Ext(\mathcal{E}(L_0, \mathcal{L}_0), \mathcal{E}(L_1, \mathcal{L}_1)) \otimes Ext(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2)) \to Ext(\mathcal{E}(L_0, \mathcal{L}_0), \mathcal{E}(L_2, \mathcal{L}_2))$$

will become the product of Floer homology

$$HF((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1)) \otimes HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \to HF((L_0, \mathcal{L}_0), (L_2, \mathcal{L}_2))$$

by the isomorphism (3)*. Here the product of Floer homology is one introduced in* [Fu1] *inspired by an idea due to Donaldson* [D] *and Segal.*

We wrote here only a part of the homological mirror symmetry conjecture. We will describe some of the other parts later.

Remark 2.2. Usually, mirror symmetry associates Kähler manifold $M^\dagger$ to a Kähler manifold M, the complex structure of $M^\dagger$ depends on the symplectic structure (Kähler form) of M, and the symplectic structure of $M^\dagger$ depends on the complex structure of M.

In "Conjecture 2.1", we assume only a symplectic structure on M and a complex structure on $M^\dagger$. The author does not know an example of a mirror pair $M, M^\dagger$ such that M is symplectic non-Kähler and/or $M^\dagger$ is complex non-Kähler.

However most of the constructions below work for general sympelctic manifold M without assuming it to be Kähler.

Rather surpringingly, Kontsevich discovered homological mirror symmetry conjecture before duality and D-brain became important in string theory (namely before 2nd string theory revolution). In fact, "Conjecture 2.1" is closely related to the notion of D-branes as we now explain.

Definition 2.3.
(1) Let (M, ω) be a symplectic manifold. A pair $(L, \mathcal{L})$ of Lagrangian submanifold L of M and a flat complex line bundle $\mathcal{L}$ on L is a *brane* (in a classical sense) of A-model compactified by (M, ω).
(2) Let $(M^\dagger, J_{M^\dagger})$ be a complex manifold. A *brane* of B-model compactified by $(M^\dagger, J_{M^\dagger})$.

Thus "Conjecture 2.1" is restated as follows.

"Conjecture 2.4". *The moduli space of branes of A model compactified by (M, ω) coincides with the moduli space of branes of B model compactified by $(M^\dagger, J_{M^\dagger})$.*

Remark 2.5. It seems unlikely that there exists a pair $(L, \mathcal{L})$ corresponding to every object of coherent sheaves on $M^\dagger$. In other words, there is not enough Lagrangian submanifolds compared to the coherent sheaves of the mirror[2].

"Conjecture 2.4" is closely related to the geomeric mirror symmetry conjecture by Strominger-Yau-Zaslow. Let $(M^\dagger, J_{M^\dagger})$ be a complex manifold and $p \in M^\dagger$. We define the *skyscraper sheaf* $\mathfrak{F}_p$ by

$$\mathfrak{F}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U, \\ 0 & \text{if } p \notin U. \end{cases}$$

The moduli space of skyscrper sheaves is exactly the space of $M^\dagger$ itself. Therefore, "Conjecture 2.4" implies the following :

"Conjecture 2.6". *The mirror of (M, ω) is a component of the moduli space of pairs $(L, \mathcal{L})$.*

To make more precise sense to "Conjectures" 2.4 and 2.6, we need to clarify what we mean by "moduli space of pairs $(L, \mathcal{L})$". In fact, the set of all such pairs is an infinite dimensional space. To obtain something of finite dimension, there are two ways. One is to restrict Lagrangian submanifold, the other is to consider an appropriate equivalence relation.

The first way was taken by Strominger-Yau-Zaslow. Namely they considered the moduli space of the pairs $(L, \mathcal{L})$ where L is a special Lagrangian submanifold and $\mathcal{L}$ is a flat vector bundle. This approach is motivated by the observation due to Becker-Becker-Strominger [BBS], that the brane whose presence does not break super symmetry and κ symmetry is a special Lagrangian submanifold. We do not discuss their approach here. (See [Gs1], [Gs2], [Mc], [SW] for related results.)

Let us describe the second way. We first recall the notion of Hamiltonian diffeomorphism.

Definition 2.7.
(1) Let h be a function on a symplectic manifold (M, ω). The Hamilton vector field V_h generated by h is defined by $i_{V_h}\omega = dh$.
(2) Let $h : M \times [0, 1] \to \mathbb{R}$ be a smooth function. We put $h_t(x) = h(x, t)$. It defines a time dependent Hamilton vector field V_{h_t}. We define $\Psi_t : M \to M$ by

$$\Psi_0(x) = x, \quad \frac{d\Psi_t}{dt}(x) = V_{h_t}(\Psi_t(x)).$$

[2] A way to amplify objects in Lagrangian side is to include A_∞ functor, which is defined in [Fu4].

It is well known that Ψ_t is a symplectic diffeomorphism.
(3) A symplectic diffeomorphism $\Psi : M \to M$ is called the *Hamiltonian diffeomorphism*, if there exists h as in (2) such that $\Psi_1 = \Psi$.
(4) The set of all Hamiltonian diffeomorphisms is a group. We denote it by $Ham(M, \omega)$.

Definition 2.8. Let $(L_0, \mathcal{L}_0)$, $(L_1, \mathcal{L}_1)$ be pairs of Lagrangian submanifolds and flat $U(1)$ bundles on it. We say that they are *Hamiltonian equivalent* and write $(L_0, \mathcal{L}_0) \sim (L_1, \mathcal{L}_1)$ if there exists $\Psi \in Ham(M, \omega)$ such that $\Psi(L_0) = L_1$ and $\Psi^*\mathcal{L}_1$ is isomorphic to $\mathcal{L}_0$.

Before going further, we mention the way to modify Definition 2.8, in case there is so called a B-field. Let B be a closed 2 form on M. We consider complexified symplex form $\widetilde{\omega} = \omega + 2\pi\sqrt{-1}B$. We then modify "Definition" 2.3(1) as follows :

Definition 2.9. A pair $(L, (\mathcal{L}, \nabla))$ of Lagrangian submanifold L of M, a complex line bundle $\mathcal{L}$ on L, and its connection ∇ is a *brane* in a classical sense of A-model compactified by $(M, \widetilde{\omega})$, if the curvature F_∇ of ∇ satisfies $F_\nabla = 2\pi\sqrt{-1}B\,|_L$.

Definition 2.10. Let $(L_0, (\mathcal{L}_0, \nabla_0))$, $(L_1, (\mathcal{L}_1, \nabla_1))$ be as in Definition 2.9. We say that they are *Hamiltonian equivalent* if the following holds.

There exists $\Psi_t : M \to M$ as in Definition 2.7. We put $L = L_0$. There exists a connection $\widetilde{\nabla}$ on $L \times [0, 1]$ with the following properties.

(1) $\Psi_0(L) = L_0$, $\Psi_1(L) = L_1$.
(2) The curvarture of $\widetilde{\nabla}$ satisfies $F_{\widetilde{\nabla}} = 2\pi\sqrt{-1}\Psi^*B$
(3) The restriction of $\widetilde{\nabla}$ to $L \times \{0\}$ coincides with $\Psi_0^*\nabla_0$. The restriction of $\widetilde{\nabla}$ to $L \times \{1\}$ coincides with $\Psi_1^*\nabla_1$.

Let $\mathfrak{Lag}^+(M, \widetilde{\omega})$ be the set of all $\sim$ equivarence classes of the branes $(L, (\mathcal{L}, \nabla))$ in the sense of Definition 2.9. We want to regard this set as the moduli space of branes on $(M, \tilde{\omega})$. However it is not known whether the quotient space is Hausdorff or not.

