LAGRANGIAN FLOER THEORY ON COMPACT TORIC MANIFOLDS I

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Contents

1. Introduction

Floer theory of Lagrangian submanifolds plays an important role in symplectic geometry since Floer's invention [Fl] of the Floer homology and subsequent generalization to the class of *monotone* Lagrangian submanifolds [Oh1]. After the introduction of A_∞ structure in Floer theory [Fu1] and Kontsevich's homological

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mirror symmetry proposal [Ko], it has also played an essential role in a formulation of mirror symmetry in string theory.

In [FOOO1], we have analyzed the anomaly $\partial^2 \neq 0$ and developed an obstruction theory for the definition of Floer homology and introduced the class of *unobstructed* Lagrangian submanifolds for which one can deform Floer's original definition of the 'boundary' map by a suitable bounding cochain denoted by b. Expanding the discussion in section 7 [FOOO1] and also motivated by the work of Cho-Oh [CO], we introduced the notion of *weakly unobstructed* Lagrangian submanifolds in Chapter 3 [FOOO2] which turns out to be the right class of Lagrangian submanifolds to look at in relation to the mirror symmetry of Fano toric A-model and Landau-Ginzburg B-model proposed by physicists (see [Ho], [HV]). In this paper, we study the relationship between this class of Lagrangian submanifolds with the earlier work of Givental [Gi1] which advocates that quantum cohomology ring is isomorphic to the Jacobian ring of a certain function, which is called the Landau-Ginzburg superpotential. Combining this study with the results from [FOOO2], we also apply this study to symplectic geometry of Lagrangian fibers of toric manifolds.

Denote by $\mathfrak{Lag}^{weak}(X,\omega)$ the set of weakly unobstructed Lagrangian submanifolds in (X,ω) and by $\mathcal{M}_{\rm weak}(L)$ the moduli space of (weak) bounding cochains for a weakly unobstructed Lagrangian submanifold L. While appearance of bounding cochains is natural in the point of view of deformation theory, explicit computation thereof has not been carried out. One of the main purposes of the present paper is to perform this calculation in the case of fibers of toric manifolds and draw its various applications. Especially we show that each fiber $L(u)$ at $u \in \mathfrak{t}^*$ is weakly unobstructed for *any* toric manifold $\pi : X \to \mathfrak{t}^*$ (see Proposition 3.2), and then show that the set of the pairs $(L(u), b)$ of a fiber $L(u)$ and a bounding cochain b with nontrivial Floer cohomology can be calculated from the quantum cohomology of the ambient toric manifold, at least in the Fano case. Namely the set of such pairs $(L(u), b)$ is identified with the set of ring homomorphisms from quantum cohomology to the relevant Novikov ring. We also show by a variational analysis that for any compact toric manifold there exists at least one pair of (u, b)'s for which the Floer cohomology of $(L(u), b)$ is nontrivial.

Now more precise statement of the main results are in order.

Let X be an n dimensional smooth compact toric manifold. We fix a T^n-equivariant Kähler form on X and let $\pi : X \to \mathfrak{t}^* \cong (\mathbb{R}^n)^*$ be the moment map. The image $P = \pi(X) \subset (\mathbb{R}^n)^*$ is called the *moment polytope*. For $u \in \operatorname{Int} P$, we denote $L(u) = \pi^{-1}(u)$. $L(u)$ is a Lagrangian torus which is an orbit of the T^n action. (See section 2. We refer readers to [Au], [Ful], for example, for the details on toric manifolds.) We study the Floer cohomology defined in [FOOO2]. According to [FOOO1, FOOO2], we need an extra data, the bounding cochain, to make the definition of Floer cohomology more flexible to allow more general class of Lagrangian submanifolds. In the current context of Lagrangian torus fibers in toric manifolds, we use *weak bounding cochains*. In this situation we first show (Proposition 3.2) that each element in $H^1(L(u);\Lambda_0)$ gives rise to a weak bounding cochain, i.e.,

$$H^1(L(u);\Lambda_0) \subset \mathcal{M}_{\rm weak}(L(u)). \tag{1.1}$$

Here we use the universal Novikov ring

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\} \tag{1.2}$$

where T is a formal parameter. (We do not use the grading parameter e used in [FOOO2] since it will not play much role in this paper.) Then Λ_0 is a subring of Λ defined by

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\Big|\, \lambda_i \geq 0 \right\}. \tag{1.3}$$

We also use another subring

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\Big|\, \lambda_i > 0 \right\}. \tag{1.4}$$

We note Λ is the field of fractions of Λ_0 and Λ_0 is a local ring with maximal ideal Λ_+. Here we take the universal Novikov ring over $\mathbb{Q}$ but we also use universal Novikov ring over $\mathbb{C}$ or other subfield F of $\mathbb{C}$ which we denote $\Lambda^{\mathbb{C}}$, Λ^F, respectively.

Remark 1.1. If we strictly follow the way taken in [FOOO2], we only get the inclusion $H^1(L(u);\Lambda_+) \subset \mathcal{M}_{\text{weak}}(L(u))$, not (1.1). However we can modify the definition of weak unobstructedness so that (1.1) follows, using the idea of Cho [Cho]. See section 11.

We next consider the quantum cohomology ring $QH(X;\Lambda)$ with the universal Novikov ring Λ as a coefficient ring. (See section 5.) It is a commutative ring for the toric case, since $QH(X;\Lambda)$ is generated by even degree cohomology classes.

Definition 1.2. (1) We define the set $\operatorname{Spec}(QH(X;\Lambda))(\Lambda^{\mathbb{C}})$ to be the set of Λ algebra homomorphisms $\varphi : QH(X;\Lambda) \to \Lambda^{\mathbb{C}}$. (In other words it is the set of all $\Lambda^{\mathbb{C}}$ valued points of the scheme $\operatorname{Spec}(QH(X;\Lambda))$.

(2) We next denote by $\mathfrak{M}(\mathfrak{Lag}(X))$ the set of all pairs (u,b), $u \in \operatorname{Int} P$, $b \in H^1(L(u);\Lambda_0^{\mathbb{C}})/H^1(L(u);2\pi\sqrt{-1}\mathbb{Z})$ such that

$$HF((L(u),b),(L(u),b);\Lambda^{\mathbb{C}}) \neq \{0\}.$$

Theorem 1.3. *If X is a Fano toric manifold then*

$$\operatorname{Spec}(QH(X;\Lambda))(\Lambda^{\mathbb{C}}) \cong \mathfrak{M}(\mathfrak{Lag}(X)).$$

If $QH(X;\Lambda)$ is semi-simple in addition, then we have

$$\sum_d \operatorname{rank}_{\mathbb{Q}} H_d(X;\mathbb{Q}) = \#\left(\mathfrak{M}(\mathfrak{Lag}(X))\right). \tag{1.5}$$

We remark that a finite dimensional commutative ring over a field (Λ in our case) is semi-simple if and only if it does not contain nilpotent element. We also remark that a compact toric manifold is Fano if and only if every nontrivial holomorphic sphere has positive Chern number.

We believe that (1.5) still holds in non Fano case but are unable to prove it at the time of writing this paper. We however can prove that there exists a fiber $L(u)$ whose Floer cohomology is nontrivial, by a method different from the proof of Theorem 1.3. Due to some technical reason, we can only prove the following slightly weaker statement.

Theorem 1.4. *We assume the Kähler form ω of X is rational. Then, there exists $u \in \operatorname{Int} P$ such that for any N there exists $b \in H^1(L(u);\Lambda_0^{\mathbb{R}})$ with*

$$HF((L(u),b),(L(u),b);\Lambda_0^{\mathbb{R}}/(T^N)) \cong H(T^n;\mathbb{R}) \otimes_{\mathbb{R}} \Lambda_0^{\mathbb{R}}/(T^N).$$

The rationality assumption in Theorem 1.4 is likely to be removed. It is also likely that we can prove $\mathfrak{M}(\mathfrak{Lag}(X))$ is nonempty but its proof is a bit cumbersome at the moment to write down. We can however derive the following theorem from Theorem 1.4, without rationality assumption.

Theorem 1.5. *Let X be an n dimensional compact toric manifold. There exists $u_0 \in \mathrm{Int}P$ such that the following holds for any Hamiltonian diffeomorphism $\psi : X \to X$.*

$$\psi(L(u_0)) \cap L(u_0) \neq \emptyset \tag{1.6}$$

If $\psi(L(u_0))$ is transversal to $L(u_0)$ in addition, then

$$\#(\psi(L(u_0)) \cap L(u_0)) \geq 2^n. \tag{1.7}$$

Theorem 1.5 is proved in section 12.

We would like to point out that (1.6) can be derived from a more general intersection result by Entov-Polterovich, Theorem 2.1 [EP1], by a different method using a very interesting notion of symplectic quasi-state constructed out of the spectral invariants constructed in [Sc], [Oh3]. (See also [Vi], [Oh2] for similar constructions in the exact Lagrangian context.)

Remark 1.6. Precisely speaking, Theorem 2.1 of [EP1] is stated under the assumption that X is semi-positive and ω is rational because the theory of spectral invariant was developed in [Oh3] under these conditions. The rationality assumption has been removed in [Oh4] and the semi-positivity assumption of ω removed by Usher [Us]. Therefore the proof of Theorem 2.1 [EP1] goes through without these assumptions and hence it implies (1.6). (See the introduction of [Us].) But the result (1.7) is new.

Our proof of Theorem 1.5 gives an explicit way of locating u_0, as we show in section 8. (The method of [EP1] is indirect and does not provide the way to find such u_0. See [EP2]. Below, we will make some remarks concerning Entov-Polterovich's approach in the perspective of homological mirror symmetry.) In various explicit examples we can find more than one element u_0 that have the properties stated in this theorem. Following terminology employed in [CO], we call any such torus fiber $L(u_0)$ as in Theorem 1.5 a *balanced* Lagrangian torus fiber. (Definition 3.10.)

A criterion for $L(u_0)$ to be balanced, for the case $b = 0$, is provided by Cho-Oh [CO] and Cho [Cho] under the Fano condition. Our proofs of Theorems 1.4, 1.5 is much based on this criterion, and on the idea of Cho [Cho] of twisting *non-unitary* complex line bundles in the construction of Floer boundary operator. This criterion in turn specializes to the one predicted by physicists [HV], [Ho], which relates the location of u_0 to the critical points of the Landau-Ginzburg superpotential.

In [FOOO2], the authors have introduced a potential function

$$\mathfrak{PO}^L : \mathcal{M}_{\mathrm{weak}}(L) \to \Lambda_0$$

for an arbitrary weakly unobstructed Lagrangian submanifold $L \subset (X, \omega)$. By varying the function $\mathfrak{PO}^L$ over $L \in \mathfrak{Lag}^{weak}(X, \omega)$, we obtain the potential function

$$\mathfrak{PO} : \bigcup_{L \in \mathfrak{Lag}^{weak}(X,\omega)} \mathcal{M}_{\mathrm{weak}}(L) \to \Lambda_0. \tag{1.8}$$

This function is constructed purely in terms of A-model data of the general symplectic manifold (X, ω) *without* using mirror symmetry.

For a toric (X,ω), the restriction of $\mathfrak{PO}$ to $H^1(L(u);\Lambda_0)$ (see (1.1) can be made explicit when combined with the analysis of holomorphic discs attached to torus fibers of toric manifolds carried out in [CO], at least in the Fano case. (In the non Fano case we can make it explicit modulo 'higher order terms'.)

Remark 1.7. In [EP3] some relationships between quantum cohomology, quasi-state, spectral invariant and displacement of Lagrangian submanifolds are discussed : Consider an idempotent $\mathbf{i}$ of quantum cohomology. The (asymptotic) spectral invariants associated to $\mathbf{i}$ gives rise to a quasi-state via the procedure concocted in [EP3], which in turn detects undisplaceability of certain Lagrangian submanifolds. (The assumption of [EP1] is weaker than ours.)

In the current context of toric manifolds, we could also relate them to Floer cohomology and mirror symmetry in the following way : Quantum cohomology is decomposed to indecomposable factors. (See Proposition 6.6.) Let $\mathbf{i}$ be the idempotent corresponding to one of the indecomposable factors.

Let $L = L(u(1,\mathbf{i}))$ be a Lagrangian torus fiber whose undisplaceability is detected by the quasi-state obtained from $\mathbf{i}$. We conjecture that Floer cohomology $HF(L(u(1,\mathbf{i}),b),(Lu(1,\mathbf{i}),b))$ is nontrivial for some b. (Conjecture 4.8.) This bounding cochain b in turn is shown to be a critical point of the potential function $\mathfrak{PO}$ defined in [FOOO2].

On the other hand, $\mathbf{i}$ also determines a homomorphism $\varphi_{\mathbf{i}} : QH(X;\Lambda) \to \Lambda$. It corresponds to some Lagrangian fiber $L(u(2,\mathbf{i}))$ by Theorem 1.3. Then this will imply via Theorem 3.9 that the fiber $L(u(2,\mathbf{i}))$ is undisplaceable.

We conjecture that $u(1,\mathbf{i}) = u(2,\mathbf{i})$. We remark that $u(2,\mathbf{i})$ is explicitly calculable. Hence in view of the way $u(1,\mathbf{i})$ is found in [EP1], $u(1,\mathbf{i}) = u(2,\mathbf{i})$ will give some information on the asymptotic behavior of the spectral invariant associated with $\mathbf{i}$.

We fix a basis the Lie algebra $\mathfrak{t}$ of T^n which induces a basis on $\mathfrak{t}^*$ and hence a coordinate of $P \subset \mathfrak{t}^*$. This in turn induces a basis $H^1(L(u);\Lambda_0)$ for each $u \in \operatorname{Int} P$ and so identification $H^1(L(u);\Lambda_0) \cong (\Lambda_0)^n$. We then regard the potential function as a function

$$\mathfrak{PO}(x_1,\cdots,x_n;u_1,\cdots,u_n) : (\Lambda_0)^n \times \operatorname{Int} P \to \Lambda_0$$

and prove in Theorem 3.9 that Floer homology $HF((L(u),x),(L(u),x);\Lambda)$ with $x = (x_1,\cdots,x_n)$, $u = (u_1,\cdots,u_n)$ is nontrivial if and only if (x,u) satisfies

$$\frac{\partial \mathfrak{PO}}{\partial x_i}(x;u) = 0, \quad i = 1,\cdots,n. \tag{1.9}$$

To study (1.9), it is useful to change the variables x_i to

$$y_i = e^{x_i}.$$

In these variables we can write potential function as a sum

$$\mathfrak{PO}(x_1,\cdots,x_n;u_1,\cdots,u_n) = \sum T^{c_i(u)} P_i(y_1,\cdots,y_n)$$

where P_i are Laurent polynomial which do not depend on u, and $c_i(u)$ are positive real valued function.

We assume X is Fano, until the end of the statement of Theorem 1.9. We can calculate the right hand side and write it as a finite sum. (See Theorem 3.4.)

We define a function $\mathfrak{PO}^u$ of y_i's by

$$\mathfrak{PO}^u(y_1, \cdots, y_n) = \mathfrak{PO}(x_1, \cdots, x_n; u_1, \cdots, u_n)$$

as a Laurent polynomial of n variables with coefficient in Λ. We denote the set of Laurent polynomials by

$$\Lambda[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}]$$

and consider its ideal generated by the partial derivatives of $\mathfrak{PO}^u$. Namely

$$\left(\frac{\partial \mathfrak{PO}^u}{\partial y_i}; i = 1, \cdots, n \right).$$

Definition 1.8. We call the quotient ring

$$Jac(\mathfrak{PO}^u) = \frac{\Lambda[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}]}{\left(\frac{\partial \mathfrak{PO}}{\partial y_i}; i = 1, \cdots, n \right)}$$

the *Jacobian ring* of $\mathfrak{PO}^u$.

We will prove that the Jacobian ring is independent of the choice of u up to isomorphism (see the end of section 5).

Theorem 1.9. *If X is Fano, then there exists a Λ algebra isomorphism*

$$\psi_u : QH(X; \Lambda) \to Jac(\mathfrak{PO})$$

from quantum cohomology ring to the Jacobian ring such that

$$\psi_u(c_1(X)) = \mathfrak{PO}^u.$$

Theorem 1.9 (or Theorem 1.12 below) enables us to explicitly determine all the pairs (u, b) with $HF((L(u), b), (L(u), b); \Lambda) \neq 0$ out of the quantum cohomology of X. More specifically Batyrev's presentation of quantum cohomology in terms of the Jacobian ring plays an essential role for this purpose. We will explain how this is done in sections 6, 7 .

Remark 1.10. (1) The idea that quantum cohomology ring coincides with the Jacobian ring begins with a celebrated paper by Givental [Gi1] Theorem 5 (1). There it was claimed also that the D module defined by an oscillatory integral with the superpotential as its kernel is isomorphic to S^1-equivariant Floer cohomology of periodic Hamiltonian system. When one takes its WKB limit, the former becomes the ring of functions on its characteristic variety, which is nothing but the Jacobian ring. The latter becomes the (small) quantum cohomology ring under the same limit. *Assuming* the Ansatz that quantum cohomology can be calculated by fixed point localization, these claims are proved in a subsequent paper [Gi2] for, at least, toric Fano manifolds. Then the required fixed point localization is made rigorous later in [GrPa]. See also Iritani [Iri1].

In physics literature, it has been advocated that Landau-Ginzburg model of superpotential (that is, the potential function $\mathfrak{PO}$ in our situation) calculates quantum cohomology of X. A precise mathematical statement thereof is our Theorem 1.9. (See for example p. 473 [MIRROR].)

Our main new idea entering in the proof of Theorem 1.3 other than those already in [FOOO2] is the way how we combine them to extract information on Lagrangian submanifolds. In fact Theorem 1.9 itself easily follows if we use the claim made by Batyrev that quantum cohomology of toric Fano manifold is a quotient of polynomial ring by relations, called quantum Stanley-Reisner relation and linear relation. (This claim is now well established.) We include this simple derivation in section 5 for completeness' sake, since it is essential to take the Novikov ring Λ as the coefficient ring in our applications the version of which does not seem to be proven in the literature in the form that can be easily quoted.

(2) The proof of Theorem 1.9 given in this paper does not contain serious study of pseudo-holomorphic spheres. The argument which we outline in Remark 5.13 is based on open-closed Gromov-Witten theory, and different from various methods that have been used to calculate Gromov-Witten invariant in the literature. In particular this argument does not use the method of fixed point localization. We will present this conceptual proof of Theorem 1.9 in a sequel to this paper.

(3) In this paper, we only involve small quantum cohomology ring but we can also include big quantum cohomology ring. Then we expect Theorem 1.9 can be enhanced to establish a relationship between Frobenius structures of the deformation theory of quantum cohomology (see, for example, [Ma]) and that of Landau-Ginzburg model (which is due to K. Saito [Sa]). This statement (and Theorems 1.3, 1.9) can be regarded as a version of mirror symmetry between the toric A-model and the Landau-Ginzburg B-model. In various literature on mirror symmetry, such as [Ab], [AKO], [Ue], the B-model is dealt for Fano or toric manifolds in which the derived category of coherent sheaves is studied while the A-model is dealt for Landau-Ginzburg A-models where the directed A_∞ category of Seidel [Se2] is studied.

(4) Even when X is not necessarily Fano we can still prove a similar isomorphism

$$\psi_u : QH^\omega(X;\Lambda) \cong Jac(\mathfrak{PO}_0) \tag{1.10}$$

where the left hand side is the Batyrev quantum cohomology ring (see Definition 5.3) and the right hand side is the Jacobian ring of some function $\mathfrak{PO}_0$: it coincides with the actual potential function $\mathfrak{PO}$ 'up to higher order terms'. (See (3.8).) In the Fano case $\mathfrak{PO}_0 = \mathfrak{PO}$. (1.10) is Proposition 5.7.

(5) During the final stage of writing this article, a paper [CL] by Chan and Leung was posted in the Archive which studies the above isomorphism via SYZ transformations. They give a proof of this isomorphism for the case where X is a product of projective spaces and with the coefficient ring $\mathbb{C}$, not with Novikov ring. Leung presented their result [CL] in a conference held in Kyoto University in January 2008 where the first named author also presented the content of this paper.

From our definition, it follows that the potential functions $\mathfrak{PO}$ and $\mathfrak{PO}_0$ can be extended to the whole product $(\mathbb{R}^n)^* \times (\Lambda_0^{\mathbb{C}})^n$ in a way that they are invariant under the translations by elements in $(2\pi\sqrt{-1}\mathbb{Z})^n$. Hence we may regard them as functions defined on

$$(\mathbb{R}^n)^* \times \left(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\right)^n \cong \mathbb{R}^n \times \left(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\right)^n$$

Definition 1.11. We denote by

$$\mathfrak{M}_{+,0}(\mathfrak{Lag}(X)), \quad \text{(respectively } \mathfrak{M}_{+}(\mathfrak{Lag}(X)))$$

the subset of pairs $(u,x) \in \mathbb{R}^n \times (\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z}))^n$ satisfying the equation

$$\frac{\partial \mathfrak{PO}_0}{\partial x_i}(x;u) = 0, \qquad \left(\text{respectively } \frac{\partial \mathfrak{PO}}{\partial x_i}(x;u) = 0\right)$$

$i = 1, \cdots, n$ at x.

Note an element $(u,x) \in \mathfrak{M}_+(\mathfrak{Lag}(X))$ gives rise to an element of $\mathfrak{M}(\mathfrak{Lag}(X))$ if $u \in \operatorname{Int} P$.

We also remark $\mathfrak{PO}_0 = \mathfrak{PO}$ in case X is Fano. The following is a more precise form of Theorem 1.3.

Theorem 1.12. (1) *There exists a bijection :*

$$\operatorname{Spec}(QH^\omega(X;\Lambda))(\Lambda^{\mathbb{C}}) \cong \mathfrak{M}_{+,0}(\mathfrak{Lag}(X)).$$

(2) *If X is Fano then we have*

$$\mathfrak{M}(\mathfrak{Lag}(X)) = \mathfrak{M}_+(\mathfrak{Lag}(X)) = \mathfrak{M}_{+,0}(\mathfrak{Lag}(X)).$$

(3) *If $QH^\omega(X;\Lambda)$ is semi-simple, then*

$$\sum_d \operatorname{rank}_\Lambda QH^\omega(X;\Lambda) = \#\left(\mathfrak{M}_{+,0}(\mathfrak{Lag}(X))\right).$$

In section 7, we illustrate by an example that the first equality of (2) does not hold in the non-Fano case.

We would like to point out that $\mathfrak{PO}_0$ is explicitly computable. But we do not know the explicit form of $\mathfrak{PO}$. However we can show that elements of $\mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$ and of $\mathfrak{M}_+(\mathfrak{Lag}(X))$ can be naturally related to each other under a mild nondegeneracy condition. (Theorem 9.4.) So we can use $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$ in most of the cases. For example we can use it to prove that the following :

Theorem 1.13. *For any k, there exists a Kähler form on $X(k)$, the k point blow up of $\mathbb{C}P^2$, that is toric and has exactly $k+1$ balanced fibers.*

See Definition 3.10 for the definition of balanced fibers. Balanced fiber satisfies the conclusions (1.6), (1.7) of Theorem 1.5. We prove Theorem 1.13 in section 9.

Remark 1.14. The cardinality of $b \in H^1(L(u);\Lambda_0^{\mathbb{C}})/H^1(L(u);2\pi\sqrt{-1}\mathbb{Z})$ with non-vanishing Floer cohomology is an invariant of Lagrangian submanifold $L(u)$. This is a consequence of [FOOO2] Theorem G.

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2. Compact toric manifolds

In this section, we summarize basic facts on the toric manifolds and set-up our notations to be consistent with those in [CO], which in turn closely follow those in Batyrev [B1] and M. Audin [Au].

2.1. Complex structure.

In order to obtain an n-dimensional compact toric manifold X, we need a combinatorial object Σ, a *complete fan of regular cones*, in an n-dimensional vector space over $\mathbb{R}$.

Let N be the lattice $\mathbb{Z}^n$, and let $M = Hom_{\mathbb{Z}}(N,\mathbb{Z})$ be the dual lattices of rank n. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Definition 2.1. A convex subset $\sigma \subset N_{\mathbb{R}}$ is called a regular k-dimensional cone ($k \geq 1$) if there exists k linearly independent elements $v_1, \cdots, v_k \in N$ such that

$$\sigma = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \in \mathbb{R}, a_i \geq 0\},$$

and the set $\{v_1, \cdots, v_k\}$ is a subset of some $\mathbb{Z}$-basis of N. In this case, we call $v_1, \cdots, v_k \in N$ the integral generators of σ.

Definition 2.2. A regular cone σ' is called a *face* of a regular cone σ (we write $\sigma' \prec \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

Definition 2.3. A finite system $\Sigma = \sigma_1, \cdots, \sigma_s$ of regular cones in $N_{\mathbb{R}}$ is called a *complete n-dimensional fan* of regular cones, if the following conditions are satisfied.

(1) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$;
(2) if σ, σ' are in Σ, then $\sigma' \cap \sigma \prec \sigma$ and $\sigma' \cap \sigma \prec \sigma'$;
(3) $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.

The set of all k-dimensional cones in Σ will be denoted by $\Sigma^{(k)}$.

Definition 2.4. Let Σ be a complete n-dimensional fan of regular cones. Denote by $G(\Sigma) = \{v_1, \cdots, v_m\}$ the set of all generators of 1-dimensional cones in Σ ($m = \text{Card } \Sigma^{(1)}$). We call a subset $\mathcal{P} = \{v_{i_1}, \cdots, v_{i_p}\} \subset G(\Sigma)$ a *primitive collection* if $\{v_{i_1}, \cdots, v_{i_p}\}$ does not generate p-dimensional cone in Σ, while for all k ($0 \leq k < p$) each k-element subset of $\mathcal{P}$ generates a k-dimensional cone in Σ.

Definition 2.5. Let $\mathbb{C}^m$ be an m-dimensional affine space over $\mathbb{C}$ with the set of coordinates $z_1, \cdots, z_m$ which are in the one-to-one correspondence $z_i \leftrightarrow v_i$ with elements of $G(\Sigma)$. Let $\mathcal{P} = \{v_{i_1}, \cdots, v_{i_p}\}$ be a primitive collection in $G(\Sigma)$. Denote by $\mathbb{A}(\mathcal{P})$ the $(m-p)$-dimensional affine subspace in $\mathbb{C}^n$ defined by the equations

$$z_{i_1} = \cdots = z_{i_p} = 0.$$

Since every primitive collection $\mathcal{P}$ has at least two elements, the codimension of $\mathbb{A}(\mathcal{P})$ is at least 2.

Definition 2.6. Define the closed algebraic subset $Z(\Sigma)$ in $\mathbb{C}^m$ as follows

$$Z(\Sigma) = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}),$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$. Put

$$U(\Sigma) = \mathbb{C}^m \setminus Z(\Sigma).$$

Definition 2.7. Let $\mathbb{K}$ be the subgroup in $\mathbb{Z}^m$ consisting of all lattice vectors $\lambda = (\lambda_1, \cdots, \lambda_m)$ such that

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0.$$

Obviously $\mathbb{K}$ is isomorphic to $\mathbb{Z}^{m-n}$ and we have the exact sequence:

$$0 \to \mathbb{K} \to \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \to 0, \tag{2.1}$$

where the map π sends the basis vectors e_i to v_i for $i = 1, \cdots, m$.

Definition 2.8. Let Σ be a complete n-dimensional fan of regular cones. Define $D(\Sigma)$ to be the connected commutative subgroup in $(\mathbb{C}^*)^m$ generated by all one-parameter subgroups

$$\begin{aligned} a_\lambda : \mathbb{C}^* &\to (\mathbb{C}^*)^m, \\ t &\mapsto (t^{\lambda_1}, \cdots, t^{\lambda_m}) \end{aligned}$$

where $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{K}$.

It is easy to see from the definition that $D(\Sigma)$ acts freely on $U(\Sigma)$. Now we are ready to give a definition of the compact toric manifold X_Σ associated with a complete n-dimensional fan of regular cones Σ.

Definition 2.9. Let Σ be a complete n-dimensional fan of regular cones. Then the quotient

$$X_\Sigma = U(\Sigma)/D(\Sigma)$$

is called the *compact toric manifold associated with* Σ.

There exists a simple open coverings of $U(\Sigma)$ by affine algebraic varieties.

Proposition 2.10. *Let σ be a k-dimensional cone in Σ generated by $\{v_{i_1}, \cdots, v_{i_k}\}$. Define the open subset $U(\sigma) \subset \mathbb{C}^m$ as*

$$U(\sigma) = \{(z_1, \cdots, z_m) \in \mathbb{C}^m \mid z_j \neq 0 \ \text{ for all } j \notin \{i_1, \cdots, i_k\}\}.$$

Then the open sets $U(\sigma)$ have the following properties:

(1)
$$U(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma);$$

(2) *if $\sigma \prec \sigma'$, then $U(\sigma) \subset U(\sigma')$;*

(3) *for any two cone $\sigma_1, \sigma_2 \in \Sigma$, one has $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$; in particular,*
$$U(\Sigma) = \sum_{\sigma \in \Sigma^{(n)}} U(\sigma).$$

Proposition 2.11. *Let σ be an n-dimensional cone in $\Sigma^{(n)}$ generated by $\{v_{i_1}, \cdots, v_{i_n}\}$, which spans the lattice M. We denote the dual $\mathbb{Z}$-basis of the lattice M by $\{u_{i_1}, \cdots, u_{i_n}\}$. i.e.*

$$\langle v_{i_k}, u_{i_l} \rangle = \delta_{k,l} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between lattices N and M.

Then the affine open subset $U(\sigma)$ is isomorphic to $\mathbb{C}^n \times (\mathbb{C}^*)^{m-n}$, the action of $D(\Sigma)$ on $U(\sigma)$ is free, and the space of $D(\Sigma)$-orbits is isomorphic to the affine space $U_\sigma = \mathbb{C}^n$ whose coordinate functions $y_1^\sigma, \cdots, y_n^\sigma$ are n Laurent monomials in $z_1, \cdots, z_m$:

$$\begin{cases} y_1^\sigma = z_1^{\langle v_1, u_{i_1} \rangle} \cdots z_m^{\langle v_m, u_{i_1} \rangle} \\ \quad \vdots \\ y_n^\sigma = z_1^{\langle v_1, u_{i_n} \rangle} \cdots z_m^{\langle v_m, u_{i_n} \rangle} \end{cases} \tag{2.3}$$

The last statement yields a general formula for the local affine coordinates $y_1^\sigma, \cdots, y_n^\sigma$ of a point $p \in U_\sigma$ as functions of its "homogeneous coordinates" $z_1, \cdots, z_m$.

2.2. **Symplectic structure.** In the last subsection, we associated a compact manifold X_Σ to a fan Σ. In this subsection, we review the construction of symplectic (Kähler) manifold associated to a convex polytope P.

Let M be a dual lattice, we consider a convex polytope P in $M_{\mathbb{R}}$ defined by

$$\{u \in M_{\mathbb{R}} \mid \langle u, v_j \rangle \geq \lambda_j \text{ for } j = 1, \cdots, m\} \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is a dot product of $M_{\mathbb{R}} \cong \mathbb{R}^n$. Namely, v_j's are inward normal vectors to the codimension 1 faces of the polytope P. We associate to it a fan in the lattice N as follows: With any face Γ of P, fix a point u_0 in the (relative) interior of Γ and define

$$\sigma_\Gamma = \bigcup_{r \geq 0} r \cdot (P - u_0).$$

The associated fan is the family $\Sigma(P)$ of dual convex cones

$$\begin{aligned} \check{\sigma}_\Gamma &= \{x \in N_{\mathbb{R}} \mid \langle y, x \rangle \geq 0 \ \ \forall y \in \sigma_\Gamma\} && (2.5) \\ &= \{x \in N_{\mathbb{R}} \mid \langle u, x \rangle \leq \langle p, x \rangle \ \ \forall p \in P, u \in \Gamma\} && (2.6) \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is dual pairing $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. Hence we obtain a compact toric manifold $X_{\Sigma(P)}$ associated to a fan $\Sigma(P)$.

