Smoothing moduli spaces and Lagrangian Floer theory of arbitrary genus.:

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$$\mathbf{L} = \{(L_\kappa, b_\kappa)\}$$ A finite set of pairs

L_κ (relatively spin) Lagrangian submanifolds

$\mathscr{L}$ A cyclic unital filtered A inifinity category

$\mathbf{L}$ set of objects

$H(L \cap L'; \Lambda_0)$ set of morphisms

F, Oh, Ohta, Ono (FOOO) (+ Abouzaid FOOO (AFOOO))

Theorem (Lagrangian Floer theory of arbitrary genus) (to be written up)

There exists $\mathscr{M}^H_{\ell,g} \in E_\ell \mathbf{B}^H$ such that BV master equation $\mathbf{B}^H = \left(B^{cyc}\mathscr{L}\right)^*$

symmetric tensor product

cyclic Bar complex

$$d\mathscr{M}^H_{\ell,g} + \frac{1}{2}\sum_{\ell_1+\ell_2=\ell+1}\sum_{g_1+g_2=g}\{\mathscr{M}^H_{\ell_1,g_1},\mathscr{M}^H_{\ell_2,g_2}\}_{\text{out}} + \}\mathscr{M}^H_{\ell-1,g}\{ + \{\mathscr{M}^H_{\ell+1,g-1}\}_{\text{int}} = 0$$

is satisfied.

dIBL structure (differential involutive bi-Lie structure) on **B**

3 kinds of operations

$d:\mathbf{B}\to\mathbf{B}$ differential $dd=0$

$\{\quad\}:\mathbf{B}\otimes\mathbf{B}\to\mathbf{B}$ Lie bracket Jacobi

$\}\{:\mathbf{B}\to\mathbf{B}\otimes\mathbf{B}$ co Lie Bracket co Jacobi

$\{\quad\}$ is a derivation

$\}\{$ is a coderivation

with respect to d

compatibility between $\{\quad\}$ and $\}\{$

$(C, \langle \ \rangle, d)$ chain complex with inner product

$$\mathbf{B} = \left(B^{\mathrm{cyc}} C\right)^*$$ (dual cyclic bar complex) has a structure of dIBL algebra (cf. Cielibak-F-Latschev)

Example

$$(C, \langle \ \rangle, d) = (H(L), \text{Poincare duality}, 0)$$

We denote $\mathbf{B}^H = \left(B^{\mathrm{cyc}} H(L)\right)^*$ in this case.

I will use de Rham theory and discuss the case with only one Lagrangian submanifold.

L compact smooth manifold.

$\Omega(L)$ its de Rham complex. This is a DGA with inner product.

But it is infinite dimension and inner product (Poincare duality) is not perfect.

Some trouble to define dIBL structure on $\mathbf{B} = (B^{\mathrm{cyc}}\Omega(L))^*$

$$\mathbf{B}_\infty = \left\{ (B^{\mathrm{cyc}}\Omega(L))^* = Hom(B^{\mathrm{cyc}}\Omega(L),\mathbb{R}) \;\middle|\; \begin{array}{l}\text{Continuous and has} \\ \text{smooth Schwartz Kernel.}\end{array} \right\}$$

$$\subset \bigoplus_k \Omega(L^k)$$

Proposition (Cielibak-Latschev-F)

$$d : \mathbf{B}_\infty \to \mathbf{B}_\infty$$

$$\{\} : \mathbf{B}_\infty \hat{\otimes} \mathbf{B}_\infty \to \mathbf{B}_\infty$$

$$\}\{ : \mathbf{B}_\infty \to \mathbf{B}_\infty \hat{\otimes} \mathbf{B}_\infty$$

Has all the properties of dIBL algebra.