In general, moduli space of geometric objects is Hausdorff only when we introduce stability and take stable objects only. The author does not yet know the best definition of stability of Lagrangian submanifolds. So we use the following tentative definition.

Tentative definition 2.11. We put the C^∞-topology on the set of the pair $(L, (\mathcal{L}, \nabla))$ as in Definition 2.9. We say that $(L, (\mathcal{L}, \nabla))$ is *stable* if there exists a neighborhood U of it in C^∞ topology such that the image of U in $\mathfrak{Lag}^+(M, \widetilde{\omega})$ with quotient topology is Hausdorff.

Note that the stability of $(L, (\mathcal{L}, \nabla))$ depends only on L. Hence we say that L is stable instead of saying $(L, (\mathcal{L}, \nabla))$ to be stable.

We next remark that stability or Hausdorffness of the moduli space $\mathfrak{Lag}^+(M, \widetilde{\omega})$ is closely related to the flux conjecture, which we review briefly. (See [Ba], [LMP] for detail.)

Let (M, ω) be a symplectic manifold and V be a vector field such that $L_V\omega = 0$. (Here L_V denotes the Lie derivative.) It is a standard fact in symplectic geometry (see [MS2] for example) that $i_V\omega$ is a closed one form. Now let $\gamma; S^1 \to \mathrm{Diff}(M, \omega)$

be a loop of symplectic diffeomorphisms. The differential $d\gamma/dt$ is a family of vector fields on M with $L_{d\gamma/dt}\omega = 0$. We put :

$$\mathrm{Flux}(\gamma) = \left[\int_{S^1} i_{d\gamma/dt}\omega dt\right] \in H^1(M;\mathbb{R}).$$

It is easy to see that Flux(γ) depends only the cohomology class of γ and hence defines a homomorphism Flux : $H_1(\mathrm{Diff}(M,\omega)) \to H^1(M;\mathbb{R})$. We call this homomorphism the *flux homomorphism*.

Conjecture 2.12. (Flux conjecture) *The image of flux homomorphism is discrete.*

The relation of flux conjectgure 2.12 to the stability of Lagrangian submanifold is described as the following lemma whose proof is easy and is omitted.

Lemma 2.13. *If flux conjecture holds true for* (M,ω) *then the diagonal* Δ_M *in* $(M \times M, pi_1^*\omega - pi_2^*\omega)$ *is a stable Lagrangian submanifold in the sense of Tentative Definition 2.11.*

Thus, if we regard the notion of Lagrangian submanifolds as a generalization of symplectic diffeomorphisms[3], then the problem of stability of Lagrangian submanifold is a natural generalizatin of flux conjecture.

The following problem seems to be essential to study the relation of our moduli space $\mathfrak{Lag}^+(M,\widetilde{\omega})$ to the moduli space of special Lagrangian submanifolds.

Problem 2.14([Fu7])**.** Let $[L,\mathcal{L}]$ be an element of $\mathfrak{Lag}^+(M,\widetilde{\omega})$. Are the following two conditions eqiuvalent to each other ?

(1) L is stable.
(2) There exists a special Lagrangian submanifold L' such that $(L,\mathcal{L})$ is $\sim$ equivalent to $(L',\mathcal{L})$.

We remark that a famous conjecture by S.Kobayashi, which was proved by Donaldson [D2] and Uhlenbeck-Yau [UY], says that a complex vector bundle E over a Kähler manifold has a Yang-Mills $U(n)$ connection if and only if E is stable. Problem 2.14 is similar to it. One might be able to regard 2.14 as a "mirror" of Kobayashi conjecture.

The following problem seems to open also.

Problem 2.15([Fu7])**.** Let $(L,\mathcal{L})$, $(L',\mathcal{L}')$ be elements of $\mathfrak{Lag}^+(M,\widetilde{\omega})$ such that L, L' are both special and $(L,\mathcal{L}) \sim (L',\mathcal{L}')$. Does $L = L'$ follows ?

Let $(L,\mathcal{L}) \in \mathfrak{Lag}^+(M,\widetilde{\omega})$. We assume that it is stable.

Proposition 2.16. *There exists a neighborhood* U_L *of zero in* $H^1(L;\mathbb{C})$ *and a homeomorphism* $\Psi_L : U_L \to \mathfrak{Lag}^+(M,\widetilde{\omega})$ *onto a neighborhood of* $(L,\mathcal{L})$ *with the following properties.*

If L_0, L_1 *are stable and* $\Psi_{L_0}(U_{L_0}) \cap \Psi_{L_1}(U_{L_1}) \neq \emptyset$, *then the coordinate change* $\Psi_{L_1L_0} = \Psi_{L_1}\Psi_{L_0}^{-1}$ *is holomorphic.*

Proof. By a theorem of Weinstein, there exists a neighborhood W of zero section in T^*L and a symplectic diffeomorphism $\Phi : W \to M$ to its image, such that $\Phi(x,0) = x$.

[3] This is a classical idea in symplectic geometry

We may choose a neighborhooh U_0 of $(L,\mathcal{L}) \in \mathfrak{Lag}^+(M,\widetilde{\omega})$ with the following properties. If $(L',\mathcal{L}') \in U_0$, then there exists a closed one form $\theta_{L'}$ on L such that $\Phi^{-1}(L')$ is equal to the graph of $\theta_{L'}$. We put

$$h_0(L',\mathcal{L}') = [\theta_{L'}] \in H^1(L;\mathbb{R}).$$

Let $\ell : S^1 \to L$ be a loop and ℓ' is the loop on L' such that $\pi\Phi^{-1}\ell' = \ell$. Let $\ell^+ : S^1 \times [0,1] \to M$ be a map such that $\ell^+(t,0) = \ell(t)$, $\ell^+(t,1) = \ell'(t)$. We consider

$$h_{1,\ell}(L',\mathcal{L}') = \log\left(\frac{hol_{\mathcal{L}}(\ell)}{hol_{\mathcal{L}'}(\ell')}\right) + 2\pi\sqrt{-1}\ell^{+*}B,$$

where $hol_{\mathcal{L}}(\ell) \in U(1)$ is a holonomy of the connection $\mathcal{L}$ along the loop ℓ. We can prove that if $(L'_0,\mathcal{L}'_0) \sim (L'_1,\mathcal{L}'_1)$, then $h_{1,\ell}(L'_0,\mathcal{L}'_0) = h_{1,\ell}(L'_1,\mathcal{L}'_1)$.

Let $\ell_1,\cdots,\ell_b$ be a basis of $H_1(L;\mathbb{Z})$ and $\ell_1^*,\cdots,\ell_b^* \in H^1(L;\mathbb{Z})$ be the dual basis. We put

$$h_1(L'_0,\mathcal{L}'_0) = \sum h_{1,\ell_k}(L'_0,\mathcal{L}'_0)\ell_k^* \in \sqrt{-1}H^1(L;\mathbb{R}).$$

$h = h_0 + h_1$ is a map from a neighborhood of $[L,\mathcal{L}]$ in $\mathfrak{Lag}^+(M,\widetilde{\omega})$ to a neighborhood of 0 in $H^1(L;\mathbb{C})$. It is easy to see that h is a homeomorphism. Let Ψ_L be the inverse of h. We can easily show that it has the required properties.