Now we define a symplectic (Kähler) form on $X_{\Sigma(P)}$ as follows. Recall the exact sequence :

$$0 \to \mathbb{K} \xrightarrow{i} \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \to 0.$$

It induces another exact sequence :

$$0 \to K \to \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^n/\mathbb{Z}^n \to 0.$$

Denote by k the Lie algebra of the real torus K. Then we have the exact sequence of Lie algebras:

$$0 \to k \to \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n \to 0.$$

And we have the dual of above exact sequence:

$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^m)^* \xrightarrow{i^*} k^* \to 0.$$

Now, consider $\mathbb{C}^m$ with symplectic form $\frac{i}{2}\sum dz_k \wedge d\overline{z}_k$. The standard action T^n on $\mathbb{C}^n$ is hamiltonian with moment map

$$\mu(z_1, \cdots, z_m) = \frac{1}{2}(|z_1|^2, \cdots, |z_m|^2). \tag{2.7}$$

For the moment map μ_K of the K action is then given by

$$\mu_K = i^* \circ \mu : \mathbb{C}^m \to k^*.$$

If we choose a $\mathbb{Z}$-basis of $\mathbb{K} \subset \mathbb{Z}^m$ as

$$Q_1 = (Q_{11}, \cdots, Q_{m1}), \cdots, Q_k = (Q_{1k}, \cdots, Q_{mk})$$

and $\{q^1, \cdots, q^k\}$ be its dual basis of $\mathbb{K}^*$. Then the map i^* is given by the matrix Q^t and so we have

$$\mu_K(z_1, \cdots, z_m) = \frac{1}{2}\left(\sum_{j=1}^m Q_{j1}|z_j|^2, \cdots, \sum_{j=1}^m Q_{jk}|z_j|^2\right) \in \mathbb{R}^k \cong k^* \tag{2.8}$$

in the coordinates associated to the basis $\{q^1, \cdots, q^k\}$. We denote again by μ_K the restriction of μ_K on $U(\Sigma) \subset \mathbb{C}^m$.

Proposition 2.12 (See Audin [Au]). *Then for any* $r = (r_1, \cdots, r_{m-n}) \in \mu_K(U(\Sigma)) \subset k^*$, *we have a diffeomorphism*

$$\mu_K^{-1}(r)/K \cong U(\Sigma)/D(\Sigma) = X_\Sigma \tag{2.9}$$

And for each (regular) value of $r \in k^*$, *we can associate a symplectic form* ω_P *on the manifold* X_Σ *by symplectic reduction* [MW].

To obtain the original polytope P that we started with, we need to choose r as follows: Consider λ_j for $j = 1, \cdots, m$ which we used to define our polytope P by the set of inequalities $\langle u, v_j \rangle \geq \lambda_j$. Then, for each $a = 1, \cdots, m-n$, let

$$r_a = -\sum_{j=1}^m Q_{ja}\lambda_j.$$

Then we have

$$\mu_K^{-1}(r_1, \cdots, r_{m-n})/K \cong X_{\Sigma(P)}$$

and for the residual $T^n \cong T^m/K$ action on $X_{\Sigma(P)}$, and for its moment map π, we have

$$\pi(X_{\Sigma(P)}) = P.$$

Using Delzant's theorem [De], one can reconstruct the symplectic form out of the polytope P (up to T^n-equivariant symplectic diffeomorphisms). In fact, Guillemin [Gu] proved the following explicit closed formula for the T^n-invariant Kähler form associated to the canonical complex structure on $X = X_\Sigma(P)$

Theorem 2.13 (Guillemin). *Let* P, $X_{\Sigma(P)}$, ω_P *be as above and*

$$\pi : X_{\Sigma(P)} \to (\mathbb{R}^m/k)^* \cong (\mathbb{R}^n)^*$$

be the associated moment map. Define the functions on $(\mathbb{R}^n)^*$

$$\begin{aligned} \ell_i(u) &= \langle u, v_i \rangle - \lambda_i \;\; \text{for} \;\; i = 1, \cdots, m \\ \ell_\infty(u) &= \sum_{i=1}^m \langle u, v_i \rangle = \left\langle u, \sum_{i=1}^m v_i \right\rangle. \end{aligned} \tag{2.10}$$

Then we have

$$\omega_P = \sqrt{-1}\partial\overline{\partial}\left(\pi^*\Big(\sum_{i=1}^m \lambda_i(\log \ell_i) + \ell_\infty\Big)\right) \tag{2.11}$$

on $\operatorname{Int} P$.

The affine functions ℓ_i will play an important role in our description of potential function as in [CO] since they also measure symplectic areas $\omega(\beta_i)$ of the canonical generators β_i of $H_2(X, L(u); \mathbb{Z})$. More precisely we have

$$\omega(\beta_i) = 2\pi\ell_i(u) \tag{2.12}$$

(see Theorem 8.1 [CO]). We also recall

$$P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \geq 0, \, i = 1, \cdots, m\} \tag{2.13}$$

by definition (2.4).

3. Potential function

In [FOOO2], we introduced the notion of weak bounding cochains of a filtered A_∞ algebra in general. The A_∞ structure canonically induces one on a canonical model of the given A_∞ algebra. In the geometric context of A_∞ algebra associated to a Lagrangian submanifold $L \subset M$ of a general symplectic manifold (M, ω), the structure is defined on the cohomology group $H^*(L; \Lambda_0)$.

An element $b \in H^1(L; \Lambda_+)$ is called a *weak bounding cochain* if it satisfies the A_∞ Mauer-Cartan equation (or master equation)

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \cdots, b) \equiv 0 \mod PD([L]) \tag{3.1}$$

where $\{\mathfrak{m}_k\}_{k=0}^\infty$ is the A_∞ operators associated to L. We denote by $\widehat{\mathcal{M}}_{\text{weak}}(L)$ the set of weak bounding cochains of L. We say L is *weakly unobstructed* if $\widehat{\mathcal{M}}_{\text{weak}}(L) \neq \emptyset$. The moduli space $\mathcal{M}_{\text{weak}}(L)$ is then defined to be the quotient space of $\widehat{\mathcal{M}}_{\text{weak}}(L)$ of suitable homotopy equivalence. (See chapter 3 and 4 [FOOO2] for more explanations.)

The main point of introducing (weak) bounding cochains is the following

Lemma 3.1 (Lemma 11.12 [FOOO2]). *If $b \in \widehat{\mathcal{M}}_{weak}(L)$ then $\delta_{b,b} \circ \delta_{b,b} = 0$, where $\delta_{b,b}$ is the deformed Floer operator defined by*

$$\delta_{b,b}(x) = \mathfrak{m}_1^b(x) =: \sum_{k,\ell} \mathfrak{m}_{k+\ell+1}(b^{\otimes k}, x, b^{\otimes \ell}).$$

For $b \in \widehat{\mathcal{M}}_{\text{weak}}(L)$, we define

$$HF(L; b) = \frac{\operatorname{Ker}(\delta_{b,b} : C \to C)}{\operatorname{Im}(\delta_{b,b} : C \to C)},$$

where C is an appropriate subcomplex of the singular chain complex of L. When L is weakly unobstructed i.e., $\widehat{\mathcal{M}}_{\text{weak}}(C) \neq \emptyset$, we define a function

$$\mathfrak{PO} : \widehat{\mathcal{M}}_{\text{weak}}(C) \to \Lambda_+$$

by the equation

$$\mathfrak{m}(e^b) = \mathfrak{PO}(b) \cdot PD([L]).$$

This is the *potential function* introduced in [FOOO2].

For the later analysis of examples, we recall from [FOOO1, FOOO2] that $\mathfrak{m}_k$ is further decomposed into

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)} e^{\mu(\beta)/2}.$$

Firstly we incorporate the grading parameter e into the ground ring and do not write it explicitly. Secondly to eliminate many appearance of 2π in front of the affine function ℓ_i in the exponents of the parameter T later in this paper, we redefine T as $T^{2\pi}$. Under this arrangement, we get the formal power series expansion

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)/2\pi} \tag{3.2}$$

which we will use throughout the paper.

Now we restrict to the case of toric manifold. Let $X = X_\Sigma$ be associated a complete regular fan Σ, and $\pi : X \to \mathfrak{t}^*$ be the moment map of the action of the torus $T^n \cong T^m/K$. We make the identifications

$$\mathfrak{t} = Lie(T^n) \cong N^n_{\mathbb{R}} \cong \mathbb{R}^n,\ \mathfrak{t}^* \cong M_{\mathbb{R}} \cong (\mathbb{R}^n)^*.$$

We will exclusively use $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ to be consistent with the standard notations in toric geometry instead of $\mathfrak{t}$ (or $\mathbb{R}^n$) and $\mathfrak{t}^*$ (or $(\mathbb{R}^n)^*$) as much as possible.

Denote the image of $\pi : X \to M_{\mathbb{R}}$ by $P \subset M_{\mathbb{R}}$ which is the moment polytope of the T^n action on X.

We will prove the following in section 10.

Proposition 3.2. *For any $u \in \mathrm{Int}P$, the fiber $L(u)$ is weakly unobstructed. Moreover we have the canonical inclusion*

$$H^1(L(u); \Lambda_+) \subset \mathcal{M}_{\text{weak}}(L(u)).$$

Choose an integral basis $\mathbf{e}_i^*$ of N and $\mathbf{e}_i$ be its dual basis on M. With this choice made, we identify $M_{\mathbb{R}}$ with $\mathbb{R}^n$ as long as its meaning is obvious from the context. Identifying $H_1(T^n; \mathbb{Z})$ with $N \cong \mathbb{Z}^n$ via $T^n = \mathbb{R}^n/\mathbb{Z}^n$, we regard $\mathbf{e}_i$ as a basis of $H^1(L(u); \mathbb{Z})$. The following immediately follows from definition.

Lemma 3.3. *We write $\pi = (\pi_1, \cdots, \pi_n) : X \to M_{\mathbb{R}}$ using the coordinate of $M_{\mathbb{R}}$ associated to the basis $\mathbf{e}_i$. Let $S^1_i \subset T^n$ be the subgroup whose orbit represents $\mathbf{e}_i^* \in H_1(T^n; \mathbb{Z})$. Then π_i is proportional to the moment map of S^1_i action on X.*

Let

$$b = \sum x_i \mathbf{e}_i \in H^1(L(u); \Lambda_+) \subset \mathcal{M}_{\text{weak}}(L(u)).$$

We study the potential function

$$\mathfrak{PO} : H^1(L(u); \Lambda_+) \to \Lambda_+.$$

Once a choice of the family of bases $\{\mathbf{e}_i\}$ on $H^1(L(u); \mathbb{Z})$ for $u \in \mathrm{Int}\, P$ is made as above starting from a basis on N , then we can regard this function as a function of $(x_1, \cdots, x_n) \in (\Lambda_+)^n$ and $(u_1, \cdots, u_n) \in P \subset M_{\mathbb{R}}$. We denote its value by $\mathfrak{PO}(x; u) = \mathfrak{PO}(x_1, \cdots, x_n; u_1, \cdots, u_n)$. We put

$$y_i = e^{x_i} = \sum_{k=0}^{\infty} \frac{x_i^k}{k!} \in \Lambda_0.$$

Let

$$\partial P = \bigcup_{i=1}^{m} \partial_i P$$

be the decomposition of the boundary of the moment polytope into its faces of codimension one. ($\partial_i P$ is a polygon in an $n-1$ dimensional affine subspace of $M_{\mathbb{R}}$.)

Let ℓ_i be the affine functions

$$\ell_i(u) = \langle u, v_i \rangle - \lambda_i \ \text{ for } \ i = 1, \cdots, m$$

appearing in Theorem 2.13. Then the followings hold from construction :

(1) $\ell_i \equiv 0$ on $\partial_i P$.
(2) $P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \geq 0, i = 1, \cdots, m\}$.

(3) The coordinates of the vectors $v_i = (v_{i,1}, \cdots, v_{i,n})$ satisfy

$$v_{i,j} = \frac{\partial \ell_i}{\partial u_j} \tag{3.3}$$

and are all integers.

Theorem 3.4. *Let $L(u) \subset X$ be as in Theorem 1.5 and ℓ_i be as above. Suppose X is Fano. Then we can take the canonical model of A_∞ structure of $L(u)$ over $u \in \operatorname{Int} P$ so that the potential function restricted to*

$$\bigcup_{u \in \operatorname{Int} P} H^1(L(u); \Lambda_+) \cong (\Lambda_+)^n \times \operatorname{Int} P$$

has the form

$$\mathfrak{PO}(x; u) = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{3.4}$$

$$= \sum_{i=1}^{m} e^{\langle v, x\rangle} T^{\ell_i(u)} \tag{3.5}$$

where $(x; u) = (x_1, \cdots, x_n; u_1, \cdots, u_n)$ and $v_{i,j}$ is as in (3.3).

Theorem 3.4 is a minor improvement of a result from [CO] (see (15.1) of [CO]) and [Cho] : The case considered in [CO] corresponds to the case where $y_i \in U(1) \subset \{z \in \mathbb{C} \mid |z| = 1\}$ and the case in [Cho] corresponds to the one where $y_i \in \mathbb{C} \setminus \{0\}$. The difference of Theorem 3.4 from the ones thereof is that y_i is allowed to contain T, the formal parameter of the universal Novikov ring encoding the energy.

For the non-Fano case, we prove the following slightly weaker statement. The proof will be given in section 10.

Theorem 3.5. *Let X be an arbitrary toric manifold and $L(u)$ be as above. Then there exist $c_j \in \mathbb{Q}$, $e_j^i \in \mathbb{Z}_{\geq 0}$ and $\rho_j > 0$, such that $\sum_{i=1}^m e_j^i > 0$ and*

$$\begin{aligned}\mathfrak{PO}(x_1, \cdots, x_n; u_1, \cdots, u_n) &- \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \\ &= \sum_{j=1} c_j y_1^{v'_{j,1}} \cdots y_n^{v'_{j,n}} T^{\ell'_j(u)+\rho_j}.\end{aligned} \tag{3.6}$$

where

$$v'_{j,k} = \sum_{i=1}^{m} e_j^i v_{i,k}, \quad \ell'_j = \sum_{i=1}^{m} e_j^i \ell_i.$$

If there are infinitely many non-zero c_j's, we have

$$\lim_{j \to \infty} \ell'_j(u) + \rho_j = \infty.$$

Moreover $\rho_j = [\omega] \cap \alpha_j$ for some $\alpha_j \in \pi_2(X)$ with nonpositive Chern number.

We note that although $\mathfrak{PO}$ is defined originally on $\Lambda_+^{\mathbb{C}} \times P$, Theorems 3.4 and 3.5 imply that $\mathfrak{PO}$ extends to a function on $(\Lambda_0^{\mathbb{C}})^n \times M_{\mathbb{R}}$. Furthermore these theorems also imply the periodicity of $\mathfrak{PO}$ in x_i's,

$$\mathfrak{PO}(x_1, \cdots, x_i + 2\pi\sqrt{-1}, \cdots, x_n; u) = \mathfrak{PO}(x_1, \cdots, x_n; u). \tag{3.7}$$

We write

$$\mathfrak{PO}_0 = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{3.8}$$

to distinguish it from $\mathfrak{PO}$. We call $\mathfrak{PO}_0$ the *leading order potential function.*

We will concern the existence of the bounding cochain b for which the Floer cohomology $HF((L(u),b),(L(u),b))$ is not zero, and prove that critical points of the $\mathfrak{PO}^u$ (as a function of $y_1,\cdots,y_n$) have this property. (Theorem 3.9.)

This leads us to study the equation

$$\frac{\partial \mathfrak{PO}^u}{\partial y_k}(\mathfrak{y}_1,\cdots,\mathfrak{y}_n) = 0, \qquad k = 1,\cdots,n, \tag{3.9}$$

where $\mathfrak{y}_i \in \Lambda_0 \setminus \Lambda_+$.

We regard $\mathfrak{PO}^u$ as either a function of x_i or of y_i. Since the variable (x_i or y_i) is clear from situation, we do not mention it occasionally.

Proposition 3.6. *We assume that the coordinates of the vertices of P are rational. Then there exists $u_0 \in \mathrm{Int} P \cap \mathbb{Q}^n$ such that for each N there exists $\mathfrak{y}_1,\cdots,\mathfrak{y}_n \in \Lambda_0 \setminus \Lambda_+$ satisfying :*

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_k}(\mathfrak{y}_1,\cdots,\mathfrak{y}_n) \equiv 0, \quad \mod T^N \qquad k = 1,\cdots,n. \tag{3.10}$$

Moreover there exists $\mathfrak{y}_1',\cdots,\mathfrak{y}_n' \in \Lambda_0 \setminus \Lambda_+$ such that

$$\frac{\partial \mathfrak{PO}_0^{u_0}}{\partial y_k}(\mathfrak{y}_1',\cdots,\mathfrak{y}_n') = 0, \qquad k = 1,\cdots,n. \tag{3.11}$$

We will prove Proposition 3.6 in section 8 .

Remark 3.7. (1) u_0 is independent of N. But $\mathfrak{y}_i$ may depend on N. (We believe it does not depend on N, but are unable to prove it at the time of writing this paper.)
(2) If $[\omega] \in H^2(X;\mathbb{R})$ is contained in $H^2(X;\mathbb{Q})$ then we may choose P so that its vertices are rational.
(3) We believe that rationality of the vertices of P is superfluous. We also believe there exists not only a solution of (3.10) or of (3.11) but also of (3.9). However then the proof seems to become more cumbersome. Since we can reduce the general case to the rational case by approximation in most of the applications, we will be content to prove the above weaker statement in this paper.

We put

$$\mathfrak{x}_i = \log \mathfrak{y}_i \in \Lambda_0.$$

Since $\mathfrak{y}_i \in \Lambda_0 \setminus \Lambda_+$, $\log \mathfrak{y}_i$ is well-defined (by using non-Archimedean topology on Λ_0) and is contained in Λ_0.

Take $\mathfrak{y}_{i,0} \in \mathbb{C} \setminus \{0\}$ such that $\mathfrak{y}_i - \mathfrak{y}_{i,0} \equiv 0 \mod \Lambda_+^{\mathbb{C}}$ and write

$$b = \sum_i \mathfrak{x}_i \mathbf{e}_i \in H^1(L(u_0);\Lambda_0).$$

If we put an additional assumption that $\mathfrak{y}_{i,0} = 1$ for $i = 1,\cdots,n$, then b lies in

$$H^1(L(u_0);\Lambda_+) \subset H^1(L(u_0);\Lambda_0).$$

Therefore Proposition 3.2 implies the Floer cohomology $HF((L(u_0),b),(L(u_0),b);\Lambda_0)$ is defined. Then (3.10) combined with the argument from [CO] (see section 12) imply

$$HF((L(u_0),b),(L(u_0),b);\Lambda_0^{\mathbb{C}}/(T^N)) \cong H(T^n;\Lambda_0^{\mathbb{C}}/(T^N)). \tag{3.12}$$

We now consider the case when $\mathfrak{y}_{i,0} \neq 1$ for some i. In this case, we follow the idea of Cho [Cho] of twisting the Floer cohomology of $L(u)$ by *non-unitary* flat line bundle and proceed as follows :

We define $\rho : H_1(L(u);\mathbb{Z}) \to \mathbb{C} \setminus \{0\}$ by

$$\rho(\mathbf{e}_i) = \mathfrak{y}_{i,0}. \tag{3.13}$$

Let $\mathfrak{L}_\rho$ be the flat complex line bundle on $L(u)$ whose holonomy representation is ρ. We use ρ to twist the operator $\mathfrak{m}_k$ in the same way as [Fu2], [Cho] to obtain a filtered A_∞ algebra, which we write $((H(L(u);\Lambda_0),\rho),\mathfrak{m}^\rho)$. It is weakly unobstructed and $\mathcal{M}_{\text{weak}}((H(L(u);\Lambda_0),\rho),\mathfrak{m}^\rho) \supseteq H^1(L(u);\Lambda_+)$. (See section 11.)

We deform the filtered A_∞ structure $\mathfrak{m}^\rho$ to $\mathfrak{m}^{\rho,b}$ using $b \in H^1(L(u);\Lambda_+)$ for which $\mathfrak{m}_1^{\rho,b}\mathfrak{m}_1^{\rho,b} = 0$ holds. Denote by $HF((L(u_0),\rho,b),(L(u_0),\rho,b),\Lambda_0^{\mathbb{C}})$ the cohomology of $\mathfrak{m}_1^{\rho,b}$. We denote the potential function of $((H(L(u);\Lambda_0),\rho),\mathfrak{m}^\rho)$ by

$$\mathfrak{PO}^u_\rho : H^1(L(u);\Lambda_+) \to \Lambda_+$$

which is defined in the same was as $\mathfrak{PO}^u$ is using $\mathfrak{m}^\rho$ instead of $\mathfrak{m}$. From the way how the definition goes, we can easily prove

Lemma 3.8.

$$\mathfrak{PO}^u_\rho(x) = \mathfrak{PO}^u\left(x + \sum_{i=1}^n \mathfrak{x}_{i,0}\mathbf{e}_i\right).$$

Here $x \in H^1(L(u);\Lambda_+)$.

We note from the remark right after Theorem 3.5 that $\mathfrak{PO}^u$ has been extended to a function on $(\Lambda_0^{\mathbb{C}})^n \times M_{\mathbb{R}}$ and hence the right hand side of the identity in this lemma has a well-defined meaning. Lemma 3.8 will be proved in section 12.

Now we have :

Theorem 3.9. *Let* $\mathfrak{x}_i$ *and* ρ *satisfy* (3.9) *and* (3.13) *respectively. We put*

$$b = \sum_{i=1}^n (\mathfrak{x}_i - \mathfrak{x}_{i,0})\mathbf{e}_i \in H^1(L(u);\Lambda_+^{\mathbb{C}}). \tag{3.14}$$

Then we have

$$HF((L(u_0),\rho,b),(L(u_0),\rho,b),\Lambda_0^{\mathbb{C}}) \cong H(T^n;\Lambda_0^{\mathbb{C}}). \tag{3.15}$$

If (3.10) *and* (3.13) *are satisfied instead then we have*

$$HF((L(u_0),\rho,b),(L(u_0),\rho,b),\Lambda_0^{\mathbb{C}}/(T^N)) \cong H(T^n;\Lambda_0^{\mathbb{C}}/(T^N)). \tag{3.16}$$

Theorem 3.9 is proved in section 12. Using this we prove Theorem 1.5 in section 12. More precisely, we will also discuss the following two points in that section :

(1) We need to study the case where ω is not necessarily rational
(2) We only have (3.16) instead of (3.15).

We define :

Definition 3.10. Let (X, ω) be a smooth compact toric manifold, P be its moment polytope. We say the fiber $L(u_0)$ at $u_0 \in P$ is *balanced* if there exists a sequence ω_i, u_i such that

(1) ω_i is an T^n invariant Kähler structure on X such that $\lim_{i\to\infty} \omega_i = \omega$.
(2) u_i is in the interior of the moment polytope P_i of P. We make an appropriate choice of moment polytope P_i so that they converge to P. Then $\lim_{i\to\infty} u_i = u_0$.
(3) For each N, there exist a sufficiently large i and $b_{i,N} \in H_1(L(u_i); \Lambda_0^{\mathbb{C}})$ such that
$$HF((L(u_i), b_{i,N}), (L(u_i), b_{i,N}; \Lambda^{\mathbb{C}}/(T^N)) \cong H(T^n; \mathbb{C}) \otimes \Lambda^{\mathbb{C}}/(T^N).$$

Theorem 3.9 implies that $L(u_0)$ in Proposition 3.6 is balanced. (Proposition 12.3.) We will prove that any balanced Lagrangian fiber satisfies the conclusion of Theorem 1.5. (Lemma 12.2.)

Denoting $b' = b + \sum \mathfrak{x}_{i,0}\mathbf{e}_i$, we sometime write $HF((L(u_0), b'), (L(u_0), b'), \Lambda_0)$ for $HF((L(u_0), \rho, b), (L(u_0), \rho, b), \Lambda_0)$ from now on.

4. Examples and conjectures

In this section, we discuss various examples of toric manifolds which illustrate the results of section 3.

Example 4.1. Consider $X = S^2$ with standard symplectic form with area 2π. The moment polytope of the standard S^1-action by rotations along an axis becomes $P = [0, 1]$ after a suitable translation. We have $\lambda_1(u) = u$, $\lambda_2(u) = 1 - u$ and
$$\mathfrak{PO}(x; u) = e^x T^u + e^{-x} T^{1-u} = yT^u + y^{-1}T^{1-u}.$$
The zero of
$$\frac{\partial \mathfrak{PO}^u}{\partial y} = T^u - y^{-2}T^{1-u}$$
is $\mathfrak{y} = \pm T^{(1-2u)/2}$. If $u \neq 1/2$ then
$$\log \mathfrak{y} = \frac{1-2u}{2} \log(\pm T)$$
is not an element of universal Novikov ring. In other words, there is no critical point in $(\Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}})^n$.

If $u = 1/2$ then $\mathfrak{y} = \pm 1$. The case $\mathfrak{y} = 1$ corresponds to $\mathfrak{x} = 0$. Namely $b = 0$. We have
$$HF((L(1/2), 0), (L(1/2), 0); \Lambda_0) \cong H(S^1; \Lambda_0^{\mathbb{C}}).$$
The other case $\mathfrak{y} = -1$, corresponds to a nontrivial flat bundle on S^1.

Example 4.2. We consider $X = \mathbb{C}P^n$. Then
$$P = \{(u_1, \cdots, u_n) \mid 0 \le u_i, u_1 + \cdots + u_n \le 1\},$$
is a simplex. We have
$$\mathfrak{PO}(x_1, \cdots, x_n; u_1, \cdots, u_n) = \sum_{i=1}^n e^{x_i} T^{u_i} + e^{-\sum x_i} T^{1-\sum u_i}.$$

We put $u = u_0 = \left(\frac{1}{n+1}, \cdots, \frac{1}{n+1}\right)$. Then

$$\mathfrak{PO}^{u_0} = (y_1 + \cdots + y_n + y_1^{-1}y_2^{-1}\cdots y_n^{-1})T^{1/(n+1)}.$$

Solutions of the equation (3.9) are given by

$$\mathfrak{y}_1 = \cdots = \mathfrak{y}_{n+1} = e^{2\pi k\sqrt{-1}/(n+1)}, \quad k = 0, \cdots, n.$$

Hence the conclusion of Theorem 1.5 holds for our torus. The case $k = 0$ corresponds to $b = 0$. The other cases correspond to an appropriate flat bundles on T^n.

Remark 4.3. The critical values of the potential function is $(n+1)e^{2\pi\sqrt{-1}k/(n+1)}$, $k = 0, \cdots, n$.

We consider the quantum cohomology ring

$$QH(\mathbb{C}P^n; \Lambda_0^{(0)}) \cong \Lambda_0^{(0)}[z,T]/(z^{n+1} - T).$$

The first Chern class c_1 is $(n+1)z$. The eigenvalues of the operator

$$c : QH(\mathbb{C}P^n) \to QH(\mathbb{C}P^n), \alpha \mapsto c_1 \cup_Q \alpha$$

are $(n+1)e^{2\pi\sqrt{-1}k/(n+1)}$, $k = 0, \cdots, n$. It coincides with the set of critical values.

Kontsevich announced this statement at the homological mirror symmetry conference at Vienna 2006. (According to some physicists, this statement is known to them before.) See [Aur]. In our situation we can prove it by using Theorem 1.9.

In the rest of this subsection, we discuss 2 dimensional examples.

Let $\mathbf{e}_1$, $\mathbf{e}_2$ be the basis of $H^1(T^2;\mathbb{Z})$ as in Lemma 3.3. We put $\mathbf{e}_{12} = \mathbf{e}_1 \cup \mathbf{e}_2 \in H^2(T^2;\mathbb{Z})$. Let $\mathbf{e}_\emptyset$ be the standard basis of $H^0(T^2;\mathbb{Z}) \cong \mathbb{Z}$. The proof of the following proposition will be postponed until section 12.

Proposition 4.4. *Let $b = \mathfrak{y} \in H^1(L(u);\Lambda_+)$. Then the boundary operator $\mathfrak{m}_1^b$ is given as follows :*

$$\begin{cases} \mathfrak{m}_1^b(\mathbf{e}_i) = \dfrac{\partial \mathfrak{PO}^u}{\partial y_i}(\mathfrak{y})\mathbf{e}_\emptyset, \\ \mathfrak{m}_1^b(\mathbf{e}_{12}) = \dfrac{\partial \mathfrak{PO}^u}{\partial y_1}(\mathfrak{y})\mathbf{e}_2 - \dfrac{\partial \mathfrak{PO}^u}{\partial y_2}(\mathfrak{y})\mathbf{e}_1, \\ \mathfrak{m}_1^b(\mathbf{e}_\emptyset) = 0. \end{cases} \tag{4.1}$$

With (4.1) in our disposal, we examine various examples.

Example 4.5. We consider $M = \mathbb{C}P^2$ again. We put $u_1 = \epsilon + 1/3$, $u_2 = 1/3$. ($\epsilon > 0$.) Using (4.1) we can easily find

$$HF^{odd}((L(u),0),(L(u),0)) \cong HF^{even}((L(u),0),(L(u),0)) \cong \Lambda_0/(T^{1/3-\epsilon}).$$

Let us apply Theorem J [FOOO2] in this situation. (See also Theorem 4.11 below.) We consider a Hamiltonian diffeomorphism $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$. We denote by $\|\psi\|$ the Hofer distance of ψ from identity. Then we have

$$\#(\psi(L(u)) \cap L(u)) \geq 4$$

if $\|\psi\| < 2\pi(\frac{1}{3} - \epsilon)$ and $\psi(L(u))$ is transversal to $L(u)$.

We remark that this fact was already proved by Chekanov [Che]. (Actually the basic geometric idea behind our proof is the same as Chekanov's.)

Example 4.6. Let $M = S^2(\frac{a}{2}) \times S^2(\frac{b}{2})$, where $S^2(r)$ denotes the 2-sphere with radius r. We assume $a < b$.

Then $B = [0,a] \times [0,b]$ and we have :

$$\mathfrak{PO}(x_1, x_2; u_1, u_2) = y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} T^{a-u_1} + y_2^{-1} T^{b-u_2}.$$

Let us take $u_1 = a/2$, $u_2 = b/2$. Then

$$\frac{\partial \mathfrak{PO}^u}{\partial y_1} = (1 - y_1^{-2}) T^{a/2}, \quad \frac{\partial \mathfrak{PO}^u}{\partial y_2} = (1 - y_2^{-2}) T^{b/2}.$$

Therefore $y_1 = \pm 1$ $y_2 = \pm 1$ are solutions of (3.9). Hence we can apply Theorem 3.9 to our torus.

We next put $u_1 = a/2$, $a < 2u_2 < b$. Then

$$\frac{\partial \mathfrak{PO}^u}{\partial y_1} = (1 - y_1^{-2}) T^{a/2}, \quad \frac{\partial \mathfrak{PO}^u}{\partial y_2} = T^{u_2} - y_2^{-2} T^{b-u_2}.$$

We put $y_1 = y_2 = 1$. Then $\frac{\partial \mathfrak{PO}^u}{\partial y_1} = 0$, $\frac{\partial \mathfrak{PO}^u}{\partial y_2} \neq 0$. We find that

$$HF^{odd}((L(u),0),(L(u),0)) \cong HF^{even}((L(u),0),(L(u),0)) \cong \Lambda_0/(T^{u_2}).$$

Let $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$ be a Hamiltonian diffeomorphism. Then, Theorem J [FOOO2] implies that

$$\#(\psi(L(u)) \cap L(u)) \geq 4$$

if $\|\psi\| < 2\pi u_2$ and $\psi(L(u))$ is transversal to $L(u)$. Note there exists a pseudo-holomorphic disc with symplectic area πa $(< 2\pi u_2)$. Hence our result improves a result from [Che].

Example 4.7. Let X be two-point blow up of $\mathbb{C}P^2$. We may take its Kähler form so that the moment polytope is given by

$$P = \{(u_1, u_2) \mid -1 \leq u_1 \leq 1, -1 \leq u_2 \leq 1, u_1 + u_2 \leq 1 + \alpha\},$$

where $-1 < \alpha < 1$ depends on the choice of Kähler form. We have

$$\begin{aligned}\mathfrak{PO}(x_1, x_2; u_1, u_2) = {} & y_1 T^{1+u_1} + y_2 T^{1+u_2} + y_1^{-1} T^{1-u_1} \\ & + y_2^{-1} T^{1-u_2} + y_1^{-1} y_2^{-1} T^{1+\alpha-u_1-u_2}.\end{aligned} \tag{4.2}$$

Note X is Fano in our case.

(Case 1: $\alpha = 0$).

In this case X is monotone. We put $u_0 = (0,0)$. $L(u_0)$ is a monotone Lagrangian submanifold. We have

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = (1 - y_1^{-2} - y_1^{-2} y_2^{-1}) T, \quad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = (1 - y_2^{-2} - y_1^{-1} y_2^{-2}) T.$$

The solutions of (3.9) are given by $y_2 = \frac{1}{y_1^2 - 1}$, $y_1^5 + y_1^4 - 2y_1^3 - 2y_1^2 + 1 = 0$ in $\mathbb{C}$. (There are 5 solutions.)

(Case 2: $\alpha > 0$).