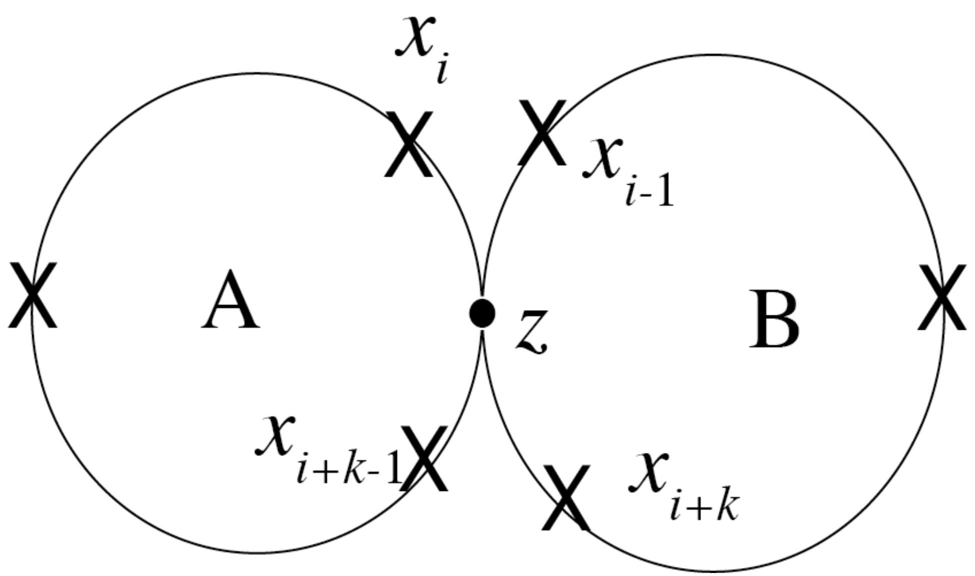
$\mathbf{B}_\infty \hat{\otimes} \mathbf{B}_\infty$

completion of tensor product.

$$A(x_1,\ldots,x_k) \in \Omega(L^k), \quad B(y_1,\ldots,y_l) \in \Omega(L^l)$$

$$\{A,B\} \in \Omega(L^{k+l-2})$$

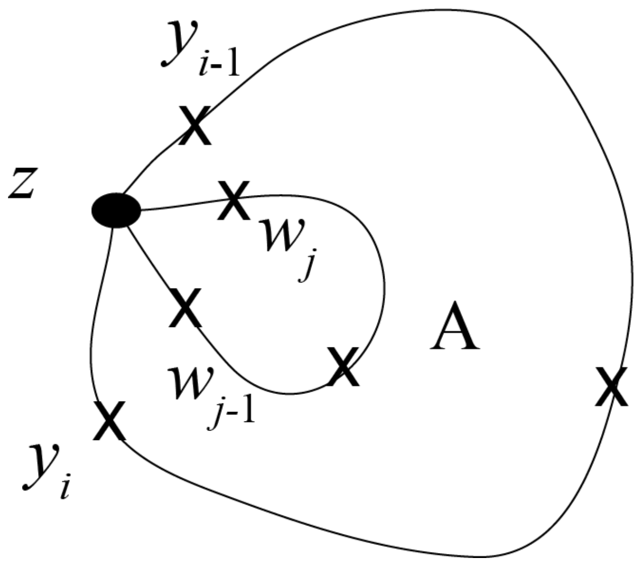
$$\{A,B\}(x_1,\ldots,x_{k+l-2})$$
$$= \sum \int_{z\in L} \pm A(z,x_i,\ldots,x_{i+k-1}) \wedge B(z,x_{i+k},\ldots,x_{i-1})$$



$$A(x_1,\ldots,x_k)\in\Omega(L^k)$$

$$\}A\{\quad\in\quad\bigoplus_l\Big(\Omega(L^l)\,\hat{\otimes}\,\Omega(L^{k-l-2})\Big)\quad=\quad\bigoplus_l\Omega(L^{k-2})$$

$$\begin{aligned}&\}A\{(y_1,\ldots,y_l;w_1,\ldots,w_{k-l-2})\\&=\sum\int_{z\in L}\pm A(z,y_i,\ldots,y_{i-1},z,w_j,\ldots,w_{j-1})\end{aligned}$$