Let us denote by $\mathfrak{Lag}^+_{st}(M,\widetilde{\omega})$ the set of the equivalence classes of stable elements. Proposition 2.16 implies that it has a complex structure. We call this complex structure the *classical complex structure.*[4]

We proved the isomorphism

$$T_{[L,\mathcal{L}]}\mathfrak{Lag}^+_{st}(M,\widetilde{\omega}) \simeq H^1(L;\mathbb{C}) \tag{2.1}$$

during the proof of Proposition 2.16. Let us compare (2.1) to its mirror by homological mirror symmetry.

Let $\mathcal{E}$ be a holomorphic vector bundle over $M^\dagger$. We assume that $\mathcal{E}$ is stable. Then Kuranishi theory of the moduli space of holomorphic vector bundle gives a map

$$s : Ext^1(\mathcal{E},\mathcal{E}) \to Ext^2(\mathcal{E},\mathcal{E}) \tag{2.2}$$

such that a neighborhood of $\mathcal{E}$ in the moduli space of stable vector bundles is isomorphic to $s^{-1}(0)$ (as a scheme in fact.) Let us denote by $\mathcal{M}(M^\dagger)$ the moduli space of stable vector bundles on $M^\dagger$.

If $\mathcal{E} = \mathcal{E}(L,\mathcal{L})$ is as in "Conjecture 2.1", then we have

$$Ext^i(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L,\mathcal{L})) \simeq HF^i(L;\mathbb{C}). \tag{2.3}$$

(Here we considered holomorphic vector bundle. But we can generalize it to (stable) objects of derived category of coherent sheaves.) Comparing (2.1), (2.2), (2.3), we find the following.

[4]In fact we need to include "quantum correction" to get correct complex structure of the mirror. See §3 and Part II on this point.

(a) $Ext^1(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L,\mathcal{L})) \simeq H^1(L;\mathbb{C})$ is related to $HF^1((L,\mathcal{L}),(L,\mathcal{L}))$ but is not necessary equal to it. Namely (P.3) may not hold in general.

(b) The Kuranishi map may be nonzero in general. Namely the moduli space of stable vector bundles might be obstructed. (An example is given in [Th].) On the other hand, the moduli space $\mathfrak{Lag}^+_{st}(M,\widetilde{\omega})$ is never obstructed. Namely we can deform the pair $(L,\mathcal{L})$ to any direction in $H^1(L;\mathbb{C})$.

In the absolute case (when there was no Lagrangian submanifold or vector bundles), the corresponding phenomenon did not occur by the following reason.

The tangent space of Kähler moduli is $H^{1,1}(M)$ which is isomorphic to the Zariski tangent space $H^1_{\overline{\partial}}(M;TM)$. (Compare (a).) A theorem of Bogomolov-Tian-Todorov [Bo],[Ti],[To] implies that the moduli space of complex structures of Calabi-Yau manifold is unobstructed. Namely the Kuranishi map : $H^1_{\overline{\partial}}(M;TM) \to H^2_{\overline{\partial}}(M;TM)$ vanishes automatically. On the other hand, the Kähler moduli is always unobstructed. (Compare (b).)

Homological mirror symmetry together with (a), (b) suggests :

(a$'$) There is a quantum effect to the Floer homology of Lagrangian submanifold as an abelian group. (In the case of quantum cohomology (or Floer homology of periodic Hamiltonian system) there is a quantum effect only to the product structure.)

(b$'$) Not all pair $(L,\mathcal{L})$ correspond to an object of the derived category on its mirror.

To clarify these points is a purpose of the next section.

Remark 2.18. The degree of Floer homology is rather delicate. Namely Floer homology is not necessary graded over $\mathbb{Z}$. As a consequence, it is rather hard to distinguish HF^1 from HF^{2k+1}. (See [Sei] for some discussion on the degree of Floer homology of Lagrangian submanifolds.) On the other hand, if we consider the moduli space of objects of derived category of coherent sheaves rather than moduli space of stable sheavs, then the tangent space of the moduli space can be regarded as Ext^{odd} rather than Ext^1. It then seems more natural to introduce extended moduli space (see [Ra]).

In fact, in this article, we did not discuss one of the most hard and important points of the study of the moduli space of Lagrangian submanifolds. Namely we did not discuss its compactification. As in the case of compactification of other moduli spaces appeared in geometry, the compactification of the moduli space of Lagrangian submanifolds are to be done by adding singular Lagrangian "submanifolds". It is very hard, however, to find an appropriate condition to be imposed to the singularity of the Lagrangian "submanifolds". In the study of geometric mirror symmetry conjecture, compactification of the moduli space of special Lagrangian submanifolds is studyed by [Gs1], [Gs2] etc. In that case, singular Lagrangian "submanifolds", which appear in the compactification, is a singular fiber of a special Lagrangian fibration[5]. To study compactification, restricting ourselves to special Lagrangian submanifolds makes problem easier to handle, since then we can use various results of minimal surface theory, especially geometric measure theory.

From the point of view of this article, it is important to study the Floer homology of singular Lagrangian submanifold. In [Fu7], [FKO$_3$] and in Part II of this article, we study Floer homology of Lagrangian submanifold obtained by Lagrangian

[5]The singular fibration whose generic fiber is a special Lagrangian tori.

surgery. This is a special case of a study of a the Floer homology of Lagrangian submanifold obtained by deforming a singular one. In the case of Lagrangian surgery, the singularity is the simplest one (normal crossing). So far, it is very hard to study more serious singurality. Another kind of Floer homomlogy of singular Lagrangian submanifolds appeared in [KO]. We also remark that Mcpherson [GM] suggested a relation of Floer homology of singular Lagrangian submanifold to intersection homology.

3. Obstruction to the Floer homology of Lagrangian submanifolds.

To develop obstruction theory to the Floer homology of Lagrangian submanifolds, we introduce the notion of A_∞ algebra. A_∞ algebra was introduced by J. Stasheff [St]. It was applied by [Fu1] to the study of Floer homology. Its relation to mirror symmetry was discoverd by Kontsevitch [Ko1], [Ko2].

Let R be a $\mathbb{Z}_2$-graded ring and C be a $\mathbb{Z}_2$-graded module over it. We define a graded R algebra ΠC by shifting degree. Namely we put $(\Pi C)^m = C^{m+1}$. We put

$$B_k\Pi C = \underbrace{\Pi C \otimes \cdots \otimes \Pi C}_{k \text{ times}}, \qquad B\Pi C = \bigoplus_k B_k \Pi C.$$

We consider a family of maps $\mathfrak{m}_k : B_k\Pi C \to \Pi C$ of degree $+1$. (Here the degree is one after we shifted.) It induces a map $d_k : B\Pi C \to B\Pi C$ by

$$d_k(x_1\cdots x_n) = \sum_{i=1}^{n-k+1} (-1)^{\deg x_1 + \cdots \deg x_{i-1}} x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i \cdots x_{i+k-1}) \otimes \cdots \otimes x_n.$$

We then put $\hat d = \sum d_k$.

Definition 3.1. (Stasheff [St]) $(C, \mathfrak{m}_k)$ for $k = 1, 2, \cdots$ is said to be an A_∞ *algebra* if $\hat d \hat d = 0$. $(C, \mathfrak{m}_k)$ for $k = 0, 1, \cdots$ is said to be a *weak* A_∞ *algebra* if $\hat d\hat d = 0$.

We define a filtration *the number filter* on $B\Pi C$ by $\mathfrak{G}_k B\Pi C = \bigoplus_{\ell \le k} B_\ell \Pi C$. We remark that d_k preserves number filter if $k > 0$. The differential $\hat d$ of weak A_∞ algebra does *not* preserve the number filter. On the other hand, the differential $\hat d$ of A_∞ algebra does preserve the number filter.