We put $u_0 = (0,0)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = (1 - y_1^{-2}) T - y_1^{-2} y_2^{-1} T^{1+\alpha}, \quad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = (1 - y_2^{-2}) T - y_1^{-1} y_2^{-2} T^{1+\alpha}.$$

We consider, for example, the case $y_1 = y_2 = \tau$. Then (3.9) becomes

$$\tau^3 - \tau - T^\alpha = 0. \tag{4.3}$$

The solution of (4.3) with $\tau \equiv 1 \mod \Lambda_+$ is given by

$$\tau = 1 + \frac{1}{2}T^{\alpha} - \frac{3}{8}T^{2\alpha} + \frac{1}{2}T^{3\alpha} + \sum_{k=4}^{\infty} c_k T^{k\alpha}.$$

Let us put $b = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ with

$$x_1 = x_2 = \log\left(1 + \frac{1}{2}T^{\alpha} - \frac{3}{8}T^{2\alpha} + \frac{1}{2}T^{3\alpha} + \cdots\right) \in \Lambda_+.$$

Then by Theorem 3.9 we have

$$HF((L(u_0), b), (L(u_0), b); \Lambda_0) \cong H(T^2; \Lambda_0).$$

We like to point out that in this example it is essential to deform Floer cohomology using an element b of $H^1(L(u_0); \Lambda_+)$ containing the formal parameter T to obtain nonzero Floer cohomology.

At u_0, there are actually 4 solutions such that

$$(y_1, y_2) \equiv (1,1), (1,-1), (-1,1), (-1,-1) \mod \Lambda_+,$$

respectively.

In the current case there is another point $u_0' = (\alpha, \alpha) \in P$ where $L(u_0')$ is balanced [1]. In fact at $u_0' = (\alpha, \alpha)$ the equation (4.3) becomes

$$0 = -(y_1^{-2}y_2^{-1} + y_1^{-2})T^{1-\alpha} + T^{1+\alpha}, \quad 0 = -(y_1^{-1}y_2^{-2} + y_2^{-2})T^{1-\alpha} + T^{1+\alpha}.$$

we put $\tau = y_1 = y_2$ to obtain

$$\tau^3 T^{2\alpha} - \tau - 1 = 0$$

This equation has a unique solution with $\tau \equiv -1 \mod \Lambda_+$. (The other solution is $T^{2\alpha/3}\tau \equiv 1 \mod \Lambda_+$, for which Theorem 3.9 is not applicable for this case.)

The total number of the solutions (u, b) is 5.

(Case 3: $\alpha < 0$).

We first consider $u_0 = (0, 0)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = -y_1^{-2}y_2^{-1}T^{1+\alpha} + (1 - y_1^{-2})T, \quad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = -y_1^{-1}y_2^{-2}T^{1+\alpha} + (1 - y_2^{-2})T.$$

We assume y_i satisfies (3.9). It is then easy to see that $y_1^{-1} \equiv 0$, or $y_2^{-1} \equiv 0 \mod \Lambda_+$. In other words, there is no y_1, y_2 to which we can apply Theorem 3.9. Actually it is easy to find a Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(L(u_0)) \cap L(u_0) = \emptyset$.

We next take $u_0' = (\alpha/3, \alpha/3)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0'}}{\partial y_1} = (1 - y_1^{-2}y_2^{-1})T^{1+\alpha/3} - y_1^{-2}T^{1-\alpha/3},$$
$$\frac{\partial \mathfrak{PO}^{u_0'}}{\partial y_2} = (1 - y_1^{-1}y_2^{-2})T^{1+\alpha/3} - y_2^{-2}T^{1-\alpha/3}.$$

By putting $y_1 = y_2 = \tau$ for example, (3.9) becomes

$$\tau^3 - T^{-2\alpha/3}\tau - 1 = 0. \tag{4.4}$$

[1] Using the method of spectral invariants and symplectic quasi-states, Entov and Polterovich discovered some undisplaceabe Lagrangian fiber which was not covered by the criterion given in [CO] (see section 9 [EP1]). Recently this example, among others, was explained by Cho [Cho] via Lagrangian Floer homology twisted by non-unitary line bundles.

Let us put $b = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ with

$$x_1 = x_2 = \log\tau = \log\left(1 + \frac{1}{3}T^{-2\alpha/3} + \frac{2}{3}T^{-4\alpha/3} + \cdots\right) \in \Lambda_+,$$

where τ solves (4.4). Theorem 3.9 is applicable. (There are actually 3 solutions of (3.9) corresponding to the 3 solutions of (4.4).)

There are two more points $u = (\alpha+1,\alpha),(\alpha,\alpha+1)$ where (3.9) has a solution in $(\Lambda_0 \setminus \Lambda_+)$. Each u has one b.

Thus the total number of the pair (u,b) is again 5. We remark

$$5 = \sum \operatorname{rank} H^k(X;\mathbb{Q}).$$

This is not just a coincidence but an example of general phenomenon stated as in Theorem 1.3.

We remark that as $\alpha \to 1$ our X blows down to $S^2(1)\times S^2(1)$. On the other hand, as $\alpha \to -1$ our X blows down to $\mathbb{C}P^2$. The situation of the case $\alpha > 0$ can be regarded as a perturbation of the situation of $S^2(1)\times S^2(1)$, by the effect of exceptional curve corresponding to the segment $u_1 + u_2 = 1 + \alpha$. The situation of the case $\alpha < 0$ can be regarded as a perturbation of the situation of $\mathbb{C}P^2$ by the effect of the two exceptional curves corresponding to the segments $u_1 = 1$ and $u_2 = 1$. An interesting phase change occurs at $\alpha = 0$.

The discussion of this section strongly suggests that Lagrangian Floer theory (Theorems G, J [FOOO2]) gives the optimal result for the study of displacement of the T^n-orbits in toric manifolds.

Conjecture 4.8. *Let X be a compact toric manifold and $L(u) = \pi^{-1}(u)$, $u \in \operatorname{Int}P$. Then the following two conditions are equivalent.*

1. *There exists no Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(L(u)) \cap L(u) = \emptyset$.*
2. *There exists $(\mathfrak{y}_1,\cdots,\mathfrak{y}_n) \in (\Lambda_0 \setminus \Lambda_+)^n$ satisfying* (3.9).

Note (2) $\Rightarrow$ (1) follows from Theorem 3.9. In many cases (including all the examples we discuss in this paper) we can prove (1) $\Rightarrow$ (2).

Using the argument employed in Example 4.6 we can discuss the relationship between the Hofer distance and displacement. First we introduce some notations for this purpose. We denote by $Ham(X,\omega)$ the group of Hamiltonian diffeomorphisms of (X,ω). For a time-dependent Hamiltonian $H : [0,1]\times X \to \mathbb{R}$, we denote by ϕ_H^t the time t-map of Hamilton's equation $\dot{x} = X_H(t,x)$. The Hofer norm of $\psi \in Ham(X,\omega)$ is defined to be

$$\|\psi\| = \inf_{H;\phi_H^1=\psi} \int_0^1 (\max H_t - \min H_t)\,dt$$

(See [H].)

Definition 4.9. Let $Y \subset X$. We define the *displacement energy* $e(Y) \in [0,\infty]$ by

$$e(Y) := \inf\{\|\psi\| \mid \psi \in Ham(X,\omega),\ \psi(Y)\cap\overline{Y} = \emptyset\}.$$

We put $e(Y) = \infty$ if there exists no $\psi \in Ham(X,\omega)$ with $\psi(Y)\cap\overline{Y} = \emptyset$.

Let us consider $\mathfrak{PO}(y_1,\cdots,y_n;u_1,\cdots,u_n) : \Lambda_0^n \times P \to \Lambda_+$ as in Theorem 3.4.

Definition 4.10. We define the number $\mathfrak{E}(u) \in (0, \infty]$ as the supremum of all λ such that there exists $\mathfrak{y}_1, \cdots, \mathfrak{y}_n \in (\Lambda_0 \setminus \Lambda_0^+)^n$ satisfying

$$\frac{\partial \mathfrak{PO}}{\partial y_i}(\mathfrak{y}_1, \cdots, \mathfrak{y}_n; u) \equiv 0 \mod T^{\lambda} \tag{4.5}$$

for $i = 1, \cdots, n$. (Here we consider universal Novikov ring with $\mathbb{C}$-coefficients.) We call $\mathfrak{E}(u)$ the $\mathfrak{PO}$-*threshold* of the fiber $L(u)$, and a point $(\mathfrak{y}_1, \cdots, \mathfrak{y}_n; u)$ satisfying (4.5) a $\mathfrak{PO}$-*threshold point* of $L(u)$.

Theorem 4.11. *For any compact toric manifold X and $L(u) = \pi^{-1}(u)$, $u \in \mathrm{Int}P$, we have*

$$e(L(u)) \geq 2\pi\mathfrak{E}(u). \tag{4.6}$$

Proof. Let us consider $(\mathfrak{y}_1, \cdots, \mathfrak{y}_n; u) \in (\Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}})^n$ be a $\mathfrak{PO}$-threshold point of $L(u)$. We associate to it a local system ρ by (3.13) and a bounding cochain b by (3.14). Then by the same way as the proof of Theorem 3.9, we prove

$$HF((L(u), \mathfrak{L}^u_\rho; b), (L(u), \mathfrak{L}^u_\rho; b); \Lambda_0^{\mathbb{C}}/(T^\lambda)) \cong H(T^n; \Lambda_0^{\mathbb{C}}/(T^\lambda)).$$

This isomorphism and Universal Coefficient Theorem imply that the Floer cohomology $HF((L(u), \mathfrak{L}^u_\rho; b), (L(u), \mathfrak{L}^u_\rho; b); \Lambda_0^{\mathbb{C}})$ contains a torsion summand $\Lambda_0^{\mathbb{C}}/(T^\lambda)$ with $\lambda' \geq \lambda$ or Λ_0. Therefore Theorem J [FOOO2], after twisting Floer cohomology by the local system ρ, implies $e(L(u)) \geq 2\pi\lambda$. This finishes the proof. □

Conjecture 4.12. *The equality always holds in* (4.6).

It is very likely that Conjecture 4.12 holds for all the examples in this paper although we did not check them all. (In the non-Fano case, we may use $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$.)

We like to remark $\mathfrak{E}(u)$ can be easily calculated once the toric manifold X is given explicitly.

Remark 4.13. Appearance of a new family of pseudo-holomorphic discs with Maslov index 2 after blow up, which we observed in Examples 4.7 can be related to the operator $\mathfrak{q}$ that we introduced in section 13 [FOOO2] in the following way.

We denote by $\mathcal{M}_{l,k+1}(\beta)$ the moduli space of stable maps $f : (\Sigma, \partial\Sigma) \to (X, L)$ from bordered Riemann surface Σ of genus zero with l interior and $k+1$ boundary marked points and in homology class β. (See section 3 [FOOO1].) Let us consider the case when Maslov index of β is 4. We assume $[f] \in \mathcal{M}_{0,0+1}(\beta)$ and $\mathcal{M}_{0,0+1}(\beta)$ is Fredholm regular at f. The virtual dimension of $\mathcal{M}_{0,0+1}(\beta)$ is $n+2$. We blow up X at a point $p = f(0) \in X$ and obtain $\widehat{X}$. (We assume $p \notin L$.) Let $[E] \in H_{2n-2}(\widehat{X})$ be the homology class of the exceptional divisor $E = \pi^{-1}(p)$. Now f induces a map $\widehat{f} : (D^2; \partial D^2) \to (\widehat{X}, L)$. The Maslov index of the homology class $[\widehat{f}] \in H_2(\widehat{X}, L)$ becomes 2. We put $\widehat{\beta} = [\widehat{f}]$.

For the case where X is toric and L is a T^n-orbit, we can take a T^n-invariant perturbation. (See section 10.) If p is a fixed point of T^n action, a T^n-invariant perturbation lifts to a perturbation of the moduli space $\mathcal{M}_{0+1}(\widehat{\beta})$.

Then any T^n-orbit of the moduli space $\mathcal{M}_{0+1}(X; \beta)$ of holomorphic discs passing through p corresponds to the T^n-orbit of $\mathcal{M}_{0,0+1}(\widehat{X}; \widehat{\beta})$ and vice versa. Namely we have an isomorphism

$$\mathcal{M}_{1,0+1}(\beta) \ {}_{ev}\times_X \{p\} \cong \mathcal{M}_{0,0+1}(\widehat{\beta}). \tag{4.7}$$

Here ev in the left hand side is the evaluation map at the interior marked point. (Actually we need to work out analytic detail of gluing construction etc.. It seems very likely that we can do it in the same way as the argument of Chapter 10 [FOOO2]. See also [LiRu].)

Using (4.7) we may prove :

$$\mathfrak{q}_{1,k;\beta}(PD([p]); b, \cdots, b) = \mathfrak{m}_{k,\widehat{\beta}}(b, \cdots, b),$$

where

$$\mathfrak{q}_{1,k;\beta}(Q; P_1, \cdots, P_k) = ev_{0*}\left(\mathcal{M}_{1,k+1}(\beta) \times_{(X\times L^k)} (Q \times P_1 \times \cdots \times P_k)\right)$$

is defined in section 13 [FOOO2]. (Here Q is a chain in X and P_i are chains in $L(u)$, and $ev_0 : \mathcal{M}_{1,k+1}(\beta) \to X$ is the evaluation map at the 0-th boundary marked point. In the right hand side, we take fiber product over $X \times L^k$.) This is an example of a blow-up formula in Lagrangian Floer theory.

5. Quantum cohomology and Jacobian ring

In this section, we prove Theorem 1.9. Let $\mathfrak{PO}_0$ be the leading order potential function. (Recall if X is Fano, we have $\mathfrak{PO}_0 = \mathfrak{PO}$.) We define the monomial

$$\overline{z}_i(u) = y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \in \Lambda_0[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}]. \tag{5.1}$$

Compare this with (2.3). It is also suggestive to write $\overline{z}_i$ as

$$\overline{z}_i(u) = e^{\langle x, v_i\rangle} T^{\ell_i(u)},\ x = (x_1, \cdots, x_n),\ y_i = e^{x_i}. \tag{5.2}$$

By definition we have

$$\mathfrak{PO}_0^u = \sum_{i=1}^m \overline{z}_i(u) \tag{5.3}$$

$$y_j \frac{\partial \overline{z}_i}{\partial y_j} = v_{i,j}\overline{z}_i(u). \tag{5.4}$$

The following is a restatement of Theorem 1.9. Let $z_i \in H^2(X;\mathbb{Z})$ be the Poincaré dual of the divisor $\pi^{-1}(\partial_i P)$.

Theorem 5.1. *If X is Fano, there exists an isomorphism*

$$\psi_u : QH(X;\Lambda) \cong Jac(\mathfrak{PO})$$

such that $\psi_u(z_i) = \overline{z}_i$.

Since $c_1(X) = \sum_{i=1}^m z_i$ (see [Ful]) and $\mathfrak{PO}_0^u = \sum_{i=1}^m \overline{z}_i(u)$ by definition, Theorem 1.9 follows from Theorem 5.1.

In the remaining section, we prove Theorem 5.1. We remark that z_i ($i = 1, \cdots, m$) generates the quantum cohomology ring $QH(X;\Lambda)$ as a Λ-algebra (see Theorem 5.5 below). Therefore it is enough to prove that the assignment $\tilde{\psi}_u(z_i) = \overline{z}_i(u)$ extends to a homomorphism $\tilde{\psi}_u : \Lambda[z_1, \cdots, z_m] \to Jac(\mathfrak{PO}_0^u)$ that induces an isomorphism in $QH(X;\Lambda)$. In other words, it suffices to show that the relations among the generators in $\Lambda[z_1, \cdots, z_m]$ and in $Jac(\mathfrak{PO}_0^u)$ are mapped to each other under the assignment $\tilde{\psi}_u(z_i) = \overline{z}_i(u)$. To establish this correspondence, we will review Batyrev's description of the relations among z_i's.

We first clarify the definition of quantum cohomology ring over the universal Novikov rings Λ_0 and Λ. Let (X,ω) be a symplectic manifold and $\alpha \in \pi_2(X)$. Let $\mathcal{M}_3(\alpha)$ be the moduli space of stable map of genus 0 with 3 marked points. Let ev :

$\mathcal{M}_3(\alpha) \to X^3$ be the evaluation map. We can define the virtual fundamental class $ev_*[\mathcal{M}_3(\alpha)] \in H_d(X^3;\mathbb{Q})$ where $d = 2(\dim_{\mathbb{C}} X + c_1(X) \cap \alpha)$. Let $a_i \in H^*(X;\mathbb{Q})$. We define $a_1 \cup_Q a_2 \in H^*(X;\Lambda_0)$ by the following formula.

$$\langle a_1 \cup_Q a_2, a_3 \rangle = \sum_{\alpha} T^{\omega\cap\alpha/2\pi} ev_*[\mathcal{M}_3(\alpha)] \cap (a_1 \times a_2 \times a_3). \tag{5.5}$$

Here $\langle \cdot, \cdot \rangle$ is the Poincaré duality. Extending this linearly we obtain the quantum product

$$\cup_Q : H(X;\Lambda_0) \otimes H(X;\Lambda_0) \to H(X;\Lambda_0).$$

Extending the coefficient ring further to Λ, we obtain the (small) quantum cohomology ring $QH(X;\Lambda)$.

Now we specialize to the case of compact toric manifolds and review Batyrev's presentation of quantum cohomology ring. We consider the exact sequence

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2(X;L(u)) \longrightarrow \pi_1(L(u)) \longrightarrow 0. \tag{5.6}$$

We note $\pi_2(X;L(u)) \cong \mathbb{Z}^m$ and choose its basis adapted to this exact sequence as follows : Consider the divisor $\pi^{-1}(\partial_i P)$ and take a small disc transversal to it. Each such disc gives rise to an element

$$[\beta_i] \in H_2(X;\pi^{-1}(\mathrm{Int}P)) \cong H_2(X;L(u)) \cong \pi_2(X,L(u)). \tag{5.7}$$

The set of $[\beta_i]$ with $i = 1,\cdots,m$ forms a basis of $\pi_2(X;L(u)) \cong \mathbb{Z}^m$. The boundary map $[\beta] \mapsto [\partial\beta] : \pi_2(X;L(u)) \to \pi_1(L(u))$ is identified with the corresponding map $H_2(X;L(u)) \to H_1(L(u))$. Using the basis chosen in Lemma 3.3 on $H_1(L(u))$ we identify $H_1(L(u)) \cong \mathbb{Z}^n$. Then this homomorphism maps $[\beta_i]$ to

$$[\partial\beta_i] \cong v_i = (v_{i,1},\cdots,v_{i,n}), \tag{5.8}$$

where $v_{i,j}$ is as in (3.3). By the exactness of (5.6), we have an isomorphism

$$H_2(X) \cong \{\beta \in H_2(X;L(u)) \mid [\partial\beta] = 0\}. \tag{5.9}$$

Lemma 5.2. *We have*

$$\omega \cap \left[\sum k_i\beta_i\right] = 2\pi \sum k_i\ell_i(u). \tag{5.10}$$

If $[\sum k_i\partial\beta_i] = 0$ *then*

$$\sum k_i \frac{d\ell_i}{du_j} = 0. \tag{5.11}$$

In particular, the right hand side of (5.10) *is independent of* u.

Proof. (5.10) follows from the area formula (2.12), $\omega(\beta_i) = 2\pi\ell_i(u)$. On the other hand if $[\sum k_i\partial\beta_i] = 0$, we have

$$\sum_{i=1}^m k_i v_i = 0.$$

By the definition of ℓ_i, $\ell_i(u) = \langle u, v_i\rangle - \lambda_i$, from Theorem 2.13, this equation is precisely (5.11) and hence the proof. □

Let $\mathcal{P} \subset \{1,\cdots,m\}$ be a primitive collection (see Definition 2.4). There exists a unique subset $\mathcal{P}' \subset \{1,\cdots,m\}$ such that $\sum_{i\in\mathcal{P}} v_i$ lies in the interior of the cone spanned by $\{v_{i'} \mid i' \in \mathcal{P}'\}$, which is a member of the fan Σ. (Since X is compact, we can choose such $\mathcal{P}'$. See section 2.4 [Ful].) We write

$$\sum_{i\in\mathcal{P}} v_i = \sum_{i'\in\mathcal{P}'} k_{i'} v_{i'}. \tag{5.12}$$

Since X is assumed to be nonsingular $k_{i'}$ are all positive integers. (See p.29 of [Ful].) We put

$$\omega(\mathcal{P}) = \sum_{i\in\mathcal{P}} \ell_i(u) - \sum_{i'\in\mathcal{P}'} k_{i'}\ell_{i'}(u). \tag{5.13}$$

It follows from (5.10) that $2\pi\omega(\mathcal{P})$ is the symplectic area of the homotopy class

$$\beta(\mathcal{P}) = \sum_{i\in\mathcal{P}} \beta_i - \sum_{i'\in\mathcal{P}'} k_{i'}\beta_{i'} \in \pi_2(X). \tag{5.14}$$

We remark that $\omega(\mathcal{P}) > 0$: In fact, since the cone spanned by $\{v_{i'} \mid i' \in \mathcal{P}'\}$ does not contain the origin it follows that

$$\bigcap_{i'\in\mathcal{P}'} \pi^{-1}(\partial_i P) \neq \emptyset.$$

Then for any $u \in \bigcap_{i'\in\mathcal{P}'} \pi^{-1}(\partial_i P)$, we have $\ell_{i'}(u) = 0$ and so obtain $\omega(\mathcal{P}) = \sum_{i\in\mathcal{P}} \ell_i(u) > 0$. The last inequality follows since $\ell_i(u) > 0$ for $u \in \operatorname{Int} P$ by definition (2.13) of P.

Now we associate $z_1, \cdots, z_m$ formal variables to $v_1, \cdots, v_m$ respectively.

Definition 5.3 (Batyrev [B1])**.** (1) The *quantum Stanley-Reisner ideal* $SR_\omega(X)$ is the ideal generated by

$$z(\mathcal{P}) = \prod_{i\in\mathcal{P}} z_i - T^{\omega(\mathcal{P})} \prod_{i'\in\mathcal{P}'} z_{i'}^{k_{i'}} \tag{5.15}$$

in the polynomial ring $\Lambda[z_1, \cdots, z_m]$. Here $\mathcal{P}$ runs over all primitive collection.

(2) We denote by $P(X)$ the ideal generated by

$$\sum_{i=1}^{m} v_{i,j} z_i \tag{5.16}$$

for $j = 1, \cdots, n$. In this paper we call $P(X)$ the *linear relation ideal.*

(3) We call the quotient

$$QH^\omega(X;\Lambda) = \frac{\Lambda[z_1, \cdots, z_m]}{(P(X) + SR_\omega(X))} \tag{5.17}$$

the *Batyrev quantum cohomology ring.*

Remark 5.4. We do not take closure of our ideal $P(X) + SR_\omega(X)$ here. See Proposition 7.4.

Theorem 5.5 (Batyrev [B1, Gi2])**.** *If X is Fano there exists a ring isomorphism from $QH^\omega(X;\Lambda)$ to the quantum cohomology ring $QH(X;\Lambda)$ of X such that z_i is sent to the Poincaré dual to $\pi^{-1}(\partial_i P)$.*

The main geometric part of the proof of Theorem 5.5 is the following.

Proposition 5.6. *The Poincaré dual to $\pi^{-1}(\partial_i P)$ satisfy the quantum Stanley-Reisner relation.*

We do not prove Proposition 5.6 in this paper. See Remarks 5.13 and 5.14. However since our choice of the coefficient ring is different from other literature, we explain here for reader's convenience how Theorem 5.5 follows from Proposition 5.6.

Proposition 5.6 implies that we can define a ring homomorphism $h : QH^\omega(X;\Lambda) \to QH(X;\Lambda)$ by sending z_i to $PD(\pi^{-1}(\partial_i P))$. Let $F^k QH(X;\Lambda)$ be the direct sum of elements of degree $\le 2k$. Let $F^k QH^\omega(X;\Lambda)$ be the submodule generated by the polynomial of degree $k/2$ on z_i. Clearly $h(F^k QH^\omega(X;\Lambda)) \subset F^k QH(X;\Lambda)$.

Since X is Fano, it follows that,

$$x \cup_Q y - x \cup y \in F^{\deg x + \deg y - 2} QH(X;\Lambda).$$

We also recall the cohomology ring $H(X;\mathbb{Q})$ is obtained by putting $T = 0$ in quantum Stanley-Reisner relation. Moreover we find that the second product of the right hand side of (5.15) has degree strictly smaller than the first since X is Fano.

Therefore the graded ring

$$gr(QH(X;\Lambda)) = \bigoplus_k F^k(QH(X;\Lambda))/F^{k-1}(QH(X;\Lambda)),$$

is isomorphic to the (usual) cohomology ring as a ring. The same holds for $QH^\omega(X;\Lambda)$. It follows that h is an isomorphism. □

In the rest of this section, we will prove the following Proposition 5.7. Theorem 5.1 follows immediately from Proposition 5.7 and Theorem 5.5.

Proposition 5.7. *There exists an isomorphism :*

$$\psi_u : QH^\omega(X;\Lambda) \cong Jac(\mathfrak{PO}_0)$$

such that $\psi_u(z_i) = \overline{z}_i$.

We remark that we do *not* assume that X is Fano in Proposition 5.7. We also remark that for our main purpose to calculate $\mathfrak{M}_0(\mathfrak{Lag}(X))$, Proposition 5.7 suffices. Proposition 5.7 is a rather simple algebraic result whose proof does not require study of pseudo-holomorphic discs or spheres.

Proof of Proposition 5.7. We start with the following proposition.

Proposition 5.8. *The assignment*

$$\widehat{\psi}_u(z_i) = \overline{z}_i(u). \tag{5.18}$$

induces a well-defined ring isomorphism

$$\widehat{\psi}_u : \frac{\Lambda\,[z_1,\cdots,z_m]}{SR_\omega(X)} \to \Lambda[y_1,\cdots,y_n,y_1^{-1},\cdots,y_n^{-1}] \tag{5.19}$$

Proof. Let $\mathcal{P}$ be a primitive collection and $\mathcal{P}'$, $k_{i'}$ be as in (5.12). We calculate

$$\prod_{i\in\mathcal{P}} \overline{z}_i(u) = \prod_{i\in\mathcal{P}} y_1^{v_{i,1}}\cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{5.20}$$

by (5.1). On the other hand,

$$\begin{aligned}\prod_{i'\in\mathcal{P}'} \overline{z}_{i'}^{k_{i'}}(u) &= \prod_{i'\in\mathcal{P}'} y_1^{k_{i'}v_{i',1}}\cdots y_n^{k_{i'}v_{i',n}} T^{k_{i'}\ell_{i'}(u)} \\ &= \prod_{i\in\mathcal{P}} y_1^{v_{i,1}}\cdots y_n^{v_{i,n}} \prod_{i'\in\mathcal{P}'} T^{k_{i'}\ell_{i'}(u)}\end{aligned}$$

by (5.12). Moreover

$$\sum_{i\in\mathcal{P}} \ell_i(u) - \sum_{i\in\mathcal{P}'} k_{i'}\ell_{i'}(u) = \omega(\mathcal{P})$$

by (5.13). Therefore

$$\prod_{i\in\mathcal{P}} \overline{z}_i(u) = T^{\omega(\mathcal{P})} \prod_{i'\in\mathcal{P}'} \overline{z}_{i'}^{k_{i'}}(u)$$

in $\Lambda[y_1,\cdots,y_n,y_1^{-1},\cdots,y_n^{-1}]$. In other words, (5.18) defines a well-defined ring homomorphism (5.19).

We now prove that $\widehat{\psi}_u$ is an isomorphism. Let

$$pr : \mathbb{Z}^m \cong \pi_2(X;L(u)) \longrightarrow \mathbb{Z}^n \cong \pi_1(L(u))$$

be the homomorphism induced by the boundary map $pr([\beta]) = [\partial\beta]$. (See (2.1).) We remark $pr(a_1,\cdots,a_m) = (b_1,\cdots,b_n)$ with $b_j = \sum_i a_i v_{i,j}$. Let $A = \sum a_i\beta_i$ be an element in the kernel of pr. We write it as

$$\sum_{i\in I} a_i\beta_i - \sum_{j\in J} b_j\beta_j$$

where a_i b_j are positive and $I \cap J = \emptyset$. We define

$$r(A) = \prod_{i\in I} z_i^{a_i} - T^{\sum_i a_i\ell_i(u) - \sum_j b_j\ell_j(u)} \prod_{j\in J} z_j^{b_j}. \tag{5.21}$$

We remark that a generator of quantum Stanley-Reisner ideal corresponds to $r(A)$ for which I is a primitive collection $\mathcal{P}$ and $J = \mathcal{P}'$. We also remark that the case $I = \emptyset$ or $J = \emptyset$ is included.

Lemma 5.9.

$$r(A) \in SR_\omega(X).$$

Proof. This lemma is proved in [B1]. We include its proof here for reader's convenience. We prove the lemma by an induction over the values

$$E(A) = \sum_{i\in I} a_i\ell_i(u_0) + \sum_{j\in J} b_j\ell_j(u_0).$$

Here we fix a point $u_0 \in \mathrm{Int}P$ during the proof of Lemma 5.9.

Since $I \cap J = \emptyset$, at least one of $\{v_i \mid i \in I\}$, $\{v_i \mid i \in J\}$ can not span a cone that is a member of the fan Σ. Without loss of generality, we assume that $\{v_i \mid i \in I\}$ does not span such a cone. Then it contains a subset $\mathcal{P} \subset I$ that is a primitive collection. We take $\mathcal{P}'$, $k_{i'}$ as in (5.12) and define

$$Z = \prod_{i\in I} z_i^{a_i} - T^{\omega(\mathcal{P})} \prod_{i\in I\setminus\mathcal{P}} z_i^{a_i} \prod_{i\in\mathcal{P}} z_i^{a_i-1} \prod_{i''\in\mathcal{P}'} z_{i''}^{k_{i''}}. \tag{5.22}$$

Then Z lies in $SR_\omega(X)$ by construction. We recall from Lemma 5.2 that the values

$$\sum_{i\in\mathcal{P}} \ell_i(u) - \sum_{i\in\mathcal{P}'} k_i\ell_i(u) = \omega(\mathcal{P})$$

are independent of u and positive. By the definitions (5.21), (5.22) of $r(A)$ and Z, we can express

$$r(A) - Z = T^{\omega(\mathcal{P})+c}\left(\prod_{h\in K} z_h^{n_h}\right) r(B)$$

for an appropriate B in the kernel of pr and a constant c. Moreover we have

$$E(B) + 2\sum_{h\in K} n_h \ell_h(u_0) + \omega(\mathcal{P}) = E(A).$$

Since $u_0 \in \operatorname{Int} P$ it follows that $\ell_h(u_0) > 0$ which in turn gives rise to $E(B) < E(A)$. The induction hypothesis then implies $r(B) \in SR_\omega(X)$. The proof of the lemma is now complete. □

Corollary 5.10. *z_i is invertible in*

$$\frac{\Lambda\,[z_1,\cdots,z_m]}{SR_\omega(X)}.$$

Proof. Since X is compact, the vector $-v_i$ is in some cone spanned by v_j $(j \in I)$. Namely

$$-v_i = \sum_{j\in I} k_j v_j$$

where k_j are nonnegative integers. Then

$$T^{\ell_i(u)+\sum_j k_j\ell_j(u)} = z_i \prod_{j\in I} z_j^{k_j} \quad \bmod SR_\omega(X)$$

by Lemma 5.9. Since $T^{\ell_i(u)+\sum_j k_j\ell_j(u)}$ is invertible in the field Λ, it follows that $\prod_{j\in I} z_j^{k_j}$ defines the inverse of z_i in the quotient ring. □

We recall from Lemma 5.2 that $\ell_i(u)+\sum_j k_j\ell_j(u)$ is independent of u. We *define*

$$z_i^{-1} = T^{-\ell_i(u)-\sum_j k_j\ell_j(u)} \prod_{j\in I} z_j^{k_j}. \tag{5.23}$$

(Note we have not yet proved that $\Lambda\,[z_1,\cdots,z_m]/SR_\omega(X)$ is an integral domain. This will follow later when we prove Proposition 5.8.)

Since $v_1,\cdots,v_m$ generates the lattice $\mathbb{Z}^n$, we can always assume the following by changing the order of v_i, if necessary.

Condition 5.11. The determinant of the $n\times n$ matrix $(v_{i,j})_{i,j=1,\cdots,n}$ is ± 1.

Let $(v^{i,j})$ be the inverse matrix of $(v_{i,j})$. Namely $\sum_j v^{i,j} v_{j,k} = \delta_{i,k}$. Condition 5.11 implies that each $v^{i,j}$ is an integer. Inverting the matrix $(v_{i,j})$, we obtain

$$y_i = T^{-c_i(u)} \prod_{j=1}^n \overline{z}_i^{v^{i,j}} \tag{5.24}$$

from (5.20) where $c_i(u) = \sum v^{i,j}\ell_j(u)$. We define using Corollary 5.10

$$\widehat{\phi}_u(y_i^{\pm 1}) = T^{-\pm c_i(u)} \prod_{j=1}^n z_j^{\pm v^{i,j}} \in \frac{\Lambda\,[z_1,\cdots,z_m]}{SR_\omega(X)}.$$

More precisely, we plug (5.23) here if $\pm v^{i,j}$ is negative.