There is a category version that is almost the same.

$$\mathbf{B} = (B^{\mathrm{cyc}} \mathscr{L})^*$$

IBL infinity structure = its homotopy everything analogue

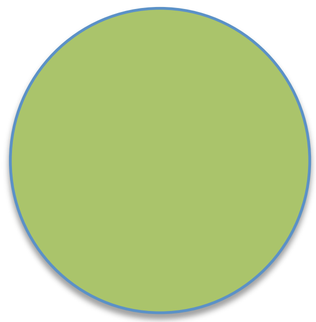
Cyclic A_∞ stucture (operations m_k) on $\mathscr{L}$

$$\Updownarrow$$

$\mathscr{M}_{1,0} \in \mathbf{B} = \left(B^{\mathrm{cyc}}\mathscr{L}\right)^*$ satisfying Maurer-Cartan equation

$$d\mathscr{M}_{1,0} + \frac{1}{2}\{\mathscr{M}_{1,0}, \mathscr{M}_{1,0}\} = 0$$

This is induced by a holomorphic DISK



Theorem (Lagrangian Floer theory of arbitrary genus) (to be written up)

There exists $\mathscr{M}^H_{\ell,g} \in E_\ell \mathbf{B}^H$ such that BV master equation $\mathbf{B}^H = \left(B^{\mathrm{cyc}}\mathscr{L}\right)^*$

$$d\mathscr{M}^H_{\ell,g} + \frac{1}{2}\sum_{\ell_1+\ell_2=\ell+1}\sum_{g_1+g_2=g}\{\mathscr{M}^H_{\ell_1,g_1},\mathscr{M}^H_{\ell_2,g_2}\}_{\mathrm{out}} + \}\mathscr{M}^H_{\ell-1,g}\{ + \{\mathscr{M}^H_{\ell+1,g-1}\}_{\mathrm{int}} = 0$$

is satisfied.

The gauge equivalence class of $\{\mathscr{M}^H_{\ell,g}\}$ is well-defined.

$$\{x_1 \cdots x_n, y_1 \cdots y_m\}_{\text{out}} = \sum_{i,j} \pm \{x_i, y_j\} x_1 \cdots \hat{x}_i \cdots x_n y_1 \cdots \hat{y}_j \cdots y_m$$

$$\{x_1 \cdots x_n\}_{\text{int}} = \sum_{i,j} \pm \{x_i, x_j\} x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots y_n$$

$$\} x_1 \cdots x_n \{ = \sum_i \pm x_1 \cdots \} x_i \{ \cdots \hat{x}_j \cdots y_n$$

Remark

Given solution of BV Master equation
we can twist the structure of dIBL structure
and obtain

IBL infinity structure.

The homotopy type of it is the invariant, that can be called Lagrangian Floer theory of higher genus.

Remark

The structure in the theorem is over Novikov ring.

If we forget all the part with positive energy and consider the structure over $\mathbb{R}$.

Then it is expected to coincides with STRING TOPOLOGY by Chas-Sullivan, via the iterated integral (Chen), in case of single L.

Remark (This is due to Cielibak-Latschev)

- This String Topology version is expected to coincides to the contact homlogy of the unit cotangent bundle. ST^*L

- Then including the effect of positive energy (non constant) disks is expected to coincides with a particular case of symplectic field theory (Eliashberg-Givental-Hofer).