Let $(C, \mathfrak{m}_k)$ be an A_∞ algebra. We apply the condition $\hat d\hat d = 0$ on each $\mathfrak{G}_k B\Pi/\mathfrak{G}_{k-1}B\Pi \simeq B_k\Pi C$ to obtain the following series of relations.

$$0 = \mathfrak{m}_1\mathfrak{m}_1, \tag{3.1}$$

$$0 = \mathfrak{m}_1\mathfrak{m}_2(x,y) + \mathfrak{m}_2(\mathfrak{m}_1(x),y) + (-1)^{\deg x}\mathfrak{m}_2(x,\mathfrak{m}_1(y)), \tag{3.2}$$

$$\begin{aligned} 0 = &\mathfrak{m}_1\mathfrak{m}_3(x,y,z) + \mathfrak{m}_3(\mathfrak{m}_1(x),y,z) \\ &+ (-1)^{\deg x}\mathfrak{m}_3(x,\mathfrak{m}_1(y),z) + (-1)^{\deg x + \deg y}\mathfrak{m}_3(x,y,\mathfrak{m}_1(z)) \\ &+ \mathfrak{m}_2(\mathfrak{m}_2(x,y),z) + (-1)^{\deg x}\mathfrak{m}_2(x,\mathfrak{m}_2(y,z)). \end{aligned} \tag{3.3}$$

(3.1) implies that $\mathfrak{m}_1$ is a boundary operator. (3.2) implies that $\mathfrak{m}_2$ is a derivation with respect to $\mathfrak{m}_1$.

(3.3) is an asociativity relation. However we need to be careful about the sign. Following sign convention by Getzler-Jones [GJ], we put $x\cdot y = (-1)^{\deg x - 1}\mathfrak{m}_2(x,y)$.

Then, in case when $\mathfrak{m}_3 = 0$, (3.3) is equivalent to $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ the associativity law. Thus A_∞ algebra is a natural generalization of differential graded algebra (hereafter abbreviated by DGA).

Actually, we need to use Novikov ring as a coefficent ring. In that case, we need to modify the definition slightly as follows. Let $\Lambda'_{nov,+}$ be the universal Novikov ring introduced in §1. Let u be a formal parameter of degree 2 and put $\Lambda_{nov,+} = \Lambda'_{nov,+}[u,u^{-1}]$. $\Lambda_{nov,+}$ is a graded algebra. We put

$$\left\|\sum c_i T^{\lambda_i} u^{n_i}\right\| = \inf_{c_i\neq 0} \lambda_i.$$

$\Lambda_{nov,+}$ is complete with respect to the metric $dist(x,y) = \exp(-\|x-y\|)$.

Definition 3.2. A *complete* $\Lambda_{nov,+}$ *module* is a graded $\Lambda_{nov,+}$ module C equipped with $\|\cdot\| : C \to \mathbb{R}$ such that :

(1) $\|xv\| \ge \|x\| + \|v\|$ and $\|v_1+v_2\| \ge \max\{\|v_1\|,\|v_2\|\}$ hold for each $x \in \Lambda_{nov,+}$ and $v, v_1, v_2 \in C$.
(2) The metric $dist(v_1,v_2) = \exp(-\|v_1-v_2\|)$ is complete.

Let C be a complete $\Lambda_{nov,+}$ module. We define $B\Pi C$ in the same way. We extend $\|\cdot\|$ to $B\Pi C$ as follows. First we put $\|v_1\otimes\cdots\otimes v_k\| = \|v_1\| + \cdots + \|v_k\|$. We then extend it so that $\|x_1+x_2\| \ge \max\{\|x_1\|,\|x_2\|\}$ holds. We use it to define a completion $\hat{B}\Pi C$ of $B\Pi C$ with respect to $dist(x,y) = \exp(-\|x-y\|)$.

Let us consider series of maps $\mathfrak{m}_k : B_k\Pi C \to \Pi C$ of degree $+1$. We assume that

$$\|\mathfrak{m}_k(x)\| \ge \|x\| \tag{3.4}$$

where C is independent of k. We define $\hat{d}_k$ in the same way as before. Then (3.4) implies that we can extend $\hat{d} : B\Pi C \to C$ to the completion $\hat{B}\Pi C$ uniquely. We write it by the same symbol.

Definition 3.3. ([FKO$_3$]) $(C,\mathfrak{m}_k)$ for $k = 1,2,\cdots$ is said to be a *filtered* A_∞ *algebra* if $\hat{d}\hat{d} = 0$. $(C,\mathfrak{m}_k)$ for $k = 0,1,\cdots$ is said to be a *weak filtered* A_∞ *algebra* if $\hat{d}\hat{d} = 0$.

From now on we assume that C is generated by elements x with $\|x\| = 0$. Let $\Lambda^0_{nov,+}$ be the ideal of $\Lambda^0_{nov,+}$ defined by $\Lambda^0_{nov,+} = \{x \in \Lambda_{nov,+} | \|x\| > 0\}$. We have $\Lambda_{nov,+}/\Lambda^0_{nov,+} = \mathbb{R}[u,u^{-1}]$. We put $R = \mathbb{R}[u,u^{-1}]$. Then $C/\Lambda^0_{nov,+}$ is a finitely generated graded R module. If $(C,\mathfrak{m}_k)$ is a (weak)filtered A_∞ algebra then $C/\Lambda^0_{nov,+}$ is a (weak) A_∞ algebra over R.

Definition 3.4. We say that a (weak) A_∞ algebra $(C,\mathfrak{m}_k)$ is a deformation of an A^∞ algebra $\overline{C}$ over $\mathbb{R}$ if $C/\Lambda^0_{nov,+}$ is isomorphic to $\overline{C}\otimes\mathbb{R}[u,u^{-1}]$ as A_∞ algebra.

We next define a notion of A_∞ maps between two A_∞ algebras $(C,\mathfrak{m}_k)$, $(C',\mathfrak{m}'_k)$. We consider a family of maps $\varphi_k : B_k\Pi C \to \Pi C'$ of degree 0. Using it we define $\hat{\varphi} : \hat{B}\Pi C \to \hat{B}\Pi C'$ by

$$\hat{\varphi}(x_1\cdots x_n) = \sum_{0=k_1\le\cdots\le k_\ell=n} \varphi_{k_2-k_1}(x_{k_1},\cdots,x_{k_2})\otimes\cdots \otimes \varphi_{k_\ell-k_{\ell-1}}(x_{k_{\ell-1}+1},\cdots,x_{k_\ell}).$$

Definition 3.5. φ_k, $k = 1, 2, \cdots$ is said to be an A^∞ *map*, if $\hat{\varphi}$ is a chain map. φ_k, $k = 0, 1, 2, \cdots$ is said to be a *weak* A^∞ *map*, if $\hat{\varphi}$ is a chain map.

An A^∞ map between filtered A_∞ algebras C, C' which deform (the same) A_∞ algebra $\overline{C}$ is said to be a *homotopy equivalence* if it induces an identity on $\overline{C}$. (In other words if $\varphi_k \equiv 0$ for $k \neq 1$ and $\varphi_1 \equiv id$.)

In case, φ_k is a weak A^∞ map, we say that it is a weak homotopy equivalence if $\varphi_k \equiv 0$ for $k \neq 1$ and $\varphi_1 \equiv id$.

A main result of [FKO$_3$] is stated as follows.

Theorem 3.6. *Let M be a symplectic manifold and $(L, \mathcal{L})$ be a pair of Lagrangian submanifold and a flat bundle on it. We assume that the second Stiefel-Whitney class $w^2(L)$ of L is in the image of $H^2(M; \mathbb{Z}_2)$.*

Then, we can associate a weak filterd A_∞ algebra $(C(L), \mathfrak{m}_k)$ which deforms the A_∞ algebra describing the rational homotopy type of L.