The identity $\widehat{\psi}_u \circ \widehat{\phi}_u = id$ is a consequence of (5.24). We next calculate $(\widehat{\phi}_u \circ \widehat{\psi}_u)(z_h) = \widehat{\phi}_u(\overline{z}_h(u))$ and prove

$$(\widehat{\phi}_u \circ \widehat{\psi}_u)(z_h) = T^{\ell_h(u)}\widehat{\phi}_u(y_1^{v_{h,1}}\cdots y_n^{v_{h,n}}) = T^{e(h;u)} \prod_{j=1}^n z_j^{m_j},$$

where $m_j \ge 0$ and

$$v_h = \sum m_j v_j, \qquad \ell_h(u) = e(h;u) + \sum m_j \ell_j(u) : \tag{5.25}$$

To see (5.25), we consider any *monomial* Z of $y_i, z_i, \overline{z}_i, T^\alpha$. We define its multiplicative valuation $\mathfrak{v}_u(Z) \in \mathbb{R}$ by putting

$$\mathfrak{v}_u(y_i) = 0, \quad \mathfrak{v}_u(z_i) = \mathfrak{v}_u(\overline{z}_i) = \ell_i(u), \quad \mathfrak{v}_u(T^\alpha) = \alpha.$$

We also define a (multiplicative) grading $\rho(Z) \in \mathbb{Z}^n$ by

$$\rho(y_i) = \mathbf{e}_i, \quad \rho(z_i) = \rho(\overline{z}_i) = v_i, \quad \rho(T^\alpha) = 0.$$

and by $\rho(ZZ') = \rho(Z) + \rho(Z')$. We remark that $\mathfrak{v}_u$ and ρ are consistent with (5.1). We next observe that both $\mathfrak{v}_u$ and ρ are preserved by $\widehat{\psi}_u$, $\widehat{\phi}_u$ and by (5.23). This implies (5.25).

Now we use Lemma 5.9 and (5.25) to conclude

$$z_h - T^{e(h;u)} \prod_{j=1}^n z_j^{m_j} \in SR_\omega(X).$$

The proof of Proposition 5.8 is now complete. □

Next we prove

Lemma 5.12. *Let $P(X)$ be the linear relation ideal defined in Definition 5.3. Then*

$$\widehat{\psi}_u(P(X)) = \left(\frac{\partial \mathfrak{PO}_0^u}{\partial y_i}; i = 1, \cdots, n \right).$$

Proof. Let $\sum_{i=1}^m v_{i,j} z_i$ be in $P(X)$. Then we have

$$\widehat{\psi}_u \left(\sum_{i=1}^m v_{i,j} z_i \right) = \sum_{i=1}^m v_{i,j} \overline{z}_i = \sum_{i=1}^m y_j \frac{\partial \overline{z}_i}{\partial y_j} = y_j \frac{\partial \mathfrak{PO}_0^u}{\partial y_i}$$

by (5.1) and (5.4). Since y_j's are invertible in $\Lambda[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}]$, this identity implies the lemma. □

The proof of Theorem 5.1 and of Proposition 5.7 is now complete. □

We define

$$\psi_{u',u} : Jac(\mathfrak{PO}_0^u) \to Jac(\mathfrak{PO}_0^{u'})$$

by

$$\psi_{u',u}(\overline{z}_i(u)) = \overline{z}_i(u') = T^{\ell_i(u') - \ell_i(u)} \overline{z}_i(u). \tag{5.26}$$

It is an isomorphism. We have

$$\psi_{u',u} \circ \psi_u = \psi_{u'}.$$

The well-definedness $\psi_{u',u}$ is proved from this formula or by checking directly.

In case no confusion can occur, we identify $Jac(\mathfrak{PO}_0^u)$, $Jac(\mathfrak{PO}_0^{u'})$ by $\psi_{u',u}$ and denote them by $Jac(\mathfrak{PO}_0)$. Since $\psi_{u',u}(\overline{z}_i(u)) = \overline{z}_i(u')$ we write them $\overline{z}_i$ when we regard it as an element of $Jac(\mathfrak{PO}_0)$. Note $\psi_{u',u}(y_i) \neq y_i$. In case we regard $y_i \in Jac(\mathfrak{PO}_0^u)$ as an element of $Jac(\mathfrak{PO}_0)$ we write it as $y_i(u) := \psi_{0,u}(y_i)$.

Remark 5.13. The above proof of Theorem 5.1 uses Batyrev's presentation of quantum cohomology ring and is not likely generalized beyond the case of compact toric manifolds. (In fact the proof is purely algebraic and do *not* contain serious study of pseudo-holomorphic curve, except Proposition 5.6, which we quote without proof and Theorem 3.4, which is a minor improvement of an earlier result of [CO].) There is an alternative way of constructing the ring homomorphism ψ_u which is less computational. (This will give a new proof of Proposition 5.6.) We will give this conceptual proof in a sequel to this paper.

We use the operations

$$\mathfrak{q}_{1,k;\beta} : H(X;\mathbb{Q})[2] \otimes B_k H(L(u);\mathbb{Q})[1] \to H(L(u);\mathbb{Q})[1]$$

which was introduced by the authors in section 13 [FOOO2]. Using the class $z_i \in H^2(X;\mathbb{Z})$ the Poincaré dual to $\pi^{-1}(\partial_i P)$ we put

$$\psi_u(z_i) = \sum_k \sum_{i=1}^m T^{\beta_i \cap \omega/2\pi} \int_{L(u)} \mathfrak{q}_{1,k;\beta_i}(z_i \otimes b^{\otimes k}). \tag{5.27}$$

Here we put $b = \sum x_i \mathbf{e}_i$ and the right hand side is a formal power series of x_i with coefficients in Λ.

Using the description of the moduli space defining the operators $\mathfrak{q}_{1,k;\beta}$ (See section 10.) it is easy to see that the right hand side of (5.27) coincides with the definition of $\overline{z}_i$ in the current case. Extending the expression (5.27) to an arbitrary homology class x of arbitrary degree we obtain

$$\psi_u(z) = \sum_k \sum_{\beta;\mu(\beta)=\deg x} T^{\beta\cap\omega/2\pi} \int_{L(u)} \mathfrak{q}_{1,k;\beta}(z \otimes b^{\otimes k}). \tag{5.28}$$

Since $\mu(\beta) = \deg z$, $\mathfrak{q}_{1,k;\beta}(z\otimes b^{\otimes k}) \in H^n(L(u);\mathbb{Q})$. One can prove that (5.28) defines a ring homomorphism from quantum cohomology to the Jacobian ring $Jac(\mathfrak{PO}^u)$. We may regard $Jac(\mathfrak{PO}^u)$ as the moduli space of deformations of Floer theories of Lagrangian fibers of X. (Note the Jacobian ring parameterizes deformations of a holomorphic function up to an appropriate equivalence. In our case the equivalence is the right equivalence, that is, the coordinate change of the *domain*.)

Thus (5.28) is a particular case of the ring homomorphism

$$QH(X) \to HH(\mathfrak{Lag}(X))$$

where $HH(\mathfrak{Lag}(X))$ is the Hochschild cohomology of Fukaya category of X. (We remark that Hochschild cohomology parameterizes deformations of A_∞ category.) Existence of such a homomorphism is a folk theorem which is verified by various people in various favorable situation. (See for example [Aur].) It is conjectured to be an isomorphism under mild conditions by various people including P. Seidel and M. Kontsevich.

This point of view is suitable for generalizing our story to more general X (to non-Fano toric manifolds, for example) and also for including big quantum cohomology group into our story. (We will then also need to use the operators $\mathfrak{q}_{\ell,k}$ mentioned above for $\ell \geq 2$.)

These points will be discussed in subsequent papers in this series of papers. In this paper we follow more elementary approach exploiting the known calculation of quantum cohomology of toric manifolds, although it is less conceptual.

Remark 5.14. There are two other approaches towards a proof of Proposition 5.6 besides the fixed point localization. One is written by Cieliebak and Salamon [CS] which uses vortex equations (gauged sigma model) and the other is written by McDuff and Tolman [MT] which uses Seidel's result [Se1].

6. Localization of quantum cohomology ring at moment polytope

In this section, we discuss applications of Theorem 1.9. In particular, we prove Theorem 1.12. (Note Theorem 1.3 is a consequence of Theorem 1.12.) The next theorem and Theorem 1.9 immediately imply (1) of Theorem 1.12.

Theorem 6.1. *There exists a bijection*

$$\mathfrak{M}_{+,0}(\mathfrak{Lag}(X)) \cong Hom(Jac(\mathfrak{PO}_0); \Lambda^{\mathbb{C}}).$$

Here the right hand side is the set of unital $\Lambda^{\mathbb{C}}$-algebra homomorphisms.
We start with the following definition

Definition 6.2. For an element $x \in \Lambda \setminus \{0\}$, we define its valuation $\mathfrak{v}_T(x)$ as the unique number $\lambda \in \mathbb{R}$ such that $T^{-\lambda}x \in \Lambda_0 \setminus \Lambda_+$.

We note that $\mathfrak{v}_T$ is multiplicative non-Archimedean valuation, i.e., satisfies

$$\begin{aligned} \mathfrak{v}_T(x+y) &\geq \min(\mathfrak{v}_T(x), \mathfrak{v}_T(y)), \\ \mathfrak{v}_T(xy) &= \mathfrak{v}_T(x) + \mathfrak{v}_T(y). \end{aligned}$$

Lemma 6.3. *For any $\varphi \in Hom(Jac(\mathfrak{PO}_0); \Lambda^{\mathbb{C}})$ there exists a unique $u \in M_{\mathbb{R}}$ such that*

$$\mathfrak{v}_T(\varphi(y_j(u))) = 0 \tag{6.1}$$

for all $j = 1, \cdots, n$.

Proof. We still assume Condition 5.11. By definition (5.1) of $\overline{z}_i$, homomorphism property of φ and multiplicative property of valuation, we obtain

$$\mathfrak{v}_T(\varphi(\overline{z}_i)) = \ell_i(u) + \sum_{j=1}^{n} v_{i,j}\mathfrak{v}_T(\varphi(y_j(u))), \tag{6.2}$$

for $i = 1, \cdots, m$. On the other hand, since $\ell_i(u) = \langle u, v_i\rangle - \lambda_i$ and $(v_{i,j})_{i,j=1,\cdots,n}$ is invertible, there is a unique u that satisfies

$$\mathfrak{v}_T(\varphi(\overline{z}_i)) = \ell_i(u) \tag{6.3}$$

for $i = 1, \cdots, n$. But by the invertibility of $(v_{i,j})_{i,j=1,\cdots,n}$ and (6.2), this is equivalent to (6.1) and hence the proof. □

We remark that obviously by the above proof the formula (6.3) automatically holds for $i = n + 1, \cdots, m$ and u in Lemma 6.3 as well.

Proof of Theorem 6.1. Consider the maps

$$\Psi_1(\varphi) = u \in M_{\mathbb{R}}, \qquad \Psi_2(\varphi) = \sum_{i=1}^{n} (\log \varphi(y_i(u)))\mathbf{e}_i \in H^1(L(u); \Lambda_0)$$

where u is obtained as in Lemma 6.3. Since $y_i(u) \in \Lambda_0 \setminus \Lambda_+$ it follows that we can define its logarithm on Λ_0 as a convergent power series with respect to the non-Archimedean norm.

Set $(u,b) = (\Psi_1(\varphi), \Psi_2(\varphi))$. Since φ is a ring homomorphism from $Jac(\mathfrak{PO}_0) \cong Jac(\mathfrak{PO}_0^u)$ it follows from the definition of the Jacobian ring that

$$\frac{\partial \mathfrak{PO}_0^u}{\partial y_i}(b) = 0.$$

Therefore by Theorem 3.9, $HF(L(u,b),(L(u),b);\Lambda) \neq 0$. We have thus defined

$$\Psi : Hom(Jac(\mathfrak{PO}_0);\Lambda^{\mathbb{C}}) \to \mathfrak{M}_{+,0}(\mathfrak{Lag}(X)).$$

Let $(u,b) \in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$. We put $b = \sum x_i \mathbf{e}_i$. We define a homomorphism $\varphi : Jac(\mathfrak{PO}_0) \to \Lambda$ by assigning

$$\varphi(y_i(u)) = e^{x_i}.$$

It is straightforward to check that φ is well defined. Then we define $\Phi(u,b) := \varphi$. It easily follows from definition that Φ is an inverse to Ψ. The proof of Theorem 6.1 is complete. □

We next work with the (Batyrev) quantum cohomology side.

Definition 6.4. For each z_i, we define a Λ-linear map $\widehat{z}_i : QH^\omega(X;\Lambda^{\mathbb{C}}) \to QH^\omega(X;\Lambda^{\mathbb{C}})$ by $\widehat{z}_i(z) = z_i \cup_Q z$, where $\cup_Q$ is the product in $QH^\omega(X;\Lambda^{\mathbb{C}})$.

Since $QH^\omega(X;\Lambda)$ is generated by even degree elements it follows that it is commutative. Therefore we have

$$\widehat{z}_i \circ \widehat{z}_j = \widehat{z}_j \circ \widehat{z}_i. \tag{6.4}$$

Definition 6.5. For $\mathfrak{w} = (\mathfrak{w}_1, \cdots, \mathfrak{w}_n) \in (\Lambda^{\mathbb{C}})^n$ we put

$$QH^\omega(X;\mathfrak{w}) = \{x \in QH^\omega(X;\Lambda^{\mathbb{C}}) \mid (\widehat{z}_i - \mathfrak{w}_i)^N x = 0 \quad \text{for } i = 1, \cdots, n \text{ and large } N.\}$$

We say that $\mathfrak{w}$ is a *weight* of $QH^\omega(X)$ if $QH^\omega(X;\mathfrak{w})$ is nonzero. We denote by $W(X;\omega)$ the set of weights of $QH^\omega(X)$.

We remark that $\mathfrak{w}_i \neq 0$ since z_i is invertible. (Corollary 5.10.)

Proposition 6.6. (1) *There exists a factorization of the ring*

$$QH^\omega(X;\Lambda^{\mathbb{C}}) \cong \prod_{\mathfrak{w} \in W(X;\omega)} QH^\omega(X;\mathfrak{w}).$$

(2) *There exists a bijection*

$$W(X;\omega) \cong Hom(QH^\omega(X;\Lambda);\Lambda^{\mathbb{C}}).$$

(3) $QH^\omega(X;\mathfrak{w})$ *is a local ring and* (1) *is the factorization to indecomposables.*

Proof. Existence of decomposition (1) as a $\Lambda^{\mathbb{C}}$-vector space is a standard linear algebra. (We remark that $\Lambda^{\mathbb{C}}$ is an algebraically closed field.) If $z \in QH^\omega(X;\mathfrak{w})$ and $z' \in QH^\omega(X;\mathfrak{w}')$ then

$$(z_i - \mathfrak{w}_i)^N \cup_Q (z \cup_Q z') = ((z_i - \mathfrak{w}_i)^N \cup_Q z) \cup_Q z' = 0,$$
$$(z_i - \mathfrak{w}'_i)^N \cup_Q (z \cup_Q z') = ((z_i - \mathfrak{w}'_i)^N \cup_Q z') \cup_Q z = 0.$$

Therefore $z \cup_Q z' \in QH^\omega(X;\mathfrak{w}) \cap QH^\omega(X;\mathfrak{w}')$. This implies that the decomposition (1) is a ring factorization.

Let $\varphi : QH^\omega(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$ be a unital $\Lambda^{\mathbb{C}}$ algebra homomorphism. It induces a homomorphism $QH^\omega(X;\Lambda) \to \Lambda^{\mathbb{C}}$ by (1). We denote this ring homomorphism by the same letter φ. Let $z \in QH^{\omega}(X;\mathfrak{w})$ be an element such that $\varphi(z) \neq 0$. Then we have

$$(\varphi(z) - \mathfrak{w}_i)^N \varphi(z) = \varphi((z_i - \mathfrak{w}_i)^N \cup_Q z) = 0.$$

Therefore

$$\mathfrak{w}_i = \varphi(z_i). \tag{6.5}$$

Since z_i generates $QH^{\omega}(X;\mathfrak{w})$, it follows from (6.5) that there is a unique $\Lambda^{\mathbb{C}}$ algebra homomorphism : $QH^{\omega}(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$. (2) follows.

Since $QH^{\omega}(X;\mathfrak{w})$ is a finite dimensional $\Lambda^{\mathbb{C}}$ algebra and $\Lambda^{\mathbb{C}}$ is algebraically closed, we have an isomorphism

$$\frac{QH^{\omega}(X;\mathfrak{w})}{\text{rad}} \cong (\Lambda^{\mathbb{C}})^k \tag{6.6}$$

for some k. (Here $\text{rad} = \{z \in QH^{\omega}(X;\mathfrak{w}) \mid z^N = 0 \quad \text{for some } N.\}$) Since there is a unique unital $\Lambda^{\mathbb{C}}$-algebra homomorphism : $QH^{\omega}(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$, it follows that $k = 1$. Namely $QH^{\omega}(X;\mathfrak{w})$ is a local ring.

It also implies that $QH^{\omega}(X;\mathfrak{w})$ is indecomposable. □

The result up to here also works for the non-Fano case. But the next theorem will require the fact that X is Fano since we use the equality $QH^{\omega}(X;\Lambda) \cong QH(X;\Lambda)$.

Theorem 6.7. *If X is Fano then* $\mathfrak{M}_+(\mathfrak{Lag}(X)) = \mathfrak{M}(\mathfrak{Lag}(X))$

Proof. Let $\mathfrak{w}$ be a weight. We take $z \in QH^{\omega}(X;\mathfrak{w}) \subset H(X;\Lambda^{\mathbb{C}}) \cong H(X;\mathbb{C}) \otimes \Lambda^{\mathbb{C}}$. We may take z so that

$$z \in (H(X;\mathbb{C}) \otimes \Lambda_0^{\mathbb{C}}) \setminus (H(X;\mathbb{C}) \otimes \Lambda_+^{\mathbb{C}}).$$

Since

$$z_i \cup_Q z \equiv z_i \cup z \mod \Lambda_+^{\mathbb{C}},$$

where $\cup$ is the classical cup product. (We use $QH^{\omega}(X;\Lambda) = QH(X;\Lambda)$ here.) It follows that

$$\mathfrak{w}_i^n z = (\widehat{z}_i)^n(z) = (z_i)^n \cup_Q z \equiv (z_i)^n \cup z \mod \Lambda_+^{\mathbb{C}}$$

Therefore $\mathfrak{w}_i \in \Lambda_+^{\mathbb{C}}$ as $(z_i)^n \cup z = 0$. (6.3) and (6.5) then imply

$$\ell_i(u) = \mathfrak{v}_T(\mathfrak{w}_i) > 0.$$

Namely $u \in \text{Int}P$. □

We are now ready to complete the proof of Theorem 1.12. (1) is Theorem 6.1. (2) is Theorem 6.7. If $QH^{\omega}(X;\Lambda^{\mathbb{C}})$ is semi-simple, then (6.6) and $k = 1$ there implies

$$QH^{\omega}(X;\Lambda^{\mathbb{C}}) \cong (\Lambda^{\mathbb{C}})^{\#W(X;\omega)} \tag{6.7}$$

as a $\Lambda^{\mathbb{C}}$ algebra. (3) follows from (6.7), Proposition 6.6 (2), and Theorem 6.1. The proof of Theorem 1.12 is complete. □

We next explain the factorization in Proposition 6.6 (1) from the point of view of Jacobian ring. Let $(u,b) \in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$.

Definition 6.8. We consider the ideal generated by

$$\frac{\partial}{\partial w_i}\mathfrak{PO}_0^u(y_1 + w_1, \cdots, y_n + w_n)$$

$i = 1, \cdots, n$, in the ring $\Lambda[[w_1, \cdots, w_n]]$ of formal power series where $b = \sum x_i(b)\mathbf{e}_i$ and $y_i = e^{x_i(b)}$. We denote its quotient ring by $Jac(\mathfrak{PO}_0; u, b)$.

Proposition 6.9. (1) *There is a direct product decomposition :*

$$Jac(\mathfrak{PO}_0) \cong \prod_{(u,b)\in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))} Jac(\mathfrak{PO}_0; u, b),$$

as a ring.

(2) *If $(u,b) \in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$ corresponds to $\mathfrak{w} \in W(X;\omega)$ via the isomorphism given in Proposition* 6.6 (2) *and Theorem* 6.1*, then ψ_u induces an isomorphism*

$$\psi_u : QH^{\omega}(X;\mathfrak{w}) \cong Jac(\mathfrak{PO}_0; u, b).$$

(3) *$Jac(\mathfrak{PO}_0; u, b)$ is one dimensional (over Λ) if and only if the Hessian*

$$\left(\frac{\partial^2 \mathfrak{PO}_0^u}{\partial y_i \partial y_j}\right)_{i,j=1,\cdots,n}$$

is invertible over Λ at b.

Proof. Let $\mathfrak{m}(u,b)$ be the ideal generated by $y_i - y_i(b)$, in $Jac(\mathfrak{PO}_0^u)$. Since $Jac(\mathfrak{PO}_0)$ is finite dimensional over Λ it follows that

$$Jac(\mathfrak{PO}_0) \cong \prod_{(u,b)\in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))} Jac(\mathfrak{PO}_0)_{\mathfrak{m}(u,b)}.$$

Here $Jac(\mathfrak{PO}_0)_{\mathfrak{m}(u,b)}$ is the localization of the ring $Jac(\mathfrak{PO}_0)$ at $\mathfrak{m}(u,b)$. Using finite dimensionality of $Jac(\mathfrak{PO}_0)$ again we have $Jac(\mathfrak{PO}_0)_{\mathfrak{m}(u,b)} \cong Jac(\mathfrak{PO}_0; u, b)$. (1) follows.

Now we prove (2). If $z \in QH^{\omega}(X;\mathfrak{w})$ then $(z_i - \mathfrak{w}_i)^N z = 0$. Let $\pi_{u,b}$: $Jac(\mathfrak{PO}_0) \to Jac(\mathfrak{PO}_0; u, b)$ be the projection. We then have

$$(T^{\ell_i(u)} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} - \mathfrak{w}_i)^N \pi_{u,b}(\psi_u(z)) = 0. \tag{6.8}$$

We remark that

$$\mathfrak{w}_i = T^{\ell_i(u')} y_1'^{v_{i,1}} \cdots y_n'^{v_{i,n}} \tag{6.9}$$

if $\mathfrak{w}_i$ corresponds (u', b') and y'_i are exponential of the coordinates of b'. We define the operator $\widehat{y}_i : Jac(\mathfrak{PO}_0; u, b) \to Jac(\mathfrak{PO}_0; u, b)$ by

$$\widehat{y}_i(x) = y_i x.$$

By definition of $Jac(\mathfrak{PO}_0; u, b)$ the eigenvalue of $\widehat{y}_i$ is e^{x_i}, where $b = \sum x_i \mathbf{e}_i$. Therefore (6.8) and (6.9) imply that $\pi_{u,b}(\psi_u(z)) = 0$ unless $(u,b) = (u,b')$. (2) follows.

(3) is a standard fact on Jacobian ring. So we omit the proof. □

We recall that a symplectic manifold (X,ω) is said to be (spherically) monotone if there exists $\lambda > 0$ such that $c_1(X) \cap \alpha = \lambda\,[\omega] \cap \alpha$ for all $\alpha \in \pi_2(X)$. Lagrangian submanifold L of (X,ω) is said to be monotone if there exists $\lambda > 0$ such that $\mu(\beta) = \lambda\omega(\beta)$ for any $\beta \in \pi_2(X, L)$. (Here μ is the Maslov index.) In the monotone case we have the following :

Theorem 6.10. *If X is a monotone compact toric manifold then there exists a unique u_0 such that*

$$\mathfrak{M}(\mathfrak{Lag}(X)) \subset \{u_0\} \times \Lambda$$

i.e., whenever $(u,b) \in \mathfrak{M}(\mathfrak{Lag}(X))$, $u = u_0$. Moreover $L(u_0)$ is monotone.

Remark 6.11. Related results are discussed in [EP1].

Proof. Since X is Fano, we have $QH^{\omega}(X;\Lambda) = QH(X;\Lambda)$. We assume $c_1(X) \cap \alpha = \lambda\,[\omega] \cap \alpha$ with $\lambda > 0$. We define $\cup_\alpha$ by

$$x \cup_Q y = x \cup y + \sum_{\alpha \in \pi_2(X) \setminus \{0\}} T^{\alpha \cap [\omega]/2\pi} x \cup_\alpha y.$$

Then

$$\deg(x \cup_\alpha y) = \deg x + \deg y - 2c_1(X) \cap \alpha = \deg x + \deg y - 2\lambda\alpha \cap [\omega]. \tag{6.10}$$

We define

$$\mathfrak{v}_{\deg}(T) = 2\lambda, \quad \mathfrak{v}_{\deg}(x) = \deg x \quad (\text{for } x \in H(X;\mathbb{Q})).$$

$\mathfrak{v}_{\deg}$ is a multiplicative non-Archimedean valuation on $QH(X;\Lambda)$ such that $\mathfrak{v}_{\deg}(a \cup_Q b) = \mathfrak{v}_{\deg}(a) + \mathfrak{v}_{\deg}(b)$, by virtue of (6.10). Moreover for $c \in \Lambda$ and $a \in QH(X;\Lambda)$ we have $\mathfrak{v}_{\deg}(ca) = 2\lambda\mathfrak{v}_T(c) + \mathfrak{v}_{\deg}(a)$. Now let $\mathfrak{w}$ be a weight and $x \in QH^{\omega}(X;\mathfrak{w})$. Since $\mathfrak{v}_{\deg}(z_i) = 2$ it follows that

$$2\lambda\mathfrak{v}_T(\mathfrak{w}_i) + \mathfrak{v}_{\deg}(x) = \mathfrak{v}_{\deg}(z_i x) = 2 + \mathfrak{v}_{\deg}(x).$$

Therefore if (u,b) corresponds to $\mathfrak{w}$ then $\ell_i(u) = \mathfrak{v}_T(\mathfrak{w}_i) = 1/\lambda$. Namely u is independent of $\mathfrak{w}$. We denote it by u_0.

For $\beta_i \in H_2(X, L(u_0))$ $(i = 1, \cdots, m)$ given by (5.7), we have $\omega(\beta_i) = \ell_i(u_0) = 1/\lambda$. Hence $\mu(\beta_i) = 2\lambda\omega(\beta_i)$. Since β_i generates $H_2(X, L(u_0))$, it follows that $L(u_0)$ is monotone, as required. □

So far we have studied Floer cohomology with $\Lambda^{\mathbb{C}}$-coefficients. We next consider the case of Λ^F coefficient where F is a finite Galois extension of $\mathbb{Q}$. We choose F so that each of the weight $\mathfrak{w}$ lies in $(\Lambda_0^F)^n$. (Since every finite extension of $\Lambda^{\mathbb{Q}}$ is contained in such Λ^F we can always find such an F.) Then we have a decomposition

$$QH^{\omega}(X;\Lambda^F) \cong \prod_{\mathfrak{w} \in W(X;\omega)} QH^{\omega}(X;\mathfrak{w};F). \tag{6.11}$$

It follows that the Galois group $Gal(F/\mathbb{Q})$ acts on $W(X;\omega)$. It induces a $Gal(F/\mathbb{Q})$ action on $\mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$. We write it as $(u,b) \mapsto (\sigma(u), \sigma(b))$. We remark the following:

Proposition 6.12. (1) *$\sigma(u) = u$.*
(2) *We write by $y_i(b)$ the exponential of the coordinates of b. Then $y_i(b) \in \Lambda^F$ and $y_i(\sigma(b)) = \sigma(y_i(b))$.*
(3) *If $QH^{\omega}(X;\Lambda^{\mathbb{Q}})$ is indecomposable, there exists u_0 such that whenever $(u,b) \in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$, $u = u_0$.*

Proof. Let $\mathfrak{w}_i(b)$ corresponds to (u,b). Then

$$\ell_i(\sigma(u)) = \mathfrak{v}_T(\mathfrak{w}_i(\sigma(u,b))) = \mathfrak{v}_T(\sigma\mathfrak{w}_i(u,b)) = \mathfrak{v}_T(\mathfrak{w}_i(u,b)) = \ell_i(u).$$

(1) follows. (2) follows from the definition and (1). (3) is a consequence of (1). □

Am monotone blow up of $\mathbb{C}P^2$ (at one or two points) gives an example where the assumption of Proposition 6.12 (3) is satisfied.

7. Further examples and remarks

In this section we show how we can use the argument of the last 2 sections to illustrate calculations of $\mathfrak{M}(\mathfrak{Lag}(X))$ in examples.

Example 7.1. We consider one point blow up X of $\mathbb{C}P^2$. We choose its Kähler form so that the moment polytope is

$$P = \{(u_1, u_2) \mid 0 \le u_1, u_2,\ u_1 + u_2 \le 1,\ u_2 \le 1 - \alpha\},$$

$0 < \alpha < 1$. The potential function is

$$\mathfrak{PO} = y_1 T^{u_1} + y_2 T^{u_2} + (y_1 y_2)^{-1} T^{1-u_1-u_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

We put $\overline{z}_1 = y_1 T^{u_1}$, $\overline{z}_2 = y_2 T^{u_2}$, $\overline{z}_3 = (y_1 y_2)^{-1} T^{1-u_1-u_2}$, $\overline{z}_4 = y_2^{-1} T^{1-\alpha-u_2}$.

The quantum Stanley-Reisner relation is

$$\overline{z}_1 \overline{z}_3 = \overline{z}_4 T^{\alpha}, \quad \overline{z}_2 \overline{z}_4 = T^{1-\alpha}, \tag{7.1}$$

and linear relation is

$$\overline{z}_1 - \overline{z}_3 = 0, \quad \overline{z}_2 - \overline{z}_3 - \overline{z}_4 = 0. \tag{7.2}$$

We put $X = \overline{z}_1$ and $Y = \overline{z}_2$ and solve (7.1), (7.2). We obtain

$$X^3 (T^{\alpha} + X) = T^{1+\alpha}, \tag{7.3}$$

with $Y = X + T^{-\alpha} X^2$. We consider valuations of both sides of (7.3). There are three different cases to consider.

(Case 1: $\mathfrak{v}_T(X) > \alpha$). (7.3) implies $3\mathfrak{v}_T(X) + \alpha = 1 + \alpha$. Namely $\mathfrak{v}_T(X) = 1/3$. So $\alpha < 1/3$. Moreover $\mathfrak{v}_T(Y) = 1/3$. We have $u_1 = \mathfrak{v}_T(X) = 1/3$, $u_2 = \mathfrak{v}_T(Y) = 1/3$. (See Lemma 6.3.) It easily follows from consideration of the leading term equation of (7.3) that we have three solutions for b : Writing $X = a_1 T^{1/3} + a_2 T^{\lambda} +$ higher order terms with $\lambda > \frac{1}{3}$ and substituting this into (7.3), we get the leading term equation $a_1^3 = 1$ which has 3 simple roots.
(Case 2: $\mathfrak{v}_T(X) < \alpha$). By taking the valuation of (7.3) we obtain $u_1 = \mathfrak{v}_T(X) = (1+\alpha)/4$. Hence $\alpha > 1/3$. Moreover $u_2 = \mathfrak{v}_T(Y) = (1-\alpha)/2$. There are four solutions.
(Case 3: $\mathfrak{v}_T(X) = \alpha$). We put $X = a_1 T^{\alpha} + a_2 T^{\lambda} +$ higher order terms where $\lambda > \alpha$.
(Case 3-1: $a_1 \ne -1$). By taking valuation of (7.3), we obtain $u_1 = \mathfrak{v}_T(X) = 1/3$. Then $\alpha = 1/3$ and $u_2 = \mathfrak{v}_T(Y) = 1/3$. (7.3) becomes

$$a_1^4 + a_1^3 - 1 = 0. \tag{7.4}$$

(In this case $X = a_1 T^{\alpha}$ has no higher term.) There are four solutions. We remark that (7.4) is irreducible over $\mathbb{Q}$. Namely the assumption of Proposition 6.12 (3) is satisfied. Actually X is monotone in the case $\alpha = 1/3$. Hence the same conclusion (uniqueness of u) follows from Theorem 6.10 also.
(Case 3-2: $a_1 = -1$). By taking valuation of (7.3), we obtain $\lambda = 1 - 2\alpha$. $\lambda > \alpha$ implies $\alpha < 1/3$. $u_2 = \mathfrak{v}_T(Y) = 1 - 2\alpha$. ($u_1 = \mathfrak{v}_T(X) = \alpha$.) There is one solution.