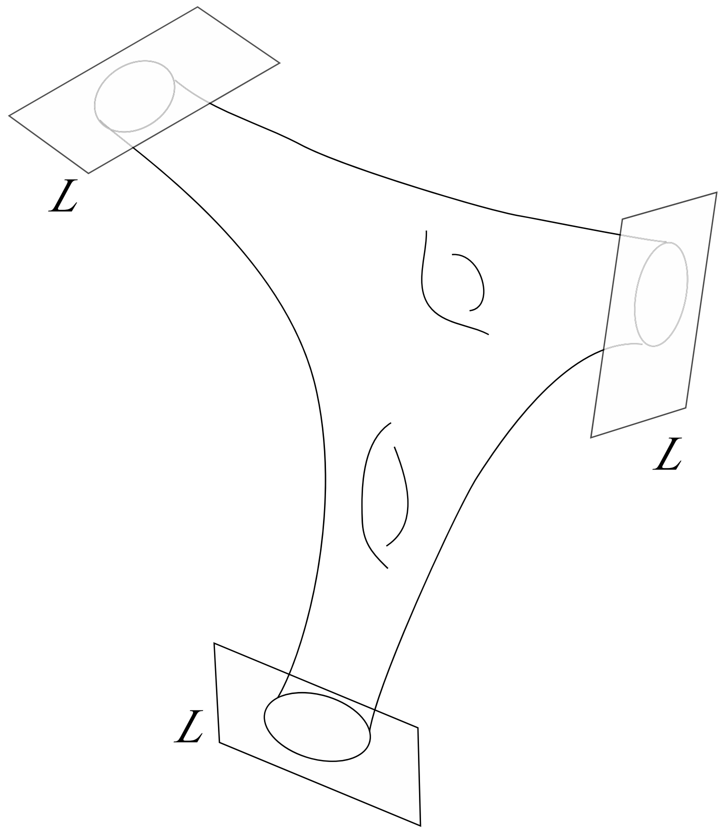
IBL infinity structure is expected to serve as algebraic basis of 3 theories

- Lagrangian Floer theory (of arbitrary genus)
- String topology
- Contact homology and SFT

Very naive idea of construction.

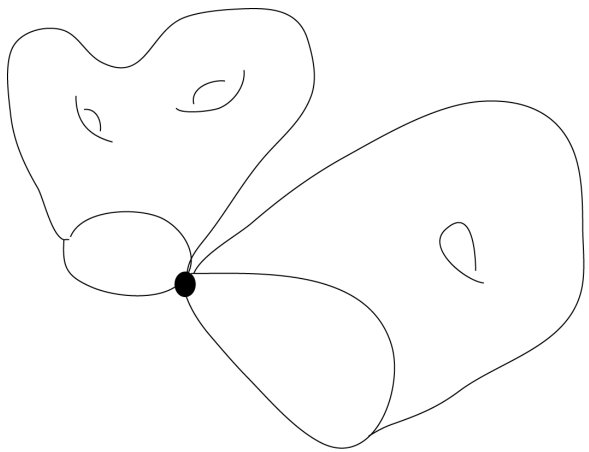
$$\mathscr{M}_{\ell,g} \in \boldsymbol{E}_\ell \mathbf{B}$$

is obtained from moduli space of genus g
bordered Riemann surface with ℓ boundary components