$(C(L), \mathfrak{m}_k)$ depends only on symplectic manifold M and L upto weak homotopy equivalence.

If $\psi : M \to M$ is a Hamiltonian diffeomorphism, then $(C(L), \mathfrak{m}_k)$ is weak homotopy equivalent to $(\psi(L)), \mathfrak{m}_k)$

Remark 3.7. Let L be a manifold and $S_*(L)$ be its singular chain complex. We may use Poincaré duality to regard $S_*(L)$ as a cochain complex. The cup product then is given by the intersection pairing. However the intersection pairing is *not* well-defined in the chain level on $S_*(L)$. Especially the intersection of P with itself is never transversal.

This causes a trouble to study rational homotopy theory using singular chains. To overcome this trouble, we need to use A_∞ algebra in place of DGA. In fact we can find an appropriate $\mathfrak{m}_k$ on a countably generated subcomplex of $S_*(L)$ which gives all higher Massey products. (See [FKO$_3$].) This is what we mean by the A_∞ algebra describing the rational homotopy type of L.

Let us now explain an outline of the costruction of A_∞ algebra in Theorem 3.6, and the properties of it. The construction is in fact similar to the construction of quantum cohomology and of Gromov-Witten invariant in symplectic geometry. We first define

$$\tilde{\mathcal{M}}_d^\partial = \{(z_1, \cdots, z_d) | z_i \in \partial D^2, z_i \neq z_j \text{ for } i \neq j, z_i \text{ respects cyclic order of } \partial D^2\}.$$

$Aut(D^2, J) \simeq PSL(2; \mathbb{Z})$ acts on it by $u(z_1, \cdots, z_d) = (u(z_1), \cdots, u(z_d))$. Let $\mathcal{M}_d^\partial$ be the quotient space. We can show that $\mathcal{M}_d^\partial$ is homeomorphic to $\mathrm{Int} D^{d-3}$ if $d \geq 3$. $\mathcal{M}_d^\partial$ has a compactification $\mathcal{CM}_d^\partial$. An element of $\partial\mathcal{CM}_d^\partial$ can be regarded as a several D^2's glued to each other at their boundaries, together with marked points on the boundary (Figure 2). (See [FOh] for these matter.)

Figure 2

Now, let (M, ω) be a symplectic manifold and L be a Lagrangian submanifold. Let $\beta \in \pi_2(M, L)$. We are going to define a moduli space $\mathcal{M}_d^\partial(M; \beta)$.

Definition 3.8. $\tilde{\mathcal{M}}_d^\partial(M; \beta)$ is the set of all pairs $((z_1, \cdots, z_d), \varphi)$. Here $(z_1, \cdots, z_d)$ is an element of $\tilde{\mathcal{M}}_d^\partial$ and φ is a pseudoholomorhpic map $\varphi : D^2 \to M$ such that $\varphi(\partial D^2) \subset L$ and the homotopy class of φ is β.

$Aut(D^2, J) \simeq PSL(2;\mathbb{Z})$ acts on $\tilde{\mathcal{M}}^{\partial}_d(M;\beta)$ by

$$u((z_1, \cdots, z_d), \varphi) = ((u(z_1), \cdots, u(z_d)), \varphi \circ u^{-1}).$$

$\mathcal{M}^{\partial}_d(M;\beta)$ denotes the quotient space.

We define the evaluation map $ev : \mathcal{M}^{\partial}_d(M;\beta) \to L^d$ by

$$ev[(z_1, \cdots, z_d), \varphi] = (\varphi(z_1), \cdots, \varphi(z_d)).$$

We put $ev = (ev_1, \cdots, ev_d)$. We can prove the following :

Proposition 3.9 ([FKO$_3$],[Sil]). *A lift of the second Stiefel-Whitney class $w^2(L)$ of L to $H^2(M;\mathbb{Z}_2)$ determines an orientation of $\tilde{\mathcal{M}}^{\partial}_d(M;\beta)$.*

In a similar way to the definition of Gromov-Witten invariant, (of symplectic manifold) ([FOn], [LiT], [Ru], [Sie]), we can find a compactification and a perturbation of $\mathcal{CM}^{\partial}_d(M;\beta)$ as a chain over $\mathbb{Q}$, and can extend the map ev there. We also obtain a map $\mathfrak{forget} : \mathcal{CM}^{\partial}_d(M;\beta) \to \mathcal{CM}^{\partial}_d$ if $d \geq 3$, which extends $[(z_1, \cdots, z_d), \varphi] \mapsto [z_1, \cdots, z_d] \in \mathcal{M}^{\partial}_d$. Moreover

$$\dim \mathcal{CM}^{\partial}_d(M;\beta) = n + \mu(\beta) + d - 2. \tag{3.4}$$

Here n is the dimension of L and μ is the Maslov index introduced in §1.

An important remark here is that $\mathcal{CM}^{\partial}_d(M;\beta)$ is not necessary a cycle. In fact, in the simplest case when M is a point, $\mathcal{CM}^{\partial}_d(M;\beta) = \mathcal{CM}^{\partial}_d$ is a disk, which has nonempty boundary. This is related to the fact that Floer homology is ill-defined in general. (Namely $\partial\partial \neq 0$.)

However we can still define a family of (multi-valued) perturbations of $\mathcal{CM}^{\partial}_d(M;\beta)$ and can define a chain over $\mathbb{Q}$. The set $\mathcal{CM}^{\partial}_d(M;\beta) - \mathcal{M}^{\partial}_d(M;\beta)$ is described by the fiber product of various $\mathcal{CM}^{\partial}_{d'}(M;\beta')$'s with $\mathcal{E}(\beta') < \mathcal{E}(\beta)$. We can make our perturbation so that it is compatible with this identification. (This is important to show A_∞ formulae.) The construction of the perturbation is very similar to the discussion in [FOn] where Floer homology of periodic Hamiltonian system is constructed. So we omit it. The proof will be given in [FKO$_3$].

Now, let P_i be a chain on L. (Technically speaking, we take geometric chain introduced in [Gr1] and used in [Fu3] systematically in a related context of Bott-Morse theory.) We put :

$$\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k) = ev_{k+1*}\left((P_1 \times \cdots \times P_k) \times_{(ev_1, \cdots, ev_k)} \mathcal{CM}^{\partial}_{k+1}(M;\beta)\right). \tag{3.5}$$

Here $\times_{(ev_1, \cdots, ev_k)}$ denotes the fiber product. Transversality theorem implies that, for generic P_i, the right hand side is a well-defined chain.

The case when $\beta = 0$ and $k = 1$ is exceptional. In that case, we put

$$\mathfrak{m}_{1,0}(P) = \partial P. \tag{3.6}$$

We take an appropriate countablly generated subcomplex of the singular chain complex of L (or more precisely the chain complex of geometric chains) so that the right hand side of (3.5) satisfies appropriate transversality for $P_i \in C(L)$ and it gives again an element of $C(L)$. We refer those points to [FKO$_3$].

We now put

$$C = C(L) \otimes \Lambda_{nov,+},$$

and define

$$\mathfrak{m}_k(P_1, \cdots, P_k) = \sum_\beta \mathfrak{m}_{k,\beta}(P_1, \cdots, P_k) \otimes T^{\mathcal{E}(\beta)} u^{\mu(\beta)}. \tag{3.7}$$

By using Gromov compactness, we can show that the right hand side of (3.7) is in C.