In summary, if $\alpha < 1/3$ there are two choices of $u = (\alpha, 1-2\alpha), (1/3, 1/3)$. On the other hand the numbers of choices of b are 1 and 3 respectively.

If $\alpha \ge 1/3$ there is the unique choice $u = ((1+\alpha)/4, (1-\alpha)/2)$. The number of choices of b is 4.

We next study the non-Fano case. We will study Hirzerbruch surface F_n. Note F_1 is one point blow up of $\mathbb{C}P^2$ which we have already studied. We leave the case F_2 to the reader.

Example 7.2. We consider Hirzerbruch surface F_n, $n \geq 3$. We take its Kähler form so that the moment polytope is

$$P = \{(u_1, u_2) \mid 0 \leq u_1, u_2,\ u_1 + nu_2 \leq n,\ u_2 \leq 1 - \alpha\},$$

$0 < \alpha < 1$. The leading order potential function is

$$\mathfrak{PO}_0 = y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} y_2^{-n} T^{n-u_1-nu_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

We put $\overline{z}_1 = y_1 T^{u_1}$, $\overline{z}_2 = y_2 T^{u_2}$, $\overline{z}_3 = y_1^{-1} y_2^{-n} T^{n-u_1-nu_2}$, $\overline{z}_4 = y_2^{-1} T^{1-\alpha-u_2}$.

The quantum Stanley-Reisner relation and linear relation gives

$$\overline{z}_1 \overline{z}_3 = \overline{z}_4^n T^{n\alpha}, \qquad \overline{z}_2 \overline{z}_4 = T^{1-\alpha}, \tag{7.5}$$

$$\overline{z}_1 - \overline{z}_3 = 0, \qquad \overline{z}_2 - n\overline{z}_3 - \overline{z}_4 = 0. \tag{7.6}$$

Let us assume n is odd. We put

$$z_1 = Z^n, \qquad \overline{z}_4 = Z^2 T^{-\alpha}.$$

(In case $n = 2n'$ is even we put $z_1 = Z^{n'}$, $\overline{z}_4 = \pm Z T^{-\alpha}$. The rest of the argument are similar and is omitted.) Then $\overline{z}_2 = T^{-\alpha} Z^2 + nZ^n$ and

$$Z^4 (nZ^{n-2} + T^{-\alpha}) = T. \tag{7.7}$$

(Case 1 : $(n-2)\mathfrak{v}_T(Z) > -\alpha$). In the first case, we have $\mathfrak{v}_T(Z) = (\alpha+1)/4$. (Then $(n-2)\mathfrak{v}_T(Z) > -\alpha$ is automatically satisfied.) Therefore $u_1 = \mathfrak{v}_T(z_1) = n(\alpha+1)/4$, $u_2 = \mathfrak{v}_T(z_2) = (1-\alpha)/2$. We also can check that there are 4 solutions. We remark that we are using $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$. However we can easily check the strongly nondegeneracy condition of Corollary 9.5 in our case and hence each critical point of $\mathfrak{PO}_0$ corresponds to a unique critical point of $\mathfrak{PO}$ with the same u. Hence $L(u)$ is balanced.

(Case 2 : $(n-2)\mathfrak{v}_T(Z) < -\alpha$). We have $v_T(z) = 1/(n+2)$. This can never occur since $1/(n+2) > 0 > -\alpha/(n-2)$.

(Case 3 : $(n-2)\mathfrak{v}_T(Z) = -\alpha$). We put $Z = a_1 T^{-\alpha/(n-2)} + a_2 T^{\lambda} +$ higher order term.

(Case 3-1 : $na_1^{n-2} \neq -1$). Then $\mathfrak{v}_T(Z) = (\alpha+1)/4$. Since $(\alpha+1)/4 \neq -\alpha/(n-2)$, this case never occur.

(Case 3-2 : $na_1^{n-2} = -1$). We have $4v_T(Z) + (n-3)v_T(Z) + \lambda = 1$. Therefore

$$\lambda = \frac{n - 2 + (n+1)\alpha}{n-2}.$$

We have

$$u_1 = v_T(z_1) = -\frac{n\alpha}{n-2}, \qquad u_2 = v_T(z_2) = 1 - \alpha - v_T(z_4) = \frac{n-2+2\alpha}{n-2}.$$

Thus (u_1, u_2) is *not* in the moment polytope.

In Example 7.2, we have

$$\mathfrak{M}(\mathfrak{Lag}(X)) \neq \mathfrak{M}_{+,0}(\mathfrak{Lag}(X)).$$

On the other hand, the order of $\mathfrak{M}(\mathfrak{Lag}(X))$ is 4 and is equal to the Betti number.

Conjecture 7.3. *Let X be a compact toric manifold which is not necessarily Fano. If $QH(X;\Lambda)$ is semi-simple then*

$$\sum_d \operatorname{rank} H_d(X;\mathbb{Q}) = \#(\mathfrak{M}(\mathfrak{Lag}(X))).$$

We next discuss the version of the above story where we substitute some explicit number into the formal variable T. We define a Laurent polynomial

$$\mathfrak{PO}^u_{0,T=t} \in \mathbb{C}[y_1,\cdots,y_n,y_1^{-1},\cdots,y_n^{-1}]$$

by substituting a complex number $t \in \mathbb{C}\setminus\{0\}$. In the same way we define the algebra $QH^\omega(X;T=t;\mathbb{C})$ over $\mathbb{C}$ by substituting $T=t$ in the quantum Stanley-Reisner relation. The argument of section 5 goes through to show

$$QH^\omega(X;T=t;\mathbb{C}) \cong Jac(\mathfrak{PO}^u_{0,T=t}). \tag{7.8}$$

In particular the right hand side is independent of u up to an isomorphism. Here the $\mathbb{C}$-algebra in the right hand side of (7.8) is the quotient of the polynomial ring $\mathbb{C}[y_1,\cdots,y_n,y_1^{-1},\cdots,y_n^{-1}]$ by the ideal generated by $\partial\mathfrak{PO}_{0,T=t}/\partial y_i$. ($i = 1,\cdots,n$.)

We remark that right hand side of (7.8) is always nonzero, for small t, by Proposition 3.6. It follows that the equation

$$\frac{\partial\mathfrak{PO}^u_{0,T=t}}{\partial y_i} = 0 \tag{7.9}$$

has a solution $y_i \neq 0$ for *any* u. Namely, as far as the Floer cohomology after $T=t$ substituted, there *always* exists $b \in H^1(X;\mathbb{C})$ with nonvanishing Floer homology. Since the version of Floer cohomology after substituting $T=t$ is *not* invariant under the Hamiltonian isotopy, this is not useful for the application to symplectic topology. (Compare this with section 14.2 [CO].)

The relation between the set of solutions of (7.9) and that of (3.9) is as follows : Let $(y_1^{(c)}(t;u),\cdots,y_n^{(c)}(t;u))$ be a branch of the solutions of (7.9) for $t\neq 0$ where c is an integer with $1\le c\le l$ for some $l\in\mathbb{N}$. We can easily show that it is a holomorphic function of t on $\mathbb{C}\setminus\{0\}$. We consider its behavior as $t\to 0$. For usual u the limit either diverges or converges to 0. However if $b=\sum x_i\mathbf{e}_i$ lies in $\mathfrak{M}_0(\mathfrak{Lag}(X))$ then there is some c such that

$$\lim_{t\to 0} y_i^{(c)}(t;u) \in \mathbb{C}\setminus\{0\} \quad \text{and that} \quad y_i^{(c)}(t;u) = e^{x_i(t)}.$$

The rest of this section owes much to the discussion with H. Iritani and also to his papers [Iri1], [Iri2]. The results we describe below will not be used in the other part of this paper.

We go back to the discussion on the difference between two moduli spaces $\mathfrak{M}_0(\mathfrak{Lag}(X))$ and $\mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$. We recall that we did *not* take closure of the ideal $(P(X)+SR_\omega(X))$ in section 5. This is actually the reason why we have $\mathfrak{M}_0(\mathfrak{Lag}(X)) \neq \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$. More precisely we have the following Proposition 7.4.

We consider the polynomial ring $\Lambda[z_1,\cdots,z_m]$. We define its norm $\|\cdot\|$ so that

$$\left\| \sum_{\vec{i}} a_{\vec{i}} z_1^{i_1}\cdots z_m^{i_m} \right\| = \exp\left(-\inf_{\vec{i}} \mathfrak{v}_T(a_{\vec{i}})\right).$$

We take the closure of the ideal $(P(X)+SR_\omega(X))$ with respect to this norm and denote it by $\mathrm{Clos}(P(X)+SR_\omega(X))$. We put

$$\overline{QH}^\omega(X;\Lambda) = \frac{\Lambda[z_1,\cdots,z_m]}{\mathrm{Clos}(P(X)+SR_\omega(X))}. \tag{7.10}$$

Let $W^{\rm geo}(X;\omega)$ be the set of all weight such that the corresponding (u,b) satisfies $u \in {\rm Int}\, P$. We remark that $\mathfrak w \in W^{\rm geo}(X;\omega)$ if and only if $\mathfrak v_T(\mathfrak w_i) > 0$ for all i.

Proposition 7.4 (Iritani)**.** *There exists an isomorphism*

$$\overline{QH}^{\omega}(X;\Lambda^{\mathbb C}) \cong \prod_{\mathfrak w \in W^{\rm geo}(M;\omega)} QH^{\omega}(X;\mathfrak w).$$

Proof. Let $\mathfrak w \in W(X;\omega) \setminus W^{\rm geo}(X;\omega)$. We first assume $\mathfrak v_T(\mathfrak w_i) = -\lambda < 0$. (The case $\mathfrak v_T(\mathfrak w_i) = 0$ will be discussed at the end of the proof.)

Then, there exists $f \in \Lambda_0 \setminus \Lambda_+$ such that $T^{\lambda} f \mathfrak w_i = 1$. Let $x \in QH^{\omega}(X;\mathfrak w)$. We have $T^{\lambda} f z_i x = x$. Since $\lim_{N\to\infty} \Vert (f z_i T^{\lambda})^N \Vert = 0$, it follows that $x = 0$ in $\overline{QH}^{\omega}(X;\Lambda^{\mathbb C})$.

We next assume $\mathfrak v_T(\mathfrak w_i) > 0$ for all i. We consider the homomorphism

$$\varphi : \Lambda[z_1,\cdots,z_m] \to Hom_{\Lambda}(QH^{\omega}(X;\mathfrak w), QH^{\omega}(X;\mathfrak w)),$$

defined by

$$\varphi(z_i)(x) = z_i \cup_Q x.$$

We have $\varphi(P(X) + SR_{\omega}(X)) = 0$. We may choose the basis of $QH^{\omega}(X;\mathfrak w)$ so that $\varphi(z_i)$ is upper triangular matrix whose diagonal entries are all $\mathfrak w_i$ and whose off diagonal entries are all 0 or 1. We use it and $\mathfrak v_T(\mathfrak w_i) > 0$ to show that $\varphi({\rm Clos}(P(X) + SR_{\omega}(X))) = 0$. Namely φ induces a homomorphism from $\overline{QH}^{\omega}(X;\Lambda)$. It follows easily that the restriction of the projection $QH^{\omega}(X;\Lambda^{\mathbb C}) \to \overline{QH}^{\omega}(X;\Lambda^{\mathbb C})$ to $QH^{\omega}(X;\mathfrak w)$ is an isomorphism to its image.

We finally show that for $u \in \partial P$, there is no critical point of $\mathfrak{PO}_0$ on $(\Lambda_0 \setminus \Lambda_+)^n$. Let

$$u \in \bigcup_{i\in I} \partial_i P \setminus \bigcup_{i \notin I} \partial_i P_i.$$

Then

$$\mathfrak{PO}_0^u \equiv \sum_{i\in I} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} \mod \Lambda_+.$$

We remark that v_i $(i \in I)$ is a part of the $\mathbb Z$ basis of $\mathbb Z^n$, since X is nonsingular toric. Hence by changing the variables to appropriate y'_i it is easy to see that there is no nonzero critical point of $\sum_{i\in I} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} = \sum_{i \in I'} y'_i$. The proof of Proposition 7.4 is now complete. □

To further discuss the relationship between contents of sections 5 and 6 and those in [Iri2], we compare the coefficient rings used here and in [Iri2]. In [Iri2] (like many of the literatures on quantum cohomology such as [Gi1]) the formal power series ring $\mathbb Q[[q_1,\cdots,q_{m-n}]]$ is taken as the coefficient ring. ($m-n$ is the rank of $H^2(X;\mathbb Q)$ and we choose a basis of it.) The superpotential in [Iri2] (which is the same as the one used in [Gi1]) is given as [2]

$$F_q = \sum_{i=1}^{m} \left(\prod_{a=1}^{m-n} q_a^{l_{a,i}} \prod_{j=1}^{n} s_j^{v_{i,j}} \right). \tag{7.11}$$

Here $l_{a,i}$ is a matrix element of a splitting of $H_2(X;\mathbb Z) \to H_2(X,T^n;\mathbb Z)$. We will show that (7.11) pulls back to our potential function $\mathfrak{PO}_0^u$ after a simple change of

[2]We change the notation so that it is consistent to ours. $m, n, v_{i,j}$ here corresponds to $r+N$, r, $x_{i,b}$ in [Iri2], respectively.

variables. Let $\alpha_a \in H_2(X;\mathbb{Z})$ be the basis we have chosen. (We choose it so that $[\omega]\cap\alpha_a$ is positive.)

Lemma 7.5. *There exists $f_j(u) \in \mathbb{R}$ ($j = 1,\cdots,n$) such that*

$$\frac{1}{2\pi}\sum_a l_{a,i}[\omega]\cap\alpha_a = \ell_i(u) - \sum_j v_{i,j} f_j(u).$$

Proof. We consider the exact sequence

$$0 \longrightarrow H_2(X;\mathbb{Z}) \xrightarrow{i_*} H_2(X,L(u);\mathbb{Z}) \longrightarrow H_1(L(u);\mathbb{Z}) \to 0.$$

$(c_1,\cdots,c_m) \in H_2(X,L(u);\mathbb{Z})$ is in the image of $H_2(X;\mathbb{Z})$ if and only if $\sum_i c_i v_i = 0$. (Here $v_i = (v_{i,1},\cdots,v_{i,n}) \in \mathbb{Z}^n$.) For given $\alpha \in H_2(X,\mathbb{Z})$ denote $i_*(\alpha) = (c_1,\cdots,c_m)$. Then we have

$$\sum_a [\omega]\cap c_i l_{a,i}\alpha_a = [\omega]\cap\alpha = 2\pi\sum c_i\ell_i(u),$$

This implies the lemma. □

We now put

$$q_a = T^{[\omega]\cap\alpha_a/2\pi}, \qquad s_j(u) = T^{f_j(u)}y_j \tag{7.12}$$

We obtain the identity

$$F_q(s_1(u),\cdots,s_n(u)) = \mathfrak{PO}^u_0(y_1,\cdots,y_n). \tag{7.13}$$

We remark that if we change the choice of Kähler form then the identification (7.12) changes. In other words, the story over $\mathbb{Q}[[q_1,\cdots,q_{m-n}]]$ corresponds to studying all the symplectic structures simultaneously, while the story over Λ focus on one particular symplectic structure.

In [Iri2] Corollary 5.12, Iritani proved semi-simplicity of quantum cohomology ring of toric manifold with coefficient ring $\mathbb{Q}[[q_1,\cdots,q_{m-n}]]$. It does not imply the semi-simplicity of our $QH^\omega(X;\Lambda)$ since the semi-simplicity in general is not preserved by the pull-back. (On the other way round, semi-simplicity follows from semi-simplicity of the pull-back.) However it is preserved by pull back at a generic point. Namely we have:

Proposition 7.6. *The set of T^n-invariant symplectic structures on X for which $Jac(\mathfrak{PO}^u_0)$ is semi-simple is open and dense.*

Proof. We give a proof for completeness, following the argument in the proof of Proposition 5.11 [Iri2]. Consider the polynomial

$$F_{w_1,\cdots,w_m} = \sum_{i=1}^m w_i y_1^{v_{i,1}}\cdots y_n^{v_{i,n}}$$

where $w_i \in \mathbb{C}\setminus\{0\}$. By Kushnirenko's theorem [Ku] the Jacobian ring of $F_{w_1,\cdots,w_m}$ is semi-simple for a generic $w_1,\cdots,w_m$. We put

$$w_i = \exp\left(\frac{1}{2\pi}\sum_a l_{a,i}[\omega]\cap\alpha + \sum_j v_{i,j}f_j(u)\right).$$

It is easy to see that when we move $[\omega]\cap\alpha_a$ and u (there are $m-n$, n parameters respectively) then w_i moves in an arbitrary way. Therefore for generic choice of ω and u, the Jacobian ring $Jac(\mathfrak{PO}^u_0)$ is semi-simple. Since $Jac(\mathfrak{PO}^u_0)$ is independent of u up to isomorphism, the proposition follows. □

Remark 7.7. Combined with Theorem 1.9, this proposition gives a partial answer to Question in section 3 [EP2].

8. Variational analysis of potential function

In this section, we prove Proposition 3.6. Let $\mathfrak{PO}$ be defined as in (3.6).

We define

$$s_1(u) = \inf\{\ell_i(u) \mid i = 1, \cdots, m\}.$$

s_1 is a continuous, piecewise affine and convex function and $s_1 \equiv 0$ on ∂P. Recall if $u \in \partial_i P$ then $\ell_i(u) = 0$ by definition.

We put

$$S_1 = \sup\{s_1(u) \mid u \in P\},$$
$$P_1 = \{u \in P \mid s_1(u) = S_1\}.$$

Proposition 8.1. *There exist s_k, S_k, and P_k with the following properties.*

(1) *P_k is a convex polyhedron in $M_{\mathbb{R}}$.* $\dim P_k \le \dim P_{k-1} - 1$.
(2) *$s_{k+1} : P_k \to \mathbb{R}$ is a continuous, convex piecewise affine function.*
(3) *$s_{k+1}(u) = \inf\{\ell_i(u) \mid \ell_i(u) > S_k\}$ for $u \in \mathrm{Int} P_k$.*
(4) *$s_{k+1}(u) = S_k$ for $u \in \partial P_k$.*
(5) *$S_{k+1} = \sup\{s_{k+1}(u) \mid u \in P_k\}$.*
(6) *$P_{k+1} = \{u \in P_k \mid s_{k+1}(u) = S_{k+1}\}$.*
(7) *$P_{k+1} \subset \mathrm{Int} P_k$.*
(8) *s_k, S_k, P_k are defined for $k = 1, 2, \cdots, K$ for some $K \in \mathbb{Z}_+$ and P_K consists of a single point.*

Example 8.2. Let $P = [0, a] \times [0, b]$ $(a < b.)$ Then $s_1(u_1, u_2) = \inf\{u_1, u_2, a - u_1, b - u_2\}$. $S_1 = a/2$, $P_1 = \{(a/2, u_2) \mid a/2 \le u_2 \le b - a/2\}$, $s_2(1/2, u_2) = \inf\{u_2, b - u_2\}$, $S_2 = b/2$, $P_2 = \{(a/2, b/2)\}$.

Proof. We define s_k, S_k, P_k inductively over k. We assume that s_k, S_k, P_k are defined for $k = 1, \cdots, k_0$ so that (1) - (7) of Proposition 8.1 are satisfied for $k = 1, \cdots, k_0 - 1$.

We define s_{k_0+1} by (3) and (4). We will prove that it satisfies (2). We use the following lemma for this purpose.

Lemma 8.3. *Let $u_j \in \mathrm{Int} P_{k_0}$ and $\lim_{j\to\infty} u_j = u_\infty \in \partial P_{k_0}$. Then*

$$\lim_{j\to\infty} s_{k_0+1}(u_j) = S_{k_0}.$$

Proof. We put

$$I_{k_0} = \{\ell_i \mid \ell_i(u_\infty) = S_{k_0}\}. \tag{8.1}$$

We take the affine space $A_l \subset M_{\mathbb{R}}$ such that $\mathrm{Int} P_{k_0}$ is an open subset of A_l. We take $\vec{u} \in T_{u_\infty} A_l$ such that $u + \epsilon\vec{u} \notin P_{k_0}$ for sufficiently small positive ϵ. We use Proposition 8.1 (7) for $k = k_0 - 1$. Then we have $u + \epsilon\vec{u} \in P_{k_0-1}$.

Therefore we have

$$s_{k_0}(u + \epsilon\vec{u}) < S_{k_0}.$$

It follows that there exists $\ell_i \in I_{k_0}$ such that

$$\ell_i(u + \epsilon\vec{u}) < \ell_i(u_\infty) < \ell_i(u - \epsilon\vec{u}). \tag{8.2}$$

Since (8.2) holds for any $\vec{u} \in T_{u_\infty} A^l$ with $u + \epsilon\vec{u} \notin P_{k_0}$, it follows that, for any sufficiently large j there exists $i \in I$ such that

$$\ell_i(u_j) > S_{k_0}. \tag{8.3}$$

Therefore

$$s_{k_0+1}(u_j) = \inf\{\ell_i(u_j) \mid \ell_i \in I_{k_0},\ \ell_i(u_j) > S_{k_0}\}. \tag{8.4}$$

This implies the lemma. □

Lemma 8.3 implies that s_{k_0+1} is continuous and piecewise linear in a neighborhood of ∂P_{k_0}. We can then check (2) easily.

We define S_{k_0+1} by (5). Then we can define P_{k_0+1} by (6). (In other words the right hand side of (6) is nonempty.) (7) is a consequence of Lemma 8.3. We can easily check that P_{k_0+1} satisfies (1). (We remark that ℓ_i is not constant.) We can continue the induction until P_{k_0} becomes 0-dimensional (namely a point). Hence we have (8). The proof of Proposition 8.1 is now complete. □

The next lemma easily follows from construction.

Lemma 8.4. *If all the vertices of P lie in $\mathbb{Q}^n$ then $u_0 \in \mathbb{Q}^n$. Here $\{u_0\} = P_K$.*

By parallelly translating the polytope, we may assume, without loss of generality, that $u_0 = \mathbf{0}$, the origin. In the rest of this subsection, we will prove that $\mathfrak{PO}^{\mathbf{0}}$ has a critical point on $(\Lambda_0 \setminus \Lambda_+)^n$. More precisely we prove Proposition 3.6 for $u_0 = \mathbf{0}$. (We remark that if P and ℓ_i are given we can easily locate u_0.)

Example 8.5. Let us consider Example 7.1 in the case $\alpha > 1/3$. At $u_0 = ((1+\alpha)/4, (1-\alpha)/2)$ we have

$$\mathfrak{PO}^{u_0} = (y_2 + y_2^{-1})T^{(1-\alpha)/2} + (y_1 + (y_1y_2)^{-1})T^{(1+\alpha)/4}.$$

Therefore the constant term $\mathfrak{y}_{i;0}$ of the coordinate y_i of the critical point is given by

$$1 - \mathfrak{y}_{2;0}^{-2} = 0, \quad 1 - \mathfrak{y}_{1;0}^{-2}\mathfrak{y}_{2;0}^{-1} = 0. \tag{8.5}$$

Note the first equation comes from the term of the smallest exponent and contains only $\mathfrak{y}_{2;0}$. The second equation comes from the term which has second smallest exponent and contains both $\mathfrak{y}_{1;0}$ and $\mathfrak{y}_{2;0}$. So we need to solve the equation inductively according to the order of the exponent. This is the situation we want to work out in general.

We remark that the affine space A_i defined above in the proof of Lemma 8.3

$$M_{\mathbb{R}} = A_0 \supset A_1 \supset \cdots \supset A_{K-1} \supset A_K = \{\mathbf{0}\}$$

is a strictly decreasing sequence of linear subspaces such that Int P_k is an open subset of A_k. Let

$$A_l^{\perp} \subset (M_{\mathbb{R}})^* \cong N_{\mathbb{R}}$$

be the annihilator of $A_l \subset M_{\mathbb{R}}$. Then we have

$$\{\mathbf{0}\} = A_0^{\perp} \subset A_1^{\perp} \subset \cdots \subset A_{K-1}^{\perp} \subset A_K^{\perp} = N_{\mathbb{R}}.$$

We recall that the formula (8.1) in our case is

$$I_k = \{\ell_i \mid \ell_i(\mathbf{0}) = S_k\}, \tag{8.6}$$

for $k = 1, \cdots, K$. We renumber $\bigcup_k I_k$ so that

$$\{\ell_{k,j} \mid j = 1, \cdots, a(k)\} = I_k. \tag{8.7}$$

By construction

$$s_k(u) = \inf_j \ell_{k,j}(u) \tag{8.8}$$

in a neighborhood of $\mathbf{0}$ in P_{k-1}. In fact $s_{k-1}(\mathbf{0}) = S_{k-1} < S_k = s_k(\mathbf{0})$ and $\{\ell_i(\mathbf{0}) \mid i = 1, \cdots, m\} \cap (S_{k-1}, S_k) = \emptyset$.

Lemma 8.6. *If $u \in A_k$ then $\ell_{k,j}(u) = S_k$.*

Proof. We may assume $k < K$. Hence $\mathbf{0} \in \mathrm{Int} P_k$. We regard $u \in A_k = T_{\mathbf{0}} A_k$. By (8.8), we have

$$s_k(\varepsilon u) = \inf\{\ell_{k,j}(\epsilon u) \mid j = 1, \cdots, a(k)\}.$$

Since $s_k(\varepsilon u) = S_k$ for $\varepsilon u \in P_k$ it follows that $\ell_{k,j}(u) = S_k$. □

Lemma 8.6 implies that the linear part $d\ell_{k,j}$ of $\ell_{k,j}$ is an element of $A_k^{\perp} \subset \mathfrak{t} = N_{\mathbb{R}}$. In fact if $\ell_{k,j} = \ell_i$, we have $d\ell_{k,j} = v_i$ from the definition of ℓ_i, $\ell_i(u) = \langle u, v_i \rangle - \lambda_i$ given in Theorem 2.13.

Lemma 8.7. *For any $v \in A_k^{\perp}$, there exists nonnegative real numbers $c_j \geq 0$, $j = 1, \cdots, a(k)$ such that*

$$v - \sum_{j=1}^{a(k)} c_j d\ell_{k,j} \in A_{k-1}^{\perp}.$$

Proof. Suppose to the contrary that

$$\left\{ v - \sum_{j=1}^{a(k)} c_j d\ell_{k,j} \,\middle|\, c_j \geq 0,\ j = 1, \cdots, a(k) \right\} \cap A_{k-1}^{\perp} = \emptyset.$$

Then we can find $u \in A_{k-1} \setminus A_k$ such that

$$d\ell_{k,j}(u) \geq 0 \tag{8.9}$$

for all $j = 1, \cdots, a(k)$.

Since $\varepsilon u \in A_{k-1} \setminus A_k$ it follows that

$$s_k(\varepsilon u) < S_k$$

for a sufficiently small ε. On the other hand, (8.9) implies $d\ell_{k,j}(\varepsilon u) \geq 0$ for all $\varepsilon > 0$ and so $\ell_{k,j}(\varepsilon u) \geq \ell_{k,j}(\mathbf{0}) = S_k$. Therefore by definition of s_k in Proposition 8.1 we have

$$\begin{aligned} s_k(\varepsilon u) &\geq \inf\{\ell_{k,j}(\varepsilon u) \mid j = 1, \cdots, a(k)\} \\ &\geq \inf\{\ell_{k,j}(\mathbf{0}) \mid j = 1, \cdots, a(k)\} = S_k. \end{aligned}$$

This is a contradiction. □

Applying Lemma 8.7 inductively downwards starting from $\ell = k$ ending at $\ell = 1$, we immediately obtain the following

Corollary 8.8. *For any $v \in A_k^{\perp}$, there exist $c_{l,j} \geq 0$ for $l = 1, \cdots, k$, $j = 1, \cdots, a(l)$ such that*

$$v = \sum_{l=1}^{k} \sum_{j=1}^{a(l)} c_{l,j} d\ell_{l,j}.$$

We denote

$$\mathfrak{I} = \{\ell_i \mid i = 1, \cdots, m\} \setminus \bigcup_{k=1}^{K} I_k. \tag{8.10}$$

It is easy to see that

$$\ell \in \mathfrak{I} \Rightarrow \ell(\mathbf{0}) > S_K. \tag{8.11}$$

Now we go back to the situation of (3.6). We use the notation of (3.6). In this case, for each $k = 1, \cdots, K$, we also associate a set $\mathfrak{I}_k$ consisting of pairs (ℓ, ρ) with an affine map $\ell : M_{\mathbb{R}} \to \mathbb{R}$ and $\rho \in \mathbb{R}_+$.

Definition 8.9. We say that a pair $(\ell, \rho) = (\ell'_j, \rho_j)$ is an element of $\mathfrak{I}_k$ if the following holds :

(1) If $e^i_j \neq 0$ then $\ell_i \in \bigcup_{l=1}^k I_l$. (Note $\ell'_j = \sum_i e^i_j \ell_i$.)
(2) (1) does not hold if we replace k by $k-1$.

A pair $(\ell, \rho) = (\ell'_j, \rho_j)$ as in (3.6) is, by definition, an element of $\mathfrak{I}_{K+1}$ if it is not contained in any of $\mathfrak{I}_k$, $k = 1, \cdots, K$.

Lemma 8.10. (1) *If $(\ell, \rho) \in \mathfrak{I}_k$ then $d\ell \in A_k^\perp$.*
(2) *If $(\ell, \rho) \in \mathfrak{I}_k$ then $\ell(\mathbf{0}) + \rho > S_k$.*
(3) *If $(\ell, \rho) \in \mathfrak{I}_{K+1}$ then $\ell(\mathbf{0}) + \rho > S_K$.*

Proof. (1) follows from Definition 8.9 (1) and Lemma 8.6.

If $(\ell, \rho) = (\ell'_j, \rho_j) \in \mathfrak{I}_k$ then there exists $e^i_j \neq 0$, $\ell_i = \ell_{k,j'}$. Then

$$\ell(\mathbf{0}) + \rho \geq e^i_j \ell_i(\mathbf{0}) + \rho_j > \ell_i(\mathbf{0}) = S_k.$$

(2) follows. The proof of (3) is the same. □

Lemma 8.11. *The vector space A_k is defined over $\mathbb{Q}$.*

Proof. A_k is defined by equalities of the type $\ell_i = S_k$ on A_{k-1}. Since the linear part of ℓ_i has integer coefficients, the lemma follows by induction on k. □

We put $d(k) = \dim A_{k-1} - \dim A_k = \dim A_k^\perp - \dim A_{k-1}^\perp$. We choose $\mathbf{e}^*_{i,j} \in Hom(M_{\mathbb{Q}}, \mathbb{Q}) \cong N_{\mathbb{Q}}$ $(i = 1, \cdots, K,\ j = 1, \cdots, d(k))$ such that the following condition holds. Here $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$

Condition 8.12. (1) $\mathbf{e}^*_{1,1}, \cdots, \mathbf{e}^*_{k,d(k)}$ is a basis of $A_k^\perp \subset N_{\mathbb{R}}$.
(2) $d\ell_{k,j} = \sum_{k',j'} v_{(k,j),(k',j')} \mathbf{e}^*_{k',j'}$ with $v_{(k,j),(k',j')} \in \mathbb{Z}$.
(3) If $(\ell, \rho) \in \mathfrak{I}_k$ or $\ell \in \mathfrak{I}$, then $d\ell = \sum_{k',j'} v_{\ell,(k',j')} \mathbf{e}^*_{k',j'}$ with $v_{\ell,(k',j')} \in \mathbb{Z}$.