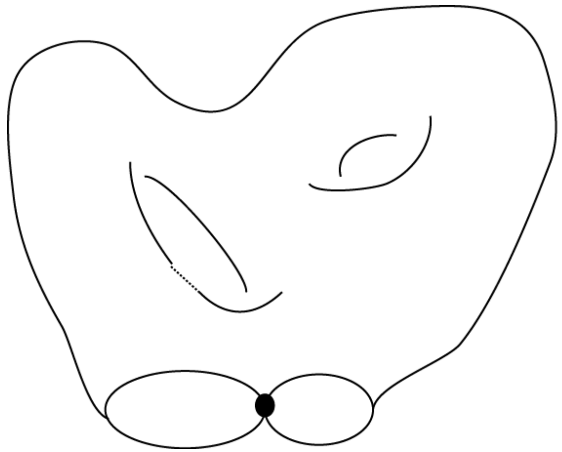


$$g = 2, \quad \ell = 3$$

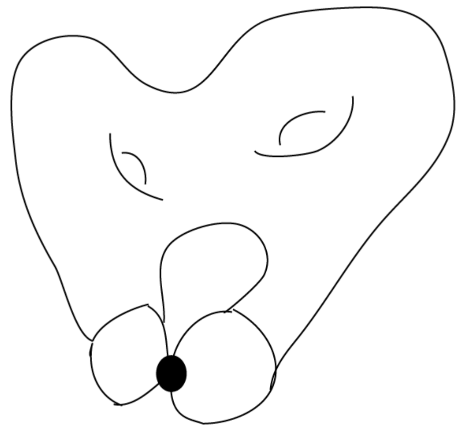
$$d\mathscr{M}_{\ell,g}+\frac{1}{2}\sum_{\ell_1+\ell_2=\ell+1}\sum_{g_1+g_2=g}\{\mathscr{M}_{\ell_1,g_1},\mathscr{M}_{\ell_2,g_2}\}_{\mathrm{out}}+\}\mathscr{M}_{\ell-1,g}\{+\{\mathscr{M}_{\ell+1,g-1}\}_{\mathrm{int}}=0$$



$\{\mathscr{M}_{\ell_1,g_1},\mathscr{M}_{\ell_2,g_2}\}_{\mathrm{out}}$



$\}\mathscr{M}_{\ell-1,g}\{$



$\{\mathscr{M}_{\ell+1,g-1}\}_{\mathrm{int}}$

Remark:

(1): In case the target space M is a point, a kind of this theorem appeared in papers by various people including Baranikov, Costello, Voronov, etc. (In Physics there is much older work by Zwieback.) However the BV master equation here is slightly different from one by Costello (The Gromov-Witten Potential associated to a TCFT)'s. (His equation has 3 terms and ours have 4 terms.)

(2): Theorem itself is also expected to hold somehow by various people including F for a long time.

(3): The most difficult part of the proof is transversality.

(4): Because of all these, the novel part of the proof of this theorem is extremely technical. So I understand that it should be written up carefully before being really established.

Therefore the main part of this talk is to explain how to work out the construction rigorously.

Use moduli space of bordered stable maps to define series of elements

$$\mathscr{M}_{\ell,g} \in \hat{E}_\ell \mathbf{B}_\infty$$

$\hat{E}_\ell \mathbf{B}_\infty$ is a completion of $E_\ell \mathbf{B}_\infty$

Theorem (Cielibak-Latschev-F)

There exists IBL infinity homomorphism

$\mathbf{B}_\infty \to \mathbf{B}^H$ that is a homotopy equivalence.

$$\mathscr{M}_{\ell,g}^H \in E_\ell \mathbf{B}^H$$

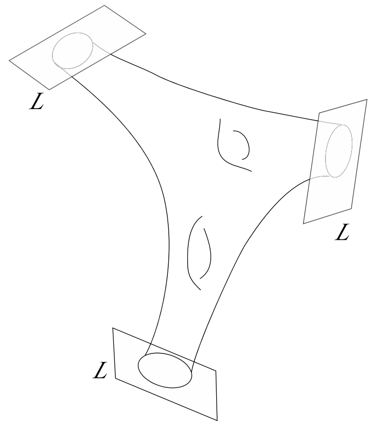
The proof is similar to perturbative Chern-Simons theory.

$$\mathbf{B}^H = (EH(L))^*$$

$\mathcal{M}_{\vec{k},g}(\beta) =$ moduli space of genus g bordered stable maps with $\vec{k} = (k_1, \ldots, k_\ell)$ marked points on the boundary and of homology class β.

$$\Downarrow$$

$$(\Sigma; (z_{i,j}); u)$$



Σ bordered semi-stable curve of genus g with ℓ boundary components

$u : (\Sigma, \partial\Sigma) \to (X, L)$ holomorphic

$z_{i,1}, \ldots, z_{i,k_i} \in \partial_i \Sigma$ i-th boundary component of Σ

respecting the cyclic order of the boundary loop

$$\mathrm{ev} : \mathcal{M}_{\vec{k},g}(\beta) \to \prod_{i=1}^{\ell} L^{k_i}$$

$$\mathrm{ev}(\Sigma;(z_{i,j});u) = (u(z_{i,j}))$$

candidate of $\mathscr{M}_{\ell,g}$

$$\mathrm{ev}!\left(\left[\mathcal{M}_{\vec{k},g}(\beta)\right]\right) \in \Omega^{-\infty}\left(\prod_{i=1}^{\ell} L^{k_i}\right)$$

is `well-defined' as a distribution
if the moduli space is transversal.