Remark 3.10. In case when we include $\mathcal{L}$ a flat line bundle on L, we modify (3.7) and define

$$\mathfrak{m}_k(P_1, \cdots, P_k) = \sum_\beta \mathfrak{m}_{k,\beta}(P_1, \cdots, P_k)(\mathrm{hol}_{\partial\beta}\mathcal{L}) \otimes \ T^{\mathcal{E}(\beta)} u^{\mu(\beta)}.$$

Here $\mathrm{hol}_{\partial\beta}\mathcal{L}$ is the holonomy of the flat connection $\mathcal{L}$ along closed curve $\partial\beta$.

In case when there is a B field and $(L, \mathcal{L})$ is as in Definitin 2.9, we modify further. (We omit the way how we modify, since it is rather a straightforward excercise.)

We remark that $\mathfrak{m}_0$ may not be 0. In fact

$$\mathfrak{m}_0(1) = \sum_\beta ev_*(\mathcal{CM}_1^\partial(M;\beta))(\mathrm{hol}_{\partial\beta}\mathcal{L}) \otimes \ T^{\mathcal{E}(\beta)} u^{\mu(\beta)}.$$

$ev_*(\mathcal{CM}_1^\partial(M;\beta))$ is the chain obtained as union of boundary values of the pseudo-holomorphic disks of homotopy class β.

The proof of A_∞ formulae goes in a similar way as the proof of associativity (and $Comm_\infty$ formulae) of quantum cohomology, and proceed roughly as follows. The formula we are going to show is

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{k_1,\beta_1}(P_1, \cdots, P_i, \mathfrak{m}_{k_2,\beta_2}(P_{i+1}, \cdots, P_{i+m_2}), P_{i+m_2+1}, \cdots, P_k) = 0.$$

We remark that

$$\begin{aligned} &\mathfrak{m}_{1,0}\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k) = \\ &\qquad \partial \left(ev_{k+1*}\left((P_1 \times \cdots \times P_k) \times_{(ev_1, \cdots, ev_k)} \mathcal{CM}_{k+1}^\partial(M;\beta) \right) \right) \end{aligned} \tag{3.8}$$

by (3.5) and (3.6). We find that the right hand side of (3.8) is the sum of

$$\sum_i \pm ev_{k+1*}\left((P_1 \times \cdots \times \partial P_i \times \cdots \times P_k) \times_{(ev_1, \cdots, ev_k)} \mathcal{CM}_{k+1}^\partial(M;\beta) \right) \tag{3.9}$$

and

$$\pm ev_{k+1*}\left((P_1 \times \cdots \times P_k) \times_{(ev_1, \cdots, ev_k)} \partial\mathcal{CM}_{k+1}^\partial(M;\beta) \right). \tag{3.10}$$

(3.6) implies that (3.9) is equal to :

$$\sum_i \pm\mathfrak{m}_{k,\beta}(P_1,\cdots,\mathfrak{m}_{1,0}(P_i),\cdots,P_k). \tag{3.11}$$

On the other hand, $\partial\mathcal{CM}^{\partial}_{k+1}(M;\beta)$ is divided into various components describing the splitting of the pseudoholomorphic disk into two disks. (Namely bubbling off of pseudoholomorphic disks.) For example Figure 3 below corresponding to the term $\mathfrak{m}_{k_1,\beta_1}(P_1,P_2,\mathfrak{m}_{k_2,\beta_2}(P_3,P_4,P_5),P_6,P_7,P_8)$.

Figure 3

Together with (3.11), the right hand sides of (3.8), (3.10) gives all the terms of A_∞ formula, completeing the proof of A_∞ formula.

In short, the proof of A_∞ formula can be described as :

$$\begin{aligned}
&\partial\left((P_1\times\cdots\times P_k)\times_{ev_1\cdots ev_k}\mathcal{CM}_{0,k+1}(M,J_M;L;\beta)\right)\\
&=\sum_i \pm(P_1\times\cdots\times\partial P_i\times\cdots\times P_k)\times_{ev_1\cdots\,ev_k}\mathcal{CM}_{0,k+1}(M,J_M;L;\beta)\\
&\quad+\sum_{\beta_1+\beta_2=\beta}\ \sum_{1\le i\le j\le k}\pm(P_1\times\cdots\times P_{i-1}\times\\
&\qquad\qquad((P_i\times\cdots\times P_j)\times_{ev}\mathcal{CM}_{0,j-i}(M,J_M;L;\beta_1))\\
&\qquad\qquad\times P_{j+1}\times\cdots\times P_k)\times_{ev}\mathcal{CM}_{0,k-j+i}(M,J_M;L;\beta_2).
\end{aligned}$$

This is an outline of the proof of Theorem 3.6.

Before describing the properties of the weak filtered A_∞ algebra in Theorem 3.6, we need a bit more homological algebra. To motivate the construction, we mention a relation of $\mathfrak{m}_0$ to the trouble in the definition of Floer homology we discussed in §1. As we mentioned before, in the case of (filtered) A_∞ algebra, $\hat{d}\hat{d}=0$ implies $\mathfrak{m}_1\mathfrak{m}_1=0$. This is not the case of weak A_∞ algebra. In that case, we have

$$\mathfrak{m}_1\mathfrak{m}_1(P)=\pm\mathfrak{m}_2(\mathfrak{m}_0(1),P)\pm\mathfrak{m}_2(P,\mathfrak{m}_0(1)). \tag{3.12}$$

Namely $\partial^2\neq 0$ if we put $\partial=\mathfrak{m}_1$. We remark that, in the case of weak filtered A_∞ algebra of Theorem 3.6, $\mathfrak{m}_0(1)$ is the homology class on L represeted by a chain obtained as union of boundary values of the pseudoholomorphic disks. In §1, we mentioned that the presence of pseudoholomorphic disk obstructs the well-definedness ($\partial\partial=0$) of the Floer homology. Formula (3.12) shows it more precisely.

Now let $(C,\mathfrak{m}_k)$ be a weak filtered A_∞ algebra. We are going to try to modify $\mathfrak{m}_k$ and to obtain a filtered A_∞ algebra.

Let $b\in\Pi C^0=C^1$. Assume $\|b\|>0$. We put

$$e^b=1+b+b\otimes b+b\otimes b\otimes b+\cdots\in\hat{B}\Pi C.$$

($\|b\|>0$ implies that the right hand side converges.)

Definition 3.11. b is said to be a *bounding chain* if $\hat{d}e^b = 0$.

The condition $\hat{d}e^b = 0$ is equivalent to

$$\mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b,b) + \mathfrak{m}_2(b,b,b) + \cdots = 0. \tag{3.13}$$

Remark 3.12. Let us assume $\mathfrak{m}_0 = \mathfrak{m}_3 = \mathfrak{m}_4 = \cdots = 0$. We write $\mathfrak{m}_1 = d$, $\mathfrak{m}_2 = \wedge$. Then(3.13) is written as

$$db + b \wedge b = 0. \tag{3.14}$$

This equation can be regarded as a version of Maurer-Cartan equation[6] or Batalin-Vilkovsky Master equation. Maurer-Cartan equation describes moduli space of vector bundle, for example. (Namely, if we replace d by $\overline{\partial}$ then (3.13) is equivalent $(\overline{\partial} + b)^2$, that is the condition that $\overline{\partial} + b$ determines a holomorphic structure on vector bundle.)