We identify $\mathbb{R}^n$ with $H^1(L(u); \mathbb{R}^n)$ in the same way as Lemma 3.3 and let $x_{k,j} \in Hom(H^1(L(u); \mathbb{R}), \mathbb{R})$ be the element corresponding to $\mathbf{e}^*_{k,j}$ by this identification. In other words, if

$$\mathbf{e}^*_{k,j} = \sum_i a_{(k,j);i} \mathbf{e}^*_i,$$

where $\mathbf{e}^*_i$ is as in Lemma 3.3, then we have

$$x_{k,j} = \sum_i a_{(k,j);i} x_i.$$

We put $y_{k,j} = e^{x_{k,j}}$. We define

$$Y(k,j) = \prod_{k'=1}^{K} \prod_{j'=1}^{d(k')} y_{k',j'}^{v_{(k,j),(k',j')}}. \tag{8.12}$$

And for $(\ell, \rho) \in \mathfrak{I}_k$ or $\ell \in \mathfrak{I}$, we define

$$Y(\ell) = \prod_{k=1}^{K} \prod_{j=1}^{d(k)} y_{k,j}^{v_{\ell,(k,j)}}. \tag{8.13}$$

By Theorem 3.5 there exists $c_{(\ell,\rho)} \in \mathbb{Q}$ such that :

$$\begin{aligned} \mathfrak{PO}^{\mathbf{0}} =& \sum_{k=1}^{K} \left(\sum_{j=1}^{a(k)} Y(k,j) \right) T^{S_k} + \sum_{\ell \in \mathfrak{I}} Y(\ell) T^{\ell(\mathbf{0})} \\ &+ \sum_{k=1}^{K+1} \sum_{(\ell,\rho)\in\mathfrak{I}_k} c_{(\ell,\rho)} Y(\ell) T^{\ell(\mathbf{0})+\rho}. \end{aligned} \tag{8.14}$$

Lemma 8.13. (1) *If $k' < k$ then*

$$\frac{\partial Y(k',j')}{\partial y_{k,j}} = 0. \tag{8.15}$$

(2) *If $(\ell, \rho) \in \mathfrak{I}_{k'}$, $k' < k$ then*

$$\frac{\partial Y(\ell)}{\partial y_{k,j}} = 0. \tag{8.16}$$

(3) *If $(\ell, \rho) \in \mathfrak{I}_k$ then $\ell(\mathbf{0}) + \rho > S_k$.*
(4) *If $(\ell, \rho) \in \mathfrak{I}_{K+1}$ then $\ell(\mathbf{0}) + \rho > S_K$.*
(5) *If $\ell \in \mathfrak{I}$ then $\ell(\mathbf{0}) > S_K$.*

Proof. Since $d\ell_{k',j'} \in A_{k'}^{\perp}$ by Lemma 8.6 it follows that $v_{(k',j'),(k,j)} = 0$ for $k > k'$. (1) follows. (2) follows from Lemma 8.10 (1) in the same way. (3) follows from Lemma 8.10 (2). (4) follows from Lemma 8.10 (3). (5) follows from (8.11). □

Now equation (3.9) becomes

$$0 = \frac{\partial \mathfrak{PO}^{\mathbf{0}}}{\partial y_{k,j}}.$$

We calculate this equation using Lemma 8.13 to find that it is equivalent to :

$$\begin{aligned} 0 =& \sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}} + \sum_{k'>k} \sum_{j'=1}^{a(k')} \frac{\partial Y(k',j')}{\partial y_{k,j}} T^{S_{k'}-S_k} \\ &+ \sum_{k'=k}^{K+1} \sum_{(\ell,\rho)\in\mathfrak{I}_{k'}} c_{(\ell,\rho)} \frac{\partial Y(\ell)}{\partial y_{k,j}} T^{\ell(\mathbf{0})+\rho-S_k} + \sum_{\ell\in\mathfrak{I}} \frac{\partial Y(\ell)}{\partial y_{k,j}} T^{\ell(\mathbf{0})-S_k}. \end{aligned} \tag{8.17}$$

Note the exponents of T in the second, third, and fourth terms of (8.17) are all strictly positive. So after putting $T = 0$ we have

$$0 = \sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}}. \tag{8.18}$$

Note that the equation (8.18) does not involve T but becomes a numerical equation. We call (8.18) the *leading term equation*.

Lemma 8.14. *There exist positive real numbers $\mathfrak{y}_{k,j;0}$, $k = 1, \cdots, K$, $j = 1, \cdots, d(k)$, solving the leading term equations for $k = 1, \cdots, K$.*

Proof. We remark the leading term equation for k, j contain the monomials involving only $y_{k',j}$ for $k' \le k$. We first solve the leading term equation for $k = 1$. Denote

$$f_1(x_{1,1}, \cdots, x_{1,d(1)}) = \sum_{j=1}^{a(1)} Y(1, j).$$

It follows from Corollary 8.8 that for any $(x_{1,1}, \cdots, x_{1,d(1)}) \ne 0$, there exists j such that

$$d\ell_{1,j}(x_{1,1}, \cdots, x_{1,d(1)}) > 0.$$

Therefore, we have

$$\lim_{t\to\infty} f_1(tx_{1,1}, \cdots, tx_{1,d(1)}) \ge \lim_{t\to\infty} C \exp(td\ell_{1,j}(x_{1,1}, \cdots, x_{1,d(1)})) = +\infty.$$

Hence $f_1(x_{1,1}, \cdots, x_{1,d(1)})$ attains its minimum at some point of $\mathbb{R}^{d(1)}$. Taking its exponential, We obtain $\mathfrak{y}_{1,j;0} \in \mathbb{R} \setminus \{0\}$.

Suppose we have already found $\mathfrak{y}_{k',j;0}$ for $k' < k$. Then we put

$$F_k(x_{1,1}, \cdots, x_{k,1}, \cdots, x_{k,d(k)}) = \sum_{j=1}^{a(k)} Y(k, j)$$

and

$$f_k(x_{k,1}, \cdots, x_{k,d(k)}) = F_k(\mathfrak{x}_{1,1;0}, \cdots, \mathfrak{x}_{k-1,d(k-1);0}, x_{k,1}, \cdots, x_{k,d(k)})$$

where $\mathfrak{x}_{k',j;0} = \log \mathfrak{y}_{k',j;0}$. Again using Corollary 8.8, we find

$$\lim_{t\to\infty} f_k(tx_{k,1}, \cdots, tx_{k,d(k)}) = +\infty.$$

for any $(x_{k,1}, \cdots, x_{k,d(k)}) \ne 0$. Hence $f_k(x_{k,1}, \cdots, x_{k,d(k)})$ attains a minimum and we obtain $\mathfrak{y}_{k,j;0}$. Lemma 8.14 now follows by induction. □

We next find the solution of our equation (3.10) or (3.11). We take a sufficiently large N and put

$$\begin{aligned} \mathfrak{PO}^{\mathbf{0}}_{k,N} = &\sum_{j=1}^{a(k)} Y(k,j) + \sum_{k'>k} \sum_{j'=1}^{a(k')} \frac{\partial Y(k',j')}{\partial y_{k,j}} T^{S_{k'}-S_k} \\ &+ \sum_{\ell \in \mathfrak{I},\ \ell(\mathbf{0}) \le N} Y(\ell) T^{\ell(\mathbf{0})-S_k} \\ &+ \sum_{k'=k+1}^{K+1} \sum_{(\ell,\rho)\in \mathfrak{I}_{k'}, \ell(\mathbf{0})+\rho \le N} c_{(\ell,\rho)} Y(\ell) T^{\ell(\mathbf{0})+\rho-S_k}. \end{aligned} \tag{8.19}$$

We remark that (3.10) is equivalent to

$$\frac{\partial \mathfrak{PO}^{\mathbf{0}}_{k,N}}{\partial y_{k,j}}(\mathfrak{y}_1, \cdots, \mathfrak{y}_n) \equiv 0 \mod T^{N-S_k} \qquad k = 1, \cdots, K,\ j = 1, \cdots, a(k). \tag{8.20}$$

We also put

$$\overline{\mathfrak{PO}}^{\mathbf{0}}_k = \sum_{j=1}^{a(k)} Y(k, j).$$

It satisfies
$$\overline{\mathfrak{PO}}^{\mathbf{0}}_k \equiv \mathfrak{PO}^{\mathbf{0}}_{k,N} \mod \Lambda_+. \tag{8.21}$$
For given positive numbers $R(1),\cdots,R(K)$ we define the discs
$$D(R(k)) = \{(x_{k,1}\cdots,x_{k,d(k)}) \mid x_{k,1}^2+\cdots+x_{k,d(k)}^2 \le R(k)\} \subset \mathbb{R}^{d(k)}$$
and the poly-discs
$$\begin{aligned} D(R(\cdot)) &= \prod_{k=1}^{K} D(R(k)) \\ &= \{(x_{1,1}\cdots,x_{K,d(K)}) \mid x_{k,1}^2+\cdots+x_{k,d(k)}^2 \le R(k),\ k=1,\cdots,K\}. \end{aligned}$$
We factorize
$$\mathbb{R}^n = \prod_{k=1}^{K} \mathbb{R}^{d(k)}.$$
Then we consider the Jacobian of $\overline{\mathfrak{PO}}^{\mathbf{0}}_k$
$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k : \mathbb{R}^n \to \mathbb{R}^{d(k)}$$
i.e., the map
$$(\mathfrak{x}_{1,1},\cdots,\mathfrak{x}_{K,d(K)}) \mapsto \left(\frac{\partial\overline{\mathfrak{PO}}^{\mathbf{0}}_k}{\partial x_{k,j}}(\mathfrak{x}_{1,1},\cdots,\mathfrak{x}_{K,d(K)})\right)_{j=1,\cdots,d(k)}. \tag{8.22}$$
We remark that $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k$ depends only on $\mathbb{R}^{d(1)}\times\cdots\times\mathbb{R}^{d(k)}$ components.

Combining all $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k$, $k=1,\cdots,K$ (8.22) induces a map
$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}} : \mathbb{R}^n \to \mathbb{R}^n$$
defined by
$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}} = (\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_1,\cdots,\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_K).$$
The next lemma is closely related to Lemma 8.14.

Lemma 8.15. *We may choose the positive numbers $R(k)$ for $k=1,\cdots,K$ such that the following holds :*

(1) $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}$ *is nonzero on* $\partial(D(R(\cdot)))$.
(2) *The map* : $\partial(D(R(\cdot))) \to S^{n-1}$
$$\mathfrak{x} \mapsto \frac{\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}}{\|\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}\|}$$
has degree 1.

Proof. We first prove the following sublemma by an upward induction on k_0.

Sublemma 8.16. *There exist $R(k)$'s for $1\le k\le K$ such that for any given $1\le k_0\le K$ we have*
$$\sum_{j=1}^{d(k_0)} x_{k_0,j}\frac{\partial\overline{\mathfrak{PO}}^{\mathbf{0}}_{k_0}}{\partial x_{k_0,j}}(x_{1,1},\cdots,x_{k_0,d(k_0)}) > 0 \tag{8.23}$$
if $(x_{k,1},\cdots,x_{k,d(k)}) \in D(R(k))$ for all $1\le k\le k_0-1$ and $(x_{k_0,1},\cdots,x_{k_0,d(k_0)}) \in \partial D(R(k_0))$.

Proof. In case $k_0 = 1$ the existence of $R(1)$ satisfying (8.23) is a consequence of Corollary 8.8. We assume that the sublemma is proved for $1, \cdots, k_0 - 1$.

For each fixed $\mathbf{x} = (x_{1,1}, \cdots, x_{k_0-1,d(k_0-1)})$ we can find $R(k_0)_{\mathbf{x}}$ such that (8.23) holds for $(x_{k_0,1}, \cdots, x_{k_0,d(k_0)}) \in \mathbb{R}^{d(k_0)} \setminus D(R(k_0)_{\mathbf{x}}/2)$. This is also a consequence of Corollary 8.8.

We take supremum of $R(k_0)_{\mathbf{x}}$ over the compact set $\mathbf{x} \in \prod_{k=1}^{k_0-1} D(R(k))$ and obtain $R(k_0)$. The proof of Sublemma 8.16 is complete. $\square$

It is easy to see that Lemma 8.15 follows from Sublemma 8.16. $\square$

We now use our assumption that the vertices of P lies in $M_{\mathbb{Q}} = \mathbb{Q}^n$ and that $\rho_j \in \mathbb{Q}$. Replacing T by $T^{1/M!}$ if necessary, we may assume that all the exponents of $y_{k,j}$ and T appearing in (8.19) are integers. Then

$$\mathfrak{PO}^{\mathbf{0}}_{k,N} = \mathfrak{PO}^{\mathbf{0}}_{k,N}(y_{1,1}, \cdots, y_{K,d(K)}; T)$$

are polynomials of $y_{k,j}$, $y_{k,j}^{-1}$ and T. Define the set $\mathfrak{X}$ by the set consisting of

$$(\mathfrak{y}_{1,1}, \cdots, \mathfrak{y}_{K,d(K)}; q) \in (\mathbb{R}_+)^n \times \mathbb{R}$$

that satisfy

$$\frac{\partial \mathfrak{PO}^{\mathbf{0}}_{k,N}}{\partial y_{k,j}}(\mathfrak{y}_{1,1}, \cdots, \mathfrak{y}_{K,d(K)}; q) = 0, \tag{8.24}$$

for $k = 1, \cdots, K$, $j = 1, \cdots, d(k)$. Clearly $\mathfrak{X}$ is a real affine algebraic variety. (Note the equation for y_i are polynomials. So we need to regard y_i (not x_i) as variables to regard $\mathfrak{X}$ as a real affine algebraic variety.)

Consider the projection

$$\pi : \mathfrak{X} \to \mathbb{R}, \quad \pi(\mathfrak{y}_{1,1}, \cdots, \mathfrak{y}_{K,d(K)}; q) = q$$

which is a morphism of algebraic varieties.

Lemma 8.17. *There exists a sufficiently small $\epsilon > 0$ such that if $|q| < \epsilon$ then*

$$\pi^{-1}(q) \cap \{(e^{x_1}, \cdots, e^{x_n}) \mid (x_1, \cdots, x_n) \in D(R(\cdot))\} \neq \emptyset.$$

Proof. We consider the real analytic q-family of polynomials

$$\mathfrak{PO}^{\mathbf{0}}_{k,N,q}(y_{1,1}, \cdots, y_{K,d(K)}) = \mathfrak{PO}^{\mathbf{0}}_{k,N}(y_{1,1}, \cdots, y_{K,d(K)}; q).$$

Then

$$\mathfrak{PO}^{\mathbf{0}}_{k,N,0} = \overline{\mathfrak{PO}}^{\mathbf{0}}_{k,N} \tag{8.25}$$

Replacing $\overline{\mathfrak{PO}}^{\mathbf{0}}_{k,N}$ by $\mathfrak{PO}^{\mathbf{0}}_{k,N,q}$, we can repeat construction of the map

$$\nabla \mathfrak{PO}^{\mathbf{0}}_{N,q} : \mathbb{R}^n \to \mathbb{R}^n$$

for each fixed $q \in \mathbb{R}$ in the same way as we defined $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}$. Then the conclusion of Lemma 8.15 holds for $\nabla \mathfrak{PO}^{\mathbf{0}}_{N,q}$ if $|q|$ is sufficiently small. (This is a consequence of Lemma 8.15 and (8.25).) Lemma 8.17 follows from elementary algebraic topology. $\square$

Lemma 8.17 implies that we can find

$$\mathfrak{y}_0 = (\mathfrak{y}_{1,1;0}, \cdots, \mathfrak{y}_{K,d(K);0}) \in (\mathbb{R} \setminus \{0\})^n$$

and a sequence

$$(\mathfrak{y}_h, q_h) = (\mathfrak{y}^h_{1,1;0}, \cdots, \mathfrak{y}^h_{K,d(K);0}; q_h) \in \mathfrak{X} \subset \mathbb{R}^{n+1}$$

$h = 1, 2, \cdots$ such that $q_h > 0$ and $\lim_{h\to\infty}(\mathfrak{y}_h, q_h) = (\mathfrak{y}_0, 0)$. Therefore by the curve selection lemma (Lemma 3.1 [Mi]) there exists a real analytic map

$$\gamma : [0, \epsilon) \to \mathfrak{X}$$

such that $\gamma(0) = (\mathfrak{y}_0, 0)$ and $\pi(\gamma(t)) > 0$ for $t > 0$. We reparameterize $\gamma(t)$, so that its q-component is $t^{a/b}$, where a and b are relatively prime integers. We put $T = t^{a/b}$ i.e., $t = T^{b/a}$ and denote the $y_{k,j}$-components of $\gamma(t)$ by

$$\mathfrak{y}_{k,j} = \mathfrak{y}_{k,j;0} + \sum_{\ell=1}^{\infty} \mathfrak{y}_{k,j;\ell} T^{b\ell/a}.$$

Since $\gamma(t) \in \mathfrak{X}$, the element $(\mathfrak{y}_{k,j})_{k,j} \in (\Lambda_0^{\mathbb{R}} \setminus \Lambda_+^{\mathbb{R}})^n$ is the required solution of (3.10).

Since $\mathfrak{PO}_0$ contains only a finite number of summands, we can take $\mathfrak{PO}_{0,N} = \mathfrak{PO}_0$. Therefore we can find a solution of (3.11) for $\mathfrak{PO}_0$.

The proof of Proposition 3.6 is now complete. □

9. Elimination of higher order term in nondegenerate cases

In this section, we prove a rather technical (but useful) result, which shows that solutions of the leading term equation (8.18) correspond to actual critical points under certain non-degeneracy condition. For this purpose, we slightly modify the argument of the last part of section 8. This result will be useful to determine $\mathfrak{M}(\mathfrak{Lag}(X))$ in the non-Fano case. In fact it shows that we can use $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$ in most practical cases. We remark that we explicitly calculate $\mathfrak{PO}_0$ but do not know the precise form of $\mathfrak{PO}$ in the non-Fano case.

In order to state the result in a general form, we prepare some notations. Let $u_0 \in \operatorname{Int} P$. (In section 8, u_0 is determined as the unique element of P_K defined in Proposition 8.1. The present situation is more general.)

We define positive real numbers $S_1 < S_2 < \cdots$ by

$$\{\ell_i(u_0) \mid i = 1, \cdots, m\} = \{S_1, S_2, \cdots, S_{m'}\} \tag{9.1}$$

and the sets

$$I_k = \{\ell_i \mid \ell_i(u_0) = S_k\}, \tag{9.2}$$

for $k = 1, \cdots$. We renumber $\bigcup_k I_k$ so that

$$\{\ell_{k,j} \mid j = 1, \cdots, a(k)\} = I_k. \tag{9.3}$$

Definition 9.1. Let $A_l^{\perp}$ be the linear subspace of $N_{\mathbb{R}}$ spanned by $d\ell_{k,j}$ $k \le l$, $j \le a(k)$. We define K to be the smallest number such that $A_K^{\perp} = N_{\mathbb{R}}$

Note our notations here are consistent with one in section 8 in case $\{u_0\} = P_K$. We define $\mathfrak{I}$ and $\mathfrak{I}_k$ by (8.10) and Definition 8.9. Then Lemma 8.10 and (8.11) hold. We choose $\mathbf{e}_{i,j}^* \in Hom(M_{\mathbb{Q}}, \mathbb{Q})$ such that Condition 8.12 is satisfied. (Note $A_l^{\perp}$ is defined over $\mathbb{Q}$.) $x_{i,j}$ and $y_{i,j}$ then are defined in the same way as section 8. We define $Y(k,j)$ by (8.12) and $Y(\ell)$ by (8.13). Then (8.14) and Lemma 8.13 hold.

We remark that Corollary 8.8 does *not* hold in general in the current situation. In fact we can write

$$v = \sum_{l=1}^{k} \sum_{j=1}^{a(l)} c_{l,j} d\ell_{l,j}.$$

under the assumption of Corollary 8.8 but we may not be able to ensure $c_{l,j} \ge 0$.

Definition 9.2. (1) We call

$$0 = \sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}}, \quad k = 1, \cdots, K, \ j = 1, \cdots, d(k)$$

the *leading term equation* at u_0. We regard it as a polynomial equation for $\mathfrak{y}_{k,j} \in \mathbb{C} \setminus \{0\}$, $k = 1, \cdots, K$, $j = 1, \cdots, d(k)$.

(2) A solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\cdots,K,\, j=1,\cdots,d(k)}$ of leading term equation is said to be *weakly nondegenerate* if it is isolated in the set of solutions.

(3) A solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\cdots,K,\, j=1,\cdots,d(k)}$ of leading term equation is said to be *strongly nondegenerate* if the matrices

$$\left(\sum_{j'=1}^{a(k)} \frac{\partial^2 Y(k,j')}{\partial y_{k,j_1} \partial y_{k,j_2}} \right)_{j_1,j_2=1,\cdots,a(k)}$$

are invertible for $k = 1, \cdots, K$, at $\mathfrak{y}^0$.

(4) We define the multiplicity of leading term equation in the standard way of algebraic geometry, in the weakly nondegenerate case.

Example 9.3. In Example 8.5, the equation (8.5) is the leading term equation.

Let $\mathfrak{PO}^{u_0}_*$ be either $\mathfrak{PO}^{u_0}_0$ or $\mathfrak{PO}^{u_0}$.

Theorem 9.4. *For any strongly nondegenerate solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})$ of leading term equation, there exists a solution $\mathfrak{y} = (\mathfrak{y}_{k,j}) \in (\Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}})^n$ of*

$$\frac{\partial \mathfrak{PO}^{u_0}_*}{\partial y_{k,j}}(\mathfrak{y}) = 0 \tag{9.4}$$

such that $\mathfrak{y}_{k,j} \equiv \mathfrak{y}_{k,j;0} \mod \Lambda_+^{\mathbb{C}}$.

If all the vertices of P and u_0 are rational, the same conclusion holds for weakly nondegenerate $\mathfrak{y}^0$.

The following corollary is an immediate consequence.

Corollary 9.5. *Let $(u,b) \in \mathfrak{M}_{+,0}(\mathfrak{Lag}(X))$ and $u \in \operatorname{Int} P$. Assume one of the following conditions :*

1. *The corresponding solution of the leading term equation is strongly nondegenerate.*
2. *P, u are rational and the corresponding solution of leading term equation is weakly nondegenerate.*

Then there exists b' such that $(u,b') \in \mathfrak{M}(\mathfrak{Lag}(X))$ and $b' \equiv b \mod \Lambda_+^{\mathbb{C}}$.

Remark 9.6. (1) Using Proposition 9.7 below, we can apply Theorem 9.4 and Corollary 9.5, for weakly nondegenerate case, without assuming rationality, to study displacement of Lagrangian fibers. See the last step of the proof of Theorem 1.5 given at the end of section 12.

(2) The authors do not know an example where the weak nondegeneracy assumption of Corollary 9.5 is not satisfied.

(3) In this section we work with $\Lambda^{\mathbb{C}}$ coefficients, while in the last section we work with $\Lambda^{\mathbb{R}}$ coefficients.

(4) If we define the multiplicity of the element of $\mathfrak{M}_0(\mathfrak{Lag}(X))$ as the dimension of the Jacobian ring $Jac(\mathfrak{PO}_0; u_0, b)$ in Definition 6.8 (namely as the Milnor number) then the sum of the multiplicities of the solutions of (9.4) converging to $\mathfrak{y}^0$ as $T \to 0$, is equal to the multiplicity of $\mathfrak{y}^0$.
(5) In the strongly nondegenerate case, the solution of (9.4) with the given leading term is unique.

Proposition 9.7. *Let (X,ω) be a compact toric manifold with moment polytope P and $u_0 \in \operatorname{Int} P$. Then there exist (X, ω^h) with moment polytope P^h and $u_0^h \in \operatorname{Int} P^h$ such that the following holds:*
(1) $\lim_{h\to\infty} \omega^h = \omega$. $\lim_{h\to\infty} u_0^h = u_0$.
(2) *The vertices of P^h and u_0^h are rational.*
(3) *The leading term equation at u_0^h is the same as the leading term equation at u_0.*

We prove Proposition 9.7 at the end of section 12.

We first derive Theorem 1.13 from Theorem 9.4 before proving Theorem 9.4.

Proof of Theorem 1.13. We start with $\mathbb{C}P^2$ and blow up a T^n fixed point to obtain $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. We take a Kähler form so that the volume of the exceptional $\mathbb{C}P^1$ is ϵ_1 which is small. We next blow up again at one of the fixed points so that the volume of the exceptional $\mathbb{C}P^1$ is ϵ_2 and is small compared with ϵ_1. We continue k times to obtain $X(k)$, whose Kähler structure depends on $\epsilon_1, \cdots, \epsilon_k$. Note $X(k)$ is non-Fano for $k > 3$.

Let $P(k)$ be the moment polytope of $X(k)$ and $\mathfrak{PO}_{0,k}$ be the leading order potential function of $X(k)$. We remark that $P(k)$ is obtained by cutting out a vertex of $P(k-1)$. (See [Ful].)

Lemma 9.8. *We may choose ϵ_i ($i = 1, \cdots, k$) so that the following holds for $l \le k$.*
(1) *The number of balanced fibers of $P(l)$ is $l+1$. We write them as $L(u^{(l,i)})$ $i = 0, \cdots, l$.*
(2) $u^{(l-1,i)} = u^{(l,i)}$ *for $i \le l-1$.* $u^{(l,0)} = (1/3, 1/3)$.
(3) $u^{(l,l)}$ *is in an $o(\epsilon_l)$ neighborhood of the vertices corresponding to the point we blow up.*
(4) *The leading term equation of $\mathfrak{PO}_{0,l-1}$ at $u^{(l-1,i)}$ is the same as the leading term equation of $\mathfrak{PO}_{0,l}$ at $u^{(l,i)}$ for $i \le l-1$.*
(5) *The leading term equations are all strongly nondegenerate.*

Proof. The proof is by induction on k. There is nothing to show for $k = 0$. Suppose that we have proved Lemma 9.8 up to $k-1$. Let w be the vertex of the polytope we cut out which corresponds to the blow up of $X(k-1)$. Let ℓ_i, $\ell_{i'}$ be the affine functions associated to the two edges containing w. It is easy to see that

$$P(k) = \{u \in P(k-1) \mid \ell_i(u) + \ell_{i'}(u) \ge \epsilon_k\}.$$

We also have :

$$\mathfrak{PO}_{0,k} = \mathfrak{PO}_{0,k-1} + T^{\ell_i(u)+\ell_{i'}(u)-\epsilon_k} y_1^{v_{i,1}+v_{i',1}} y_2^{v_{i,2}+v_{i',2}}.$$

Therefore if we choose ϵ_k sufficiently small, the leading term equation at $u^{(k-1,i)}$ does not change.

We take $u^{(k,k)}$ such that

$$\ell_i(u^{(k,k)}) = \ell_{i'}(u^{(k,k)}) = \epsilon_k.$$

It is easy to see that there exists such $u^{(k,k)}$ uniquely if ϵ_k is sufficiently small. We put

$$y_1' = y_1^{v_{i,1}} y_2^{v_{i,2}}, \quad y_2' = y_1^{v_{i',1}} y_2^{v_{i',2}}.$$

(We remark that v_i and $v_{i'}$ are $\mathbb{Z}$ basis of $\mathbb{Z}^2$, since $X(k-1)$ is smooth toric.) Then we have

$$\mathfrak{PO}_{0,k}^{u^{(k,k)}} \equiv (y_1' + y_2' + y_1'y_2')T^{\epsilon_k} \mod T^{\epsilon_k}\Lambda_+.$$

Therefore the leading term equation is

$$1 + y_1' = 1 + y_2' = 0$$

and hence has a unique solution $(-1,-1)$. In particular it is strongly nondegenerate. We can also easily check that there is no other solution of leading term equation. The proof of Lemma 9.8 now follows by Theorem 9.4. □

Theorem 1.13 immediately follows from Lemma 9.8. □

Note that Theorem 1.13 can be generalized to $\mathbb{C}P^n$ by the same proof.

We are now ready to give the proof of Theorem 9.4.

Proof of Theorem 9.4. We first consider the weakly nondegenerate case. Let $\mathfrak{m}$ be the multiplicity of $\mathfrak{y}^0$. We choose δ such that the ball $B_\delta(\mathfrak{y}^0)$ centered at $\mathfrak{y}^0$ and of radius δ does not contain a solution of the leading term equation other than $\mathfrak{y}^0$. For $y \in \partial B_\delta(\mathfrak{y}^0)$ we define

$$\nabla\overline{\mathfrak{PO}}(y) = \left(\sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}}(y)\right)_{k=1,\cdots,K,\, j=1,\cdots,d(k)} \in \mathbb{C}^n.$$

The map

$$y \mapsto \frac{\nabla\overline{\mathfrak{PO}}(y)}{\|\nabla\overline{\mathfrak{PO}}(y)\|} \in S^{2n-1}$$

is well defined and of degree $\mathfrak{m} \neq 0$ by the definition of multiplicity.

We define $\mathfrak{PO}^{u_0}_{*,k,N}$ in the same way as (8.19). For $q \in \mathbb{C}$, we define $\mathfrak{PO}^{u_0}_{*,k,N}(\cdots;q)$ by substituting q to T. Then in the same way as the proof of Lemma 8.17 we can prove the following.

Lemma 9.9. *There exists $\epsilon > 0$ such that if $|q| < \epsilon$, the equation*

$$0 = \frac{\partial}{\partial y_{k,j}} \mathfrak{PO}^{u_0}_{*,k,N}(\cdots;q) \tag{9.5}$$

has a solution in $B_\delta(\mathfrak{y}^0)$. The sum of multiplicities of the solutions of (9.5) *converging to $\mathfrak{y}_{k,j;0}$ is $\mathfrak{m}$.*

(9.5) is a polynomial equation. Hence multiplicity of its solution is well-defined in the standard sense of algebraic geometry.

Now we assume that all the vertices of P and u_0 are contained in $\mathbb{Q}^n$. (9.5) also depends polynomially on $q' = T'$, where $T' = T^{1/C!}$ for a sufficiently large C. (We remark that C is determined by the denominators of the coordinates of the vertices of P and of u_0. In particular it can be taken to be independent of N.)

We denote $y = (y_1, \cdots, y_n)$ and put

$$\mathfrak{X} = \{(y,q') \mid y \in B_\delta(\mathfrak{y}^0),\ q' \text{ with } |q'| < \epsilon \text{ and } q = (q')^{C!} \text{ satisfying (9.5)}\}.$$

We consider the projection

$$\pi_{q'} : \mathfrak{X} \to \{q' \in \mathbb{C} \mid |q'| < \epsilon\}. \tag{9.6}$$

By choosing a sufficiently small $\epsilon > 0$, we may assume that (9.6) is a local isomorphism on the punctured disc $\{q' \mid 0 < |q'| < \epsilon\}$. Namely, $\pi_{q'}$ is an etalé covering over the punctured disc.

We remark that for each q' the fiber consists of at most $\mathfrak{m}$ points, since the multiplicity of the leading term equation is $\mathfrak{m}$. We put $q'' = (q')^{1/\mathfrak{m}!}$. Then the pull-back

$$\pi_{q''} : \mathfrak{X}' \to \{q'' \in \mathbb{C} \mid 0 < |q''| < \epsilon\} \tag{9.7}$$

of (9.6) becomes a trivial covering space. Namely there exists a single valued section of $\pi_{q''}$ on $\{q'' \mid 0 < |q''| < \epsilon\}$. It extends to a holomorphic section of $\{q'' \mid |q''| < \epsilon\}$.

In other words there exists a holomorphic family of solutions of (9.5) which is parameterized by $q'' \in \{q'' \mid |q''| < \epsilon\}$. We put $T'' = (T')^{1/\mathfrak{m}!}$. Then by taking the Taylor series of the q''-parameterized family of solutions at 0, we obtain the following :

Lemma 9.10. *If all the vertices of P and u_0 are rational, then for each N there exists $\mathfrak{y}^{(N)} = (\mathfrak{y}^{(N)}_{k,j})$*

$$\mathfrak{y}^{(N)}_{k,j} = \sum_{l=0}^{N} \mathfrak{y}^{(N)}_{k,j;l} (T'')^l$$

$(\mathfrak{y}^{(N)}_{k,j;l} \in \mathbb{C})$ such that

$$\frac{\partial \mathfrak{PO}^{u_0}_*}{\partial y_{k,j}}(\mathfrak{y}^{(N)}_{k,j}) \equiv 0 \mod (T'')^N \tag{9.8}$$

and that $\mathfrak{y}^{(N)}_{k,j;0} \equiv \mathfrak{y}_{k,j;0}$.

We remark that Lemma 9.10 is sufficient for most of the applications. In fact it implies that $L(u_0)$ is balanced if there exists a solution of leading term equation at u_0. Hence we can apply Lemma 12.2.

For completeness we prove the slightly stronger statement made for the weakly nondegenerate case in Theorem 9.4. The argument is similar to one in subsection 30.11 [FOOO2].

For each N, we denote by $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); N)$ the set of all $(\mathfrak{y}^{(N)}_{k,j;l})_{k,j;l} \in \mathbb{C}^{nN}$, where $k = 1, \cdots, K$, $j = 1, \cdots, a(k)$, $l = 1, \cdots, N$, such that

$$\mathfrak{y}^{(N)}_{k,j} = \mathfrak{y}_{k,j;0} + \sum_{l=1}^{N} \mathfrak{y}^{(N)}_{k,j;l} (T'')^l$$

satisfies (9.8).

By definition, $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); N)$ is the set of $\mathbb{C}$-valued points of certain complex affine algebraic variety (of finite dimension). Lemma 9.10 implies that $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); N)$ is nonempty. For $N_1 > N_2$ there exists an obvious morphism

$$I_{N_1,N_2} : \widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); N_1) \to \widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); N_2)$$

of complex algebraic variety.