Problem starts from the simplest case.

$$\mathrm{ev} : \mathcal{M}_{3,0}(0) \to L^3$$

has the small diagonal $L \subset L^3$ as the image

and $\mathrm{ev}!\left(\left[\mathcal{M}_{3,0}(0)\right]\right)$ is not a smooth form.

$\boxed{\text{Schwartz Kernel of the wedge product operator is not a smooth form.}}$

Plan of solution of the problem.

Smoothen the distributions

$$\mathrm{ev}!\left(\left[\mathscr{M}_{\vec{k},g}(\beta)\right]\right) \in \Omega^{-\infty}\left(\prod_{i=1}^{\ell} L^{k_i}\right)$$

inductively so that all the required relations are satisfied.

Use family of multisections for smoothing

$$\mathrm{ev}!\left(\left[\mathcal{M}_{\vec{k},g}(\beta)\right]\right) \in \Omega^{-\infty}\left(\prod_{i=1}^{\ell} L^{k_i}\right)$$

Kuranishi structure (F-Ono 1996)

Moduli space such as $\mathcal{M}_{\vec{k},g}(\beta)$ locally described as

$$\mathcal{M}_{\vec{k},g}(\beta) \cap \mathcal{U} = s^{-1}(0) \big/ \Gamma$$

$\mathcal{E} \to \mathcal{U}$ Γ equivariant bundle = obstruction bundle

$s : \mathcal{U} \to \mathcal{E}$ section = Kuranishi map

Such description can be glued in an appropriate sense.

Kuranishi structure

maps such as

$$\mathrm{ev} : \mathcal{M}_{\vec{k},g}(\beta) \to \prod_{i=1}^{\ell} L^{k_i} = N$$

N is a manifold

can be extended to its Kuranishi neighborhood $\mathcal{U}$

$$\mathcal{M}_{\vec{k},g}(\beta) \cap \mathcal{U} = s^{-1}(0)/\Gamma$$

Let $\Gamma = \{1\}$ for simplicity.

$$\mathcal{M}_{\vec{k},g}(\beta) \cap \mathcal{U} = \mathfrak{s}^{-1}(0)$$

Take $\mathfrak{s}_w$ a family of perturbation of $\mathfrak{s}$ parametrized by $w \in W$

W is a parameter space of huge dimension,

may assume $\{(x,w) \in \mathcal{U} \times W \mid \mathfrak{s}_w(x) = 0\} \xrightarrow{\mathrm{ev}} N$ is a submersion.

then we may assume $\mathcal{E} \to \mathcal{U}$ is a submersion.

Replace obstruction $\mathrm{ev}: \mathcal{U} \to N$ by a bigger bundle

$$
\begin{tikzcd}[column sep=small]
 & \{(x,w) \in \mathcal{U} \times W \mid \mathscr{s}_w(x) = 0\} \arrow[dl, "proj"'] \arrow[dr, "\mathrm{ev}"] & \\
W \quad \omega_W & & N
\end{tikzcd}
$$

ω_W is a compact support smooth form of top degree with $\int_W \omega_W = 1$

$\mathrm{ev}!(proj^*(\omega_W))$ smooth form on $N = \prod_{i=1}^{\ell} L^{k_i}$

smoothing of $\mathrm{ev}!\left(\left[\mathscr{M}_{\vec{k},g}(\beta)\right]\right) \in \Omega^{-\infty}\left(\prod_{i=1}^{\ell} L^{k_i}\right)$

Do it in a way compatible with gluing we obtain smoothing of

$$\mathrm{ev}!\left(\left[\mathcal{M}_{\vec{k},g}(\beta)\right]\right) \in \Omega^{-\infty}\left(\prod_{i=1}^{\ell} L^{k_i}\right)$$

Definition:

$$\mathscr{M}_{\ell,g} = \sum_{\beta} T^{\omega\cap\beta}\mathrm{ev}!\left(\left[\mathcal{M}_{\vec{k},g}(\beta)\right]\right)_{\mathrm{smooth}}$$

To see how it works in a bit more explicitly.