Let b a bounding chain of a weak filtered A_∞ algebra $(C, \mathfrak{m}_k)$. We define $\mathfrak{m}_k^b$ by

$$\mathfrak{m}_k^b(x_1, \cdots, x_k) = \sum_{\ell_0, \cdots, \ell_k} \mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b, \cdots, b}_{\ell_0}, x_1, \underbrace{b, \cdots, b}_{\ell_1}, \cdots, \underbrace{b, \cdots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \cdots, b}_{\ell_k}).$$

Proposition 3.13. *If b is a bounding chain, then $(C, \mathfrak{m}_k^b)$ is a filtered A_∞ algebra. In particular $\mathfrak{m}_0^b = 0$.*

The proof is easy. (See [FKO$_3$].)

Definition 3.14. Let $(C, \mathfrak{m}_k)$ be a weak filtered A_∞ algebra. Then $\hat{\mathcal{M}}(C)$ denotes the set of all bounding chains of it. In case when $(C, \mathfrak{m}_k)$ is the weak filtered A_∞ algebra of Theorem 3.6, we put $\hat{\mathcal{M}}(L)$ or $\hat{\mathcal{M}}(L, \mathcal{L})$ in place of $\hat{\mathcal{M}}(C)$.

We can define a gauge equivalence $\sim$ between two elements of $\hat{\mathcal{M}}(C)$. ([FKO$_3$]. In [Ko3], gauge equivalence is defined in a similar context of L_∞ algebra.) Let $\mathcal{M}(C)$, $\mathcal{M}(L)$ be the set of $\sim$ equivalence classes.

Lemma 3.15. *If $b \sim b'$ then $(C, \mathfrak{m}_k^b)$ is homotopy equivalent to $(C, \mathfrak{m}_k^{b'})$.*

The proof is in [FKO$_3$]. We also have the following :

Lemma 3.16. *Let $\varphi_k : B_k \Pi C \to \Pi C'$ be a weak homotopy equivalence between weak filtered A_∞ algebras. Then there exists a map $\varphi_* : \hat{\mathcal{M}}(C) \to \hat{\mathcal{M}}(C')$ such that, for each $b \in \hat{\mathcal{M}}(C)$, the filtered A_∞ algebra $(C, \mathfrak{m}_k^b)$ is homotopy equivalent to $(C', \mathfrak{m}_k'^{\varphi_*(b)})$.*

We define φ_* by

$$\varphi_*(b) = \varphi_0(1) + \varphi_1(b) + \varphi_2(b,b) + \cdots = \varphi(e^b).$$

We can easily check that $\varphi_*(b) \in \mathcal{M}(C')$. We refer [FKO$_3$] for the rest of the proof of Lemma 3.16.

Theorem 3.6 together with the results above implies the following. Let L be as in Theorem 3.6 and $\mathcal{L}$ is a flat complex vector bundle on it.

[6]A relation of Maurer-Cartan equation is known to many people. For example, it is in [BK], [Sch], [SS]. A relation of Maurer-Cartan equation with Floer homology is discovered in [Fu7].

Theorem 3.17. *For each $b \in \mathcal{M}(L,\mathcal{L})$ we have a (homotopy type) of A_∞ algebra $(C(L)\otimes\Lambda_{nov,+},\mathfrak{m}_k^b)$. If L is Hamiltonian equivalent to L', then there exists an isomorphism $I : \mathcal{M}(L,\mathcal{L}) \to \mathcal{M}(L',\mathcal{L}')$ (of sets) such that $(C(L)\otimes\Lambda_{nov,+},\mathfrak{m}_k^b)$ is homotopy equivalent to $(C(L')\otimes\Lambda_{nov,+},\mathfrak{m}_k^{I(b)})$.*

We put

$$HF((L,\mathcal{L},b),(L,\mathcal{L},b)) = \mathrm{Ker}\mathfrak{m}_1^b/\,\mathrm{Cokernel}\,\mathfrak{m}_1^b, \tag{3.14}$$

and call it the Floer homology of Lagrangian submanifold.

Remark 3.18. Floer homology $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$ does depend on the choice of bounding chain b. While we move b on $\mathcal{M}(L,\mathcal{L})$, Floer homology $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$ jumps suddenly at some wall $\subset \mathcal{M}(L,\mathcal{L})$. This phenomenon is similar to wall crossing of Donaldson invariant of 4 manifold M with $b_2^+ = 1$[7] discovered by Donaldson in [D1].

As we remarked before, Floer homology $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$ is related to but is different from the homology group of L. The precise relation between them is described by the following :

Theorem 3.19 ([FKO$_3$])**.** *There exists a spectral sequence E_*^* with the following properties.*

(1) *E_*^* converges to the Floer homology $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$.*
(2) *$E_*^2 \simeq H(L;\Lambda_{nov,+})$.*
(3) *The image of the differential $d_k : E^k \to E^k$ lies in the image of* $\mathrm{Ker} i_*$*. Here $i : L \to M$ is the natural inclusion and i_* is the map induced by i to homology.*[8]

(3) means in particular that if i_* is injective then Floer homology coincides with usual homology of L.

We can also define a Floer homology of a pair of $((L_i,\mathcal{L}_i),b_i)$, $(i=1,2)$. Namely we have the following Theorem 3.20. We put $\Lambda_{nov} = \Lambda'_{nov}[u,u^{-1}]$.

Theorem 3.20 ([FKO$_3$])**.** *Suppose that there exists $w \in H^2(M;\mathbb{Z}_2)$ which restricts to the second Stiefel-Whitney class of both of L_0 L_1.*

Then, there exists Floer homology group $HF((L_0,\mathcal{L}_0,b_0),(L_1,\mathcal{L}_1,b_1);w)$ which is a finitely generated module over Λ_{nov}. It satisfies the following.

(1) *If L_0 is transversal to L_1, and if $L_0\cap L_1$ has m elements, then $HF((L_0,\mathcal{L}_0,b_0),(L_1,\mathcal{L}_1,b_1);w)$ is generated by m elements over Λ_{nov}.*
(2) *We assume $(L_0,\mathcal{L}_0)\sim(L'_0,\mathcal{L}'_0)$, $(L_1,\mathcal{L}_1)\sim(L'_1,\mathcal{L}'_1)$. Let $I_i : \mathcal{M}(L_i,\mathcal{L}_i) \to \mathcal{M}(L'_i,\mathcal{L}'_i)$ be an isomorphism in Theorem 3.17. Then*

$$HF((L_0,\mathcal{L}_0,b_0),(L_1,\mathcal{L}_1,b_1);w) \simeq HF((L'_0,\mathcal{L}'_0,b'_0),(L'_1,\mathcal{L}'_1,b'_1);w).$$

(3) *If $(L_0,\mathcal{L}_0,b_0) = (L_1,\mathcal{L}_1,b_1) = (L,\mathcal{L},b)$ then the Floer homology of Theorem 3.20 is the Floer homology of* (3.15) $\otimes_{\Lambda_{nov}^+}\Lambda_{nov}$*.*

Now we can state a part of the Conjecture 2.1 more precise.

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[7]This means that the intersection form on $H^2(M;\mathbb{Q})$ has exactly one negative eigenvalue.

[8]We remark that there exists a subspace $K_k \subset E_*^2 \simeq H_*(L;\mathbb{Q})$ such that E^k is a quotiend space of K_k. (3) means that image of d_k is contained in the quotient of $K_k \cap \mathrm{Ker} i_*$.