To complete the proof of Theorem 9.4 in the weakly nondegenerate case, it suffices to show that the projective limit

$$\varprojlim(\widetilde{\mathfrak{M}}(\mathfrak{y}_{k,j;0};N)) \tag{9.9}$$

is nonempty.

Lemma 9.11.

$$\bigcap_{N>1} \mathrm{Im} I_{N,1} \neq \emptyset.$$

Proof. By a theorem of Chevalley (see Chapter 6 [Mat]), each $\mathrm{Im}I_{N,1}$ is a constructible set. It is nonempty and its dimension $\dim \mathrm{Im}I_{N,1}$ is nonincreasing as $N \to \infty$. Therefore we may assume $\dim \mathrm{Im}I_{N,1} = d$ for $N \geq N_1$.

We consider the number of d dimensional irreducible components of $\mathrm{Im}I_{N,1}$. This number is nonincreasing for $N \geq N_1$. Therefore, there exists N_2 such that for $N \geq N_2$ the number of d dimensional irreducible components of $\mathrm{Im}I_{N,1}$ is independent of N. It follows that there exists X_N a sequence of d dimensional irreducible components of $\mathrm{Im}I_{N,1}$ such that $X_{N+1} \subset X_N$. Since $\dim(X_N \setminus X_{N+1}) < d$, it follows from Baire's category theorem that $\cap_N X_N \neq \emptyset$. Hence the lemma. □

Lemma 9.12. *There exists a sequence* $(\mathfrak{y}^{(n)}_{k,j;l})_{k,j;l}$ $n = 1,2,3,\cdots m$ *such that*

$$I_{n,n-1}((\mathfrak{y}^{(n)}_{k,j;l})_{k,j;l} = (\mathfrak{y}^{(n-1)}_{k,j;l})_{k,j;l})$$

for $n = 2,\cdots,m$ *and that*

$$(\mathfrak{y}^{(m)}_{k,j;l})_{k,j;l} \in \bigcap_{N>m} \mathrm{Im} I_{N,m}.$$

Proof. The proof is by induction on m. The case $m = 1$ is Lemma 9.11. Each of the inductive step is similar to the proof of Lemma 9.11 and so it omitted. □

Lemma 9.12 implies that the projective limit (9.9) is nonempty. The proof of weakly nondegenerate case of Theorem 9.4 is complete.

We next consider the strongly nondegenerate case. We prove the following lemma by induction on M. Let G be a submonoid of $(\mathbb{R}_{\geq 0},+)$ generated by the numbers appearing in the exponent of (8.17). Namely it is generated by

$$\begin{gathered} S_{k'} - S_k \ (k' > k), \quad \ell(u_0) + \rho - S_k \ ((\ell,\rho) \in \mathfrak{I}_{k'},\ k' \geq k), \\ \ell(u_0) - S_k \ (\ell \in \mathfrak{I}). \end{gathered} \tag{9.10}$$

We define $0 < \lambda_1 < \lambda_2 < \cdots$ by

$$\{\lambda_i \mid i = 1,2,\dots\} = G.$$

Lemma 9.13. *We assume that* $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\cdots,K,\, j=1,\cdots,d(k)}$ *is a strongly nondegenerate solution of the leading term equation. Then, there exists*

$$\mathfrak{y}^{(M)}_{k,j} = \mathfrak{y}_{k,j;0} + \sum_{l=1}^{M} \mathfrak{y}_{k,j;l} T^{\lambda_l}.$$

such that

$$\sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}} (\mathfrak{y}^{(M)}_{k,1},\cdots,\mathfrak{y}^{(M)}_{K,d(K)}) \equiv 0 \mod T^{\lambda_{M+1}}. \tag{9.11}$$

Moreover we may choose $\mathfrak{y}_{k,j}^{(M)}$ *so that*

$$\mathfrak{y}_{k,j}^{(M)} \equiv \mathfrak{y}_{k,j}^{(M+1)} \mod T^{\lambda_{M+1}}.$$

Proof. The proof is by induction on M. There is nothing to show in the case $M = 0$. Assume we have proved the lemma up to $M-1$. Then we have

$$\sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}}(\mathfrak{y}_{k,1}^{(M-1)}, \cdots, \mathfrak{y}_{K,d(K)}^{(M-1)}) \equiv c_{k,j,M}T^M \mod T^{\lambda_{M+1}}.$$

Consider $\mathfrak{y}_{k,j}^{(M)}$ of the form

$$\mathfrak{y}_{k,j}^{(M)} = \mathfrak{y}_{k,j}^{(M-1)} + \Delta_{k,j,M}T^{\lambda_M}$$

for some $\Delta_{k,j,M}$. Then we can write

$$\begin{aligned}\sum_{j'=1}^{a(k)} &\frac{\partial Y(k,j')}{\partial y_{k,j}}(\mathfrak{y}_{k,1}^{(M-1)} + \Delta_{k,1,M}T^{\lambda_M}, \cdots, \mathfrak{y}_{K,d(K)}^{(M-1)} + \Delta_{K,d(K),M}T^{\lambda_M}) \\ &\equiv \left(c_{k,j,M} + \sum_{j',j''=1}^{a(k)} \frac{\partial^2 Y(k,j')}{\partial y_{k,j}\partial y_{k,j''}}\Delta_{k,j'',M}\right)T^{\lambda_M} \mod T^{\lambda_{M+1}}.\end{aligned}$$

Since $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\cdots,K,\, j=1,\cdots,d(k)}$ is strongly nondegenerate, we can find $\Delta_{k,j'',N} \in \mathbb{C}$ so that the right hand side become zero module $T^{\lambda_{M+1}}$. The proof of Lemma 9.13 is complete. □

By Lemma 9.13, the limit $\lim_{M\to\infty} \mathfrak{y}_{k,j}^{(M)}$ exists. We set

$$\mathfrak{y}_{k,j} := \lim_{M\to\infty} \mathfrak{y}_{k,j}^{(M)}.$$

This is the required solution of (9.4). The proof of Theorem 9.4 is complete □

10. Calculation of potential function

In this section, we prove Theorems 3.4 and 3.5. We begin with a review of [CO]. Let $\pi : X \to P$ be the moment map and $\partial P = \bigcup_{i=1}^m \partial_i P$ be the decomposition of the boundary of P into $n-1$ dimensional faces. Let $\beta_i \in H_2(X, L(u);\mathbb{Z})$ be the elements such that

$$\beta_i \cap [\pi^{-1}(\partial P_j)] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

The Maslov index $\mu(\beta_i)$ is 2. (Theorem 5.1 [CO].)

Let $\beta \in \pi_2(X, L(u))$ and $\mathcal{M}_{k+1}^{\text{main}}(L(u),\beta)$ be the moduli space of stable maps from bordered Riemann surfaces of genus zero with $k+1$ boundary marked points in homology class β. (See [FOOO1] section 3. We require the boundary marked points to respect the cyclic order of $S^1 = \partial D^2$. (In other words we consider the main component in the sense of [FOOO1] section 3.)) Let $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u),\beta)$ be its subset consisting of all maps from a disc. (Namely the stable map without disc or sphere bubble.) The next theorem easily follows from the results of [CO].

Theorem 10.1. (1) *If* $\mu(\beta) < 0$*, or* $\mu(\beta) = 0$*,* $\beta \ne 0$*, then* $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u),\beta)$ *is empty.*

(2) *If* $\mu(\beta) = 2$, $\beta \ne \beta_1, \cdots, \beta_m$, *then* $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ *is empty.*

(3) *For* $i = 1, \cdots, m$, *we have*

$$\mathcal{M}_1^{\text{main,reg}}(L(u), \beta_i) = \mathcal{M}_1^{\text{main}}(L(u), \beta_i). \tag{10.1}$$

Moreover $\mathcal{M}_1(L(u), \beta_i)$ *is Fredholm regular. Furthermore the evaluation map*

$$ev : \mathcal{M}_1^{\text{main}}(L(u), \beta_i) \to L(u)$$

is an orientation preserving diffeomorphism.

(4) *For any* β, *the moduli space* $\mathcal{M}_1^{\text{main,reg}}(L(u), \beta)$ *is Fredholm regular. Moreover*

$$ev : \mathcal{M}_1^{\text{main,reg}}(L(u), \beta) \to L(u)$$

is a submersion.

(5) *If* $\mathcal{M}_1^{\text{main}}(L(u), \beta)$ *is not empty then there exists* $k_i \in \mathbb{Z}_{\ge 0}$ *and* $\alpha_j \in H_2(X; \mathbb{Z})$ *such that*

$$\beta = \sum_i k_i \beta_i + \sum_j \alpha_j$$

and α_j *is realized by holomorphic sphere. There is at least one nonzero* k_i.

Proof. For reader's convenience and completeness, we explain how to deduce Theorem 10.1 from the results in [CO].

By Theorems 5.5 and 6.1 [CO], $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ is Fredholm regular for any β. Since the complex structure is invariant of the T^n action and $L(u)$ is T^n invariant, it follows that T^n acts on $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ and

$$ev : \mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta) \to L(u)$$

is T^n equivariant. Since the T^n action on $L(u)$ is free and transitive, it follows that ev is a submersion if $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ is nonempty. (4) follows.

We assume $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ is nonempty. Since ev is a submersion it follows that

$$n = \dim L(u) \le \dim \mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta) = n + \mu(\beta) - 2$$

if $\beta \ne 0$. Therefore $\mu(\beta) \ge 2$. (1) follows.

We next assume $\mu(\beta) = 2$, and $\mathcal{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ is nonempty. Then by Theorem 5.3 [CO], we find $\beta = \beta_i$ for some i. (2) follows.

We next prove (5). It suffices to consider

$$[f] \in \mathcal{M}_1^{\text{main}}(L(u), \beta) \setminus \mathcal{M}_1^{\text{main,reg}}(L(u), \beta).$$

We decompose the domain of u into irreducible components and restrict f there. Let $f_j : D^2 \to M$ and $g_k : S^2 \to M$ be the restriction of f to disc or sphere components respectively. We have

$$\beta = \sum [f_j] + \sum [g_k].$$

Theorem 5.3 [CO] implies that each of f_j is homologous to the sum of the element of β_i. It implies (5).

To prove (10.1) and complete the proof of Theorem 10.1, it remains to prove $\mathcal{M}_1^{\text{main,reg}}(L(u),\beta_{i_0}) = \mathcal{M}_1^{\text{main}}(L(u),\beta_{i_0})$. (Here $i_0 \in \{1,\cdots,m\}$.) Let $[f] \in \mathcal{M}_1^{\text{main}}(L(u),\beta_{i_0})$. We take k_i and α_j as in (5). (Here $\beta = \beta_{i_0}$). We have

$$\partial\beta_{i_0} = \sum_i k_i \partial\beta_i.$$

Using the convexity of P, (5) and $k_i \geq 0$, we show the inequality

$$\beta_{i_0} \cap \omega \leq \sum_i k_i \beta_i \cap \omega. \tag{10.2}$$

holds and that the equality holds only if $k_i = 0$ $(i \neq i_0)$, $k_{i_0} = 1$, as follows : By (5) we have

$$\ell_{i_0} = \sum_{i=1}^m k_i \ell_i + c.$$

where c is a constant. Since $k_i \geq 0$ and $\ell_{i_0}(u') = 0$ for $u' \in \partial_{i_0} P$, it follows that $c \leq 0$. (Note $\ell_i \geq 0$ on P.) Since $\beta_i \cap \omega = \ell_i(u)$, we have inequality (10.2). Let us assume that the equality holds. If there exists $i \neq j$ with $k_i, k_j > 0$ then

$$\partial_{i_0} P = \{u' \in P \mid \ell_{i_0}(u') = 0\} \subseteq \{u' \in P \mid \ell_i(u') = \ell_j(u') = 0\} \subseteq \partial_i P \cap \partial_j P.$$

This is a contradiction since $\partial_{i_0} P$ is codimension 1. Therefore there is only one nonzero k_i. It is easy to see that $i = i_0$ and $k_{i_0} = 1$.

On the other hand since $\alpha_j \cap \omega > 0$ it follows that

$$\beta_{i_0} \cap \omega \geq \sum_i k_i \beta_i \cap \omega.$$

Therefore there is no sphere bubble (that is α_j). Moreover the equality holds in (10.2). Hence the domain of our stable map is irreducible. Namely

$$\mathcal{M}_1^{\text{main,reg}}(L(u),\beta_{i_0}) = \mathcal{M}_1^{\text{main}}(L(u),\beta_{i_0}).$$

The proof of Theorem 10.1 is now complete. □

Next we discuss one delicate point to apply Theorem 10.1 to the proofs of Theorems 3.4 and 3.5. (This point was already mentioned in section 16 [CO].) Let us consider the case where there exists a holomorphic sphere $g : S^2 \to X$ with

$$c_1(X) \cap g_*[S^2] < 0.$$

We assume moreover that there exists a holomorphic disc $f : (D^2, \partial D^2) \to (X, L(u))$ such that

$$f(0) = g(1).$$

We glue D^2 and S^2 at $0 \in D^2$ and $1 \in S^2$ to obtain Σ. f and g induce a stable map $h : (\Sigma, \partial\Sigma) \to (X, L(u))$.

In general h will *not* be Fredholm regular since g may not be Fredholm regular or the evaluation is not transversal at the interior nodes. In other words, elements of $\mathcal{M}_1^{\text{main}}(L(u),\beta) \setminus \mathcal{M}_1^{\text{main,reg}}(L(u),\beta)$ may not be Fredholm regular in general. Moreover replacing g by its multiple cover, we obtain an element of $\mathcal{M}_1^{\text{main}}(L(u),\beta) \setminus \mathcal{M}_1^{\text{main,reg}}(L(u),\beta)$ such that $\mu(\beta)$ is negative. Theorem 10.1 says that all the holomorphic disc without any bubble are Fredholm regular. However we can not expect that all stable maps in $\mathcal{M}_1^{\text{main}}(L(u),\beta)$ are Fredholm regular.

In order to prove Theorem 3.5, we need to find appropriate perturbations of those stable maps. For this purpose we use the T^n action and proceed as follows.

(We remark that in the Fano case, where there exists no holomorphic sphere g with $c_1(M) \cap g_*[S^2] \le 0$, for which many of the arguments below are much simplified.)

We equip each of $\mathcal{M}_1(L(u),\beta)$ with Kuranishi structure. (See [FO] for the general theory of Kuranishi structure and section 17-18 [FOOO1] for its construction in the context we currently deal with.) We may construct Kuranishi neighborhoods and obstruction bundles that carry T^n actions induced by the T^n action on X, and choose T^n-equivariant Kuranishi maps. We note that the evaluation map

$$ev : \mathcal{M}_1(L(u),\beta) \to L(u)$$

is T^n-equivariant. Since the complex structure of X is T^n-invariant and $L(u)$ is a free T^n-orbit it is easy to find such a Kuranishi structure.

We remark that the T^n action on the Kuranishi neighborhood is free since the T^n action on $L(u)$ is free and ev is T^n equivariant. We take a perturbation (that is, a multisection) of the Kuranishi map that is T^n equivariant. We can find such a multisection which is also transversal to 0 as follows : Since the T^n action is free, we can take the quotient of Kuranishi neighborhood, obstruction bundle etc. to obtain a space with Kuranishi structure. Then we take a transversal multisection of the quotient Kuranishi structure and lift it to a multisection of the Kuranishi neighborhood of $\mathcal{M}_1(L(u),\beta)$. Let $\mathfrak{s}_\beta$ be such a multisection and let $\mathcal{M}_1(L(u),\beta)^{\mathfrak{s}_\beta}$ be its zero set. We remark that the evaluation map

$$ev : \mathcal{M}_1(L(u),\beta)^{\mathfrak{s}_\beta} \to L(u) \tag{10.3}$$

is a submersion. This follows from the T^n equivariance. This makes our construction of system of multisections much simpler than the general one in section 30 [FOOO2] since the fiber product appearing in the inductive construction is automatically transversal. (See section 30.2 [FOOO2] for the reason why this is crucial.) More precisely we prove the following Lemma 10.2. Let

$$\mathfrak{forget}_0 : \mathcal{M}^{\text{main}}_{k+1}(L(u),\beta) \to \mathcal{M}^{\text{main}}_{1}(L(u),\beta) \tag{10.4}$$

be the forgetful map which forgets the first, ..., k-th marked points. (In other words, only the 0-th marked point remains.) We can construct our Kuranishi structure so that it is compatible with $\mathfrak{forget}_0$ in the same sense as Lemma 31.8 [FOOO2].

Lemma 10.2. *There exists a system of multisections $\mathfrak{s}_{\beta,k+1}$ on $\mathcal{M}^{\text{main}}_{k+1}(L(u),\beta)$ with the following properties :*

1. *They are transversal to* 0.
2. *They are invariant under the T^n action.*
3. *The multisection $\mathfrak{s}_{\beta,k+1}$ is the pull-back of the multisection $\mathfrak{s}_{\beta,1}$ by the forgetful map* (10.4).
4. *The restriction of $\mathfrak{s}_{\beta,1}$ to the boundary of $\mathcal{M}^{\text{main}}_{1}(L(u),\beta)$ is the fiber product of the multisections $\mathfrak{s}_{\beta',k'}$ with respect to the identification of the boundary given in Proposition* 29.2 [FOOO2].
5. *We do not perturb $\mathcal{M}^{\text{main}}_{1}(L(u),\beta_i)$ for $i = 1,\cdots,m$.*

Proof. We construct multisections inductively over $\omega \cap \beta$. Since (2) implies that fiber products of the perturbed moduli spaces which we have already constructed in the earlier stage of induction are automatically transversal, we can extend them so that (1), (2), (3), (4) are satisfied by the method we already explained above.

We recall from Theorem 10.1 (3) that

$$\mathcal{M}_1^{\text{main}}(L(u),\beta_i) = \mathcal{M}_1^{\text{main,reg}}(L(u),\beta_i)$$

and it is Fredholm regular and its evaluation map is surjective to $L(u)$. Therefore when we perturb the multisection we do not need to worry about compatibility of it with other multisections we have already constructed in the earlier stage of induction. This enable us to leave the moduli space $\mathcal{M}_1^{\text{main}}(L(u),\beta_i)$ unperturbed for all β_i. The proof of Lemma 10.2 is complete. □

Remark 10.3. Actually we need to stop the inductive construction of the multisections at some finite stage, by the reason we explained in section 30.3 [FOOO2]. However we can go around this trouble in the same way as explained in section 30 [FOOO2]. So we will not mention this point any more. (Alternatively we can stop the construction at some finite stage and calculate the potential function only modulo T^N. This will suffice for most of the applications.) In the Fano case, we do not need to study this point seriously since there are only finitely many moduli spaces involved in that case.

Remark 10.4. We explain one delicate point of the proof of Lemma 10.2. Let $\alpha \in \pi_2(X)$ be represented by a holomorphic sphere with $c_1(X) \cap \alpha < 0$. We consider the moduli space $\mathcal{M}_1(\alpha)$ of holomorphic sphere with one marked point and in homology class α. Let us consider $\beta \in \pi_2(X; L(u))$ and the moduli space $\mathcal{M}_{1,k+1}(\beta)$ of holomorphic discs with one interior and $k+1$ exterior marked points and of homotopy class β. The fiber product

$$\mathcal{M}_1(\alpha) \times_X \mathcal{M}_{1,k+1}(\beta)$$

taken by the evaluation maps at interior marked points are contained in $\mathcal{M}_{1,k+1}(\beta+\alpha)$. If we want to define a multisection compatible with the embedding

$$\mathcal{M}_1(\alpha) \times_X \mathcal{M}_{1,k+1}(\beta) \subset \mathcal{M}_{1,k+1}(\beta+\alpha) \tag{10.5}$$

then it is impossible to make it both transversal and T^n equivariant in general : This is because the nodal point of such a singular curve could be contained in the part of X with non-trivial isotropy group.

Our perturbation constructed above satisfies (1) and (2) of Lemma 10.2 and so may *not* be compatible with the embedding (10.5). Our construction of the perturbation given in Lemma 10.2 exploits the fact that the T^n action acts freely on the Lagrangian fiber $L(u)$ and carried out by induction on the number of *disc* components (and of energy) only, regardless of the number of sphere components.

The following corollary is an immediate consequence of Lemma 10.2.

Corollary 10.5. *If $\mu(\beta) < 0$ or $\mu(\beta) = 0$, $\beta \neq 0$, then $\mathcal{M}_1^{\text{main}}(L(u),\beta)^{s_\beta}$ is empty.*

We now use our perturbed moduli space to define de Rham version $(\Omega(L(u)), \mathfrak{m}_{k,\beta}^{dR})$ of the filtered A_∞ algebra in the same way as section 37.4 [FOOO2] as follows.

We consider the evaluation map

$$ev = (ev_1, \cdots, ev_k, ev_0) : \mathcal{M}_{k+1}^{\text{main}}(L(u),\beta)^{\mathfrak{s}_\beta} \to L(u)^{k+1}.$$

Let $\rho_1, \cdots, \rho_k$ be differential forms on $L(u)$. We define

$$\mathfrak{m}_{k,\beta}^{dR}(\rho_1, \cdots, \rho_k) = (ev_0)!(ev_1, \cdots, ev_k)^*(\rho_1 \wedge \cdots \wedge \rho_k). \tag{10.6}$$

We remark that integration along fiber $(ev_0)!$ is well defined and gives a smooth form, since ev_0 is a submersion. (It is a consequence of T^n equivariance.) Using the compatibility Lemma 10.2 (4) we can prove that (10.6) defines a filtered A_∞ structure.

We next go to the canonical model $(H(L(u);\Lambda_0), \mathfrak{m}_k^{dR,can})$ of $(\Omega(L(u)), \mathfrak{m}_{k,\beta}^{dR})$. (See section 23 [FOOO2].)

Lemma 10.6. *For $b \in H^1(L(u),\Lambda_0)$, we have*

$$\mathfrak{m}_{k,\beta_i}^{dR,can}(b,\cdots,b) = \frac{1}{k!}(\partial\beta_i \cap b)^k \cdot PD([L(u)]).$$

where $PD([L(u)])$ is the Poincaré dual to the fundamental class $[L(u)] \in H_n(L(u);\mathbb{Z})$.

Proof. Let $[\rho]$ be a harmonic representative of b. Then using Lemma 10.2 and (10.6), we can prove

$$\int_{L(u)} \mathfrak{m}_{k,\beta_i}^{dR}(\rho,\cdots,\rho) = \frac{1}{k!}(\partial\beta_i \cap b)^k \tag{10.7}$$

in the same way as the proof of Lemma 37.47 [FOOO2]. We recall this calculation briefly now. Let

$$C_k = \{(t_1,\cdots,t_k) \mid 0 \le t_1 \le \cdots \le t_k \le 1\}. \tag{10.8}$$

We identify $S^1 \cong \mathbb{R}/\mathbb{Z} \cong \partial D^2$. Lemma 10.2 (2) implies that

$$\mathcal{M}_{k+1}^{\text{main}}(L(u),\beta) \cong \mathcal{M}_1^{\text{main}}(L(u),\beta) \times C_k. \tag{10.9}$$

Moreover the evaluation map ev is

$$ev_i(\mathfrak{u};t_1,\cdots,t_k) = [t_i\partial\beta]\cdot ev(\mathfrak{u}). \tag{10.10}$$

Here $\partial\beta \in H_1(L(u);\mathbb{Z})$ is identified to an element of the universal cover $\widetilde{L}(u) \cong \mathbb{R}^n$ of $L(u)$ and $[t_i\partial\beta] \in L(u)$ acts as a multiplication on the torus. $ev(\mathfrak{u})$ is defined by the evaluation map $ev : \mathcal{M}_1^{\text{main}}(L(u),\beta) \to L(u)$. We also have :

$$ev_0(\mathfrak{u};t_1,\cdots,t_k) = ev(\mathfrak{u}). \tag{10.11}$$

We remark that $ev : \mathcal{M}_1^{\text{main}}(L(u),\beta_i) \to L(u)$ is a diffeomorphism (See Theorem 10.1 (3)). Now we have

$$\int_{L(u)} \mathfrak{m}_{k,\beta_i}^{dR}(\rho,\cdots,\rho) = \text{Vol}(C_k)\left(\int_{\partial\beta_i}\rho\right)^k = \frac{1}{k!}(\partial\beta_i \cap b)^k,$$

as required.

We next use (10.7) to calculate operations in the canonical model. According to the construction of section 23.4 [FOOO2], we have

$$\mathfrak{m}_{k,\beta_i}^{can}(b,\cdots,b) = \sum_\Gamma \mathfrak{m}_\Gamma(\rho,\cdots,\rho)$$

where Γ runs on a set of trees with $k+1$ exterior vertices and each of its interior vertex v is assigned with $\beta(v) \in \pi_2(X)$ such that $\mathcal{M}_1^{\text{main}}(L(u),\beta(v))$ is nonempty and the sum $\sum\beta(v)$ is β_i. It is then easy to see that only the following tree $\Gamma_{k,0}(\beta_i)$ gives a nontrivial contribution : Here $\Gamma_{k,0}(\beta_i)$ is a tree with $k+1$ exterior vertices and only one interior vertex to which β_i is assigned. Then we obtain

$$\mathfrak{m}_{\Gamma_{k,0}(\beta_i)}(\rho,\cdots,\rho) = \mathfrak{m}_{k,\beta_i}^{dR}(\rho,\cdots,\rho) = \frac{1}{k!}(\partial\beta_i \cap b)^k \cdot PD[L(u)]$$

from (10.7). The proof of Lemma 10.6 is now complete. ∎

The proof of the following lemma is the same as that of Lemma 37.54 [FOOO2] and is omitted here. We refer readers thereto for the details.

Lemma 10.7. $(\Omega(L(u)), \mathfrak{m}^{dR}_{k,\beta})$ *is homotopy equivalent to the filtered* A_∞ *algebra in* [FOOO2] *Theorem* A.

In fact we do not need to use Lemma 10.7 to prove Theorem 1.5 if we use de Rham version in all the steps of the proof of Theorem 1.5 without involving the singular homology version.

Proof of Proposition 3.2. Proposition 3.2 immediately follows from Corollary 10.5, Lemma 10.6 and Lemma 10.7 : We just take the sum

$$\begin{aligned}\sum_{k=0}^{\infty} \mathfrak{m}_k^{can}(b,\cdots,b) &= \sum_{k=0}^{\infty} \sum_{\beta\in\pi_2(X,L(u))} T^{\omega\cap\beta/2\pi}\mathfrak{m}^{can}_{k,\beta}(b,\cdots,b) \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{m} T^{\omega\cap\beta_i/2\pi}\mathfrak{m}^{can}_{k,\beta_i}(b,\cdots,b) \\ &= \sum_{i=1}^{m}\sum_{k=0}^{\infty} \frac{1}{k!}(\partial\beta_i\cap b)^k T^{\ell_i(u)}\cdot PD([L(u)]) \qquad (10.12)\end{aligned}$$

Since b is assumed to lie in $H^1(L(u),\Lambda_+)$ not just in $H^1(L(u),\Lambda_0)$, the series appearing as the scalar factor in (10.12) converges in non-Archimedean topology of Λ_0 and so the sum $\sum_{k=0}^{\infty}\mathfrak{m}_k^{can}(b,\cdots,b)$ is a multiple of $PD([L(u)])$. Hence $b\in\widetilde{\mathcal{M}}^{weak}(L(u))$ by definition (3.1). We remark that the gauge equivalence relation in Chapter 4 [FOOO2] is trivial on $H^1(L(u);\Lambda_0)$ and so $H^1(L(u);\Lambda_+)\hookrightarrow \mathcal{M}^{weak}(L(u))$. □

Proof of Theorem 3.4. Suppose that there is no nontrivial holomorphic sphere whose Maslov index is nonpositive. Then Theorem 10.1 (5) implies that if $\mu(\beta)\le 2$, $\beta\neq\beta_i$, $\beta\neq 0$ then $\mathcal{M}_1^{\rm main}(L(u),\beta)$ is empty. Therefore again by dimension counting as in Corollary 10.5, we obtain

$$\sum_{k=0}^{\infty}\mathfrak{m}_k^{dR,can}(x,\cdots,x) = \sum_{i=1}^{m}\sum_{k=0}^{\infty} T^{\omega\cap\beta_i/2\pi}\mathfrak{m}^{dR,can}_{k,\beta_i}(x,\cdots,x)$$

for $x\in H^1(L(u),\Lambda_+)$. On the other hand, we obtain

$$\begin{aligned}\mathfrak{PO}(x;u) &= \sum_{i=1}^{m}\sum_{k=0}^{\infty}\frac{1}{k!}(\partial\beta_i\cap x)^k T^{\ell_i(u)} \\ &= \sum_{i=1}^{m}\sum_{k=0}^{\infty}\frac{1}{k!}\langle v_i,x\rangle^k T^{\ell_i(u)} = \sum_{i=1}^{m} e^{\langle v_i,x\rangle}T^{\ell_i(u)}\end{aligned}$$

from (5.8), (10.12) and the definition of $\mathfrak{PO}$. Writing $x=\sum_{i=1}^n x_i\mathbf{e}_i$ and recalling $y_i = e^{x_i}$, we obtain $e^{\langle v_i,x\rangle} = y_1^{v_{i,1}}\cdots y_n^{v_{i,n}}$ and hence the proof of Theorem 3.4. □

Proof of Theorem 3.5. We consider the contribution of $\beta\neq\beta_i$ with $\mu(\beta)=2$. By Corollary 10.5, the virtual fundamental chain of $\mathcal{M}_1^{\rm main}(L(u),\beta)$ is a cycle. We put

$$c_\beta = ev_*([\mathcal{M}_1^{\rm main}(L(u),\beta)])\in H_n(L(u);\mathbb{Q})\cong\mathbb{Q}^n.$$

In the same way as Lemma 10.6 we can prove

$$\mathfrak{m}^{dR,can}_{k,\beta}(b,\cdots,b) = \frac{c_\beta}{k!}(\partial\beta\cap b)^k PD([L(u)]). \tag{10.13}$$

Theorem 10.1 (5) implies that

$$\partial\beta = \sum k_i\partial\beta_i, \quad \beta = \sum_i k_i\beta_i + \sum_j \alpha_j.$$

Hence

$$\sum_k T^{\beta\cap\omega/2\pi}\mathfrak{m}^{dR,can}_{k,\beta}(b,\cdots,b)$$

becomes one of the terms of the right hand side of (3.6). We remark that class β with $\mu(\beta)\ge 4$ does not contribute to $\mathfrak{m}^{dR,can}_k(b,\cdots,b)$ by the degree reason.

When all the vertices of P lie in $\mathbb{Q}^n$, then the symplectic volume of all α_j are rational. Moreover $\omega\cap\beta_i$ are rational. Therefore the exponents $\beta\cap\omega$ are rational.

The proof of Theorem 3.5 is complete. □

Remark 10.8. We remark that the number c_β in (10.13) is independent of the choice of the system of T^n invariant multisections in Lemma 10.2 : If there are two such systems, we can find a T^n invariant homotopy between them which is also transversal to 0. By a dimension counting argument applied to the parameterized version of $\mathcal{M}^{\rm main}_1(\beta)$ and its perturbation, we will have the parameterized version of Corollary 10.5. This in turn implies that the perturbed (parameterized) moduli space defines a compact cobordism between the perturbed moduli spaces of $\mathcal{M}^{\rm main}_1(\beta)$ associated to the two such systems. This implies the invariance of c_β. It follows that the potential function in Theorem 3.5 is independent of the choice of T^n invariant transversal multisection. However we do not know how to calculate it.

Remark 10.9. We used de Rham cohomology to go around the problem of transversality among chains in the classical cup product. One drawback of this approach is that we lose control of the rational homotopy type. Namely we do *not* prove here that the filtered A_∞ algebra (partially) calculated above is homotopy equivalent to the one in Theorem A [FOOO2] over $\mathbb{Q}$. (Note all the operations we obtain is defined over $\mathbb{Q}$, however.) We however confirm that they are indeed homotopy equivalent over $\mathbb{Q}$. There may be several possible ways to prove this statement, one of which is to use the rational de Rham forms used by Sullivan.

The $\mathbb{Q}$-structure is actually interesting in our situation. See for example Proposition 6.12. However the homotopy equivalence of $\mathbb{Q}$ version of Lemma 10.7 is not used in the statement of Proposition 6.12 or in its proof.

11. Non-unitary flat connection on $L(u)$

In this section we explain how we can include (not necessarily unitary) flat bundles on Lagrangian submanifolds in Lagrangian Floer theory following [Fu2], [Cho].