$L \subset L^3$ diagonal.

This is a Schwartz kernel of the wedge product.

$$m_2^{\beta=0} : \Omega(L) \otimes \Omega(L) \to \Omega(L)$$

$$m_2^{\beta=0}(a,b) = \pm a \wedge b$$

Smooth approximation gives

$$ {}_{\varepsilon}m_2^{\beta=0}(a,b) = \pm a \wedge_{\varepsilon} b $$

This has a smooth form as Schwartz kernel.

However this is NOT associative.

$$ {}_{\varepsilon}m_2^{\beta=0}({}_{\varepsilon}m_2^{\beta=0}(a,b),c) \pm {}_{\varepsilon}m_2^{\beta=0}(a,{}_{\varepsilon}m_2^{\beta=0}(b,c)) \neq 0 $$

Obtain

$$
{}_{\varepsilon}m_3^{\beta=0} : \Omega(L) \otimes \Omega(L) \otimes \Omega(L) \to \Omega(L)
$$

that has a smooth form on L^4 as Schwartz kernel
and

$$
{}_{\varepsilon}m_2^{\beta=0}({}_{\varepsilon}m_2^{\beta=0}(a,b),c) \pm {}_{\varepsilon}m_2^{\beta=0}(a,{}_{\varepsilon}m_2^{\beta=0}(b,c))
$$

$$
= [d,{}_{\varepsilon}m_3^{\beta=0}](a,b,c)
$$

We continue to obtain

$$ {}_{\varepsilon}m_k^{\beta=0} : \Omega(L)^{k\otimes} \to \Omega(L) \qquad k = 1,\ldots,K $$

that has a smooth form on L^{k+1} as Schwartz kernel and satisfies A_K relation.

Construction of A_K structure in de Rham version.

New points appearing in our higher loop version.

$$\mathcal{M}_4^{\beta=0} \in \Omega(L^4) \qquad \text{Schwartz kernel of} \qquad {}_{\varepsilon}m_3^{\beta=0}$$

$$\Big(\} \mathcal{M}_4^{\beta=0} \{ \Big)(x,y) = \int_{z\in L} \mathcal{M}_4^{\beta=0}(x,z,y,z) \in \Omega(L^2)$$

Master equation requires that there exists

$$\mathcal{M}_{(1,1);g=0}^{\beta=0} \in \Omega(L^2) \qquad \text{that bounds} \qquad \Big(\} \mathcal{M}_4^{\beta=0} \{ \Big)$$

2^{nd} of BV-Master

$$\pm\}\mathscr{M}_{1,0}\{ \quad = d\mathscr{M}_{2,0} + \{\mathscr{M}_{2,0}, \mathscr{M}_{1,0}\}_{\mathrm{out}}$$

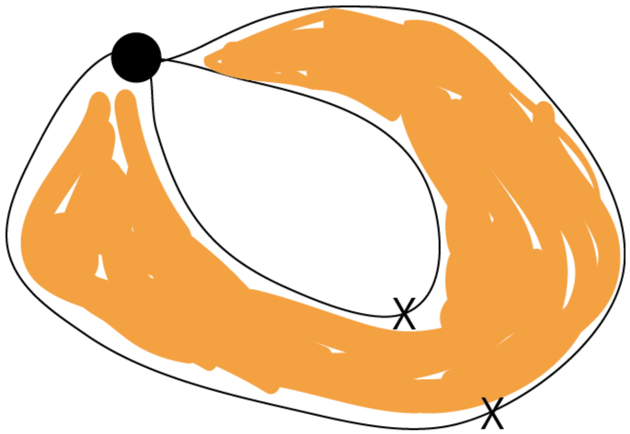
$\}\boldsymbol{\mathcal{M}}_4^{\beta=0}\{$