Conjecture 3.21. *Let $[L,\mathcal{L}] \in \mathfrak{Lag}^+_{st}(M,\omega)$. Suppose that the second Stiefel-Whitney class $w^2(L)$ of L is in the image of $H^2(M;\mathbb{Z}_2)$. We assume also that $\mathcal{M}(L,\mathcal{L})$ is nonempty. Then there exists an object $\mathcal{E}(L,\mathcal{L})$ of derived category of coherent sheaves of $M^\dagger$ such that :*

(1) $\mathcal{M}(L,\mathcal{L})$ is isomorphic to a formal neighborhood of the (extended) moduli space of $\mathcal{E}(L,\mathcal{L},b)$.
(2) $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$ is isomorphic to $Ext(\mathcal{E}(L,\mathcal{L},b),\mathcal{E}(L,\mathcal{L},b))$.
(3) The product on $HF((L,\mathcal{L},b),(L,\mathcal{L},b))$ induced by $\mathfrak{m}_2^b$ coincides with Yoneda product on $Ext(\mathcal{E}(L,\mathcal{L},b),\mathcal{E}(L,\mathcal{L},b))$. (Higher) Massey product coincides also.

Remark 3.22. Let us consider the special case when M is Calabi-Yau 3 fold. Suppose L is a Lagrangian submanifold such that $\mu : \pi_2(M,L) \to \mathbb{Z}$ is 0. (This is in particular the case when L is Hamiltonian equivalent to a special Lagrangian submanifold.) In this case, the virtual dimension of the moduli space $\mathcal{M}_0^\partial(L)$ is 0. Hence

$$\mathfrak{m}_0(1) = \sum \sharp\mathcal{M}_0^\partial(L)\ (\mathrm{hol}_{\partial\beta}\mathcal{L}) \otimes T^{\beta\cap\omega}[\partial\beta] \in H^2(L;\Lambda_{nov}). \tag{3.16}$$

We put $T = e^{-1}$ and regard the coefficient

$$(\mathrm{hol}_{\partial\beta}\mathcal{L})e^{-\beta\cap\omega} \tag{3.17}$$

as a $\mathbb{C}$-valued function of $[L,\mathcal{L}] \in \mathfrak{Lag}^+_{st}(M,\omega)$. Here we are assuming that the right hand side of (3.17) converges.[9]

The classical complex structure defined in §2 is designed so that (3.17) is a holomorphic function. Thus if $\sharp\mathcal{M}_0^\partial(L)$ is locally constant, then $\mathfrak{m}_0(1)$ is holomorphic. Therefore, $\mathcal{M}(L,\mathcal{L})$ which is the zero point set of $\mathfrak{m}_0(1)$, is a complex variety.

However, $\sharp\mathcal{M}_0^\partial(L)$ may not be constant and may jump at a set which is a countably many union of codimension one subsets of $\mathcal{M}(L,\mathcal{L})$. (An example of it is in [Fu7], in that case when M is a symplectic torus. This happens also in case M is a K3 surface and L is a Lagrangian torus.) Therefore, $\mathfrak{m}_0(1)$ is not holomorphic. If Conjecture 3.21 holds then $\mathcal{M}(L,\mathcal{L})$ will become a moduli space of objects of derived category of coherent sheaves of the mirror, which is a complex variety. So we need to find an appropriate complex structure of $\mathfrak{Lag}^+_{st}(M,\omega)$ so that $[L,\mathcal{L}] \mapsto [\mathfrak{m}_0(1)] \in H^1(L;\mathbb{C})$ is holomorphic. Such a complex structure is neccesary singular at the point where $\sharp\mathcal{M}_0^\partial(L)$ jumps. Thus a family version of our obstruction class gives the "quantum effect" to the complex structure. We will discuss it a bit more in Part II.

One can continue and can write a conjecture corresponding to (3),(4) of Conjecture 2.1. We leave it to the reader.

Remark 3.23. The author does not know a precise definition of an appropriate structure to be put on $\mathcal{M}(L,\mathcal{L})$. It should be called "super formal scheme". In general, extended moduli space should have such a structure.

We finally give two examples of the construction of this section without proof. (The proof is given in [Fu7] and [FKO$_3$].)

[9] The convergence is yet an open question. The converging version of Floer homology has some other points to study. We will discuss it in Part II.

Let us consider the torus $T^4 = \mathbb{C}^2/\mathbb{Z}[\sqrt{-1}]^2$. Let $z_i = x_i + \sqrt{-1}y_i$ $(i = 1,2)$ be complex coordinate of $\mathbb{C}^2$. Let $A = (a_{ij})$, $B = (b_{ij})$ be symmetric real 2×2 matrixes. We assume that A is positive definite. We put

$$\omega = \sum a^{ij} dx_i \wedge dy_j, \quad B = \sum b^{ij} dx_i \wedge dy_j.$$

For $v = (v_1, v_2) \in \mathbb{R}^2$, we define $L_1(v)$ by $x_1 = v_1$, $x_2 = v_2$. We also define L_2, L_3 by $y_1 = y_2 = 0$, $x_1 - y_1 = x_2 - y_2 = 0$ respectively. We perform Lagrangian surgery of L_2, L_3 at $L_2 \cap L_3 = \{(0,0)\}$ to obtain a Lagrangian submanifold L_0 diffeomorphic to genus 2 Riemann surface.[10] Since $\pi_2(T^4, L_0) = \pi_2(T^4, L_1(v)) = 0$. Using this fact, we can define Floer homology, without introducing bounding chain. Let us calculate it. We put $\{p(v)\} = L_1(v) \cap L_2 \subset L_0 \cap L_1(v)$, $\{q(v)\} = L_1(v) \cap L_3 \subset L_0 \cap L_1(v)$. Now we have

Proposition 3.24.

$$\partial[p] = \vartheta(v,0)[q]$$

where $\vartheta(v,0)$ is a theta function :

$$\vartheta(v,0) = \sum_{n \in \mathbb{Z}^2} \exp\left(-\frac{1}{2}\left\langle (v+n), (A + 2\pi\sqrt{-1}B)(v+n)\right\rangle\right). \tag{3.18}$$

Remark 3.25. We can include flat line bundle on L. It will then corresponds to the imaginary part of the variable of the theta function.

Remark 3.26. In (3.17) we put $T = e^{-1}$ as in (3.17). In this case, the power series converges.

Remark 3.27. A relation of theta function to Floer homology was discovered by Kontsevich [Ko2], in the case of elliptic curve. Polishchuk-Zaslow [PZ] studyed the case of elliptic curve in more detail. Their results are partially generalized by [Fu6], [Fu7] to higher dimension.

We next give an example where obstruction class (3.17) does not vanish. We consider the torus $T^6 = \mathbb{C}^3/\mathbb{Z}[\sqrt{-1}]^3$. $z_i = x_i + \sqrt{-1}y_i$ be coordinate and A, B be symmetric real 3×3 matrixes such that A is positive definite. We define $L_1(v)$ by $x_i = v_i$ $(i = 1,2,3)$, L_2, L_3 by $y_i = 0$, $x_i - y_i = 0$ respectively. We perform Lagrangian surjery of three Lagrangian submanifolds at three points $\{p_{ij}\} = L_i \cap L_j$ $((i,j) = (1,2),(2,3),(3,1))$ to obtain L. Let ℓ be the loop deforming the triangle $\Delta p_{12}p_{23}p_{31}$, and $\beta \in \pi_2(M, L)$ bounding ℓ.

Proposition 3.28.

$$\mathfrak{m}_{0,\beta}(1) = \vartheta(v,0)[\ell]$$

where $\vartheta(v,0)$ is as in (3.18). *(But $n \in \mathbb{Z}^3$ this time.)*

In this case, $\mathcal{M}(L)$ is empty.

We will discuss these example more in Part II.

[10] In fact, there are two choices of Lagrangian surgery up to Hamiltonian equivalence. We take one so that the Maslov index will be as described below.

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