Remark 11.1. We need to use flat complex line bundle for our purpose by the following reason. In [FOOO2] we assumed that our bounding cochain b is an element of $H(L;\Lambda_+)$ since we want the series

$$\mathfrak{m}^b_1(x) = \sum_{k,\ell}\mathfrak{m}_{k+\ell+1}(b^{\otimes k},x,b^{\otimes \ell})$$

to converge. There we used convergence with respect to the non-Archimedean norm. For the case of T^n orbits in toric manifold, the above series converges for $b \in H^1(L;\Lambda_0)$. The convergence is the usual (classical Archimedean) topology on $\mathbb{C}$ on each coefficient of T^λ.

This is not an accident and was expected to happen in general. (See Conjecture 11.46 [FOOO2].) However for this convergence to occur, we need to choose the perturbations on $\mathcal{M}^{\rm main}_{k+1}(L,\beta)$ so that it is consistent with $\mathcal{M}^{\rm main}_{k'+1}(L,\beta)$ $(k' \ne k)$ via the forgetful map. We can make this choice for the current toric situation by Lemma 10.2 (3). In a more general situation, we need to regard $\mathcal{M}^{\rm main}_{1}(L,\beta)$ as a chain in the free loop space. (See [Fu3].)

On the other hand, if we use the complex structure other than the standard one, we do not know whether Lemma 10.2 (3) holds or not. So in the proof of independence of Floer cohomology under the various choices made, there is a trouble to use a bounding cochain b lying in $H^1(L;\Lambda_0)$. The idea, which is originally due to Cho [Cho] as far as we know, is to change the leading order term of b by twisting the construction using *non-unitary* flat bundles on L.

Let X be a symplectic manifold and L be its relatively spin Lagrangian submanifold. Let $\rho : H_1(L;\mathbb{Z}) \to \mathbb{C} \setminus \{0\}$ be a representation and $\mathfrak{L}_\rho$ be the flat $\mathbb{C}$ bundle induced by ρ.

We replace the formula (10.7) [FOOO2] by

$$\mathfrak{m}^\rho_k = \sum_{\beta \in H_2(M,L)} \rho(\partial\beta)\ \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)/2\pi}.$$

(Compare this with (3.2) in section 3.)

Proposition 11.2. *$(C(L),\mathfrak{m}^\rho_k)$ is a filtered A_∞ algebra.*

Proof. Suppose that $[f] \in \mathcal{M}^{\rm main}_{k+1}(L,\beta)$ is a fiber product of $[f_1] \in \mathcal{M}^{\rm main}_{\ell+1}(L,\beta_1)$ and $[f_2] \in \mathcal{M}^{\rm main}_{k-\ell}(L,\beta_2)$. Namely $\beta_1+\beta_2=\beta$ and $ev_0(f_1) = ev_i(f_2)$. Then it is easy to see that

$$\rho(\partial\beta) = \rho(\partial\beta_1)\rho(\partial\beta_2). \tag{11.1}$$

Combined with this fact, the proof of Theorem 10.11 [FOOO2] goes through and proves Proposition 11.2. In fact it is proved there that

$$\begin{aligned} 0 = &\left(\overline{\mathfrak{m}}_1 \circ \mathfrak{m}_{k,\beta} + \mathfrak{m}_{k,\beta} \circ \widehat{\overline{\mathfrak{m}}}_1\right)(x_1,\cdots,x_k) \\ &+ \sum_{\beta_1+\beta_2=\beta}\ \sum_{k_1+k_2=k+1}\ \sum_{l=1}^{k_1} (-1)^* \mathfrak{m}_{k_1,\beta_1}(x_1,\cdots,\mathfrak{m}_{k_2,\beta_2}(x_l,\cdots),\cdots,x_k) \end{aligned} \tag{11.2}$$

$(* = \deg x_1 + \cdots + \deg x_{l-1} + l.)$ (11.1) and (11.2) imply the filtered A_∞ relation. □

The unitality can also be proved in the same way. The well definedness (that is, independence of various choices up to homotopy equivalence) can also be proved in the same way. In particular we have a canonical model $(H^*(L;\Lambda_0);\mathfrak{m}^{\rho,can}_k)$.

Remark 11.3. We have obtained our twisted filtered A_∞ structure on the (untwisted) cohomology group $H^*(L;\Lambda_0)$. This is because the flat bundle $Hom(\mathfrak{L}_\rho,\mathfrak{L}_\rho)$ is trivial. In more general situation where we consider a flat bundle $\mathfrak{L}$ of higher rank, we obtain a filtered A_∞ structure on cohomology group with local coefficients with values in $Hom(\mathfrak{L},\mathfrak{L})$.

The filtered A_∞ structure $\mathfrak{m}_k^{\rho,can}$ is different from $\mathfrak{m}_k^{can}$ in general as we can see from the expression of the potential function given in Lemma 3.8.

Next we explain the case of a relatively spin pair $(L^{(0)}, L^{(1)})$ of Lagrangian submanifolds of X. We assume that they are of clean intersection. Let $\rho_i : H_1(L^{(i)};\mathbb{Z}) \to \mathbb{C}\setminus\{0\}$ be homomorphisms which induce flat (non-unitary) complex line bundles $\mathfrak{L}_{\rho_i}$ on $L^{(i)}$. Let R_h be a connected component of $L^{(1)}\cap L^{(0)}$. We define the group $\widehat{CF}_{BM}((L^{(1)},\mathfrak{L}_1),(L^{(0)},\mathfrak{L}_0);\Lambda^{\mathbb{C}})$ as the completion of

$$\bigoplus_h \Big(C(R_h; Hom(\mathfrak{L}_{\rho_1}|_{R_h},\mathfrak{L}_{\rho_0}|_{R_h})\otimes \Theta^-_{R_h})\otimes_{\mathbb{Q}}\Lambda_h\Big)\otimes_{\mathbb{C}}\Lambda^{\mathbb{C}},$$

(See [FOOO2] right before Definition 12.71.) We define an equivalence relation $\sim$ similar to Definition 12.52 [FOOO2] to obtain a chain complex

$$CF_{BM}((L^{(1)},\mathfrak{L}_{\rho_1}),(L^{(0)},\mathfrak{L}_{\rho_0});\Lambda^{\mathbb{C}}).$$

We take its energy ≥ 0 part and obtain $CF_{BM}((L^{(1)},\mathfrak{L}_{\rho_1}),(L^{(0)},\mathfrak{L}_{\rho_0});\Lambda_0^{\mathbb{C}})$.

We next modify Definition 12.71 [FOOO2]. Let (S,α) be a singular chain of R_h with $Hom(\mathfrak{L}_{\rho_1}|_{R_h},\mathfrak{L}_{\rho_0}|_{R_h})\otimes\Theta^-_{R_h}$ coefficient. Here $S=(|S|,f)$ be a smooth singular chain and $\alpha\in(Hom(\mathfrak{L}_{\rho_1}|_{R_h},\mathfrak{L}_{\rho_0}|_{R_h})\otimes\Theta^-_{R_h})_{f(x_0)}$, where $x_0\in|S|$ is the base point. By the flat structure, the element α determines an element $\alpha(x)\in(Hom(\mathfrak{L}_{\rho_1}|_{R_h},\mathfrak{L}_{\rho_0}|_{R_h})\otimes\Theta^-_{R_h})_{f(x)}$ for any x in a canonical way. Since we can use $\Theta^-_{R_h}$ to handle the orientation in the same way as section 12 [FOOO2], we consider the case when $\Theta^-_{R_h}$ are all trivial to simplify the notation. Let

$$u:\mathbb{R}\times[0,1]\to X$$

be a map satisfying (12.54) [FOOO2]. We put

$$\lim_{\tau\to-\infty}u(t,\tau)=f(x)\in R_h,\quad \lim_{\tau\to+\infty}u(t,\tau)=y\in R_{h'}$$

We define

$$u_*(\alpha)\in Hom(\mathfrak{L}_{\rho_1}|_{R_{h'}},\mathfrak{L}_{\rho_0}|_{R_{h'}})_y$$

by

$$u_*(\alpha)=\mathrm{Hol}|_{u|_{\tau=0}}\circ\alpha(x)\circ(\mathrm{Hol}|_{u|_{\tau=1}})^{-1}. \tag{11.3}$$

Here

$$\mathrm{Hol}|_{u|_{\tau=0}}:(\mathfrak{L}_{\rho_0})_{f(x)}\to(\mathfrak{L}_{\rho_0})_y$$

and

$$\mathrm{Hol}|_{u|_{\tau=1}}:(\mathfrak{L}_{\rho_1})_{f(x)}\to(\mathfrak{L}_{\rho_1})_y$$

are the parallel transport, with respect to the given flat connection, along the path $\tau\mapsto u(\tau,0)$, $\tau\mapsto u(\tau,1)$, respectively. We use (11.3) to twist Definition 12.71 [FOOO2] and we obtain

$$\begin{aligned}\mathfrak{n}^{\rho_1,\rho_0}_{k_1,k_0}:&B_{k_1}C(L^{(1)},\Lambda_0^C)[1]\otimes CF_{BM}((L^{(1)},\mathfrak{L}_{\rho_1}),(L^{(0)},\mathfrak{L}_{\rho_0});\Lambda_0^{\mathbb{C}})\\ &\otimes B_{k_0}C(L^{(0)},\Lambda_0^C)[1]\to CF_{BM}((L^{(1)},\mathfrak{L}_{\rho_1}),(L^{(0)},\mathfrak{L}_{\rho_0});\Lambda_0^{\mathbb{C}}).\end{aligned}$$

Lemma 11.4. *$(CF_{BM}((L^{(1)},\mathfrak{L}_{\rho_1}),(L^{(0)},\mathfrak{L}_{\rho_0});\Lambda_0^{\mathbb{C}}),\mathfrak{n}^{\rho_1,\rho_0}_{k_1,k_0})$ is a filtered A_∞ bimodule over $(C(L^{(1)}),\mathfrak{m}_k^{\rho_1})$ - $(C(L^{(0)}),\mathfrak{m}_k^{\rho_0})$.*

The proof is the same as the proof of Theorem 27.72 [FOOO2] and is omitted. Various results such as unitality, well definedness etc. for Floer cohomology of pairs of Lagrangian submanifolds are generalized to the twisted version in an obvious way. In particular we have

$$\mathrm{rank}_{\Lambda^{\mathbb{C}}} HF((L^{(1)},\rho_1,b_1),(L^{(0)},\rho_0,b_0);\Lambda^{\mathbb{C}})) \le \#(L^{(1)}\cap L^{(0)})$$

in the same way. Here $b_i \in \mathcal{M}(C(L^{(i)}),\mathfrak{m}_k^{\rho_i})$. In other words we can use the twisted version to study Lagrangian intersection in the same way as the untwisted one.

12. Floer cohomology at a critical point of potential function

In this section we prove Theorem 3.9 etc. and complete the proof of Theorem 1.5.

We first prove Lemma 3.8. Let $\beta \in H_2(X,L(u_0))$ with $\mu(\beta)=2$ and $\mathcal{M}_1^{\mathrm{main}}(L(u_0),\beta)$ be nonempty. We have $\beta = \sum_{i=1}^m c_i\beta_i + \sum_j \alpha_j$ by Theorem 10.1 (5). Let ρ be as in (3.13). We have $\rho(\partial\beta) = \prod \rho(\partial\beta_i)^{c_i}$. Note $\partial\beta_i = \sum_j w_{ij}\mathbf{e}_j$. Thus we have

$$\begin{cases} \rho(\partial\beta_i) = \mathfrak{y}_{1,0}^{w_{i,1}}\cdots\mathfrak{y}_{n,0}^{w_{i,n}}, \\ \rho(\partial\beta) = \displaystyle\prod_i\prod_j \mathfrak{y}_{j,0}^{c_i w_{i,j}}. \end{cases} \tag{12.1}$$

Therefore for $b\in H^1(L(u_0);\Lambda_+^{\mathbb{C}})$ we have

$$\begin{aligned} \sum_{k=0}^{\infty}\mathfrak{m}_{k,\beta_i}^{\rho,can}(b,\cdots,b) &= \sum_{k=0}^{\infty}\mathfrak{y}_{1,0}^{w_{i,1}}\cdots\mathfrak{y}_{n,0}^{w_{i,n}}\mathfrak{m}_{k,\beta_i}^{can}(b,\cdots,b) \\ &= \sum_{k=0}^{\infty} e^{\mathfrak{x}_{1,0}w_{i,1}}\cdots e^{\mathfrak{x}_{n,0}w_{i,n}}\frac{1}{k!}(b\cap\partial\beta_i)^k\cdot[PD(L)] \\ &= \sum_{k=0}^{\infty}\mathfrak{m}_{k,\beta_i}^{can}\left(b+\sum_{j=1}^n\mathfrak{x}_{j,0}\mathbf{e}_j,\cdots,b+\sum_{j=1}^n\mathfrak{x}_{j,0}\mathbf{e}_j\right). \end{aligned}$$

In a similar way we have

$$\sum_{k=0}^{\infty}\mathfrak{m}_{k,\beta}^{\rho,can}(b,\cdots,b) = \sum_{k=0}^{\infty}\mathfrak{m}_{k,\beta}^{can}\left(b+\sum_j\mathfrak{x}_{j,0}\mathbf{e}_j,\cdots,b+\sum_j\mathfrak{x}_{j,0}\mathbf{e}_j\right).$$

However it follows from Theorems 3.4 and 3.5 that the left and the right sides of this identity corresponds to those in Lemma 3.8 respectively. This finishes the proof of Lemma 3.8. □

We next prove Theorem 3.9. Let $x=(x_1,\cdots,x_n)$, $x_1,\cdots,x_n\in\Lambda_+$. We put

$$f_+(x) = \sum_i(\mathfrak{x}_{i,0}+x_i)\mathbf{e}_i, \quad f(x) = \sum_i x_i\mathbf{e}_i.$$

From Lemma 3.8 we derive

$$\begin{aligned} \mathfrak{PO}_\rho^{u_0}(f(x)) &= \sum\mathfrak{m}_k^{\rho,can}(f(x),\cdots,f(x))\cap[L(u_0)] \\ &= \sum\mathfrak{m}_k^{can}(f_+(x),\cdots,f_+(x))\cap[L(u_0)] = \mathfrak{PO}^{u_0}(f_+(x)). \end{aligned}$$

Let b be as in (3.14) and so

$$f_+(b) = \sum_i \mathfrak{x}_i \mathbf{e}_i =: \mathfrak{x}.$$

Then we have

$$\frac{\partial}{\partial x_i}\mathfrak{PO}^{u_0}_{\rho}(f(x))\Big|_{x=b} = \frac{\partial}{\partial x_i}\mathfrak{PO}^{u_0}(f_+(x))\Big|_{x=b} = \frac{\partial \mathfrak{PO}^{u_0}}{\partial x_i}(\mathfrak{x}) = 0$$

where the last equality follows from the definition of b in (3.14). On the other hand, we have

$$\begin{aligned}&\frac{\partial}{\partial x_i}\mathfrak{PO}^{u_0}_{\rho}(f(x))\Big|_{x=b}\\ &= \sum_k \sum_\ell \mathfrak{m}_k^{\rho,can}((f(b))^{\otimes \ell}, \mathbf{e}_i, (f(b))^{\otimes(k-\ell-1)}) \cap [L(u_0)] \qquad (12.2)\\ &= \mathfrak{m}_1^{\rho,can,b}(\mathbf{e}_i) \cap [L(u_0)].\end{aligned}$$

Note here and hereafter we write $\mathfrak{m}_1^{\rho,can,b}$ in place of $\mathfrak{m}_1^{\rho,can,f(b)}$.

Hence we obtain

$$\mathfrak{m}_1^{\rho,can,b}(\mathbf{e}_i)\begin{cases} = 0 & \text{if (3.11) is satisfied} \\ \equiv 0 \mod T^N & \text{if (3.10) is satisfied.}\end{cases} \qquad (12.3)$$

We remark that by the degree reason $\mathfrak{m}_1^{\rho,can,b}(\mathbf{e}_i)$ is proportional to $[L(u_0)]$.

We next prove the vanishing of $\mathfrak{m}_1^{\rho,can,b}(\mathbf{f})$ for the classes $\mathbf{f}$ of higher degree. Namely we prove

Lemma 12.1. *For $\mathbf{f} \in H^*(L(u_0); \Lambda_0^{\mathbb{C}})$ we have :*

$$\mathfrak{m}_{1,\beta}^{\rho,can,b}(\mathbf{f})\begin{cases} = 0 & \textit{if (3.11) is satisfied} \\ \equiv 0 \mod T^N & \textit{if (3.10) is satisfied.}\end{cases}$$

Proof. The proof is similar to the proof of Theorem 27.35 [FOOO2] which uses the spectral sequence argument. Let $d = \deg \mathbf{f}$ and $2\ell = \mu(\beta)$. We say $(d,\ell) < (d',\ell')$ if $\ell < \ell'$ or $\ell = \ell'$, $d < d'$. We prove the lemma by upward induction on (d,ℓ). The case $d = 1$ is (12.3). We remark that $\mathfrak{m}_{k,\beta} = 0$ if $\mu(\beta) \le 0$.

We assume that the lemma is proved for (d',ℓ') smaller than (d,ℓ) and will prove the case of (d,ℓ). Since the case $d = 1$ is already proved, we may assume that $d \ge 2$. Let $\mathbf{f} = \mathbf{f}_1 \cup \mathbf{f}_2$ where $\deg \mathbf{f}_i \ge 1$. By the A_∞-relation we have

$$\begin{aligned}\mathfrak{m}_{1,\beta}^{\text{main},\rho,b}(\mathbf{f}_1 \cup \mathbf{f}_2) = &\sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\text{main},\rho,b}(\mathfrak{m}_{1,\beta_2}^{\text{main},\rho,b}(\mathbf{f}_1), \mathbf{f}_2)\\ &+ \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\text{main},\rho,b}(\mathbf{f}_1, \mathfrak{m}_{1,\beta_2}(\mathbf{f}_2))\\ &+ \sum_{\beta_1+\beta_2=\beta,\beta_2\ne 0} \pm \mathfrak{m}_{1,\beta_1}^{\text{main},\rho,b}(\mathfrak{m}_{2,\beta_2}(\mathbf{f}_1, \mathbf{f}_2)).\end{aligned}$$

We remark that $\mathfrak{m}_{1,\beta_0}^{\text{main},\rho,b} = 0$ since we are working on a canonical model.

The first two terms of the right hand side vanishes by the induction hypothesis since $\deg \mathbf{f}_i < \deg \mathbf{f}$ and $\mu(\beta_i) \le \mu(\beta)$. The third term also vanishes since $\mu(\beta_1) < \mu(\beta)$. The proof of Lemma 12.1 is complete. □

Lemma 12.1 immediately implies Theorem 3.9. □

Proof of Proposition 4.4. Let us specialize to the case of 2 dimension. In case $\dim L(u_0) = 2$, we can prove $\mathfrak{m}^{\rho,can,b}_{1,\beta} = 0$ for $\mu(\beta) \geq 4$ also by dimension counting. We can use that to prove Proposition 4.4 in the same way as above. □

Now we are ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. We first consider the case where the vertices of P are contained in $\mathbb{Q}^n$. Proposition 3.6 and Theorem 3.9 imply that $L(u_0)$ is balanced in the sense of Definition 3.10. Therefore the next lemma implies Theorem 1.5 in our case.

Lemma 12.2. *If $L(u_0)$ is a balanced Lagrangian fiber then the following holds for any Hamiltonian diffeomorphism $\psi : X \to X$.*

$$\psi(L(u_0)) \cap L(u_0) \neq \emptyset. \tag{12.4}$$

If $\psi(L(u_0))$ is transversal to $L(u_0)$ in addition, then

$$\#(\psi(L(u_0)) \cap L(u_0)) \geq 2^n. \tag{12.5}$$

Proof. Let ω_i, u_i, $b_{i,N}$ be as in Definition 3.10. We assume $\psi : X \to X$ does not satisfy (12.4) or (12.5) and will deduce a contradiction. We use the same (time dependent) Hamiltonian as ψ to obtain $\psi_i : (X, \omega_i) \to (X, \omega_i)$. Take an integer N such that $\|\psi_i\| < 2\pi N$ for large i. Then for sufficiently large i, $L(u_0^i)$ and ψ_i does not satisfy (12.4) or (12.5). In fact if $\psi(L(u_0)) \cap L(u_0) = \emptyset$ then for sufficiently large i, we have $\psi_i(L(u_0^i)) \cap L(u_0^i) = \emptyset$. If $\psi(L(u_0))$ is transversal to $L(u_0)$ and if (12.5) is not satisfied, then

$$\#(\psi(L(u_0)) \cap L(u_0)) \geq \#(\psi_{\varepsilon_i}(L(u_0^i)) \cap L(u_0^i)).$$

On the other hand, by Theorem 3.9 we have

$$HF((L(u_i), b_{i,k}), (L(u_i), b_{i,k}); \Lambda_0^{\mathbb{C}}/(T^N)) \cong H(T^n; \Lambda_0^{\mathbb{C}}/(T^N)).$$

It follows from Universal Coefficient Theorem that

$$HF((L(u_i), b_{i,k}), (L(u_i), b_{i,k}); \Lambda_0^{\mathbb{C}}) \cong \Lambda_0^{\oplus a} \oplus \bigoplus_{i=1}^{b} \Lambda_0/(T^{c(i)}) \tag{12.6}$$

such that $c(i) \geq N$ and $a + 2b \geq 2^n$. This contradicts to Theorem J [FOOO2]. (In fact Theorem J [FOOO2] and (12.6) imply that (12.4) and (12.5) hold for $L(u_i)$ and ψ_i with $\|\psi_i\| < 2\pi N$.) Lemma 12.2 is proved. □

If the leading term equation is strongly nondegenerate, Theorem 1.5 also follows from Theorem 3.9, Theorem 9.4 and Lemma 12.2.

We finally present an argument to remove rationality assumption. In view of Lemma 12.2, it suffices to prove the following Proposition 12.3.

Proposition 12.3. *In the situation of Theorem* 1.5*, there exists u_0 such that $L(u_0)$ is a balanced Lagrangian fiber.*

Proof. Let $\pi : X \to P$ be as in Theorem 1.5. Let us consider s_k, S_k, P_k as in section 8. We obtain $u_0 \in P$ such that $\{u_0\} = P_K$. We will prove that $L(u_0)$ is balanced.

We perturb the Kähler form ω of X a bit so that we obtain ω'. Let P' be the corresponding moment polytope and $s_k^{\omega'}$, $S_k^{\omega'}$, $P_k^{\omega'}$, be the corresponding piecewise affine function, number, subset of $P_{\omega'}$ obtained for ω', $P_{\omega'}$ as in section 8.

Proposition 12.4. *We can choose ω_h so that ω_h is rational and $\lim_{h\to\infty} s_k^{\omega_h} = s_k$, $\lim_{h\to\infty} S_k^{\omega_h} = S_k$, $\lim_{h\to\infty} P_k^{\omega_h} = P_k$, $\dim P_k^{\omega_h} = \dim P_k$.*

Proof. We write $I_k^{\omega'}$ the set I_k defined in (8.6) section 8 for ω', P'. We will prove the following lemma. We remark that the set $\mathfrak{K}$ of T^n invariant Kähler structure ω' is regarded as an open set of an affine space defined on $\mathbb{Q}$ (that is the Kähler cone). We may regard $\mathfrak{K}$ as a moduli space of moment polytope as follows : We consider a polyhedron P' each of whose edges is parallel to a corresponding edge of P. Translation defines an $\mathbb{R}^n$ action on the set of such P'. The quotient space can be identified with $\mathfrak{K}$.

Lemma 12.5. *There exists a subset $\mathfrak{P}_k$ of $\mathfrak{K}$ which is a nonempty open subset of an affine subspace defined over $\mathbb{Q}$ such that any element $\omega' \in \mathfrak{P}_k$ has the following properties :*

(1) $\dim P_l^{\omega'} = \dim P_l^{\omega}$ *for* $l \le k$.
(2) $I_l^{\omega'} = I_l^{\omega}$ *for* $l \le k$.

Remark 12.6. In the case of Example 7.1, the set $P_k^{\omega'}$ etc. jumps at the point $\alpha = 1/3$ in the Kähler cone. Hence the set $\mathfrak{P}_k$ may have strictly smaller dimension than $\mathfrak{K}$.

Proof. Let $A_l^{\omega'}$ be the affine space defined in section 8. (We put ω' to specify the symplectic form.) We write ℓ_i^{ω}, $\ell_i^{\omega'}$ in place of ℓ_i to specify symplectic form and moment polytope. We remark that the linear part of ℓ_i^{ω} is equal to the linear part of $\ell_i^{\omega'}$.

The proof of Lemma 12.5 is by induction on k. Let us first consider the case $k = 1$. We put

$$\widehat{A}_1^{\omega'} = \{u \in M_{\mathbb{R}} \mid \ell_{1,1}^{\omega'}(u) = \cdots = \ell_{1,a_1}^{\omega'}(u)\}.$$

We remark that $\{\ell_{1,1}^{\omega}, \cdots, \ell_{1,a_1}^{\omega}\} = I_1^{\omega}$ and so $\widehat{A}_1^{\omega} = A_1^{\omega}$.

We put

$$\mathfrak{P}_1' = \{\omega' \mid \dim \widehat{A}_1^{\omega'} = \dim A_1^{\omega}\}.$$

It is easy to see that $\mathfrak{P}_1'$ is a nonempty affine subset of $\mathfrak{K}$ and is defined over $\mathbb{Q}$.

Sublemma 12.7. *If $\omega' \in \mathfrak{P}_1'$ and is sufficiently close to ω, then $P_1^{\omega'}$ is an equi-dimensional polyhedron in $\widehat{A}_1^{\omega'}$. In particular $\widehat{A}_1^{\omega'} = A_1^{\omega'}$.*

Proof. The tangent space of $\widehat{A}_1^{\omega'}$ is parallel to the tangent space of A_1^{ω}. Therefore $\ell_{1,j}^{\omega'}$ is constant on $\widehat{A}_1^{\omega'}$. We put

$$\widehat{S}_1^{\omega'} = \ell_{1,1}^{\omega'}(u)$$

for some $u \in \widehat{A}_1^{\omega'}$.

On the other hand, if $\ell_i^{\omega} \notin I_1^{\omega}$ then $\ell_i^{\omega}(u) > S_1^{\omega}$ on Inf P_1^{ω}. Therefore if ω' is sufficiently close to ω we have $\ell_i^{\omega'}(u) > \widehat{S}_1^{\omega'}$ on a neighborhood of a compact subset of Int P_1^{ω}, which we identify with a subset of P'. This implies the sublemma. □

The Condition (1), (2) of Lemma 12.4 in the case $k = 1$ follows from Sublemma 12.7 easily.

Let us assume that Lemma 12.5 is proved up to $k-1$. We remark $\{\ell_{k,1}^{\omega}, \cdots, \ell_{k,a_k}^{\omega}\} = I_k^{\omega}$. We put

$$\widehat{A}_k^{\omega'} = \{u \in A_{k-1}^{\omega'} \mid \ell_{k,1}^{\omega'}(u) = \cdots = \ell_{k,a_k}^{\omega'}(u)\}.$$

and

$$\mathfrak{P}'_k = \{\omega' \in \mathfrak{P}'_{k-1} \mid \dim \widehat{A}_k^{\omega'} = \dim A_k^{\omega}\}.$$

$\mathfrak{P}'_k$ is a nonempty affine subset of $\mathfrak{K}$ and is defined over $\mathbb{Q}$. We can show that a sufficiently small open neighborhood $\mathfrak{P}_k$ of ω in $\mathfrak{P}'_k$ has the required properties in the same way as the first step of the induction. The proof of Lemma 12.5 is complete. □

Proposition 12.4 follows immediately from Lemma 12.5. In fact the set of rational points are dense in $\mathfrak{P}_K$. □

Proposition 12.3 follows from Proposition 12.4, Proposition 3.6 and Theorem 3.9. □

The proof of Theorem 1.5 is now complete. □

Proof of Proposition 9.7. The proof is similar to the proof of Proposition 12.4. Let I_k be as in (9.2). We write it as $I_k(P, u_0)$, where P is the moment polytope of (X, ω). We define $I_k(P', u'_0)$ as follows.

Let P' be a polytope which is a small perturbation of P and such that each of the faces are parallel to the corresponding face of P. Let $u'_0 \in \mathrm{Int}\, P'$. Let us consider the set $\mathfrak{K}^+$ of all such pairs (P', u'_0). It is an open set of an affine space defined over $\mathbb{Q}$. Each of such P' is a moment polytope of certain Kähler form on X. (We remark that Kähler form on X determine P' only up to translation.)

For each P', we take corresponding Kähler form on X and it determines a potential function. Therefore $I_k(P', u'_0)$ is determined by (9.2). We define $A_l^{\perp}(P', u'_0)$ in the same way as Definition 9.1.

Lemma 12.8. *There exists a subset $\mathfrak{Q}_k$ of $\mathfrak{K}^+$ which is a nonempty open set of an affine subspace defined over $\mathbb{Q}$. All the elements (P', u'_0) of $\mathfrak{Q}_k$ have the following properties.*

(1) $\dim A_l^{\perp}(P', u'_0) = \dim A_l^{\perp}(P, u_0)$ *for* $l \le k$.
(2) $I_l(P', u'_0) = I_l(P, u_0)$ *for* $l \le k$.

The proof is the same as the proof of Lemma 12.5 and is omitted.

Now we take a sequence of rational points $(P_h, u_0^h) \in \mathfrak{Q}_k$ converging to (P, u_0). Lemma 12.8 (2) implies that the leading term equation at u_0^h is the same as the leading term equation at u_0. The proof of Proposition 9.7 is complete. □

Remark 12.9. We say that $L(u_0)$ is *strongly balanced* if there exists $b \in H^1(L; \Lambda_0^{\mathbb{C}})$ such that $HF(L(u_0), b), (L(u_0), b); \Lambda_0^{\mathbb{C}}) \cong H(T^n; \mathbb{Q}) \otimes \Lambda_0^{\mathbb{C}}$.

Clearly strongly balanced implies balanced. We conjecture that the converse is true.

Remark 12.10. We can replace Definition 3.10 (3) by

$$HF((L(u_i), b_{i,N}), (L(u_i), b_{i,N}); \Lambda^{\mathbb{C}}/(T^N)) \supseteq \Lambda^{\mathbb{C}}/(T^N).$$

In fact the following three conditions are equivalent to one another :

(1) $HF((L(u), b), (L(u), b); \Lambda^{\mathbb{C}}/(T^N)) \cong H(T^n; \mathbb{C}) \otimes \Lambda^{\mathbb{C}}/(T^N)$.
(2) $HF((L(u), b), (L(u), b); \Lambda^{\mathbb{C}}/(T^N)) \supseteq \Lambda^{\mathbb{C}}/(T^N)$.
(3) $\frac{\partial \mathfrak{PO}^u}{\partial y_k} \equiv 0, \mod T^N \qquad k = 1, \cdots, n$, at b.

(1) $\Rightarrow$ (2) is obvious. (3) $\Rightarrow$ (1) is Theorem 3.9. Let us prove (2) $\Rightarrow$ (3). Suppose (3) does not hold. We put $\frac{\partial \mathfrak{PO}^u}{\partial y_k} \equiv cT^{\lambda} \mod T^{\lambda}\Lambda_+$, where $c \in \mathbb{C}\setminus\{0\}$ and $0 \le \lambda < N$. Then (12.2) implies $T^{N-\lambda}PD[L(u)] = 0$ in $HF((L(u),b),(L(u),b);\Lambda^{\mathbb{C}}/(T^N))$. Since $PD[L(u)]$ is a unit, (2) does not hold.

Remark 12.11. The proof of Theorem 3.9 (or of Lemma 12.1) does *not* seem to imply

$$\mathfrak{m}_{k,\beta}(\rho_1,\cdots,\rho_k) = 0 \tag{12.7}$$

for $\mu(\beta) \ge 4$. We however believe that (12.7) holds in our situation. In fact the homology group $H(\mathcal{L}(T^n);\mathbb{Q})$ of the free loop space $\mathcal{L}(T^n)$ is trivial of degree $> n$. On the other hand, $\dim \mathcal{M}_1^{\text{main}}(L(u_0);\beta) = n + \mu(\beta) - 2$. Hence if $\mu(\beta) \ge 4$ there is no nonzero homology class on the corresponding degree in the free loop space. Using the argument of [Fu3] it may imply that those classes do not contribute to the filtered A_∞ structure. If (12.7) holds, then we can find the filtered A_∞ structure of our Lagrangian torus (the fiber of toric manifolds) by a direct calculation.

On the other hand, pseudo-holomorphic disc with Maslov index ≥ 4 certainly contributes to the operator $\mathfrak{q}_{\ell,k,\beta}$ introduced in section 13 [FOOO2] : Namely $\mathfrak{q}_{\ell,k,\beta}$ is the operator that involves a cohomology class of the ambient symplectic manifold X. (See Remark 5.13) It seems that tropical geometry will play a role in this calculation.

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