$d\boldsymbol{\mathcal{M}}^{\beta=0}_{(1,1);g=0}$

do not exists by stability

2 marked points $\beta = 0$

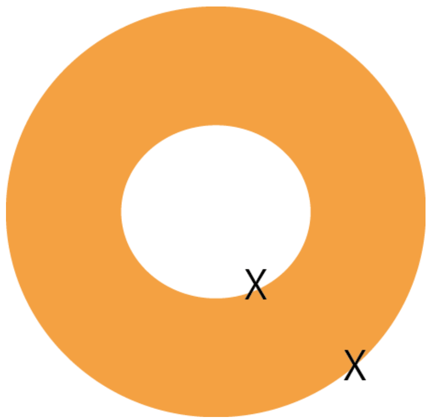
$$\}\mathcal{M}_4^{\beta=0}\{$$



is realized by the moduli space of
`holomorphic' maps from

This is a boundary of the moduli space of the `holomorphic' maps from

$$\mathcal{M}^{\beta=0}_{(1,1);g=0}$$



$$\pm\}\mathscr{M}_{1,0}\{ \quad = d\mathscr{M}_{2,0} + \{\cancel{\mathscr{M}_{2,0},\mathscr{M}_{1,0}}\}_{\mathrm{out}}$$

Two more problems

(1) Running out !

(2) To include weak obstruction classes and handle unit.

(1) `Running out' problem

Can construct

$$\mathrm{ev}\left(\mathcal{M}^{\beta}_{(k_1,\dots,k_l);g}\right) \in \Omega^{\infty}(L^{\sum k_i}) \qquad \beta \in H_2(X,L;\mathbb{Z})$$

for $$\sum k_i < K \qquad g < G \qquad \omega \cap \beta < E$$

satisfying Master equation for any but fixed K, G, E

(Cannot perturb infinitely many moduli spaces at once.)

Solution： Use homological algebra of IBL infinity structure

Lemma Assume we have

$$K_\alpha \to \infty \quad G_\alpha \to \infty \quad E_\alpha \to \infty$$

$$\mathscr{M}^{\beta}_{(k_1,\dots,k_l);g}(\alpha) \in \Omega^\infty(L^{\sum k_i})$$ satisfying Master equation

$$\sum k_i < K_\alpha \quad g < G_\alpha \quad \omega \cap \beta < E_\alpha$$

$$\left\{\mathscr{M}^{\beta}_{(k_1,\dots,k_l);g}(\alpha)\right\}$$ is gauge equivalent to $$\left\{\mathscr{M}^{\beta}_{(k_1,\dots,k_l);g}(\alpha')\right\}$$

Then we have $$\left\{\mathscr{M}^{\beta}_{(k_1,\dots,k_l);g}\right\}$$ its homotopy limit.

(2) To include weak bounding chain and handle unit.

We need to study curved A infinity algebra.

$$m_1 m_1 \neq 0 \qquad m_0 \neq 0$$

Including weak bounding chain b (FOOO).

$$m_0^b(1) = \sum_{k=0}^{\infty} m_k(b, \ldots, b) = c1$$

1 is unit.

Need the STRICT unitality for this.

$$\mathrm{ev} : \mathcal{M}_{\vec{k},g}(\beta)_{\mathrm{smooth}} \to \prod_{i=1}^{\ell} L^{k_i} = N$$

- must be a submersion (family version) for well-defined-ness.
- must be compatible with forgetful map of the marked points for STRICT unitality.

These two are INCONSISTENT.

This was mentioned in several of my earlier papers and said to be a reason why story works only in genus 0 (and 1).

Solution： is actually quite technical

Distinguish two different kinds of marked points

- marked points where the ev is a submersion. unforgetable marked points
- marked points which are compatible with forgetful map.

  forgetable marked points

Apply bounding chain b only forgetable marked points.
Output and input of the operations are always unforgetable marked points